Santilli's autotopisms of associative partial quasigroups.

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Abstract

In this paper, given an isotopism $\Theta \in \mathcal{I}_n$, we study some properties of the set $SAPLS(\Theta)$ of those partial Latin squares being the multiplication table of an associative partial quasigroup having Θ as a Santilli's autotopism.

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1 INTRODUCTION

A *quasigroup* is a nonempty set G endowed with a product $\cdot$, such that if any two of the three symbols a, b, c in the equation $a \cdot b = c$ are given as elements of G, the third one is uniquely determined as an element of G. It is equivalent to say that G is endowed with left and right division. Specifically, quasigroups are, in general, non-commutative and non-associative algebraic structures. Two quasigroups $(G, \cdot)$ and $(H, \circ)$ are *isotopic* if there are three bijections α, β, γ from H to G, such that $\gamma(a \circ b) = \alpha(a) \cdot \beta(b)$, for all $a, b \in H$. The triple $\Theta = (\alpha, \beta, \gamma)$ is called an *isotopism* from $(G, \cdot)$ to $(H, \circ)$.

The multiplication table of a quasigroup is a Latin square. A *Latin square* L of order n is an $n \times n$ array with elements chosen from a set of n distinct symbols $\{x_1, ..., x_n\}$, such that each symbol occurs precisely once in each row and each column. The set of Latin squares of order n is denoted by $LS(n)$. An exhaustive study about Latin squares and their applications is given by [3].

In this paper, for any given $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, 2, ..., n\}$. Specifically, we assume that the set of symbols of any Latin square of order n is $[n]$. The symmetric group on $[n]$ is denoted by S_n. Every permutation of S_n can be written as the composition of pairwise disjoint cycles $\delta = C_1^\delta \circ C_2^\delta \circ ... \circ C_{k_\delta}^\delta$, where:

i) For all $i \in [k_\delta]$, one has $C_i^\delta = \left(c_{i,1}^\delta \; c_{i,2}^\delta \; ... \; c_{i,\,\lambda_i^\delta}^\delta\right)$, with $\lambda_i^\delta \leq n$ and $c_{i,1}^\delta = \min_j\{c_{i,j}^\delta\}$. If $\lambda_i^\delta = 1$, then C_i^δ is a cycle of lenght 1 and so, $c_{i,1}^\delta \in Fix(\delta)$.

ii) $\sum_i \lambda_i^\delta = n$.

iii) For all $i, j \in [k_\delta]$, one has $\lambda_i^\delta \geq \lambda_j^\delta$, whenever $i \leq j$.

iv) Given $i, j \in [k_\delta]$, with $i < j$ and $\lambda_i^\delta = \lambda_j^\delta$, one has $c_{i,1}^\delta < c_{j,1}^\delta$.

Given a permutation $\delta \in S_n$, it is defined the set of its *fixed points* $Fix(\delta) = \{i \in [n] : \delta(i) = i\}$. The *cycle structure of* δ is the sequence $\mathbf{l}_\delta = (\mathbf{l}_1^\delta, \mathbf{l}_2^\delta, ..., \mathbf{l}_n^\delta)$, where $\mathbf{l}_i^\delta$ is the number of cycles of length i in δ, for all $i \in [n]$. On the other hand, given $L = (l_{i,j}) \in LS(n)$, the *orthogonal array representation of* L is the set of n^2 triples $\{(i, j, l_{i,j}) : i, j \in [n]\}$. The previous set is identified with L and then, it is written $(i, j, l_{i,j}) \in L$, for all

$i, j \in [n]$. Given $\sigma \in S_3$, one defines the *conjugate Latin square* $L^\sigma \in LS(n)$ of L, such that if $T = (i, j, l_{i,j}) \in L$, then $(\pi_{\sigma(1)}(T), \pi_{\sigma(2)}(T), \pi_{\sigma(3)}(T)) \in L^\sigma$, where π_i gives the i^{th} coordinate of T, for all $i \in [3]$. In this way, each Latin square L has six conjugate Latin squares associated with it: $L^{Id} = L$, $L^{(12)} = L^t$, $L^{(13)}$, $L^{(23)}$, $L^{(123)}$ and $L^{(132)}$.

Since a Latin square is the multiplication table of a quasigroup, an *isotopism* of a Latin square $L \in LS(n)$ is therefore a triple $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_n = S_n \times S_n \times S_n$. In this way, α, β and γ are permutations of rows, columns and symbols of L, respectively. The resulting square L^Θ is also a Latin square and it is said to be *isotopic* to L. In particular, if $L = (l_{i,j})$, then $L^\Theta = \{(i, j, \gamma\left(l_{\alpha^{-1}(i),\beta^{-1}(j)}\right) : i, j \in [n]\}$. If $\gamma = \epsilon$, the identity map on $[n]$, Θ is called a *principal isotopism*. The *cycle structure* of an isotopism $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_n$ is the triple $(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$, where $\mathbf{l}_\delta$ is the cycle structure of δ, for all $\delta \in \{\alpha, \beta, \gamma\}$. The set of isotopisms of Latin squares of order n having $(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$ as their cycle structures is denoted by $\mathcal{I}_n(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$.

An isotopism which maps L to itself is an *autotopism*. $(\epsilon, \epsilon, \epsilon)$ is called the *trivial autotopism*. The possible cycle structures of the set of non-trivial autotopisms of Latin squares of order up to 11 have been obtained by [2]. The stabilizer subgroup of L in $\mathcal{I}_n$ is its *autotopism group*, $\mathcal{U}(L) = \{\Theta \in \mathcal{I}_n : L^\Theta = L\}$. Given $L \in LS(n)$, $\Theta = (\alpha, \beta, \gamma) \in \mathcal{U}(L)$ and $\sigma \in S_3$, it is verified that $\Theta^\sigma = (\pi_{\sigma(1)}(\Theta), \pi_{\sigma(2)}(\Theta), \pi_{\sigma(3)}(\Theta)) \in \mathcal{U}(L^\sigma)$, where π_i gives the i^{th} component of Θ, for all $i \in [3]$. Given $\Theta \in \mathcal{I}_n$, the set of all Latin squares L such that $\Theta \in \mathcal{U}(L)$ is denoted by $LS(\Theta)$.

