0/1-Polytopes related to Latin squares autotopisms.*

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Abstract

The set $LS(n)$ of Latin squares of order n can be represented in $\mathbb{R}^{n^3}$ as a $(n-1)^3$-dimensional 0/1-polytope. Given an autotopism $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{A}_n$, we study in this paper the 0/1-polytope related to the subset of $LS(n)$ having Θ in their autotopism group. Specifically, we prove that this polyhedral structure is generated by a polytope in $\mathbb{R}^{((\mathbf{n}_\alpha-\mathbf{l}_\alpha^1)\cdot n^2+\mathbf{l}_\alpha^1\cdot\mathbf{n}_\beta\cdot n)-(\mathbf{l}_\alpha^1\cdot\mathbf{l}_\beta^1\cdot(n-\mathbf{l}_\gamma^1)+\mathbf{l}_\alpha^1\cdot\mathbf{l}_\gamma^1\cdot(\mathbf{n}_\beta-\mathbf{l}_\beta^1)+\mathbf{l}_\beta^1\cdot\mathbf{l}_\gamma^1\cdot(\mathbf{n}_\alpha-\mathbf{l}_\alpha^1))}$, where $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$ are the number of cycles of α and β, respectively, and $\mathbf{l}_\delta^1$ is the number of fixed points of δ, for all $\delta \in \{\alpha, \beta, \gamma\}$. Moreover, we study the dimension of these two polytopes for Latin squares of order up to 9.

Key words: 0/1-polytope, Latin Square, Autotopism group.

1 Introduction

A 0/1-*polytope* [9] in $\mathbb{R}^d$ is the convex hull $\mathcal{P}$ of a finite set of points with 0/1-coordinates. Equivalently, it is a polytope with all its vertices in the vertex set of the unit cube $C_d = [0,1]^d$. Thus, if we consider these vertices as the column vectors of a matrix $V \in \{0,1\}^{d\times n}$, it is verified that $\mathcal{P} = \mathcal{P}(V) = conv(V) = \{V\cdot(x_1, x_2, ..., x_n)^t \mid x_i \geq 0, \forall i \in [n]$ and $\sum_{i\in[n]} x_i = 1\}$, where $[n]$ will denote from now on the set $\{1, 2, ..., n\}$. The *dimension* of $\mathcal{P}$ is the maximum number of affinely independent points in $\mathcal{P}$ minus 1. Permuting coordinates and *switching* (replacing x_i by $1 - x_i$) coordinates transform 0/1-polytopes into 0/1-polytopes. Two 0/1-polytopes are said to be 0/1-equivalent if there exists a sequence of the two previous operations transforming one of them into the other one. In combinatorial optimization there are several examples of 0/1-polytopes like the salesman polytope [8], the cut polytope [2] or the Latin square polytope [3]. In this paper, we are interested in the last one, which appears in the *3-dimensional planar assignment problem* ($3PAP_n$):

$$\min \sum_{i\in I, j\in J, k\in K} w_{ijk}\cdot x_{ijk},\ s.t. \begin{cases} \sum_{i\in I} x_{ijk} = 1, \forall j \in J, k \in K. & (1.1)\\ \sum_{j\in J} x_{ijk} = 1, \forall i \in I, k \in K. & (1.2)\\ \sum_{k\in K} x_{ijk} = 1, \forall i \in I, j \in J. & (1.3)\\ x_{ijk} \in \{0,1\}, \forall i \in I, j \in J, k \in K. & (1.4)\end{cases} \quad (1)$$

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where w_{ijk} are real weights and I, J, K are three disjoint n-sets.

Euler et al. [3] observed that there exists a 1-1 correspondence between the set $LS(n)$ of Latin squares of order n and the set $FS(n)$ of feasible solutions of the $3PAP_n$. Specifically, a *Latin square* L of order n is an $n \times n$ array with elements chosen from a set of n distinct symbols such that each symbol occurs precisely once in each row and each column. From now on, we will assume $[n]$ as this set of symbols. Given $L = (l_{i,j}) \in LS(n)$, the *orthogonal array representation of* L is the set of n^2 triples $\{(i, j, l_{i,j}) \mid i, j \in [n]\}$. So, by taking $I = J = K = [n]$ and by considering the lexicographical order in $I \times J \times K$, it can be defined the 1-1 correspondence $\Phi : LS(n) \to FS(n) \subseteq \mathbb{R}^{n^3}$, such that, given $L = (l_{i,j}) \in LS(n)$, it is $\Phi(L) = (x_{111}, x_{112}, ..., x_{1nn}, x_{211}, ..., x_{nnn})$, where $x_{ijk} = \begin{cases} 1, & \text{if } l_{i,j} = k, \\ 0, & \text{otherwise.} \end{cases}$. Moreover, if A is the constraint matrix of the system of equations (1), it is defined the *Latin square polytope*, $\mathcal{P}_{LS(n)} = conv\{FS(n)\} = conv\{\mathbf{x} \in \{0,1\}^{n^3} \mid A \cdot \mathbf{x} = \mathbf{e}\}$, where $\mathbf{e} = (1, ..., 1)^t$ with $3 \cdot n^2$ entries. Thus, every point of $\mathcal{P}_{LS} \cap C_{n^3}$ is a Latin square of order n and vice versa. By obtaining the minimal equation system for P_{LS}, Euler et al. proved that this polytope is $(n-1)^3$-dimensional and they gave some general results about its facial structure.

