

A particular case of extended isotopisms: Santilli's isotopisms.

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Abstract

Due to a mathematical necessity, it has been proved that it is convenient to give a new interpretation of the multiplicity of Santilli's isounity $\hat{I} = \hat{I}(x, v, t, \mu, \rho, \dots)$ of any Santilli's isotopism, as a family of classical Bruck's isotopisms. In this paper we prove that every Santilli's isotopism is indeed an extended isotopism.

1 Introduction

A *partial algebra* is a nonempty set S endowed with binary operations which have as domain a subset \mathfrak{D} of $S \times S$. S is an *algebraic structure* if all these binary operations have domain $\mathfrak{D} = S \times S$.

A *quasigroup* [1] is an algebraic structure Q endowed with a product \cdot , such that if any two of the three symbols a, b, c in the equation $a \cdot b = c$ are given as elements of Q , the third one is uniquely determined as an element of Q . It is equivalent to say that Q is endowed with left / and right \ division. A *loop* is a quasigroup with an unit element. Two quasigroups (Q_1, \cdot) and (Q_2, \circ) are *isotopic* [2] if there are three bijections α, β, γ from Q_2 to Q_1 , such that:

$$\gamma(a \circ b) = \alpha(a) \cdot \beta(b), \text{ for all } a, b \in Q_2. \quad (1)$$

The triple $\Theta = (\alpha, \beta, \gamma)$ is called an *isotopism* from (Q_1, \cdot) to (Q_2, \circ) and it is denoted $(Q_1, \cdot)^\Theta = (Q_2, \circ)$. If $Q_1 = Q_2$ and $\alpha = \beta = \gamma$, the isotopism is indeed an *isomorphism*. If $Q_1 = Q_2$ and $\cdot \equiv \circ$, Θ is called an *autotopism*. Let us also recall that the concept of isotopism can be extended in the same way as in other algebraic structures as a triple of bijections of such a structure, which verifies an equality analogous to (1).

If we consider the multiplication table of a quasigroup, we obtain a Latin square. A *Latin square*, L , of order n , is a $n \times n$ array with elements chosen from a set of n symbols $N = \{x_1, \dots, x_n\}$, such that each symbol occurs precisely once in each row and each column. The set of Latin squares of order n is denoted by $LS(n)$. A *partial Latin square*, P , of order n , is a $n \times n$ array with elements chosen from a set of n symbols, such that each symbol occurs at most once in each row and in each column. The set of partial Latin squares of order n is denoted as $PLS(n)$. The algebraic structure

having a partial Latin square as its multiplication table is called a *partial quasigroup*.

From now on, we will consider $N = \{0, 1, \dots, n-1\}$. So, if $L = (l_{ij})$, the *orthogonal array representation of L* is the set of n^2 triples $\{(i, j, l_{ij}) : 0 \leq i, j \leq n-1\}$. An *isotopism* of a Latin square L is a triple $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_n = S_n \times S_n \times S_n$, where S_n is the symmetric group on N and so, α, β and γ are respectively, permutations of rows, columns and symbols of L . The resulting square L^Θ is also a Latin square and it is said to be *isotopic* to L . In particular, if $L = (l_{ij})$, then $L^\Theta = \{(i, j, \gamma^{-1}(l_{\alpha(i)\beta(j)})) : 0 \leq i, j \leq n-1\}$. The set of all Latin squares isotopic to L is called the *isotopy class of L* . Fixed another Latin square $L' \in LS(n)$, the set of all isotopisms from L to L' is denoted by $\mathcal{U}(L, L') = \{\Theta \in \mathcal{I}_n : L^\Theta = L'\}$. An isotopism which maps L to itself is an *autotopism*. The stabilizer subgroup of L in \mathcal{I}_n is its *autotopism group*, $\mathcal{U}(L) = \{\Theta \in \mathcal{I}_n : L^\Theta = L\}$. Given $P \in PLS(n)$, contained in L , an *isotopism of P* is a triple $\Theta = (\alpha, \beta, \gamma) \in \mathcal{I}_n$, where $\gamma(\emptyset) = \emptyset$. Given $\mathfrak{F} \subseteq \mathcal{U}(L)$, it is defined the *extended autotopy* [6] $P^\mathfrak{F} = \bigcup_{\Theta \in \mathfrak{F}} P^\Theta \in PLS(n)$. The set \mathfrak{F} is called an *extended autotopism* from P to P^Θ . If (Q_1, \cdot) and (Q_2, \circ) are, respectively, the partial quasigroups associated to P and $P^\mathfrak{F}$, then it is denoted $(Q_1, \cdot)^\mathfrak{F} = (Q_2, \circ)$. This concept of extended autotopism can be analogously defined in any multiplication table. So, it can be considered not only in partial quasigroups but in any partial algebra.

