Partial Latin squares related to a given autotopism.

Raúl M. Falcón

Department of Applied Mathematics I
University of Seville (Spain)
rafalgan@us.es

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Introduction

$\mathcal{CS}_n$: Cycle structures of a permutation.

- $\pi = (1234)(56)(78)(9) \Rightarrow z_\pi = 4^1 \cdot 2^2 \cdot 1.$
- $|\mathcal{CS}_n| = p(n) \equiv$ Number of partitions of n.

Two permutations are conjugate if and only if they have the same cycle structure.

$$\pi_2 = \pi'^{-1}\pi_1\pi'$$

$\mathcal{LS}_n$: Latin squares.

- Latin square: A **Latin square** L of order n, is a $n \times n$ array with elements chosen from the set $[n] = \{1, 2, \ldots, n\}$, such that each symbol occurs precisely once in each row and each column.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{pmatrix} \in \mathcal{LS}_4$$

- $O(L) = \{(r \equiv \text{ row}, c \equiv \text{ column}, s \equiv \text{ symbol})\}$.

- Parastrophism: $\pi \in S_n$

$$O(L^\pi) = \{(l_{\pi(1)}, l_{\pi(2)}, l_{\pi(3)}) \mid (l_1, l_2, l_3) \in O(L)\}.$$

- Isotopism $(\sim)$: $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_n = S_n^3$

$$O(L^\Theta) = \{(\alpha(r), \beta(c), \gamma(s)) \mid (r, c, s) \in O(L)\} \rightsquigarrow [L]$$

- $\Theta = ((12)(34), (12)(3)(4), (1234)) \Rightarrow z_\Theta = (2^2, 2 \cdot 1^2, 4)$.

$\mathfrak{A}_n$: Autotopisms of a Latin square.

- Autotopism: $L^{\Theta} = L \rightsquigarrow \mathfrak{A}_n$.
- $\mathcal{LS}_{\Theta} = \{L \in \mathcal{LS}_n \mid L^{\Theta} = L\}$.
- $\mathcal{CS}_{\mathfrak{A}_n}$ is known for $n \leq 17$ [Falcón, 2007; Stones, Vojtěchovský and Wanless, 2011]
- $\Delta(\Theta) = |\mathcal{LS}_{\Theta}|$ only depends on $z_{\Theta} \rightsquigarrow \Delta(z)$.
- $\Delta(z)$ is known for $n \leq 7$ [Falcón, Martín-Morales, 2007] and the majority of $n = 8$ and $n = 9$ [Falcón, Martín-Morales, 2008].

Latin Squares and the $3PAP_n$

- *3-dimensional planar assignment problem* ($3PAP_n$):

$$\begin{array}{ll} \min \sum_{r,c,s\in[n]} w_{rcs}\cdot x_{rcs}, & \\ \text{s.t.} \quad \sum_{r\in[n]} x_{rcs} = 1, \forall c,s\in[n], & \\ \quad\quad \sum_{c\in[n]} x_{rcs} = 1, \forall r,s\in[n], & (3PAP_n) \\ \quad\quad \sum_{s\in[n]} x_{rcs} = 1, \forall r,c\in[n], & \\ \quad\quad x_{rcs}\in\{0,1\}, \forall r,c,s\in[n], & \end{array}$$

- It can be defined the 1-1 correspondence:

$$x_{rcs} = \begin{cases} 1, & \text{if } (r,c,s)\in O(L), \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{pmatrix} \longleftrightarrow \begin{array}{l} (1,0,0,0,\ 0,1,0,0,\ 0,0,1,0,\ 0,0,0,1, \\ 0,1,0,0,\ 1,0,0,0,\ 0,0,0,1,\ 0,0,1,0, \\ 0,0,1,0,\ 0,0,0,1,\ 0,1,0,0,\ 1,0,0,0, \\ 0,0,0,1,\ 0,0,1,0,\ 1,0,0,0,\ 0,1,0,0) \end{array}$$

Partial Latin Squares $\mathcal{PLS}_n$.