A *partial Latin square* $P = (p_{i,j})$ of order n is a $n \times n$ array with elements chosen from a set of n symbols, such that each symbol occurs at most once in each row and in each column. In this way, P is the multiplication table of a *partial quasigroup*. The *size* of P is the number of its *filled cells*, that is, the number of triples $(i, j, k) \in P$ such that $k \neq \emptyset$. The set of partial Latin squares of order n is denoted by $PLS(n)$. *Isotopisms* of partial Latin squares are defined in a similar way as those of Latin squares, although now $\gamma(\emptyset) = \emptyset$. In particular, the sets $\mathcal{U}(P)$ and $PLS(\Theta)$ are similarly defined. It is said that an isotopism $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_n$ is a *Santilli's autotopism* of $P = (p_{i,j}) \in PLS(\Theta)$ if $\Theta \in \mathcal{U}(P)$ and there exist $\mathfrak{j}_\alpha, \mathfrak{j}_\beta, \mathfrak{j}_\gamma \in [n]$ such that $\alpha(i) = p_{i,\mathfrak{j}_\alpha}$, $\beta(i) = p_{i,\mathfrak{j}_\beta}$ and $\gamma(i) = p_{i,\mathfrak{j}_\gamma}$, for all $i \in [n]$. The triple $(\mathfrak{j}_\alpha, \mathfrak{j}_\beta, \mathfrak{j}_\gamma)$ is denoted by $S(\Theta, P)$. By the other way, $SPLS(\Theta)$ denotes the set of all partial Latin squares having Θ as a Santilli's autotopism and $\mathcal{SI}_n$ denotes the set of all $\Theta \in \mathcal{I}_n$ such that $SPLS(\Theta) \neq \emptyset$.

Several properties of those partial Latin squares of order up to 5 having in their autotopism groups a Santilli's isotopism fixing at least one symbol have been studied in [1]. However, an exhaustive study of these isotopisms is still open. In this way, we prove in this paper some more properties of these Santilli's isotopisms, being interested in those ones related to associative partial quasigroups. Specifically, the structure of the paper is the following: In Section 2, some general properties of the set $\mathcal{SI}_n$ of Santilli's autotopisms are studied. In Section 3, we study the concrete case in which Θ is a Santilli's autotopism of a partial Latin square corresponding to an associative partial quasigroup. Finally, in Section 4, a classification of all these autotopisms of partial Latin squares of order up to 5 is given.

2 SANTILLI'S AUTOTOPISMS

In this section, we are interested in studying some general properties of $\mathcal{SI}_n$. In this way, the following results are verified:

Lemma 2.1. *Let* $\Theta = (\alpha, \beta, \gamma) \in \mathcal{SI}_n$, $P = (p_{i,j}) \in SPLS(\Theta)$ *and* $(\mathrm{j}_\alpha, \mathrm{j}_\beta, \mathrm{j}_\gamma) = S(\Theta, P)$. *It must be* $\mathrm{j}_\gamma = \beta(\mathrm{j}_\alpha)$.

Proof. Given $h \in [n]$, let $i \in [n]$ be such that $\alpha(i) = h$. It is verified that $\gamma(h) = \gamma(\alpha(i)) = \gamma(p_{i,\mathrm{j}_\alpha}) = p_{\alpha(i),\beta(\mathrm{j}_\alpha)} = p_{h,\beta(\mathrm{j}_\alpha)}$. Since h is arbitrary, it must be $\mathrm{j}_\gamma = \beta(\mathrm{j}_\alpha)$. □

Proposition 2.2. *Let* $\Theta = (\alpha, \beta, \gamma) \in \mathcal{SI}_n$, $P = (p_{i,j}) \in SPLS(\Theta)$ *and* $(\mathrm{j}_\alpha, \mathrm{j}_\beta, \mathrm{j}_\gamma) = S(\Theta, P)$. *If* $\alpha = \gamma$ *and* $\mathbf{l}_1^\beta = 1$, *then* $Fix(\beta) = \{\mathrm{j}_\alpha\}$.

Proof. Since $\alpha = \gamma$, from Lemma 2.1, it must be $\mathrm{j}_\alpha = \mathrm{j}_\gamma \in Fix(\beta)$. Since $\mathbf{l}_1^\beta = 1$, it must be $Fix(\beta) = \{\mathrm{j}_\alpha\}$. □

Lemma 2.3. *Let* $\Theta = (\alpha, \beta, \gamma) \in \mathcal{SI}_n$. *If there exist* $\delta, \delta' \in \{\alpha, \beta, \gamma\}$ *and* $i_0 \in [n]$ *such that* $\delta(i_0) = \delta'(i_0)$, *then, it must be* $\delta = \delta'$.