By the other way, Santilli proposed in 1978 [7] a possible model of isotopism, which he called *Santilli's isotopism*, which allows to construct the named *isostructures*, based on an isounit I . For a historical vision of the development of the isothory and a wide bibliography related to it, the reader can consult [3]. Particularly, Santilli's isotopic model is based on the generalization of the initial unit: $I \rightarrow \widehat{I} = \widehat{I}(x, v, t, \mu, \rho, \dots)$, where $x, v, t, \mu, \rho, \dots$ are variables with values in fixed domains X, V, T, M, R, \dots , as coordinates,

velocity, temperature, density, etc. So, fixed any mathematical structure E , endowed with a binary law \times with unit element $I \in E$, this model considers a set $V \supseteq E$, endowed with an associative binary law $*$ with unit element I , where the restriction of the law $*$ to E coincides with \times , $\widehat{I} \in V$ for all factors $x, v, t, \mu, \rho, \dots$ (where here the domain of the coordinate factor is $X = E$) and $T = T(x, v, t, \mu, \rho, \dots) = \widehat{I}^{-I}$. V, T and \widehat{I} are respectively called *general set*, *isotopic element* and *isounity* of the isotopism. So, it is defined an algebraic structure \widehat{E} , endowed with the law $\widehat{\times}$ with unit \widehat{I} as:

$$E \rightarrow \widehat{E} : x \rightarrow \widehat{x} = x * \widehat{I}; \quad \widehat{a} \widehat{\times} \widehat{b} = \widehat{a} * T * \widehat{b} = \widehat{a * b}, \text{ for all } a, b \in \widehat{E}. \quad (2)$$

From a mathematical point of view, it has been proved [4] that it can be difficult to obtain explicitly the general set V . By the other way, when the isounit \widehat{I} has more than an unique value (that is, if \widehat{I} depends on external factors), it has been also seen that calculus and notations can be arduous. In this paper, we will see that these difficulties can be avoided by reinterpreting Santilli's isotopisms as a family of classical Bruck's isotopisms. In this way, the multiplicity of the isounity \widehat{I} can be displaced to a multiplicity of isotopisms. So, in the second section of this paper we compare Santilli's isotopisms with the classical Bruck's ones. To do it, we generalize the concept of extended autotopism to extended isotopism and we prove that every Santilli's isotopism is indeed an extended isotopism. In the third section we give a concrete example of this result, by working with quasigroups.

2 Comparison between Santilli's and Bruck's isotopisms

Fixed two Latin squares $L, L' \in LS(n)$, the concept of extended autotopism can be easily generalized to the set $\mathcal{U}(L, L') \subseteq \mathcal{I}_n$:

Definition 2.1. *Fixed a partial Latin square $P \in PLS(n)$ contained in L and given $\mathfrak{F} \subseteq \mathcal{U}(L, L')$, we will define the extended isotopy of P associated to \mathfrak{F} as:*

$$P^{\mathfrak{F}} = \bigcup_{\theta \in \mathfrak{F}} P^{\theta} \in PLS(n).$$

The set \mathfrak{F} is called an extended isotopism from P to P^{θ} . We will denote $(Q_1, \cdot)^{\mathfrak{F}} = (Q_2, \circ)$ if (Q_1, \cdot) and (Q_2, \circ) are, respectively, the partial quasi-groups associated to P and $P^{\mathfrak{F}}$.

It is easy to see that $P^{\mathfrak{F}}$ is indeed contained in L' . This concept of extended isotopism can be generalized to any multiplication table and so, to any algebraic structure. In particular, we will use it to prove that every algebraic structure obtained by means of a Santilli's isotopism is an extended isotopism of the corresponding initial algebraic structure.

In our study, we will consider the Santilli's isounit \widehat{I} of the Santilli's isotopism (2) as the subset:

$$\widehat{I} = \{\widehat{I}(x, v, t, \mu, \rho, \dots) : x \in X, v \in V, t \in T, \mu \in M, \rho \in R, \dots\} \subseteq V.$$

In this way, $|\widehat{I}|$ denotes the cardinal of \widehat{I} .

Lemma 2.2. *If $|\widehat{I}| = 1$, then the Santilli's isotopism (2) is indeed an isomorphism.*

Proof.