In this paper, we are interested in obtaining a similar construction than the above one, in the case of adding some extra conditions to the $3PAP_n$. Specifically, we want to study those 0/1-polytopes related to Latin squares having some symmetrical restrictions. To expose the problem, some previous considerations are needed: The permutation group on $[n]$ is denoted by S_n. Every permutation $\delta \in S_n$ can be uniquely written as a composition of pairwise disjoint cycles, $\delta = C_1^\delta \circ C_2^\delta \circ ... \circ C_{\mathbf{n}_\delta}^\delta$, where for all $i \in [\mathbf{n}_\delta]$, one has $C_i^\delta = \left(c_{i,1}^\delta \; c_{i,2}^\delta \; \cdots \; c_{i,\;\lambda_i^\delta}^\delta\right)$, with $c_{i,1}^\delta = \min_j\{c_{i,j}^\delta\}$. The *cycle structure of* δ is the sequence $\mathbf{l}_\delta = (\mathbf{l}_1^\delta, \mathbf{l}_2^\delta, ..., \mathbf{l}_n^\delta)$, where $\mathbf{l}_i^\delta$ is the number of cycles of length i in δ, for all $i \in [n]$. Thus, $\mathbf{l}_1^\delta$ is the cardinal of the set of *fixed points* of δ, $Fix(\delta) = \{i \in [n] \mid \delta(i) = i\}$. An *isotopism* of a Latin square $L = (l_{i,j}) \in LS(n)$ is a triple $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_n = S_n \times S_n \times S_n$. In this way, α, β and γ are permutations of rows, columns and symbols of L, respectively. The resulting square $L^\Theta = \{(\alpha(i), \beta(j), \gamma(l_{i,j})) \mid i, j \in [n]\}$ is also a Latin square. The *cycle structure* of Θ is the triple $(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$.

An isotopism which maps L to itself is an *autotopism*. The stabilizer subgroup of L in $\mathcal{I}_n$ is its *autotopism group*, $\mathfrak{A}(L) = \{\Theta \in \mathcal{I}_n \mid L^\Theta = L\}$. The set of all autotopisms of Latin squares of order n is denoted by $\mathfrak{A}_n$. Given $\Theta \in \mathfrak{A}_n$, the set of all Latin squares L such that $\Theta \in \mathfrak{A}(L)$ is denoted by $LS(\Theta)$ and the cardinality of $LS(\Theta)$ is denoted by $\Delta(\Theta)$. Specifically, if Θ_1 and Θ_2 are two autotopisms with the same cycle structure, then $\Delta(\Theta_1) = \Delta(\Theta_2)$. The possible cycle structures of the set of non-trivial autotopisms of Latin squares of order up to 11 were obtained in [4].

Gröbner bases were used in [5] to describe an algorithm that allows one to obtain the number $\Delta(\Theta)$ in a computational way. This algorithm was implemented in SINGULAR [7] to get the number of Latin squares of order up to 7 related to any autotopism of a given cycle structure. Specifically, the authors followed the ideas implemented by Bayer [1] to solve the problem of an n-colouring a graph, since every Latin square of order n is equivalent to an n-coloured bipartite graph $K_{n,n}$. More recently, Falcón and Martín-Morales [6] have studied the case $n > 7$ by implementing in a new algorithm the 1-1 correspondence between the $3PAP_n$ and the set $LS(n)$. As an immediate consequence, the set of vertices of C_{n^3} related to $LS(\Theta)$ can be obtained.

In Section 2, given $\Theta \in \mathfrak{A}_n$, we study the set of constraints which can be added to the $3PAP_n$ to get a set of feasible solutions equivalent to the set $LS(\Theta)$. In Section 3, we define the 0/1-polytope in $\mathbb{R}^{n^3}$ related to $LS(\Theta)$. Moreover, we prove the existence of a 0/1-subpolytope of the previous one which can generate it. We see that these two polytopes do not depend on the autotopism Θ but on the cycle structure of the autotopism. Finally, we study the dimensions of these polytopes and we give a classification for polytopes related to autotopisms of Latin squares of order up to 9.

