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Isotopisms of Lie algebras.

O.J. Falcón[1], R.M. Falcón[2] and J. Núñez[1]

[1]Dpto. de Geometría y Topología. [2]Dpto. de Matemática Aplicada I.

oscfalgan@yahoo.es, rafalgan@us.es, jnvaldes@us.es

Abstract. The distribution of algebras into equivalence classes is usually done according to the concept of isomorphism. However, such a distribution can also be done into isotopism classes. The concept of isotopism was explicitly introduced in 1942 by Abraham Adrian Albert [1] to classify non-associative algebras. In this poster we deal with the study of isotopisms of Lie algebras. The reasons for using both criteria, isotopisms and isomorphisms, to classify Lie algebras is due to that classifications by isotopisms are different from those by isomorphisms, which involves obtaining new information about these algebras. On a sake of example, we indicate some recent results obtained by ourselves, which are related to the distribution into isomorphism and isotopism classes of filiform Lie algebras over finite fields.

Introduction.

Two algebras $\mathfrak{g}$ and $\mathfrak{h}$ are ***isotopic*** [1] if there exist three non-singular linear transformations f, g and h from $\mathfrak{g}$ to $\mathfrak{h}$ such that

$$[f(u), g(v)]_{\mathfrak{h}} = h([u,v]_{\mathfrak{g}}), \text{ for all } u, v \in \mathfrak{g}. \qquad (1)$$

It is also said $\mathfrak{g}$ to be an ***isotope*** of $\mathfrak{h}$. The tuple (f,g,h) is called an ***isotopism*** of algebras. It is said to be ***principal*** if h is the identity transformation. To be isotopic is an equivalence relation among algebras. If $f = g = h$, then the algebras $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic. Isotopisms are therefore a generalization of isomorphisms that can be used to gather together non-isomorphic algebras. Since the original paper of Albert [1], it has been analyzed the isotopisms of a wide variety of types of algebras like division, Jordan, alternative or structural algebras. Nevertheless, there barely exists any result about isotopisms of Lie algebras, apart from the next two results of Albert and Bruck.

Lemma. [1] *A principal isotope $\mathfrak{g}$ of a Lie algebra $\mathfrak{h}$ with respect to an isotopism (f,g,Id) is a Lie algebra if and only if the following two conditions are verified.*

i. $[f(u), g(v)]_{\mathfrak{h}} = -[f(v), g(u)]_{\mathfrak{h}}$, for all $u, v \in \mathfrak{g}$.

ii. $[f([f(u),g(v)]_{\mathfrak{h}}), g(w)]_{\mathfrak{h}} - [f([f(u),g(w)]_{\mathfrak{h}}), g(v)]_{\mathfrak{h}} - [f(u), g([f(v),g(w)]_{\mathfrak{h}})]_{\mathfrak{h}} = 0$, for all $u, v, w \in \mathfrak{g}$.

Theorem. [4] *It is verified that:*

i. The Lie algebra of order $n(n-1)/2$, consisting of all skew-symmetric matrices, over any subfield of the field of all reals, under the multiplication $A \circ B = AB - BA$, is isotopically simple.

ii. The Lie algebra of order $n(n-1)$, consisting of all skew-hermitian matrices in any field $R(i)$ (where R is a subfield of the reals and $i^2 = -1$), under the multiplication $A \circ B = AB - BA$, is an isotopically simple algebra over R.

In the last years, the concept of isotopism of Lie algebras has reappeared in the literature. In 2008, Jiménez-Gestal and Pérez-Iquierdo [6] study the relationship that exists between the isotopisms of a finite-dimensional real division algebra and the Lie algebra of its ternary derivations. Shortly after, Allison et al. [2, 3] study isotopes of a class of graded Lie algebras called Lie tori, but the notion of isotopism that they use is quite different from the conventional one. More recently, in 2014, Falcón et al. [5] return to the conventional notion of isotopism in order to study the distribution of filiform Lie algebras over finite fields into isotopism classes. In the rest of the poster, we expose precisely some results in this regard.

Preliminaries on Lie algebras.

An n-dimensional algebra $\mathfrak{g}$ is a ***Lie algebra*** if its second inner law is bilinear and anti-commutative and satisfies the ***Jacobi identity***

$$J(u,v,w) = [u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0.$$

for all $u, v, w \in \mathfrak{g}$. The ***centralizer*** of a subset $\mathfrak{h} \subseteq \mathfrak{g}$ is the set $\mathrm{Cen}_{\mathfrak{g}}(\mathfrak{h}) = \{u \in \mathfrak{g} \mid [u,v] = 0, \text{ for all } v \in \mathfrak{h}\}$. Given $m \leq n$, we define $d_m(\mathfrak{g}) = \max\{\dim \mathrm{Cen}_{\mathfrak{g}}(\mathfrak{h}) : \ \mathfrak{h} \textit{ is an } m\textit{-dimensional ideal}\}$. The sequence $d(\mathfrak{g}) = \{d_1(\mathfrak{g}), \ldots, d_n(\mathfrak{g})\}$ is an isotopism invariant of Lie algebras.