It is sufficient to take $\alpha : E \rightarrow \widehat{E}$, such that $\alpha(x) = x * \widehat{I}$. The associativity of $*$ and the existence of the inverse $T = \widehat{I}^{-1}$ imply that α is a bijection. \square

When $|\widehat{I}| > 1$, we must explicit what elements of \widehat{I} are used in the definition of $\widehat{\times}$ in (2). In this way, all the possible values of the variable factors of which \widehat{I} depends were denoted by F in [5], that is:

$$F = \{f = (x, v, t, \mu, \rho, \dots) : x \in E, v \in V, t \in T, \mu \in M, \rho \in R, \dots\}.$$

So:

$$\widehat{I} = \{\widehat{I}(f) : f \in F\} \subseteq V.$$

Now, fixed $f \in F$, let us denote:

$$\widehat{x}_f = x * \widehat{I}(f), \text{ for all } x \in V;$$

$$\widehat{E}_f = \{\widehat{x}_f : x \in E\}.$$

Therefore:

$$\widehat{E} = \{\widehat{x}_f : x \in E, f \in F\} = \bigcup_{f \in F} \widehat{E}_f.$$

By the other way, fixed any map:

$$\Phi : F \times F \rightarrow F : (f_1, f_2) \rightarrow \Phi(f_1, f_2),$$

it was defined in [5] the law $\widehat{\times}$ of (2) in \widehat{E} as:

$$\widehat{a}_{f_1} * \widehat{b}_{f_2} = \left\{ \widehat{x * y}_{\Phi(f_s, f_t)} : \widehat{x}_{f_s} = \widehat{a}_{f_1} \text{ and } \widehat{y}_{f_t} = \widehat{b}_{f_2} \right\}_{\substack{x, y \in E \\ f_s, f_t \in F}}, \text{ for all } a, b \in E.$$

However, we are not interested in algebraic multi-structures, that is, the set of the right side of the previous equality must be of cardinal one. For this reason, we must choose Φ in such a way that this set is unitary. We will say then that Φ is *compatible with* $(E, *)$. In the case in which in the

previous equality we can swap all the elements a, b, x, y in the complete set V , we will say that Φ is *compatible with* $(V, *)$ and we can extend the law $\widehat{*}$ from \widehat{E} to V .

Therefore, let us suppose from now on that Φ is compatible with $(V, *)$ and let us fix $f_1, f_2 \in F$. We obtain then that:

$$\widehat{a}_{f_1} \widehat{*} \widehat{b}_{f_2} = \widehat{a * b}_{\Phi(f_1, f_2)}, \text{ for all } a, b \in V. \quad (3)$$

Now, if $T(f_1) = \widehat{I}(f_1)^{-I}$ and $T(f_2) = \widehat{I}(f_2)^{-I}$, let us observe that (3) is equivalent to:

$$\widehat{a}_{f_1} \widehat{*} \widehat{b}_{f_2} = \left((\widehat{a}_{f_1} * T(f_1)) * (\widehat{b}_{f_2} * T(f_2)) \right) * \widehat{I}(\Phi(f_1, f_2)).$$

Finally, if $T(\Phi(f_1, f_2)) = \widehat{I}(\Phi(f_1, f_2))^{-I}$, the previous equality is equivalent to:

$$\left(\widehat{a}_{f_1} \widehat{*} \widehat{b}_{f_2} \right) * T(\Phi(f_1, f_2)) = (\widehat{a}_{f_1} * T(f_1)) * (\widehat{b}_{f_2} * T(f_2)).$$

Let us now define the bijections on V , $\alpha_{f_1}, \beta_{f_2}$ and $\gamma_{\Phi(f_1, f_2)}$, such that:

$$\begin{aligned} \alpha_{f_1}(\widehat{x}_{f_1}) &= \widehat{x}_{f_1} * T(f_1) = x, \\ \beta_{f_2}(\widehat{x}_{f_2}) &= \widehat{x}_{f_2} * T(f_2) = x, \\ \gamma_{\Phi(f_1, f_2)}(\widehat{x}_{\Phi(f_1, f_2)}) &= \widehat{x}_{\Phi(f_1, f_2)} * T(\Phi(f_1, f_2)) = x, \end{aligned}$$

So, we finally obtain that:

$$\gamma_{\Phi(f_1, f_2)} \left(\widehat{a}_{f_1} \widehat{*} \widehat{b}_{f_2} \right) = \alpha_{f_1}(\widehat{a}_{f_1}) * \beta_{f_2}(\widehat{b}_{f_2}), \text{ for all } a, b \in V. \quad (3')$$