2 Constraints related to a Latin square autotopism

Given a autotopism $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{A}_n$, let $(1)_\Theta$ be the set of constraints obtained by adding to (1) the n^3 constraints:

$$x_{ijk} = x_{\alpha(i)\beta(j)\gamma(k)}, \forall i \in I, j \in J, k \in K. \qquad (1.5)_\Theta$$

The following results hold:

Theorem 2.1 *There exists a 1-1 correspondence between $LS(\Theta)$ and the set $FS(\Theta)$ of feasible solutions related to a combinatorial optimization problem having $(1)_\Theta$ as the set of constraints.*

Proof. It is enough to consider the restriction to $LS(\Theta)$ of the correspondence Φ between $LS(n)$ and $FS(n)$, because then, given $L = (l_{i,j}) \in LS(n)$, it is verified that $L \in LS(\Theta)$ if and only if, for all $i, j, k \in [n]$: $l_{i,j} = k \Leftrightarrow l_{\alpha(i),\beta(j)} = \gamma(k)$. But this last condition is equivalent to say that $x_{ijk} = 1$ if and only if $x_{\alpha(i)\beta(j)\gamma(k)} = 1$. That is to say, $x_{ijk} = x_{\alpha(i)\beta(j)\gamma(k)}$. □

Corollary 2.2 *Every feasible solution of $FS(\Theta)$ verifies that $x_{ijk} = 0$, for all $i, j, k \in [n]$ such that one of the following assertions is verified:*

a) $i \in Fix(\alpha), j \in Fix(\beta)$ and $k \notin Fix(\gamma)$.

b) $i \in Fix(\alpha), k \in Fix(\gamma)$ and $j \notin Fix(\beta)$.

c) $j \in Fix(\beta), k \in Fix(\gamma)$ and $i \notin Fix(\alpha)$.

Proof. From the conjugacy of rows, columns and symbols in Latin squares, it is enough to consider assertion (a). So, let us consider a feasible solution of $FS(\Theta)$ such that $x_{ijk} = 1$, for some $i, j, k \in [n]$ verifying assertion (a). From Theorem 2.1, there exists an unique $L = (l_{i,j}) \in LS(\Theta)$ being equivalent with such a feasible solution. Specifically, it must be $l_{i,j} = k$ and therefore, $k = l_{i,j} = l_{\alpha(i),\beta(j)} = \gamma(l_{i,j}) = \gamma(k)$, which is a contradiction, because $k \notin Fix(\gamma)$. □

Let $S_{Fix(\Theta)}$ be the set of triples $(i, j, k) \in [n]^3$ such that one of the assertions of Corollary 2.2 is verified. Since the $\mathbf{l}_\alpha^1 \cdot \mathbf{l}_\beta^1 \cdot (n - \mathbf{l}_\gamma^1) + \mathbf{l}_\alpha^1 \cdot \mathbf{l}_\gamma^1 \cdot (n - \mathbf{l}_\beta^1) + \mathbf{l}_\beta^1 \cdot \mathbf{l}_\gamma^1 \cdot (n - \mathbf{l}_\alpha^1)$ variables x_{ijk} related to $S_{Fix(\Theta)}$ are all nulls, we can reduce the number of variables of the system $(1)_\Theta$ in order to obtain a $1-1$ correspondence between $FS(\Theta)$ and $LS(\Theta)$. Given $s, t \in [n]$, the following sets will be useful:

$$S_{Fix(\Theta)}^{(1,s,t)} = \{i \in [n] \mid (i, s, t) \in S_{Fix(\Theta)}\}, \qquad S_{Fix(\Theta)}^{(2,s,t)} = \{j \in [n] \mid (s, j, t) \in S_{Fix(\Theta)}\},$$

$$S^{(3,s,t)}_{Fix(\Theta)} = \{k \in [n] \mid (s,t,k) \in S_{Fix(\Theta)}\}.$$

Moreover, the symmetrical structure given by the autotopism Θ can also be used to reduce the number of variables of $(1)_\Theta$. To see it, let us consider:

$$S_\Theta = \left\{ (i,j) \mid i \in S_\alpha, j \in \begin{cases} [n], & \text{if } i \notin Fix(\alpha), \\ S_\beta, & \text{if } i \in Fix(\alpha). \end{cases} \right\}$$

as a set of $(\mathbf{n}_\alpha - \mathbf{l}^1_\alpha) \cdot n + \mathbf{l}^1_\alpha \cdot \mathbf{n}_\beta$ multi-indices, where $S_\alpha = \{c^\alpha_{i,1} \mid i \in [\mathbf{n}_\alpha]\}$ and $S_\beta = \{c^\beta_{j,1} \mid j \in [\mathbf{n}_\beta]\}$. The following result is verified:

Proposition 2.3 *Let $L = (l_{i,j}) \in LS(\Theta)$ be such that all the triples of the Latin subrectangle $R_L = \{(i,j,l_{i,j}) \mid (i,j) \in S_\Theta\}$ of L are known. Then, all the triples of L are known. Indeed, given $i,j \in [n]$, there exists an unique element $(i_\Theta, j_\Theta) \in S_\Theta$ such that $l_{i,j}$ can be obtained starting from l_{i_Θ,j_Θ}.*