Proof. Let $P = (p_{i,j}) \in SPLS(\Theta)$. There must exist $j_\delta, j_{\delta'} \in [n]$ such that $p_{i,j_\delta} = \delta(i)$ and $p_{i,j_{\delta'}} = \delta'(i)$, for all $i \in [n]$. Specifically, $p_{i_0,j_\delta} = \delta(i_0) = \delta'(i_0) = p_{i_0,j_{\delta'}}$. Since P is a partial Latin square, it must be $j_\delta = j_{\delta'}$ and therefore, $\delta(i) = p_{i,j_\delta} = p_{i,j_{\delta'}} = \delta'(i)$, for all $i \in [n]$. □

Proposition 2.4. *Let $\Theta = (\alpha, \beta, \gamma) \in \mathcal{SI}_n$ be such that $\mathbf{l}_1^{\beta} = 0$. Then, $\alpha(i) \neq \gamma(i)$, for all $i \in [n]$.*

Proof. Let $i \in [n]$ be such that $\alpha(i) = \gamma(i)$. From Lemma 2.3, it must be $\alpha = \gamma$. Thus, from Lemma 2.1, $\mathrm{j}_{\alpha} = \mathrm{j}_{\gamma} = \beta(\mathrm{j}_{\alpha})$. So, $\mathrm{j}_{\alpha} \in Fix(\beta) = \emptyset$, which is a contradiction. $\square$

Lemma 2.5. *Let $\Theta = (\alpha, \beta, \gamma) \in \mathcal{SI}_n$, $P = (p_{i,j}) \in SPLS(\Theta)$ and $(\mathrm{j}_{\alpha}, \mathrm{j}_{\beta}, \mathrm{j}_{\gamma}) = S(\Theta, P)$. If there exist $m \in \mathbb{N}$ and $i_0 \in [n]$ such that $\gamma^m(\beta(i_0)) = \beta(\alpha^m(i_0))$, then $\mathrm{j}_{\beta} \in Fix(\beta^m)$. As a consequence, if there exists $m \in \mathbb{N}$ such that $\mathbf{l}_1^{\beta^m} = 0$, then $\gamma^m(\beta(i)) \neq \beta(\alpha^m(i))$, for all $i \in [n]$.*

Proof. We have that $p_{\alpha^m(i_0),\beta^m(\mathrm{j}_{\beta})} = \gamma^m(p_{i_0,\mathrm{j}_{\beta}}) = \gamma^m(\beta(i_0)) = \beta(\alpha^m(i_0)) = p_{\alpha^m(i_0),\mathrm{j}_{\beta}}$. Since $P \in PLS(n)$, it must be $\mathrm{j}_{\beta} \in Fix(\beta^m)$. The consequence is immediate. $\square$

Lemma 2.6. *Let $\Theta = (\alpha, \beta, \gamma) \in \mathcal{SI}_n$ be such that $\mathbf{l}_1^{\alpha} \cdot \mathbf{l}_1^{\beta} \cdot \mathbf{l}_1^{\gamma} > 0$. Let $P = (p_{i,j}) \in SPLS(\Theta)$. If there exist $i_0 \in Fix(\alpha), k_0 \in Fix(\gamma)$ and $\delta \in \{\alpha, \beta, \gamma\}$ such that $\delta(i_0) = k_0$, then $j_{\delta} \in Fix(\beta)$. Moreover, if $\mathrm{j}_{\beta} \in Fix(\beta)$, then $\beta(i) \in Fix(\gamma)$, for all $i \in Fix(\alpha)$.*

Proof. Since $p_{i_0,j_{\delta}} = \delta(i_0) = k_0 \in Fix(\gamma)$, we have that $p_{i_0,j_{\delta}} = k_0 = \gamma(k_0) = \gamma(p_{i_0,j_{\delta}}) = p_{\alpha(i_0),\beta(j_{\delta})} = p_{i_0,\beta(j_{\delta})}$. So, it must be $j_{\delta} \in Fix(\beta)$.

Now, let us suppose that $\mathrm{j}_{\beta} \in Fix(\beta)$ and let $i \in Fix(\alpha)$. Then, it is verified that $\beta(i) = p_{i,\mathrm{j}_{\beta}} = p_{\alpha(i),\beta(\mathrm{j}_{\beta})} = \gamma(p_{i,\mathrm{j}_{\beta}}) = \gamma(\beta(i))$. Therefore, $\beta(i) \in Fix(\gamma)$. $\square$

3 SANTILLI'S AUTOTOPISMS OF ASSOCIATIVE PARTIAL QUASIGROUPS

In this Section, we will work with partial Latin squares corresponding to associative partial quasigroups. Thus, given $\Theta \in \mathcal{SI}_n$, $SAPLS(\Theta)$ will denote the set of partial Latin squares being the multiplication table of an associative partial quasigroup having Θ as a Santilli's autotopism. The following results are verified:

Lemma 3.1. *Let* $\Theta = (\alpha, \beta, \gamma) \in \mathcal{SI}_n$*,* $P \in SAPLS(\Theta)$ *and* $(\mathfrak{j}_\alpha, \mathfrak{j}_\beta, \mathfrak{j}_\gamma) = S(\Theta, P)$*. If* P *is the multiplication table of an associative partial quasigroup* $([n], \cdot)$*, then* $\gamma = \beta \circ \alpha$*.*

Proof. Let $i, j \in [n]$. It is verified that:

$$\gamma(i\cdot j) = (i\cdot j)\cdot \mathfrak{j}_\gamma = (i\cdot j)\cdot \beta(\mathfrak{j}_\alpha) = (i\cdot j)\cdot (\mathfrak{j}_\alpha \cdot \mathfrak{j}_\beta) = ((i\cdot j)\cdot \mathfrak{j}_\alpha)\cdot \mathfrak{j}_\beta = \beta(\alpha(i\cdot j)).$$

Specifically, $\gamma(h) = \beta(\alpha(h))$, for all $h \in [n]$. Therefore, $\gamma = \beta \circ \alpha$. □

Proposition 3.2. *Let* $\Theta = (\alpha, \beta, \gamma) \in \mathcal{SI}_n$*,* $P \in SAPLS(\Theta)$ *and* $(\mathfrak{j}_\alpha, \mathfrak{j}_\beta, \mathfrak{j}_\gamma) = S(\Theta, P)$*. If* P *is the multiplication table of an associative partial quasigroup* $([n], \cdot)$ *and* $Fix(\alpha) \neq \emptyset$*, then* $\beta = \gamma$*. Moreover, it must be* $\alpha = \epsilon$*.*

