Abstract

Transformation between geodetic coordinates \((\phi, \lambda, h)\) (geodetic latitude \(\phi\), geodetic longitude \(\lambda\) and ellipsoidal/geodetic height \(h\)) and cartesian coordinates has been studied over the years, being a subject of interest in many fields as Geodesy, Aerospace Engineering, Building Engineering, Architecture etc. Most recent works related to this topic are presented in [1] and [2], where iteration processes for the transformation on a biaxial or triaxial ellipsoid are considered.

Given a reference triaxial ellipsoid, of semi-axes \(a_x \geq a_y \geq a_z\),
- the geodetic latitude \(\phi\) is defined as the angle between the ellipsoidal normal vector and the \(X - Y\) plane,
- the geodetic longitude \(\lambda\) is the angle, measured in the \(X - Y\) plane, between a line parallel to the \(X\) axis and the projection of the ellipsoidal normal vector onto the \(X - Y\) plane, and
- the geodetic height \(h\) is measured along the ellipsoidal normal.

Given a point \(P_{Xe}\) on the triaxial ellipsoid, its cartesian coordinates \((X, Y, Z)\) verifies the ellipsoid equation,

\[
\frac{x^2}{a_x^2} + \frac{y^2}{a_y^2} + \frac{z^2}{a_z^2} = 1
\]

and its geodetic and Cartesian coordinates are related as follows (as in [2]):

\[
X = v \cos \phi \cos \lambda
\]
\[
Y = v (1 - \epsilon_2^2) \cos \phi \sin \lambda
\]
\[
Z = v (1 - \epsilon_z^2) \sin \phi
\]

where \(v\) is equal to the radius of the prime vertical,
\[
v = \frac{a_x}{\sqrt{1 - \epsilon_2^2 \sin^2 \phi - \epsilon_z^2 \cos^2 \phi \sin^2 \lambda}}.
\]

and the first eccentricities squared are
\[
\epsilon_2^2 = \frac{a_x^2 - a_z^2}{a_z^2}, \quad \epsilon_z^2 = \frac{a_x^2 - a_y^2}{a_y^2}, \quad \epsilon_2^2 = \frac{a_x^2 - a_y^2}{a_y^2}
\]

Obviously, if \(\phi\) and \(\lambda\) are given, one has \((X, Y, Z)\). Vice versa, if the coordinates \((X, Y, Z)\) are given, then
\[
\lambda = \arctan \left( \frac{1}{(1 - \epsilon_2^2) X} \right), \quad \phi = \arctan \left( \frac{(1 - \epsilon_2^2) Z / \sqrt{(1 - \epsilon_2^2)^2 X^2 + Y^2}}{(1 - \epsilon_z^2) Y} \right)
\]

However, suppose now that we have a point \(P_{W}\) outside the triaxial ellipsoid. In this case, introducing the ellipsoidal height \(h > 0\), the formulas relating Cartesian and geodetic coordinates are:

\[
X_G = (v + h) \cos \phi \cos \lambda
\]
\[
Y_G = (v (1 - \epsilon_2^2) + h) \cos \phi \sin \lambda
\]
\[
Z_G = (v (1 - \epsilon_z^2) + h) \sin \phi
\]

In this case, obtaining the geodetic coordinates \((\phi, \lambda, h)\) from the cartesian ones involves firstly computing the intersection point of the ellipsoidal normal vector passing through \(P_{W}\) and the ellipsoid. This point will be named \(P_{E}\) with cartesian coordinates \((X_E, Y_E, Z_E)\). Secondly, we apply formulas (1) with \(P_{E}\) in order to obtain the geodetic coordinates. Iterative methods for this transformation are given in [1,2].

Our ongoing work consists in developing a method which directly determines the point \(P_{E}\) by using Computer Algebra techniques (mainly Gröbner bases).

The shape parameters of the celestial bodies considered in our numerical tests are defined in kms as follows (see [3]):

- Moon: \(a_x = 1735.55, \quad a_y = 1735.324, \quad a_z = 1734.898\)
- Io: \(a_x = 1829.4, \quad a_y = 1819.3, \quad a_z = 1815.7\)
- Europa: \(a_x = 1564.13, \quad a_y = 1561.23, \quad a_z = 1560.93\)
- Mimas: \(a_x = 207.4, \quad a_y = 196.8, \quad a_z = 190.6\)
- Enceladus: \(a_x = 256.6, \quad a_y = 251.4, \quad a_z = 248.3\)

Following [2], for a given altitude \(h\), we run a loop over latitudes \(0^\circ - 89.75^\circ\) and longitudes \(0^\circ - 90^\circ\) with 0.25° increment and compute Cartesian coordinates \((X_G, Y_G, Z_G)\) from (2), (3) and (4) (129,960 triplets of coordinates for every single altitude \(h\)). After this step, the Cartesian coordinates of the points corresponding to geodetic altitude \(h\) and set of latitudes \(\phi\) and longitudes \(\lambda\) are obtained. For every satellite, the different altitude considered are \(0, a_z/50, a_z/25, a_z/10, a_z/15\) and \(a_z/10\).

2. Bibliography