Proof. Let $(i,j,l_{i,j}) \in L$ be such that $i > \mathbf{n}_\alpha$ and let $r \in [\mathbf{n}_\alpha]$ and $u \in [\lambda^\alpha_r]$ be such that $c^\alpha_{r,u} = i$. Then, $(\alpha^{1-u}(i), \beta^{1-u}(j)) \in S_\Theta$, and, therefore, $l_{\alpha^{1-u}(i),\beta^{1-u}(j)}$ is known. Thus, $l_{i,j} = \gamma^{u-1}(l_{\alpha^{1-u}(i),\beta^{1-u}(j)})$.

Now, let $(i,j,l_{i,j}) \in L$ be such that $i \in Fix(\alpha)$ and $j > \mathbf{n}_\beta$. Let $s \in [\mathbf{n}_\beta]$ and $v \in [\lambda^\beta_s]$ be such that $c^\beta_{s,v} = j$. From the hypothesis, the triple $(i,\ c^\beta_{s,1},\ l_{i,c^\beta_{s,1}})$ is known. Thus, $l_{i,j} = \gamma^{v-1}(l_{i,c^\beta_{s,1}})$.

The final assertion is therefore an immediate consequence of the election of the cyclic decomposition of Θ. Specifically, it is verified that $(i_\Theta, j_\Theta) = (\alpha^{m_{i,j}}(i), \beta^{m_{i,j}}(j))$, where $m_{i,j} = \min\{t \geq 0 \mid (\alpha^t(i), \beta^t(j)) \in S_\Theta\}$. □

Given $i,j,k \in [n]$, let us define $k_\Theta = \gamma^m(k)$, where $m \in [n]$ is such that $(i_\Theta, j_\Theta) = (\alpha^m(i), \beta^m(j)) \in S_\Theta$. Thus, from the cyclic decomposition of Θ, let us observe that $(i_\Theta, j_\Theta, k_\Theta) = (\alpha^t(i)_\Theta, \beta^t(j)_\Theta, \gamma^t(k)_\Theta)$, for all $i,j \in [n]$ and for all $t \in [n]$. The following result holds:

Theorem 2.4 *There exists a 1-1 correspondence between $FS(\Theta)$ and the set of feasible solutions $FS'(\Theta)$ of the following system of equations in $d_\Theta = ((\mathbf{n}_\alpha - \mathbf{l}^1_\alpha) \cdot n^2 + \mathbf{l}^1_\alpha \cdot \mathbf{n}_\beta \cdot n) - (\mathbf{l}^1_\alpha \cdot \mathbf{l}^1_\beta \cdot (n - \mathbf{l}^1_\gamma) + \mathbf{l}^1_\alpha \cdot \mathbf{l}^1_\gamma \cdot (\mathbf{n}_\beta - \mathbf{l}^1_\beta) + \mathbf{l}^1_\beta \cdot \mathbf{l}^1_\gamma \cdot (\mathbf{n}_\alpha - \mathbf{l}^1_\alpha))$ variables:*

$$\begin{cases} \sum_{i \in [n] \setminus S^{(1,j,k)}_{Fix(\Theta)}} x_{i_\Theta j_\Theta k_\Theta} = 1, \forall j,k \in [n]. & (2.1)_\Theta \\ \sum_{j \in [n] \setminus S^{(2,i,k)}_{Fix(\Theta)}} x_{i_\Theta j_\Theta k_\Theta} = 1, \forall i,k \in [n]. & (2.2)_\Theta \\ \sum_{k \in [n] \setminus S^{(3,i,j)}_{Fix(\Theta)}} x_{i_\Theta j_\Theta k_\Theta} = 1, \forall i,j \in [n]. & (2.3)_\Theta \\ x_{ijk} \in \{0,1\}, \forall (i,j,k) \in S_\Theta \times [n] \setminus S_{Fix(\Theta)}. & (2.4)_\Theta \end{cases} \qquad (2)_\Theta$$