- Partial Latin square: A **partial Latin square** P of order n, is a $n \times n$ array with elements chosen from $[n]$, such that each symbol occurs at most once in each row and each column.

$$\begin{pmatrix} 1 & \cdot & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & \cdot & 1 \\ \cdot & 3 & 1 & \cdot \end{pmatrix} \in \mathcal{PLS}_4$$

- $O(P) = \{(r, c, s) \mid s \neq \emptyset\}$.

- $\mathcal{PLS}_\Theta = \{P \in \mathcal{PLS}_n \mid P^\Theta = P\}$.

Partial Latin Squares $\mathcal{PLS}_n$.

- P can be completed to a Latin square L if $O(P) \subseteq O(L)$.
- P-coefficient of symmetry of Θ: The computation of $\Delta(z)$ can be simplified if a multiplicative factor $c_P \in \mathbb{N}$ is found s.t.

$$\Delta(z) = c_P \cdot |\mathcal{LS}_{\Theta,P}|.$$

Main problem: c_P becomes crucial in the processing of high orders, but none exhaustive study has been developed.

A comprehensive analysis of $\mathfrak{A}_{\mathcal{P}_n}$ and $\mathcal{PLS}_\Theta$ has not been properly done until now.

The set $\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$.

LCM_n

Lemma

$\Theta \in \mathfrak{A}_{\mathcal{P}\,n} \Leftrightarrow z_\Theta \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}\,n}}$.

$$\mathrm{LCM}_n = \{(i,j,k) \in [n]^3 \mid \mathrm{lcm}(i,j) = \mathrm{lcm}(i,k) = \mathrm{lcm}(j,k) = \mathrm{lcm}(i,j,k)\}.$$

Lemma

(Generalization for $\mathcal{PLS}_\Theta$ of the necessary condition [Stones, Vojtěchovský and Wanless, 2011] for membership in $\mathfrak{A}_n$).

- $z = (z_1, z_2, z_3) \in \mathcal{CS}_{\mathfrak{I}_n}$.

$z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}\,n}} \Leftrightarrow \exists (i,j,k) \in \mathrm{LCM}_n$ s.t. $z_{1i} \cdot z_{2j} \cdot z_{3k} > 0$.

LCM$_n$

Lemma

$\mathfrak{A}_n \subset \mathfrak{A}_{\mathcal{P}\,n} \subset \mathfrak{I}_n,\ \forall n > 1.$

$$n > 1:\ (1^2, 1^2, n^1) \notin \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}\,n}}.$$

$$(2,2,2) \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}\,2}} \setminus \mathcal{CS}_{\mathfrak{A}\,2}.$$

$$(2 \cdot 1^{n-2}, 2 \cdot 1^{n-2}, 1^n) \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}\,n}} \setminus \mathcal{CS}_{\mathfrak{A}\,n}.$$

$\mathcal{CS}_{n,m}$

$$\mathcal{CS}_{n,m} = \{n^{z_n} \cdot \ldots \cdot 2^{z_2} \cdot 1^{z_1} \in \mathcal{CS}_n \mid z_m > 0 \text{ and } z_i = 0, \forall i \in [m-1]\},$$

Lemma

$$|\mathcal{CS}_{n,m}| = \begin{cases} 1, \text{ if } m = n, \\ 0, \text{ if } m \in \{\lceil \frac{n}{2} \rceil, \ldots, n-1\}, \\ p(n-m) - \sum_{i=1}^{m-1} |\mathcal{CS}_{n-m,i}|, \text{ otherwise.} \end{cases}$$

Proposition

$$|\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}\,n}| \geq \sum_{(i,j,k) \in \mathrm{LCM}_n} |\mathcal{CS}_{n,i}| \cdot |\mathcal{CS}_{n,j}| \cdot |\mathcal{CS}_{n,k}|.$$

Theorem

$$\lim_{n \to \infty} \frac{|\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}\,n}|}{|\mathcal{CS}_{\mathfrak{I}\,n}|} = 1.$$