That is, the triple $\Theta_{1,2} = (\alpha_{f_1}, \beta_{f_2}, \gamma_{\Phi(f_1, f_2)})$ is a classical isotopism from $(V, *)$ to $(V, \widehat{*})$. In a similar way, we can obtain a classical isotopism $\Theta_{i,j}$ from $(V, *)$ to $(V, \widehat{*})$ starting from any $(f_i, f_j) \in F \times F$. So, we can define the family:

$$\mathfrak{F} = \{ \Theta_{i,j} = (\alpha_{f_i}, \beta_{f_j}, \gamma_{\Phi(f_i, f_j)}) : (f_i, f_j) \in F \times F \}.$$

Particularly:

Lemma 2.3. *It is verified that:*

$$\mathfrak{F} \subseteq \mathcal{U}((V, *), (V, \widehat{*})).$$

Proof.

The result is immediate by keeping in mind the associativity of $*$ and the existence in any case of the isotopic element as the inverse of the corresponding isounity. \square

Now, let us consider $(E, *)$ as a partial substructure of $(V, *)$ in the following sense:

$$a * b = \begin{cases} a \times b, & \text{if } (a, b) \in E \times E, \\ \emptyset, & \text{if } (a, b) \notin E \times E. \end{cases}$$

We obtain then the following:

Proposition 2.4. *They are verified:*

- a) $(E, *)^\Theta$ is a partial substructure of $(V, \widehat{*})$ for all $\Theta \in \mathfrak{F}$.
- b) $(E, *)^{\mathfrak{F}} = (\widehat{E}, \widehat{*})$ is an algebraic substructure of $(V, \widehat{*})$.

In this way, $(\widehat{E}, \widehat{*})$ is indeed the extended isotopy of $(E, *)$ associated to \mathfrak{F} .

Proof

Fixed $f_1, f_2 \in F$ and $\Theta_{f_1, f_2} = (\alpha_{f_1}, \beta_{f_2}, \gamma_{\Phi(f_1, f_2)}) \in \mathfrak{F}$, we can consider $(E, *)^\Theta$ as a partial substructure of $(V, \widehat{*})$ in the following sense:

$$x \widehat{*} y = \begin{cases} \gamma_{\Phi(f_1, f_2)}^{-1} (\alpha_{f_1}(x) * \beta_{f_2}(y)), & \text{if } x \in \widehat{E}_{f_1} \text{ and } y \in \widehat{E}_{f_2}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

By the other way, fixed $x, y \in \widehat{E} = \bigcup_{f \in F} \widehat{E}_f \subseteq V$, we can find $a, b \in E$ and $f_i, f_j \in \mathfrak{F}$, such that $\alpha_{f_i}(a) = x$ and $\beta_{f_j}(b) = y$. Particularly, the triple $\Theta_{f_i, f_j} = (\alpha_{f_i}, \beta_{f_j}, \gamma_{\Phi(f_i, f_j)}) \in \mathfrak{F}$ allows to define the product $x \widehat{*} y \in \widehat{E}$. Therefore, the law $\widehat{*}$ is defined in $\widehat{E} \times \widehat{E}$ and $(E, *)^{\mathfrak{F}} = (\widehat{E}, \widehat{*})$ is an algebraic substructure of $(V, \widehat{*})$. \square

So, as a consequence of the previous construction, the following result holds:

Theorem 2.5. *Every Santilli's isotopism is an extended isotopism.* \square

3 An example of Santilli's isotopism

Let us consider $E = \{0, 1\}$, $V = \{0, 1, 2, 3\}$ and the laws \times in E and $*$ in V , given by the followings multiplication tables:

| | | |
|----------|---|---|
| \times | 0 | 1 |
| 0 | 0 | 1 |
| 1 | 1 | 0 |

| | | | | |
|-----|---|---|---|---|
| $*$ | 0 | 1 | 2 | 3 |
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

So, $(E, *) = (E, \times)$. Now, let us consider:

$$F = \{f_1, f_2\},$$

$$\widehat{I}(f_1) = 0; \quad \widehat{I}(f_2) = 2.$$

That is to say:

$$\widehat{I} = \{0, 2\}.$$

Therefore:

$$\begin{aligned} \widehat{0}_{f_1} &= 0, & \widehat{1}_{f_1} &= 1, & \widehat{2}_{f_1} &= 2, & \widehat{3}_{f_1} &= 3; \\ \widehat{0}_{f_2} &= 2, & \widehat{1}_{f_2} &= 3, & \widehat{2}_{f_2} &= 0, & \widehat{3}_{f_2} &= 1. \end{aligned}$$