Proof. Let us define the map $\Psi_\Theta : FS'(\Theta) \subseteq \mathbb{R}^{d_\Theta} \to FS(\Theta) \subseteq \mathbb{R}^{n^3}$, such that $\Psi_\Theta((x_{ijk})_{(i,j,k) \in S_\Theta \times [n] \setminus S_{Fix(\Theta)}}) = (X_{uvw})_{(u,v,w) \in [n]^3} = \begin{cases} 0, & \text{if } (u,v,w) \in S_{Fix(\Theta)}, \\ x_{u_\Theta v_\Theta w_\Theta}, & \text{otherwise}. \end{cases}$. Thus, Ψ_Θ is a 1-1 correspondence between $FS'(\Theta)$ and $FS(\Theta)$. Specifically, from Corollary 2.2 and Proposition 2.3, equations $(1.1), (1.2)$ and (1.3) and conditions (1.4) in $FS(\Theta)$ are equivalent to $(2.1)_\Theta$, $(2.2)_\Theta, (2.3)_\Theta$ and (2.4) in $FS'(\Theta)$, respectively. Now, let us consider $(x_{ijk})_{(i,j,k)\in S_\Theta\times[n]\setminus S_{Fix(\Theta)}} \in FS'(\Theta)$ and $(X_{uvw})_{(u,v,w)\in[n]^3} = \Psi_\Theta((x_{ijk})_{(i,j,k)\in S_\Theta\times[n]\setminus S_{Fix(\Theta)}})$. Given $u, v, w \in [n]$, it is verified that $X_{uvw} = \begin{cases} 0 = X_{\alpha(u)\beta(v)\gamma(w)}, & \text{if } (u,v,w) \in S_{Fix(\Theta)}, \\ x_{u_\Theta v_\Theta w_\Theta} = X_{\alpha(u)\beta(v)\gamma(w)}, & \text{otherwise.} \end{cases}$. Therefore equations $(1.5)_\Theta$ are also verified. □

In general, many of the expressions of $(2)_\Theta$ are the same equation and so, they are redundant. An immediate consequence of Theorem 2.4 is the following:

Corollary 2.5 $\Psi_\Theta^{-1} \circ \Phi_{|LS(\Theta)}$ *is a 1-1 correspondence between* $LS(\Theta)$ *and* $FS'(\Theta)$. □

3 0/1-polytopes related to a Latin square autotopism

Given a autotopism $\Theta \in \mathfrak{A}_n$, let A_Θ and A'_Θ be the constraint matrices of $(1)_\Theta$ and $(2)_\Theta$, respectively. Let us define the following 0/1-polytopes:

$$\mathcal{P}_{LS(\Theta)} = conv\{FS(\Theta)\} = conv\{\mathbf{x} \in \{0,1\}^{n^3} \mid A_\Theta \cdot \mathbf{x} = \mathbf{e}_\Theta\} \subseteq \mathbb{R}^{n^3},$$

$$\mathcal{P}'_{LS(\Theta)} = conv\{FS'(\Theta)\} = conv\{\mathbf{x} \in \{0,1\}^{n^3} \mid A'_\Theta \cdot \mathbf{x} = \mathbf{e}'_\Theta\} \subseteq \mathbb{R}^{d_\Theta},$$

where $\mathbf{e}_\Theta = (1,...,1)^t$ and $\mathbf{e}'_\Theta = (1,...,1)^t$ have $3 \cdot n^2 + n^3$ and $3 \cdot n^2$ entries, respectively. The following results hold:

Corollary 3.1 *Both* 0/1*-polytopes,* $\mathcal{P}_{LS(\Theta)}$ *and* $\mathcal{P}'_{LS(\Theta)}$*, have* $\Delta(\Theta)$ *vertices.*

Proof. It is enough to consider the 1-1 correspondences of Theorem 2.1 and Corollary 2.5. □

Theorem 3.2 $\dim(\mathcal{P}_{LS(\Theta)}) = \dim(\mathcal{P}'_{LS(\Theta)}) \leq d_\Theta - rank(A'_\Theta)$.

Proof. The inequality is an immediate consequence of the definition of $\mathcal{P}'_{LS(\Theta)}$. Besides, from the definition of Ψ_Θ given in the proof of Theorem 2.4, it is immediate to see that a set of m affinely vertices of $\mathcal{P}'_{LS(\Theta)}$ induces a set of m affinely vertices of $\mathcal{P}_{LS(\Theta)}$, because we can identify all the coordinates of the first ones in the second ones. So, $\dim(\mathcal{P}'_{LS(\Theta)}) \leq \dim(\mathcal{P}_{LS(\Theta)})$.

Now, let $\{V_1, ..., V_m\}$ be a set of m affinely independent vertices of $\mathcal{P}_{LS(\Theta)}$, where $V_i = (v_{i,1}, ..., v_{i,n^3})$, for all $i \in [m]$. From Theorem 2.4, $V'_i = \Psi_\Theta^{-1}(V_i) = (v'_{i,1}, ..., v'_{i,d_\Theta})$ is a vertex of $\mathcal{P}'_{LS(\Theta)}$, for all $i \in [m]$. Let us suppose that there exist $\lambda_1, ..., \lambda_m \in \mathbb{R}$, such that $\sum_{i=1}^m \lambda_i = 1$ and $\sum_{i=1}^m \lambda_i \cdot V'_i = \mathbf{0}$. From the definition of Ψ_Θ, given $j \in [n^3]$ non corresponding to a triple of $S_{Fix(\Theta)}$, there exists $k \in [d_\Theta]$, such that $v_{i,j} = v'_{i,k}$, for all $i \in [m]$. Thus, $\sum_{i=1}^m \lambda_i \cdot V_i = \mathbf{0}$, which is a contradiction. Therefore, $\dim(\mathcal{P}_{LS(\Theta)}) \leq \dim(\mathcal{P}'_{LS(\Theta)})$. □