ORDER $n \leq 17$

n	$\lvert\mathcal{CS}_{\mathfrak{A}_n}\rvert$	$\lvert\mathcal{CS}_{n,m}\rvert$ m = 1	2	3	4	5	6	7	8	$\lvert\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}_n}}\rvert$	$\lvert[\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}_n}}]\rvert$
1	1									1	1
2	4	1								5	3
3	6	2								15	7
4	19	3	1							65	22
5	8	5	1							223	60
6	45	7	2	1						869	197
7	12	11	2	1						2535	526
8	87	15	4	1	1					7663	1492
9	43	22	4	2	1					21156	3937
10	89	30	7	2	1	1				60264	10850
11	21	42	8	3	1	1				150953	26628
12	407	56	12	4	2	1	1			385538	66984
13	27	77	14	5	2	1	1			915452	157398
14	141	101	21	6	3	1	1	1		2193225	374127
15	150	135	24	9	3	2	1	1		4928696	836154
16	503	176	34	10	5	2	1	1	1	11209311	1893607
17	40	231	41	13	5	3	1	1	1	24406191	4110132

The size of a partial Latin square related to an autotopism.

Θ-DECOMPOSITION

- $z = (z_1, z_2, z_3) \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}\,n}}$.
- $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_z$.
- $P \in \mathcal{PLS}_\Theta$.

Θ-decomposition of P: P can be decomposed into $n_{z_1} \cdot n_{z_2}$ blocks P_{ij} whose rows and columns are respectively determined by the elements of the cycle α_i of α and the cycle β_j of β:

$$O(P_{ij}) = \{(r, c, s) \in O(P) \mid r \in \alpha_i \text{ and } c \in \beta_j\}.$$

$$\begin{pmatrix} \cdot & \cdot & * & \circ \\ \cdot & \cdot & * & \circ \\ \diamond & \diamond & \triangle & \nabla \\ \square & \square & \triangleleft & \triangleright \end{pmatrix}$$

$$\Theta = ((12)(3)(4), (12)(3)(4), (34)(1)(2))$$

Sizes

$$\mathrm{LCM}_z = \{(i,j) \in [n]^2 \mid \exists k \in [n] \text{ s.t. } (i,j,k) \in \mathrm{LCM}_n \text{ and } z_{1i} \cdot z_{2j} \cdot z_{3k} > 0\}.$$

Lemma

- $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}\,n}$.
- $\Theta \in \mathfrak{I}_z$.
- $P \in \mathcal{PLS}_\Theta$.
- $i \times j$-block B of the Θ-decomposition of P.

There exists $\omega_B \in [\gcd(i,j)] \cup \{0\}$ such that $|B| = \omega_B \cdot \mathrm{lcm}(i,j)$.

Specifically, $\omega_B = 0$ if $(i,j) \notin \mathrm{LCM}_z$.

Sizes

Proposition

- $z = (z_1, z_2, z_3) \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}\,n}}$.
- $P \in \mathcal{PLS}_z$.

It is $\mathfrak{l}_z \leq |P| \leq \mathfrak{u}_z$, where:

$$\mathfrak{l}_z = \min_{(i,j)\in \mathrm{LCM}_z} \{\mathrm{lcm}(i,j)\},$$

$$\mathfrak{u}_z = \min\{ \sum_{(i,j)\in \mathrm{LCM}_z} z_{1i}\cdot z_{2j}\cdot i\cdot j, \sum_{(i,k)\in \mathrm{LCM}_{z(23)}} z_{1i}\cdot z_{3k}\cdot i\cdot k, \sum_{(k,j)\in \mathrm{LCM}_{z(13)}} z_{2j}\cdot z_{3k}\cdot j\cdot k\}.$$

$$\mathrm{Sizes}(z) = \left\{ \sum_{(i,j)\in \mathrm{LCM}_z} \omega_{ij}\cdot \mathrm{lcm}(i,j) \leq \mathfrak{u}_z \mid \omega_{ij} \in [z_{1i}\cdot z_{2j}\cdot \mathrm{gcd}(i,j)] \right\}.$$

Sizes

- $z = (6, 3\cdot 2\cdot 1, 4\cdot 2) \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}6}$
- $\Theta = ((123456), (123)(45)(6), (1234)(56)) \in \mathcal{PLS}_z$.

$$\begin{pmatrix} \cdot & \cdot & \cdot & * & * & \circ \\ \cdot & \cdot & \cdot & * & * & \circ \\ \cdot & \cdot & \cdot & * & * & \circ \\ \cdot & \cdot & \cdot & * & * & \circ \\ \cdot & \cdot & \cdot & * & * & \circ \\ \cdot & \cdot & \cdot & * & * & \circ \end{pmatrix}.$$

$\mathrm{LCM}_z = \{(6,3)\}$, $\mathrm{LCM}_{z^{(23)}} = \{(6,2)\}$ and $\mathrm{LCM}_{z^{(13)}} = \{(2,3)\}$.