Proof. Let $i \in Fix(\alpha)$ and let us consider $j \in [n]$. It is verified that:

$$i \cdot \beta(j) = \alpha(i) \cdot \beta(j) = \gamma(i \cdot j) = (i \cdot j) \cdot \mathfrak{j}_\gamma = i \cdot (j \cdot \mathfrak{j}_\gamma) = i \cdot \gamma(j).$$

So, $\beta(j) = \gamma(j)$ and, therefore, $\beta = \gamma$ because of the arbitrariness in the choice of j. As an immediate consequence, from Lemma 3.1, it must be $\alpha = \epsilon$. □

Lemma 3.3. *Let* $\Theta = (\alpha, \beta, \gamma) \in \mathcal{SI}_n$*,* $P \in SAPLS(\Theta)$ *and* $(\mathfrak{j}_\alpha, \mathfrak{j}_\beta, \mathfrak{j}_\gamma) = S(\Theta, P)$*. If* P *is the multiplication table of an associative partial quasigroup* $([n], \cdot)$*, then* $\Theta^t = (\alpha^t, \beta^t, \gamma^t) \in \mathcal{SI}_n$*, for all* $t \in \mathbb{N}$*.*

Proof. Let $t \in \mathbb{N}$. Since $P \in PLS(\Theta)$, it is verified that $P \in PLS(\Theta^t)$. Besides, since $([n], \cdot)$ is associative, then $\delta^t(i) = i \cdot \mathfrak{j}_\delta^t$, for all $\delta \in \{\alpha, \beta, \gamma\}$ and for all $i \in [n]$. So, $S(\Theta^t, P) = (\mathfrak{j}_\alpha^t, \mathfrak{j}_\beta^t, \mathfrak{j}_\gamma^t)$. □

Definition 3.4. *Let* $\Theta \in \mathcal{SI}_n$*. We will define the* orbit *of* Θ *as the set:*

$$\mathfrak{o}(\Theta) = \{\Theta' \in \mathcal{SI}_n : \exists t \in \mathbb{N} \text{ such that } \Theta' = \Theta^t\}.$$

Given $\Theta_1, \Theta_2 \in \mathcal{SI}_n$*, we will define the equivalence relation* $\Theta_1 \sim \Theta_2$ *if and only if they have the same orbit. The equivalence class of* $\Theta \in \mathcal{SI}_n$ *will be denoted by* $[\Theta]$*.*

The following result is verified:

Proposition 3.5. *Given* $\Theta_1, \Theta_2 \in \mathcal{SI}_n$ *such that* $\Theta_1 \sim \Theta_2$*, it is* $SAPLS(\Theta_1) = SAPLS(\Theta_2)$.

Proof. Let $P \in SAPLS(\Theta_1)$. Since $\mathfrak{o}(\Theta_1) = \mathfrak{o}(\Theta_2)$, it must be $\Theta_2 \in \mathfrak{o}(\Theta_1)$, because $\Theta_2 \in \mathfrak{o}(\Theta_2)$. So, there exists $t \in \mathbb{N}$ such that $\Theta_2 = \Theta_1^t$. From the proof of Lemma 3.3, it must be $P \in SAPLS(\Theta_2)$. Therefore, $SAPLS(\Theta_1) \subseteq SAPLS(\Theta_2)$, being analogous the reciprocal. $\square$

Definition 3.6. *Given* $[\Theta_1], [\Theta_2] \in \mathcal{SI}_n/\sim$*, we will define the equivalence relation* $[\Theta_1] \sim' [\Theta_2]$ *if and only if* $SAPLS(\Theta_1) = SAPLS(\Theta_2)$.

4 CLASSIFICATION OF ASSOCIATIVE PARTIAL QUASIGROUPS OF ORDER UP TO 5.

By following the classification given by Falcón [2], we have been implemented in a computer program all the results of the previous sections to generate all the possible cycle structures of the set of non-trivial Santilli's autotopisms of partial Latin squares of order up to 5, corresponding to associative partial quasigroups. For each cycle structure of a Latin square of order n, we show all the equivalence classes $[\Theta]$ of $\mathcal{SI}_n/\sim$, a partial Latin square $P \in SAPLS(\Theta)$ and the corresponding triple $S(\Theta, P)$. Finally, we show in the last column the set $SAPLS(\Theta)$, which is denoted by the capital letters $A_n, B_n, C_n, ...$ (described in Tables 11 and 12), for each order $n \in \mathbb{N}$. In this way, we can use this last column to show those equivalence classes of $\mathcal{SI}_n/\sim$ corresponding to the same equivalence class of $(\mathcal{SI}_n/\sim)/\sim'$.

$(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$	$[\Theta]$	P	$S(\Theta, P)$	$SAPLS(\Theta)$
$((0,1),(0,1),(2,0))$	$[((12),(12),\epsilon)]$	$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	$(2,2,1)$	A_2
$((0,1),(2,0),(0,1))$	$[((12),\epsilon,(12))]$	$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	$(2,1,2)$	A_2
$((2,0),(0,1),(0,1))$	$[(\epsilon,(12),(12))]$	$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	$(1,2,2)$	A_2

Table 1: Autotopisms of $\mathcal{SI}_2$ of associative partial quasigroups.

$(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$	$[\Theta]$	P	$S(\Theta, P)$	$SAPLS(\Theta)$
$((0,0,1),$ $(0,0,1),$ $(0,0,1))$	$[((123),(123),(132))]$	$\begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$	$(1,1,2)$	A_3
$((0,0,1),$ $(0,0,1),$ $(3,0,0))$	$[((123),(132),\epsilon)]$	$\begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$	$(1,2,3)$	A_3
$((0,0,1),$ $(3,0,0),$ $(0,0,1))$	$[((123),\epsilon,(123))]$	$\begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$	$(1,3,1)$	A_3
$((3,0,0),$ $(0,0,1),$ $(0,0,1))$	$[(\epsilon,(123),(123))]$	$\begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$	$(3,1,1)$	A_3

Table 2: Autotopisms of $\mathcal{SI}_3$ of associative partial quasigroups.