Theorem 3.3 *Let* $(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$ *be the cycle structure of a Latin square autotopism and let us consider* $\Theta_1 = (\alpha_1, \beta_1, \gamma_1), \Theta_2 = (\alpha_2, \beta_2, \gamma_2) \in \mathfrak{A}_n(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$*. Then,* $\mathcal{P}_{LS(\Theta_1)}$ *and* $\mathcal{P}_{LS(\Theta_2)}$ *are* 0/1*-equivalents. Analogously,* $\mathcal{P}'_{LS(\Theta_1)}$ *and* $\mathcal{P}'_{LS(\Theta_2)}$ *are* 0/1*-equivalents.*

Proof. Let us prove the first assertion, the other case follows analogously. So, since Θ_1 and Θ_2 have the same cycle structure, we can consider the isotopism $\Theta = (\sigma_1, \sigma_2, \sigma_3) \in \mathcal{I}_n$, where:

i) $\sigma_1(c_{i,j}^{\alpha_1}) = c_{i,j}^{\alpha_2}$, for all $i \in [k_{\alpha_1}]$ and $j \in [\lambda_i^{\alpha_1}]$,

ii) $\sigma_2(c_{i,j}^{\beta_1}) = c_{i,j}^{\beta_2}$, for all $i \in [k_{\beta_1}]$ and $j \in [\lambda_i^{\beta_1}]$,

iii) $\sigma_3(c_{i,j}^{\gamma_1}) = c_{i,j}^{\gamma_2}$, for all $i \in [k_{\gamma_1}]$ and $j \in [\lambda_i^{\gamma_1}]$.

Let $L \in LS(\Theta_1)$ and $(x_{ijk})_{i,j,k\in[n]} = \Phi(L)$. From [5], we know that $L \in LS(\Theta_1)$ if and only if $L^\Theta \in LS(\Theta_2)$. Thus, if $(X_{ijk})_{i,j,k\in[n]} = \Phi(L^\Theta)$, then it must be $x_{ijk} = x_{\sigma_1(i)\sigma_2(j)\sigma_3(k)}$, for all $i, j, k \in [n]$. So, the permutation of coordinates $\pi(x_{ijk}) = x_{\sigma_1(i)\sigma_2(j)\sigma_3(k)}$ is a 1-1 correspondence between $FS(\Theta_1)$ and $FS(\Theta_2)$, which are the set of vertices of $\mathcal{P}_{LS(\Theta_1)}$ and $\mathcal{P}_{LS(\Theta_2)}$, respectively. Thus, π transforms $\mathcal{P}_{LS(\Theta_1)}$ into $\mathcal{P}_{LS(\Theta_2)}$. $\square$

From Theorem 3.3, the dimension of $\mathcal{P}_{LS(\Theta)}$ and $\mathcal{P}'_{LS(\Theta)}$ only depends on the cycle structure of Θ. Moreover, since rows, columns and symbols have an interchangeable role in Latin squares and since affine independence does not depend on these interchanges, we can suppose that the cycles α, β and γ of Θ verify that $\mathbf{n}_\alpha \leq \mathbf{n}_\beta \leq \mathbf{n}_\gamma$. Thus, let us finish this paper by following the classification of all possible cycle structures given in [4], in order to show in Tables 1 and 2 the dimensions of all possible polytopes related to any autotopisms of order up to 9. Specifically, the exact dimension is shown when the set $LS(\Theta)$ is known. As an upper bound we show the difference between d_Θ and $rank(A'_\Theta)$, which indeed can not be reached, as we can observe in Table 1. As a lower bound, we study the subsets of $LS(\Theta)$ given in [6].