So: $6 \leq |P| \leq \min\{18, 12, 6\} = 6$. Thus, $\mathrm{Sizes}(z) = \{6\}$ and $|P| = 6$.

$$\begin{pmatrix} 5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & 5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$\begin{pmatrix} \cdot & 6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & 5 & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 5 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The number of partial Latin squares related to an autotopism.

$\mathcal{PLS}_{\Theta,[P]}$ AND $\mathcal{PLS}_{\Theta,s}$

- $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}\,n}}$.
- $\Theta \in \mathfrak{I}_z$.
- $P \in \mathcal{PLS}_{\Theta}$.
- $s \in [n]$.

$$\mathcal{PLS}_{\Theta,[P]} = \mathcal{PLS}_{\Theta} \cap [P]$$

$$\mathcal{PLS}_{\Theta,s} = \mathcal{PLS}_{\Theta} \cap \mathcal{PLS}_{n,s}.$$

LEMMA

The number of isotopic partial Latin squares related to an autotopism only depends on the **parastrophic class of the cycle structure** of the latter.

Incidence structures.

Proposition

- $P \in \mathcal{PLS}_n$.
- $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}\, n}$.

$([P], \mathfrak{I}_z, I_{\mathcal{P}_n})$ is $\Delta_{[P]}(z)$-uniform and regular incidence structure.

$(\mathcal{PLS}_{n,s}, \mathfrak{I}_z, I_{\mathcal{P}_n})$ is $\Delta_s(z)$-uniform incidence structure.

$(\mathcal{PLS}_n, \mathfrak{I}_z, I_{\mathcal{P}_n})$ is $\Delta_{\mathcal{P}}(z)$-uniform incidence structure.

All their blocks have the same multiplicity.

Theorem

- $P \in \mathcal{PLS}_n$.
- $Q \in [P]$.

$|\mathfrak{A}_Q| = |\mathfrak{A}_P| = |\mathfrak{I}_{P,Q}|$
Moreover, given $\Theta \in \mathfrak{A}_P$, it is $\mathfrak{A}_Q = \{\Theta' \Theta {\Theta'}^{-1} \mid \Theta' \in \mathfrak{I}_{P,Q}\}$.

$\Delta_s(z)$.

Proposition

$$\Delta_s((n,n,1^n)) = \begin{cases} \frac{n!^2}{k!\cdot(n-k)!^2}, & \text{if } \exists k \in [n] \text{ s.t. } s = k\cdot n,\\ 0, & \text{otherwise.}\end{cases}$$

Proposition

$\Delta_n((n,n,n)) = n^2$

If $n > 2$: $\Delta_{2n}((n,n,n)) = \frac{n^2\cdot(n-1)\cdot(n-2)}{2}$.

Theorem

$$\Delta_{\mathfrak{l}_z}(z) = \sum_{\substack{(i,j)\in \mathrm{LCM}_z\\ \text{s.t. } \mathrm{lcm}(i,j)=\mathfrak{l}_z}} z_{1i}\cdot z_{2j}\cdot \gcd(i,j)\cdot \sum_{\substack{k\in[n]\\ \text{s.t. } (i,j,k)\in \mathrm{LCM}_n}} k\cdot z_{3k}.$$

Corollary

$$\Delta_{[P]}(z) = \Delta_1(z) = z_{11}\cdot z_{21}\cdot z_{31}.$$

3-PAP.

Proposition

- $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_n$.

There exists a bijection between $\mathcal{PLS}_\Theta$ and the set of feasible solutions of:

$$\begin{cases} \sum_{r\in[n]} x_{rcs} \leq 1, \forall c, s \in [n], \\ \sum_{c\in[n]} x_{rcs} \leq 1, \forall r, s \in [n], \\ \sum_{s\in[n]} x_{rcs} \leq 1, \forall r, c \in [n], \\ x_{rcs} = x_{\alpha(r)\beta(c)\gamma(s)}, \forall r, c, s \in [n], \\ x_{rcs} \in \{0, 1\}, \forall r, c, s \in [n]. \end{cases}$$

$$\sum_{r,c,s\in[n]} x_{rcs} = m \rightsquigarrow \mathcal{PLS}_{\Theta,m}$$

Gröbner bases.