$(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$	$[\Theta]$	P	$S(\Theta, P)$	$SAPLS(\Theta)$
$((0,0,0,1),$ $(0,0,0,1),$ $(0,2,0,0))$	$[((1234),$ $(1234),$ $(13)(24))]$	$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$(1,1,2)$	A_4
	$[((1243),$ $(1243),$ $(14)(23))]$	$\begin{pmatrix} 2 & 4 & 1 & 3 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$	$(1,1,2)$	B_4
	$[((1324),$ $(1324),$ $(12)(34))]$	$\begin{pmatrix} 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$(1,1,3)$	C_4
$((0,0,0,1),$ $(0,2,0,0),$ $(0,0,0,1))$	$[((1234),$ $(13)(24),$ $(1432))]$	$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$(1,2,3)$	A_4
	$[((1243),$ $(14)(23),$ $(1342))]$	$\begin{pmatrix} 2 & 4 & 1 & 3 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$	$(1,2,4)$	B_4
	$[((1324),$ $(12)(34),$ $(1423))]$	$\begin{pmatrix} 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$(1,3,2)$	C_4

Table 3: Autotopisms of $\mathcal{SI}_4$ of associative partial quasigroups.

$(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$	$[\Theta]$	P	$S(\Theta, P)$	$SAPLS(\Theta)$
$((0,2,0,0), (0,0,0,1), (0,0,0,1))$	$[((13)(24), (1234), (1432))]$	$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$	$(1,4,2)$	A_4
	$[((14)(23), (1243), (1342))]$	$\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 1 & 4 & 2 \\ 2 & 4 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$(1,3,2)$	B_4
	$[((12)(34), (1324), (1423))]$	$\begin{pmatrix} 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \end{pmatrix}$	$(1,4,3)$	C_4
$((0,0,0,1), (0,0,0,1), (4,0,0,0))$	$[((1234), (1432), \epsilon)]$	$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$(1,3,4)$	A_4
	$[((1243), (1342), \epsilon)]$	$\begin{pmatrix} 2 & 4 & 1 & 3 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$	$(1,4,3)$	B_4
	$[((1324), (1423), \epsilon)]$	$\begin{pmatrix} 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$(1,2,4)$	C_4
$((0,0,0,1), (4,0,0,0), (0,0,0,1))$	$[((1234), \epsilon, (1234))]$	$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$(1,4,1)$	A_4
	$[((1243), \epsilon, (1243))]$	$\begin{pmatrix} 2 & 4 & 1 & 3 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$	$(1,3,1)$	B_4
	$[((1324), \epsilon, (1324))]$	$\begin{pmatrix} 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$(1,4,1)$	C_4
$((4,0,0,0), (0,0,0,1), (0,0,0,1))$	$[(\epsilon, (1234), (1234))]$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$	$(1,2,2)$	A_4
	$[(\epsilon, (1243), (1243))]$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$	$(1,2,2)$	B_4
	$[(\epsilon, (1324), (1324))]$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{pmatrix}$	$(1,3,3)$	C_4

Table 4: Autotopisms of $\mathcal{SI}_4$ of associative partial quasigroups.

$(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$	$[\Theta]$	P	$S(\Theta, P)$	$SAPLS(\Theta)$
$((0,2,0,0),(0,2,0,0),(0,2,0,0))$	$[((12)(34),(13)(24),(14)(23))]$	$\begin{pmatrix} 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix}$	$(1,4,3)$	D_4
	$[((12)(34),(14)(23),(13)(24))]$	$\begin{pmatrix} 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix}$	$(1,3,4)$	D_4
	$[((13)(24),(12)(34),(14)(23))]$	$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$	$(1,4,2)$	D_4
	$[((13)(24),(14)(23),(12)(34))]$	$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$	$(1,2,4)$	D_4
	$[((14)(23),(12)(34),(13)(24))]$	$\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$(1,3,2)$	D_4
	$[((14)(23),(13)(24),(12)(34))]$	$\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$(1,2,3)$	D_4
$((0,2,0,0),(0,2,0,0),(4,0,0,0))$	$[((12)(34),(12)(34),\epsilon)]$	$\begin{pmatrix} 2 & 1 & * & * \\ 1 & 2 & * & * \\ 4 & 3 & * & * \\ 3 & 4 & * & * \end{pmatrix}$	$(1,1,2)$	E_4
	$[((13)(24),(13)(24),\epsilon)]$	$\begin{pmatrix} 3 & * & 1 & * \\ 4 & * & 2 & * \\ 1 & * & 3 & * \\ 2 & * & 4 & * \end{pmatrix}$	$(1,1,3)$	F_4
	$[((14)(23),(14)(23),\epsilon)]$	$\begin{pmatrix} 4 & * & * & 1 \\ 3 & * & * & 2 \\ 2 & * & * & 3 \\ 1 & * & * & 4 \end{pmatrix}$	$(1,1,4)$	G_4
$((0,2,0,0),(4,0,0,0),(0,2,0,0))$	$[((12)(34),\epsilon,(12)(34))]$	$\begin{pmatrix} 2 & 1 & * & * \\ 1 & 2 & * & * \\ 4 & 3 & * & * \\ 3 & 4 & * & * \end{pmatrix}$	$(1,2,1)$	E_4
	$[((13)(24),\epsilon,(13)(24))]$	$\begin{pmatrix} 3 & * & 1 & * \\ 4 & * & 2 & * \\ 1 & * & 3 & * \\ 2 & * & 4 & * \end{pmatrix}$	$(1,3,1)$	F_4
	$[((14)(23),\epsilon,(14)(23))]$	$\begin{pmatrix} 4 & * & * & 1 \\ 3 & * & * & 2 \\ 2 & * & * & 3 \\ 1 & * & * & 4 \end{pmatrix}$	$(1,4,1)$	G_4

Table 5: Autotopisms of $\mathcal{SI}_4$ of associative partial quasigroups.