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n	$\mathbf{l}_\alpha$	$\mathbf{l}_\beta$	$\mathbf{l}_\gamma$	d_Θ	$\Delta(\Theta)$	Lower bound [6]	$\dim(\mathcal{P}'_{LS(\Theta)})$	d_Θ - rank(A'_Θ)
2	(0,1)	(0,1)	(0,1)	4	2	-	1	1
3	(0,0,1)	(0,0,1)	(0,0,1)	9	3	-	2	2
			(3,0,0)		6	-	4	4
	(1,1,0)	(1,1,0)	(1,1,0)	11	4	-	3	3
4	(0,0,0,1)	(0,0,0,1)	(0,2,0,0)	16	8	-	5	7
			(2,1,0,0)		8	-	5	8
			(4,0,0,0)		24	-	9	9
	(0,2,0,0)	(0,2,0,0)	(0,2,0,0)	32	32	-	12	13
			(2,1,0,0)		32	-	13	14
			(4,0,0,0)		96	-	15	15
	(1,0,1,0)	(1,0,1,0)	(1,0,1,0)	19	9	-	8	8
	(2,1,0,0)	(2,1,0,0)	(2,1,0,0)	24	16	-	4	7
5	(0,0,0,0,1)	(0,0,0,0,1)	(0,0,0,0,1)	25	15	-	12	12
			(5,0,0,0,0)		120	-	16	16
	(1,0,0,1,0)	(1,0,0,1,0)	(1,0,0,1,0)	29	32	-	15	15
	(1,2,0,0,0)	(1,2,0,0,0)	(1,2,0,0,0)	57	256	-	27	28
	(2,0,1,0,0)	(2,0,1,0,0)	(2,0,1,0,0)	35	144	-	15	15
6	(0,0,0,0,0,1)	(0,0,0,0,0,1)	(0,0,2,0,0,0)	36	72	-	19	21
			(1,1,1,0,0,0)		72	-	20	22
			(2,2,0,0,0,0)		144	-	22	23
			(3,0,1,0,0,0)		144	-	21	23
			(4,1,0,0,0,0)		288	-	24	24
			(6,0,0,0,0,0)		720	-	25	25
	(0,0,0,0,0,1)	(0,0,2,0,0,0)	(0,3,0,0,0,0)	36	288	-	23	23
	(0,0,2,0,0,0)	(0,0,2,0,0,0)	(0,0,2,0,0,0)	72	648	-	41	41
			(3,0,1,0,0,0)		2592	-	43	43
			(6,0,0,0,0,0)		25920	34	-	45
	(1,0,0,0,1,0)	(1,0,0,0,1,0)	(1,0,0,0,1,0)	41	75	-	24	24
	(0,3,0,0,0,0)	(0,3,0,0,0,0)	(2,2,0,0,0,0)	108	36864	37	-	63
			(4,1,0,0,0,0)		110592	38	-	64
			(6,0,0,0,0,0)		460800	27	-	65
	(2,0,0,1,0,0)	(2,0,0,1,0,0)	(2,0,0,1,0,0)	48	768	-	25	25
	(2,2,0,0,0,0)	(2,2,0,0,0,0)	(2,2,0,0,0,0)	88	20480	20	-	44
	(3,0,1,0,0,0)	(3,0,1,0,0,0)	(3,0,1,0,0,0)	63	2592	-	20	28
7	(0,0,0,0,0,0,1)	(0,0,0,0,0,0,1)	(0,0,0,0,0,0,1)	49	133	-	30	30
			(7,0,0,0,0,0,0)		5040	31	-	36
	(1,0,0,0,0,1,0)	(1,0,0,0,0,1,0)	(1,0,0,0,0,1,0)	55	288	-	35	35
	(1,0,2,0,0,0,0)	(1,0,2,0,0,0,0)	(1,0,2,0,0,0,0)	109	42768	25	-	68
	(1,1,0,1,0,0,0)	(1,1,0,1,0,0,0)	(1,1,0,1,0,0,0)	109	512	-	24	52
	(2,0,0,0,1,0,0)	(2,0,0,0,1,0,0)	(2,0,0,0,1,0,0)	63	4000	20	-	37
	(1,3,0,0,0,0,0)	(1,3,0,0,0,0,0)	(1,3,0,0,0,0,0)	163	6045696	30	-	101
	(3,0,0,1,0,0,0)	(3,0,0,1,0,0,0)	(3,0,0,1,0,0,0)	79	41472	27	-	41
	(3,2,0,0,0,0,0)	(3,2,0,0,0,0,0)	(3,2,0,0,0,0,0)	131	1327104	20	-	66

Table 1: Number of vertices and dimensions of polytopes related to $\mathfrak{A}_n$, for $n \leq 7$.