$\mathcal{PLS}_{\Theta,m}$ is determined by $2n^3+3n^2+1$ polynomial equations of degree 1 and 2 in n^3 variables:

Corollary

- $\Theta=(\alpha,\beta,\gamma)\in\mathfrak{I}_n$.
- $m\in[n^2]$.

$\mathcal{PLS}_{\Theta,m}$ is the set of zeros of the ideal I:

$$\langle\,(\textstyle\sum_{r\in[n]}x_{rcs})\cdot(1-\sum_{r\in[n]}x_{rcs})=0\mid c,s\in[n]\,\rangle+$$
$$\langle\,(\textstyle\sum_{c\in[n]}x_{rcs})\cdot(1-\sum_{c\in[n]}x_{rcs})\mid r,s\in[n]\,\rangle+$$
$$\langle\,(\textstyle\sum_{s\in[n]}x_{rcs})\cdot(1-\sum_{s\in[n]}x_{rcs})\mid r,c\in[n]\,\rangle+$$
$$\langle\,x_{rcs}\cdot(1-x_{rcs})\mid r,c,s\in[n]\,\rangle+$$
$$\langle\,x_{rcs}-x_{\alpha(r)\beta(c)\gamma(s)}\mid r,c,s\in[n]\,\rangle+$$
$$\langle\,m-\textstyle\sum_{r,c,s\in[n]}x_{rcs}\,\rangle\subseteq\mathbb{Q}[\mathbf{x_n}]=\mathbb{Q}[x_{111},\ldots,x_{nnn}]$$

$\Delta_s(z)$ AND $\Delta_{\mathcal{P}}(z)$. ORDER $n \leq 3$

n	z	$\Delta_s(z)$									$\Delta_{\mathcal{P}}(z)$
		s									
		1	2	3	4	5	6	7	8	9	
1	$(1,1,1)$	1									1
2	$(2,2,2)$		4		0						4
	$(2,2,1^2)$		4		2						6
	$(1^2,1^2,1^2)$	8	16	8	2						34
3	$(3,3,3)$			9			9			3	21
	$(3,3,2\cdot1)$			3							3
	$(3,3,1^3)$			9			18			6	33
	$(2\cdot1,2\cdot1,2\cdot1)$	1	10	10	24	24	20	20	4	4	117
	$(2\cdot1,2\cdot1,1^3)$	3	6	18	6	18					51
	$(2\cdot1,1^3,1^3)$	9	18	6							33
	$(1^3,1^3,1^3)$	27	270	1278	3078	3834	2412	756	108	12	11775

$\Delta_s(z)$ and $\Delta_{\mathcal{P}}(z)$. Order $n = 4$

z	$\Delta_s(z)$																$\Delta_{\mathcal{P}}(z)$
	s																
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
$(4,4,4)$				16				48				32				0	96
$(4,4,3\cdot1)$				4													4
$(4,4,2^2)$				16				56				32				8	112
$(4,4,2\cdot1^2)$				16				64				64				8	152
$(4,4,1^4)$				16				72				96				24	208
$(3\cdot1,3\cdot1,3\cdot1)$	1		18	18		90	90		165	165		99	99		9	9	763
$(3\cdot1,3\cdot1,2\cdot1^2)$	2		6	12		6	12										38
$(3\cdot1,3\cdot1,1^4)$	4		12	48		36	144		24	96							364
$(3\cdot1,2^2,2^2)$		8		8													16
$(3\cdot1,2^2,2\cdot1^2)$		4															4
$(3\cdot1,2\cdot1^2,2\cdot1^2)$	4	4	8	4													20
$(3\cdot1,2\cdot1^2,1^4)$	8	12															20
$(3\cdot1,1^4,1^4)$	16	72	96	24													208
$(2^2,2^2,2^2)$		32		352		1664		3552		3328		1408		256		32	10624
$(2^2,2^2,2\cdot1^2)$		32		360		1792		4152		4416		2048		384		32	13216
$(2^2,2^2,1^4)$		32		368		1920		4800		5760		3264		768		96	17008
$(2^2,2\cdot1^2,2\cdot1^2)$		24		192		640		880		416		32					2184
$(2^2,2\cdot1^2,1^4)$		16		72		96		24									208
$(2\cdot1^2,2\cdot1^2,2\cdot1^2)$	8	32	136	336	752	1440	1904	2856	2400	2608	1504	1056	448	224	64	16	15784
$(2\cdot1^2,2\cdot1^2,1^4)$	16	88	272	736	1344	1632	1728	1008									6824
$(2\cdot1^2,1^4,1^4)$	28	352	2208	6504	9792	7104	2112	216									28352
$(1^4,1^4,1^4)$	64	1728	25920	239760	1437696	5728896	15326208	27534816	32971008	25941504	13153536	4215744	847872	110592	9216	576	127545136