$(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$	$[\Theta]$	P	$S(\Theta, P)$	$SAPLS(\Theta)$
$((4,0,0,0), (0,2,0,0), (0,2,0,0))$	$[(\epsilon, (12)(34), (12)(34))]$	$\begin{pmatrix} 2 & 1 & * & * \\ 1 & 2 & * & * \\ 4 & 3 & * & * \\ 3 & 4 & * & * \end{pmatrix}$	$(1,2,1)$	E_4
	$[(\epsilon, (13)(24), (13)(24))]$	$\begin{pmatrix} 3 & * & 1 & * \\ 4 & * & 2 & * \\ 1 & * & 3 & * \\ 2 & * & 4 & * \end{pmatrix}$	$(1,3,1)$	F_4
	$[(\epsilon, (14)(23), (14)(23))]$	$\begin{pmatrix} 4 & * & * & 1 \\ 3 & * & * & 2 \\ 2 & * & * & 3 \\ 1 & * & * & 4 \end{pmatrix}$	$(1,4,1)$	G_4

Table 6: Autotopisms of $\mathcal{SI}_4$ of associative partial quasigroups.

$(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$	$[\Theta]$	P	$S(\Theta, P)$	$SAPLS(\Theta)$
$((0,0,0,0,1), (0,0,0,0,1), (0,0,0,0,1))$	$[((12345), (12345), (13524))]$	$\begin{pmatrix} 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(1,1,2)$	A_5
	$[((12345), (13524), (14253))]$	$\begin{pmatrix} 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(1,2,3)$	A_5
	$[((12345), (14253), (15432))]$	$\begin{pmatrix} 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(1,3,4)$	A_5
	$[((12354), (12354), (13425))]$	$\begin{pmatrix} 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 5 & 4 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}$	$(1,1,2)$	B_5
	$[((12354), (13425), (15243))]$	$\begin{pmatrix} 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 5 & 4 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}$	$(1,2,3)$	B_5
	$[((12354), (15243), (14532))]$	$\begin{pmatrix} 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 5 & 4 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}$	$(1,3,5)$	B_5
	$[((12453), (12453), (14325))]$	$\begin{pmatrix} 2 & 4 & 1 & 5 & 3 \\ 4 & 5 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$	$(1,1,2)$	C_5

Table 7: Autotopisms of $\mathcal{SI}_5$ of associative partial quasigroups.

$(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$	$[\Theta]$	P	$S(\Theta, P)$	$SAPLS(\Theta)$
$((0,0,0,0,1),$ $(0,0,0,0,1),$ $(0,0,0,0,1))$	$[((12453),$ $(14325),$ $(15234))]$	$\begin{pmatrix} 2 & 4 & 1 & 5 & 3 \\ 4 & 5 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$	$(1,2,4)$	C_5
	$[((12453),$ $(15234),$ $(13542))]$	$\begin{pmatrix} 2 & 4 & 1 & 5 & 3 \\ 4 & 5 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$	$(1,4,5)$	C_5
	$[((12435),$ $(12435),$ $(14523))]$	$\begin{pmatrix} 2 & 4 & 5 & 3 & 1 \\ 4 & 3 & 1 & 5 & 2 \\ 5 & 1 & 4 & 2 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(1,1,2)$	D_5
	$[((12435),$ $(13254),$ $(15342))]$	$\begin{pmatrix} 2 & 4 & 5 & 3 & 1 \\ 4 & 3 & 1 & 5 & 2 \\ 5 & 1 & 4 & 2 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(1,4,3)$	D_5
	$[((12435),$ $(14523),$ $(13254))]$	$\begin{pmatrix} 2 & 4 & 5 & 3 & 1 \\ 4 & 3 & 1 & 5 & 2 \\ 5 & 1 & 4 & 2 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(1,2,4)$	D_5
	$[((12543),$ $(12543),$ $(15324))]$	$\begin{pmatrix} 2 & 5 & 1 & 3 & 4 \\ 5 & 4 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 3 & 5 & 2 & 1 \end{pmatrix}$	$(1,1,2)$	E_5
	$[((12543),$ $(14235),$ $(13452))]$	$\begin{pmatrix} 2 & 5 & 1 & 3 & 4 \\ 5 & 4 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 3 & 5 & 2 & 1 \end{pmatrix}$	$(1,5,4)$	E_5
	$[((12543),$ $(15324),$ $(14235))]$	$\begin{pmatrix} 2 & 5 & 1 & 3 & 4 \\ 5 & 4 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 3 & 5 & 2 & 1 \end{pmatrix}$	$(1,2,5)$	E_5
	$[((12534),$ $(12534),$ $(15423))]$	$\begin{pmatrix} 2 & 5 & 4 & 1 & 3 \\ 5 & 3 & 1 & 2 & 4 \\ 4 & 1 & 5 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 5 & 1 \end{pmatrix}$	$(1,1,2)$	F_5
	$[((12534),$ $(13245),$ $(14352))]$	$\begin{pmatrix} 2 & 5 & 4 & 1 & 3 \\ 5 & 3 & 1 & 2 & 4 \\ 4 & 1 & 5 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 5 & 1 \end{pmatrix}$	$(1,5,3)$	F_5
	$[((12534),$ $(15423),$ $(13245))]$	$\begin{pmatrix} 2 & 5 & 4 & 1 & 3 \\ 5 & 3 & 1 & 2 & 4 \\ 4 & 1 & 5 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 5 & 1 \end{pmatrix}$	$(1,2,5)$	F_5

Table 8: Autotopisms of $\mathcal{SI}_5$ of associative partial quasigroups.