Then:

$$\widehat{E} = \widehat{E}_{f_1} \cup \widehat{E}_{f_2} = V = \widehat{V}.$$

We define now the map:

$$\Phi : F \times F \rightarrow F$$

$$\Phi(f_1, f_1) = f_2,$$

$$\Phi(f_1, f_2) = f_1,$$

$$\Phi(f_2, f_1) = f_1,$$

$$\Phi(f_2, f_2) = f_2.$$

So, fixed $a, b \in V$ and $f_i, f_j \in F$, we can define the law $\widehat{*}$ as:

$$\widehat{a}_{f_i} \widehat{*} \widehat{b}_{f_j} = \widehat{a * b}_{\Phi(f_i, f_j)}.$$

It can be then proved that $\widehat{*}$ is given by the following multiplication table:

| | | | | |
|---|---|---|---|---|
| * | 0 | 1 | 2 | 3 |
| 0 | 2 | 3 | 0 | 1 |
| 1 | 3 | 2 | 1 | 0 |
| 2 | 0 | 1 | 2 | 3 |
| 3 | 1 | 0 | 3 | 2 |

So, $(\widehat{E}, \widehat{*}) = (V, \widehat{*})$. However, this Santilli's isotopism can be seen as an extended isotopism. To do it, we must observe that the isotopic elements of this isotopism are $T(f_1) = 0 = \widehat{I}(f_1)$ and $T(f_2) = 2 = \widehat{I}(f_2)$. So, it is sufficient to consider the bijections on V , $\alpha_{f_1} = \beta_{f_1} = \gamma_{f_1}$ and $\alpha_{f_2} = \beta_{f_2} = \gamma_{f_2}$, such that:

$$\alpha_{f_1}(a) = a * 0 = a, \quad \alpha_{f_2}(a) = a * 2, \quad \text{for all } a \in V.$$

That is:

$$\alpha_{f_1} = Id, \quad \alpha_{f_2} = (02)(13).$$

Therefore, we must consider the family of isotopisms from $(V, *)$ to $(V, \widehat{*})$:

$$\begin{aligned} \mathfrak{F} = \{ & \Theta_{f_1, f_1} = (Id, Id, (02)(13)), \Theta_{f_1, f_2} = (Id, (02)(13), Id), \\ & \Theta_{f_2, f_1} = ((02)(13), Id, Id), \Theta_{f_2, f_2} = ((02)(13), (02)(13), (02)(13)) \} \end{aligned}$$

We give the partial Latin square associated to each $(E, *)^{\Theta_{f_i, f_j}}$ in the following table:

| (i, j) | $(E, *)^{\Theta_{f_i, f_j}}$ | (i, j) | $(E, *)^{\Theta_{f_i, f_j}}$ |
|----------|--|----------|--|
| $(1, 1)$ | $\begin{pmatrix} 2 & 3 & - & - \\ 3 & 2 & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}$ | $(1, 2)$ | $\begin{pmatrix} - & - & 0 & 1 \\ - & - & 1 & 0 \\ - & - & - & - \\ - & - & - & - \end{pmatrix}$ |
| $(2, 1)$ | $\begin{pmatrix} - & - & - & - \\ - & - & - & - \\ 0 & 1 & - & - \\ 1 & 0 & - & - \end{pmatrix}$ | $(2, 2)$ | $\begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & - & 2 & 3 \\ - & - & 3 & 2 \end{pmatrix}$ |

So, finally, $(E, *)^{\mathfrak{F}}$ is associated to:

$$\begin{pmatrix} 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{pmatrix}.$$

And we obtain in this way the Santilli's isostructure $(\widehat{E}, \widehat{*})$ as an extended isotopy of $(E, *)$ associated to \mathfrak{F} .

4 Final remarks

This reinterpretation of Santilli's isotopisms allows to unify Santilli's theory to classical Bruck's theory without the multiplicity of values of \widehat{I} . However, the primordial sense of Santilli's theory is not lost, because this multiplicity is displaced to a determined number of classical isotopisms. Besides, it gives a way to see the role of the general set V in all this process, which was sometimes a difficulty to find concrete examples. However, a study for a possible reinterpretation of other aspects of Santilli's theory is still necessary. So for example, a study of infinite algebraic isostructures and, therefore, in functional isoanalysis or isodifferential calculus would be an open problem in this sense.

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