Θ-completability

C_Θ AND $C_{\Theta,s}$.

- $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}}{}_n$
- $\Theta \in \mathfrak{I}_z$
- $P \in \mathcal{PLS}_\Theta$

P is *Θ-completable* if $\mathcal{LS}_{\Theta,P} \neq \emptyset$.

$$\Theta = ((12)(3),(12)(3),(12)(3)) \in \mathfrak{A}_3.$$

$$\begin{pmatrix} 3 & \cdot & 2 \\ \cdot & 3 & 1 \\ 2 & 1 & \cdot \end{pmatrix}.$$

$$\Theta = ((12)(34),(12)(34),(12)(3)(4)) \in \mathfrak{A}_4.$$

$$\begin{pmatrix} 3 & 4 & \cdot & \cdot \\ 4 & 3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

C_Θ AND $C_{\Theta,s}$.

- $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}\, n}$ and $\Theta \in \mathfrak{I}_z$

$C_\Theta = \{\Theta\text{-completable partial Latin squares}\}$
$C_{\Theta,s} = C_\Theta \cap \mathcal{PLS}_{\Theta,s}$.

Lemma

- $\Theta_1, \Theta_2 \in \mathfrak{I}_n$.
- $[z_{\Theta_1}] = [z_{\Theta_2}]$

$$|C_{\Theta_1,s}| = |C_{\Theta_2,s}|, \text{ for all } s \in [n^2].$$

$$|C_{\Theta_1}| = |C_{\Theta_2}|.$$

$\mathfrak{c}_z$ AND $\mathfrak{c}_{z,s}$.

- $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}\,n}$ and $\Theta \in \mathfrak{I}_z$

$\mathfrak{c}_z = |C_\Theta|$.
$\mathfrak{c}_{z,s} = |C_{\Theta,s}|$.

THEOREM

- $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}\,n}$.

$$P \in C_\Theta \Leftrightarrow \mathcal{PLS}_{\Theta,[P]} \subseteq C_{\Theta,|P|}$$

$$\mathfrak{c}_{z,s} = \sum_{\substack{[P] \in \mathcal{PLS}_{\Theta,s}/\sim \\ \text{s.t. } [P] \cap C_\Theta \neq \emptyset}} \Delta_{[P]}(z).$$

$\mathfrak{c}_z$ AND $\mathfrak{c}_{z,s}$. ORDER $n \leq 4$

n	z	$\mathfrak{c}_{z,s}$																$\mathfrak{c}_z$
		s																
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
1	$(1,1,1)$	1																1
2	$(2,2,1^2)$		4		2													6
3	$(3,3,3)$			9			9			3								21
	$(3,3,1^3)$			9			18			6								33
	$(2{\cdot}1,2{\cdot}1,2{\cdot}1)$	1	10	10	24	24	16	16	4	4								109
4	$(4,4,2^2)$				16				40				32				8	96
	$(4,4,2{\cdot}1^2)$				16				40				32				8	96
	$(4,4,1^4)$				16				72				96				24	208
	$(3{\cdot}1,3{\cdot}1,3{\cdot}1)$	1		18	18		90	90		90	90		45	45		9	9	505
	$(2^2,2^2,2^2)$		32		352		1408		2144		1792		896		256		32	6912
	$(2^2,2^2,2{\cdot}1^2)$		32		336		1344		2144		1792		896		256		32	6832
	$(2^2,2^2,1^4)$		32		368		1728		3792		4224		2496		768		96	13504
	$(2{\cdot}1^2,2{\cdot}1^2,2{\cdot}1^2)$	8	32	136	200	112	784	1328	1560	1760	1568	1248	800	448	192	64	16	10240

Basis of $\mathcal{LS}_\Theta$.