$(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$	$[\Theta]$	P	$S(\Theta, P)$	$SAPLS(\Theta)$
$((0,0,0,0,1),$ $(0,0,0,0,1),$ $(5,0,0,0,0))$	$[((12345),$ $(15432),$ $\epsilon)]$	$\begin{pmatrix} 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(1,4,5)$	A_5
	$[((12354),$ $(14532),$ $\epsilon)]$	$\begin{pmatrix} 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 5 & 4 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}$	$(1,5,4)$	B_5
	$[((12453),$ $(13542),$ $\epsilon)]$	$\begin{pmatrix} 2 & 4 & 1 & 5 & 3 \\ 4 & 5 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$	$(1,5,3)$	C_5
	$[((12435),$ $(15234),$ $\epsilon)]$	$\begin{pmatrix} 2 & 4 & 5 & 3 & 1 \\ 4 & 3 & 1 & 5 & 2 \\ 5 & 1 & 4 & 2 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(1,3,5)$	D_5
	$[((12543),$ $(13452),$ $\epsilon)]$	$\begin{pmatrix} 2 & 5 & 1 & 3 & 4 \\ 5 & 4 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 3 & 5 & 2 & 1 \end{pmatrix}$	$(1,4,3)$	E_5
	$[((12534),$ $(14352),$ $\epsilon)]$	$\begin{pmatrix} 2 & 5 & 4 & 1 & 3 \\ 5 & 3 & 1 & 2 & 4 \\ 4 & 1 & 5 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 5 & 1 \end{pmatrix}$	$(1,3,4)$	F_5
$((0,0,0,0,1),$ $(5,0,0,0,0),$ $(0,0,0,0,1))$	$[((12345),$ $\epsilon,$ $(12345))]$	$\begin{pmatrix} 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(1,5,1)$	A_5
	$[((12354),$ $\epsilon,$ $(12354))]$	$\begin{pmatrix} 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 5 & 4 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}$	$(1,4,1)$	B_5
	$[((12453),$ $\epsilon,$ $(12453))]$	$\begin{pmatrix} 2 & 4 & 1 & 5 & 3 \\ 4 & 5 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$	$(1,3,1)$	C_5
	$[((12435),$ $\epsilon,$ $(12435))]$	$\begin{pmatrix} 2 & 4 & 5 & 3 & 1 \\ 4 & 3 & 1 & 5 & 2 \\ 5 & 1 & 4 & 2 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(1,5,1)$	D_5
	$[((12543),$ $\epsilon,$ $(12543))]$	$\begin{pmatrix} 2 & 5 & 1 & 3 & 4 \\ 5 & 4 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 3 & 5 & 2 & 1 \end{pmatrix}$	$(1,3,1)$	E_5
	$[((12534),$ $\epsilon,$ $(12534))]$	$\begin{pmatrix} 2 & 5 & 4 & 1 & 3 \\ 5 & 3 & 1 & 2 & 4 \\ 4 & 1 & 5 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 5 & 1 \end{pmatrix}$	$(1,4,1)$	F_5

Table 9: Autotopisms of $\mathcal{SI}_5$ of associative partial quasigroups.

$(\mathbf{l}_\alpha, \mathbf{l}_\beta, \mathbf{l}_\gamma)$	$[\Theta]$	P	$S(\Theta, P)$	$SAPLS(\Theta)$
$((5,0,0,0,0),$ $(0,0,0,0,1),$ $(0,0,0,0,1))$	$[(\epsilon,$ $(12345),$ $(12345))]$	$\begin{pmatrix} 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(5,1,1)$	A_5
	$[(\epsilon,$ $(12354),$ $(12354))]$	$\begin{pmatrix} 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 5 & 4 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}$	$(4,1,1)$	B_5
	$[(\epsilon,$ $(12453),$ $(12453))]$	$\begin{pmatrix} 2 & 4 & 1 & 5 & 3 \\ 4 & 5 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$	$(3,1,1)$	C_5
	$[(\epsilon,$ $(12435),$ $(12435))]$	$\begin{pmatrix} 2 & 4 & 5 & 3 & 1 \\ 4 & 3 & 1 & 5 & 2 \\ 5 & 1 & 4 & 2 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(5,1,1)$	D_5
	$[(\epsilon,$ $(12543),$ $(12543))]$	$\begin{pmatrix} 2 & 5 & 1 & 3 & 4 \\ 5 & 4 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 3 & 5 & 2 & 1 \end{pmatrix}$	$(3,1,1)$	E_5
	$[(\epsilon,$ $(12534),$ $(12534))]$	$\begin{pmatrix} 2 & 5 & 4 & 1 & 3 \\ 5 & 3 & 1 & 2 & 4 \\ 4 & 1 & 5 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 5 & 1 \end{pmatrix}$	$(4,1,1)$	F_5

Table 10: Autotopisms of $\mathcal{SI}_5$ of associative partial quasigroups.