The ***lower central series*** of $\mathfrak{g}$ is defined as $\mathfrak{g}^1 = \mathfrak{g}$, $\mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}]$, $\ldots$, $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$, $\ldots$. A basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$ is ***compatible with respect to its lower central series*** if $\mathfrak{g}^2 = \langle e_2, \ldots, e_{n-1}\rangle$, $\mathfrak{g}^3 = \langle e_2, \ldots, e_{n-2}\rangle, \ldots, \mathfrak{g}^{n-1} = \langle e_2 \rangle$, $\mathfrak{g}^n = 0$.

The Lie algebra $\mathfrak{g}$ is ***filiform*** if $\dim \mathfrak{g}^k = n - k$, for all $k \in \{2, \ldots, n\}$. We define a ***filiform basis*** of $\mathfrak{g}$ as a compatible basis $\{e_1, \ldots, e_n\}$ with respect to its lower central series such that either $[e_1, e_i] = e_{i-1}$ or $[e_i, e_n] = e_{i-1}$, for $3 \leq i \leq n$. Every finite-dimensional filiform Lie algebra has a filiform basis. If the only brackets distinct of zero are those of the form $[e_1, e_i] = e_{i-1}$, then $\mathfrak{g}$ is called ***model*** and is not isotopic to any other filiform Lie algebra of the same dimension. In fact, the model algebra is the only isomorphism (isotopism) class of filiform Lie algebras of dimension $n \leq 4$. For $n = 5$, there exist two isomorphism (isotopism) classes of filiform Lie algebras: the model algebra and that having an adapted basis satisfying the bracket $[e_4, e_5] = e_2$.

Isotopisms of filiform Lie algebras over finite fields.

It is verified that

a) Given a six-dimensional filiform Lie algebra $\mathfrak{g}$ over $\mathbb{K}$, there exist $a, b, c \in \mathbb{K}$ such that

$$\mathfrak{g} \cong \mathfrak{g}^6_{abc} \equiv \begin{cases} [e_1, e_{i+1}] = e_i, \text{ for all } i > 1, \\ [e_4, e_5] = ae_2, \\ [e_4, e_6] = be_2 + ae_3, \\ [e_5, e_6] = ce_2 + be_3 + ae_4. \end{cases}$$

b) Given a seven-dimensional filiform Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ of characteristic distinct of two, there exist $a, b, c, d \in \mathbb{K}$ and a filiform basis such that

$$\mathfrak{g} \cong \mathfrak{g}^7_{abcd} \equiv \begin{cases} [e_1, e_{i+1}] = e_i, \text{ for all } i > 1, \\ [e_4, e_7] = ae_2, \\ [e_5, e_6] = be_2, \\ [e_5, e_7] = ce_2 + (a+b)e_3, \\ [e_6, e_7] = de_2 + ce_3 + (a+b)e_4. \end{cases}$$

If the characteristic is two, then $\mathfrak{g}$ can also be isomorphic to either

$$\mathfrak{g}^7_a \equiv \begin{cases} [e_1, e_3] = [e_4, e_6] = [e_5, e_7] = e_2, \\ [e_4, e_7] = [e_5, e_6] = e_3, \\ [e_1, e_5] = e_4, \\ [e_1, e_6] = e_5, \\ [e_1, e_7] = e_6, \\ [e_6, e_7] = e_3 + ae_4, \text{ where } a \in \{0,1\}. \end{cases}$$

or

$$\mathfrak{h}^7_a \equiv \begin{cases} [e_3, e_7] = [e_4, e_6] = e_2, \\ [e_1, e_4] = [e_5, e_6] = e_3, \\ [e_5, e_7] = e_4, \\ [e_6, e_7] = e_5, \\ [e_1, e_7] = e_6, \\ [e_4, e_7] = ae_2, \text{ where } a \in \{0,1\}. \end{cases}$$

Theorem. *It is verified that*

a) There exist 5 isotopism classes of six-dimensional filiform Lie algebras:

$$\mathfrak{g}^6_{000}, \mathfrak{g}^6_{001}, \mathfrak{g}^6_{010}, \mathfrak{g}^6_{100} \textit{ and } \mathfrak{g}^6_{110}.$$

b) There exist 10 isotopism classes of seven-dimensional filiform Lie algebras over a field of characteristic two:

$$\mathfrak{g}^7_{0000}, \mathfrak{g}^7_{0001}, \mathfrak{g}^7_{0010}, \mathfrak{g}^7_{0100}, \mathfrak{g}^7_{1000}, \mathfrak{g}^7_{1100}, \mathfrak{g}^7_{1110}, \mathfrak{g}^7_0, \mathfrak{g}^7_1 \textit{ and } \mathfrak{h}^7_0.$$

c) There exist 8 isotopism classes of 7-dimensional filiform Lie algebras over an algebraically closed field of characteristic distinct of two and also over the finite field $\mathbb{F}_p$, where $p \neq 2$:

$$\mathfrak{g}^7_{0000}, \mathfrak{g}^7_{0001}, \mathfrak{g}^7_{0010}, \mathfrak{g}^7_{0100}, \mathfrak{g}^7_{1000}, \mathfrak{g}^7_{1100}, \mathfrak{g}^7_{1(-1)00} \textit{ and } \mathfrak{g}^7_{1(-1)10}.$$

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