- Basis of $\mathcal{LS}_\Theta$: $\{P_1, P_2, ..., P_m\}$, with P_i Θ-completable s.t.

$$\bigcup_{i\in[m]} \mathcal{LS}_{\Theta,P_i} = \mathcal{LS}_\Theta$$

$$\mathcal{LS}_{\Theta,P_i} \cap \mathcal{LS}_{\Theta,P_j} = \emptyset, \text{ whenever } i \neq j$$

$$\Downarrow$$

$$\Delta(z) = \sum_{i\in[m]} |\mathcal{LS}_{\Theta,P_i}|$$

Basis of $\mathcal{LS}_\Theta$.

Lemma

- $S \subseteq [n]^2$.
- $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{A}_n$.

Each of the following sets is non-empty if and only if it is a basis of $\mathcal{LS}_\Theta$:

$$S_{RC} = \{P \in C_\Theta \mid (r,c,s) \in O(P) \Leftrightarrow (r,c) \in S\},$$

$$S_{RS} = \{P \in C_\Theta \mid (r,c,s) \in O(P) \Leftrightarrow (r,s) \in S\},$$

$$S_{CS} = \{P \in C_\Theta \mid (r,c,s) \in O(P) \Leftrightarrow (c,s) \in S\}.$$

$$\Theta = ((12)(3)(4), (12)(3)(4), (34)(1)(2))$$

$$S = \{(3,3), (3,4), (4,3), (4,4)\}$$

$$S_{RC} = \left\{ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 \\ \cdot & \cdot & 2 & 1 \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & 1 & 2 \end{pmatrix} \right\}$$

Homogeneous basis of $\mathcal{LS}_\Theta$.

- Homogeneous basis of $\mathcal{LS}_\Theta$: $\{P_1, P_2, \ldots, P_m\}$ basis of $\mathcal{LS}_\Theta$ s.t.

$$|\mathcal{LS}_{\Theta,P_i}| = |\mathcal{LS}_{\Theta,P_j}|, \text{ for all } i, j \in [m].$$

$$\Downarrow$$

$$\Delta(z) = m \cdot |\mathcal{LS}_{\Theta,P_i}|, \text{ for all } i \in [m].$$

$$\Downarrow$$

$$m \equiv P_i\text{-coefficient of symmetry of } \Theta.$$

Homogeneous basis of $\mathcal{LS}_\Theta$.

Theorem

- $z = (z_1, z_2, z_3) \in \mathcal{CS}_{\mathfrak{A}_n}$
- $z_{11} \cdot z_{21} \cdot z_{31} \neq 0$.
- $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_z$
- $S = \{(i,j) \in [n]^2 \mid i \in \alpha_\infty, j \in \beta_\infty\}$.

S_{RC} is an homogeneous basis of $\mathcal{LS}_\Theta$ of cardinality $|\mathcal{LS}_{z_{11}}|$.

$$\begin{pmatrix} \cdot & \cdot & * & \circ \\ \cdot & \cdot & * & \circ \\ \diamond & \diamond & 1 & 2 \\ \square & \square & 2 & 1 \end{pmatrix} \in \mathcal{PLS}_4$$

$$\Theta = ((12)(3)(4), (12)(3)(4), (34)(1)(2))$$

$$S = \{(3,3), (3,4), (4,3), (4,4)\}$$

Further work.

- Application to the calculus of $\Delta(z)$.
- $|\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}\,n}}|$ has been bounded. General expression?
- General expression for $\Delta_{[P]}(z)$? $\rightsquigarrow$ Isotopic classes of $\mathcal{PLS}_n$.
- Examples of order $n > 4$.
- Comprehensive study of homogeneous bases and coefficients of symmetry.

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THANK YOU VERY MUCH!!

Raúl M. Falcón

Department of Applied Mathematics I
University of Seville (Spain)
rafalgan@us.es

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