A_2	$\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \ , \ \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}$
A_3	$\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}$
A_4	$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix} \right\}$
B_4	$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 1 & 3 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 4 & 2 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 4 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 1 & 4 & 2 \\ 2 & 4 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \right\}$
C_4	$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \right\}$
D_4	$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \right\}$
E_4	$\left\{ \begin{pmatrix} 1 & 2 & * & * \\ 2 & 1 & * & * \\ 3 & 4 & * & * \\ 4 & 3 & * & * \end{pmatrix}, \begin{pmatrix} 2 & 1 & * & * \\ 1 & 2 & * & * \\ 4 & 3 & * & * \\ 3 & 4 & * & * \end{pmatrix}, \begin{pmatrix} * & * & 1 & 2 \\ * & * & 2 & 1 \\ * & * & 3 & 4 \\ * & * & 4 & 3 \end{pmatrix}, \begin{pmatrix} * & * & 2 & 1 \\ * & * & 1 & 2 \\ * & * & 4 & 3 \\ * & * & 3 & 4 \end{pmatrix} \right\} \cup$ $C_4 \cup D_4$
F_4	$\left\{ \begin{pmatrix} 1 & * & 3 & * \\ 2 & * & 4 & * \\ 3 & * & 1 & * \\ 4 & * & 2 & * \end{pmatrix}, \begin{pmatrix} 3 & * & 1 & * \\ 4 & * & 2 & * \\ 1 & * & 3 & * \\ 2 & * & 4 & * \end{pmatrix}, \begin{pmatrix} * & 1 & * & 3 \\ * & 2 & * & 4 \\ * & 3 & * & 1 \\ * & 4 & * & 2 \end{pmatrix}, \begin{pmatrix} * & 3 & * & 1 \\ * & 4 & * & 2 \\ * & 1 & * & 3 \\ * & 2 & * & 4 \end{pmatrix} \right\} \cup$ $A_4 \cup D_4$
G_4	$\left\{ \begin{pmatrix} 1 & * & * & 4 \\ 2 & * & * & 3 \\ 3 & * & * & 2 \\ 4 & * & * & 1 \end{pmatrix}, \begin{pmatrix} 4 & * & * & 1 \\ 3 & * & * & 2 \\ 2 & * & * & 3 \\ 1 & * & * & 4 \end{pmatrix}, \begin{pmatrix} * & 1 & 4 & * \\ * & 2 & 3 & * \\ * & 3 & 2 & * \\ * & 4 & 1 & * \end{pmatrix}, \begin{pmatrix} * & 4 & 1 & * \\ * & 3 & 2 & * \\ * & 2 & 3 & * \\ * & 1 & 4 & * \end{pmatrix} \right\} \cup$ $B_4 \cup D_4$

Table 11: Sets of partial Latin squares of order $2 \leq n \leq 4$ corresponding to associative partial quasigroups related to Santilli's autotopisms.

A_5	$\left\{ \begin{pmatrix} 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \right\}$
B_5	$\left\{ \begin{pmatrix} 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 5 & 4 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 2 & 5 & 3 \\ 5 & 4 & 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 2 & 5 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \\ 5 & 4 & 1 & 3 & 2 \\ 3 & 5 & 4 & 2 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 3 & 5 & 4 & 2 & 1 \\ 5 & 4 & 1 & 3 & 2 \\ 4 & 1 & 2 & 5 & 3 \\ 2 & 3 & 5 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 4 & 1 & 3 & 2 \\ 4 & 1 & 2 & 5 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 2 & 1 \\ 2 & 3 & 5 & 1 & 4 \end{pmatrix} \right\}$
C_5	$\left\{ \begin{pmatrix} 2 & 4 & 1 & 5 & 3 \\ 4 & 5 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 1 & 5 & 2 & 4 \\ 4 & 5 & 2 & 3 & 1 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 2 & 3 & 1 \\ 5 & 3 & 4 & 1 & 2 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 1 & 5 & 2 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 3 & 1 & 5 & 2 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \\ 2 & 4 & 1 & 5 & 3 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 3 & 4 & 1 & 2 \\ 3 & 1 & 5 & 2 & 4 \\ 4 & 5 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix} \right\}$
D_5	$\left\{ \begin{pmatrix} 2 & 4 & 5 & 3 & 1 \\ 4 & 3 & 1 & 5 & 2 \\ 5 & 1 & 4 & 2 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 3 & 1 \\ 3 & 5 & 2 & 1 & 4 \\ 4 & 3 & 1 & 5 & 2 \\ 5 & 1 & 4 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 2 & 1 & 4 \\ 5 & 1 & 4 & 2 & 3 \\ 2 & 4 & 5 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 5 & 1 & 4 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \\ 2 & 4 & 5 & 3 & 1 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 3 & 1 & 5 & 2 \\ 3 & 5 & 2 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 2 & 3 \\ 2 & 4 & 5 & 3 & 1 \end{pmatrix} \right\}$
E_5	$\left\{ \begin{pmatrix} 2 & 5 & 1 & 3 & 4 \\ 5 & 4 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 3 & 5 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 3 & 5 & 2 & 1 \\ 5 & 4 & 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 5 & 4 & 2 & 1 & 3 \\ 4 & 3 & 5 & 2 & 1 \\ 2 & 5 & 1 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 4 & 3 & 5 & 2 & 1 \\ 3 & 1 & 4 & 5 & 2 \\ 5 & 4 & 2 & 1 & 3 \\ 2 & 5 & 1 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 4 & 5 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 2 & 1 \\ 5 & 4 & 2 & 1 & 3 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix} \right\}$
F_5	$\left\{ \begin{pmatrix} 2 & 5 & 4 & 1 & 3 \\ 5 & 3 & 1 & 2 & 4 \\ 4 & 1 & 5 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 5 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \\ 3 & 4 & 2 & 5 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 5 & 3 & 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 2 & 5 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 2 & 5 & 4 & 1 & 3 \\ 5 & 3 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 5 & 3 & 1 & 2 & 4 \\ 3 & 4 & 2 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \\ 4 & 1 & 5 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 5 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 2 & 4 \\ 3 & 4 & 2 & 5 & 1 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix} \right\}$

Table 12: Sets of partial Latin squares of order 5 corresponding to associative partial quasigroups related to Santilli's autotopisms.

5 FINAL REMARKS

Although in Section 4 we give all the cycle structures of Santilli's autotopisms of the Latin squares of order up to 5 corresponding to associative partial quasigroups, let us remark that the properties of Sections 2 and 3 can be implemented in an algorithm to obtain all the cycle structures of Santilli's autotopisms of the Latin squares of greater orders.

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