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n	$\mathbf{l}_\alpha = \mathbf{l}_\beta$	$\mathbf{l}_\gamma$	d_Θ	$\Delta(\Theta)$	Lower bound [6]	$\dim(\mathcal{P}'_{LS(\Theta)})$	d_Θ - rank(A'_Θ)
8	(0,0,0,0,0,0,0,1)	(0,0,0,2,0,0,0,0)	64	1152	-	43	43
		(0,2,0,1,0,0,0,0)		1408	-	44	44
		(0,4,0,0,0,0,0,0)		3456	32	-	45
		(2,1,0,1,0,0,0,0)		1408	-	44	45
		(2,3,0,0,0,0,0,0)		3456	45	-	46
		(4,0,0,1,0,0,0,0)		3456	35	-	46
		(4,2,0,0,0,0,0,0)		8064	38	-	47
		(6,1,0,0,0,0,0,0)		17280	39	-	48
		(8,0,0,0,0,0,0,0)		40320	41	-	49
	(0,0,0,2,0,0,0,0)	(0,0,0,2,0,0,0,0)	128	106496	38	-	85
		(0,2,0,1,0,0,0,0)		188416	43	-	86
		(0,4,0,0,0,0,0,0)		811008	36	-	87
		(2,1,0,1,0,0,0,0)		253952	34	-	87
		(2,3,0,0,0,0,0,0)		1007616	38	-	88
		(4,0,0,1,0,0,0,0)		712704	41	-	88
		(4,2,0,0,0,0,0,0)		2727936	35	-	89
		(6,1,0,0,0,0,0,0)		7741440	26	-	90
		(8,0,0,0,0,0,0,0)		23224320	41	-	91
	(0,1,0,0,0,1,0,0)	(2,0,0,0,0,1,0,0)	128	3456	34	-	58
		(2,0,2,0,0,0,0,0)		19008	32	-	59
	(1,0,0,0,0,0,1,0)	(1,0,0,0,0,0,1,0)	71	931	-	48	48
	(0,2,0,1,0,0,0,0)	(0,2,0,1,0,0,0,0)	192	16384	17	-	112
		(2,1,0,1,0,0,0,0)		16384	18	-	113
		(4,0,0,1,0,0,0,0)		147456	19	-	114
	(2,0,0,0,0,1,0,0)	(2,0,0,0,0,1,0,0)	80	19584	35	-	51
	(0,4,0,0,0,0,0,0)	(0,4,0,0,0,0,0,0)	256	-	-	-	171
		(2,3,0,0,0,0,0,0)		-	-	-	172
		(4,2,0,0,0,0,0,0)		-	-	-	173
		(6,1,0,0,0,0,0,0)		-	-	-	174
		(8,0,0,0,0,0,0,0)		828396011520	41	-	175
	(2,0,2,0,0,0,0,0)	(2,0,2,0,0,0,0,0)	152	12985920	25	-	96
	(2,1,0,1,0,0,0,0)	(2,1,0,1,0,0,0,0)	152	8192	15	-	74
	(3,0,0,0,1,0,0,0)	(3,0,0,0,1,0,0,0)	97	388800	19	-	56
	(2,3,0,0,0,0,0,0)	(2,3,0,0,0,0,0,0)	224	-	-	-	141
	(4,0,0,1,0,0,0,0)	(4,0,0,1,0,0,0,0)	128	7962624	21	-	69
	(4,2,0,0,0,0,0,0)	(4,2,0,0,0,0,0,0)	192	509607936	15	-	100
9	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,1)	81	2025	-	56	56
		(0,0,3,0,0,0,0,0,0)		7128	43	-	58
		(3,0,2,0,0,0,0,0,0)		12960	45	-	60
		(6,0,1,0,0,0,0,0,0)		71280	47	-	62
		(9,0,0,0,0,0,0,0,0)		362880	49	-	64
	(0,0,1,0,0,1,0,0,0)	(0,0,1,0,0,1,0,0,0)	162	15552	30	-	86
		(0,3,1,0,0,0,0,0,0)		124416	32	-	88
		(3,0,0,0,0,0,1,0,0)		62208	33	-	88
		(3,3,0,0,0,0,0,0,0)		1244160	24	-	90
	(1,0,0,0,0,0,0,1,0)	(1,0,0,0,0,0,0,1,0)	89	4096	30	-	63
	(0,0,3,0,0,0,0,0,0)	(0,0,3,0,0,0,0,0,0)	243	-	-	-	170
		(3,0,2,0,0,0,0,0,0)		-	-	-	172
		(6,0,1,0,0,0,0,0,0)		-	-	-	174
		(9,0,0,0,0,0,0,0,0)		948109639680	49	-	176
	(1,0,0,2,0,0,0,0,0)	(1,0,0,2,0,0,0,0,0)	177	12189696	14	-	124
	(1,1,0,0,0,1,0,0,0)	(1,1,0,0,0,1,0,0,0)	177	69120	16	-	84
	(2,0,0,0,0,0,1,0,0)	(2,0,0,0,0,0,1,0,0)	99	438256	32	-	67
	(3,0,0,0,0,1,0,0,0)	(3,0,0,0,0,1,0,0,0)	117	3110400	13	-	73
	(1,4,0,0,0,0,0,0,0)	(1,4,0,0,0,0,0,0,0)	353	-	-	-	246
	(3,0,2,0,0,0,0,0,0)	(3,0,2,0,0,0,0,0,0)	207	-	-	-	130
	(4,0,0,0,1,0,0,0,0)	(4,0,0,0,1,0,0,0,0)	149	199065600	18	-	87
	(3,3,0,0,0,0,0,0,0)	(3,3,0,0,0,0,0,0,0)	297	-	-	-	187

Table 2: Number of vertices and dimensions of polytopes related to $\mathfrak{A}_8$ and $\mathfrak{A}_9$.