Manuscript submitted to
AIMS' Journals
Volume **XX**, Number **0X**, Month **XXXX**

Website: http://AIMsciences.org

pp. **X–XX**

DYNAMICS OF A CLASS OF ODES VIA WAVELETS

Hildebrando M. Rodrigues(1), Tomás Caraballo(2) and Marcio Gameiro(1)

(1)Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo, Caixa Postal 668, 13560-970, São Carlos, SP, Brazil

(2)Dpto. Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla, Spain

(Communicated by ?????)

Abstract. The objective of this paper is to study a perturbed linear hyperbolic differential equation. The first part of this work is dedicated to study perturbation of the equilibrium (special solution) of a perturbed hyperbolic system. On the second part we analyze the stable and the unstable manifolds of a perturbed semilinear differential equation. We assume that the perturbed forcing function belongs to an L_2 class and that it is developed in a series of wavelets. Then we analyze the effect of this development on the special solution of the perturbed equation. Similar study is provided for the stable and unstable manifolds of this special solutions.

1. **Introduction.** The object of this paper is to study the dynamics of a class of ODEs in infinite dimension. Assuming a hyperbolic structure of the linear part we first study a special bounded solution in the real line of the forced equation with a forcing $f(t)$, that is a perturbation of the equilibrium. In the weakly nonlinear, case if the vector field $f(t, x, \varepsilon)$ in the variable t belongs to an L_2 space we develop it in a series using a wavelet basis. Then we analyze how this special bounded solution inherits properties of the vector field. We also consider similar questions for the stable and unstable manifolds of that solution.

Many classical works in Nonlinear Oscillations study this problem when the forcing term is periodic, almost periodic or almost automorphic, respectively. Unfortunately these classes of functions are not robust with respect to local perturbation in time.

In a previous paper Kloeden and Rodrigues [15] studied an alternative class of functions extending periodic and almost periodic functions which has the property that a bounded solution of a nonautonomous ordinary differential equation belongs to this class when the forcing term is introduced here. Specifically, the class of functions consists of uniformly continuous functions, defined on the real line and taking values in a Banach space. This class includes periodic, almost periodic or almost automorphic and many nonrecurrent functions. Assuming a hyperbolic

---

2000 *Mathematics Subject Classification.* 34G20, 35B15, 42C40, 42B25.

Key words and phrases. Wavelet and scaling functions, multiresolution analysis, perturbation of equilibria, stable and unstable manifolds, special solutions, pullback attractor.

H.R. was partially supported by FAPESP grant 2015/19165-5. T.C. was partially supported by the projects MTM2015-63723-P (MINECO/ FEDER, EU) and P12-FQM-1492 (Junta de Andalucía). M.G. was partially supported by FAPESP grants 2013/07460-7 and 2016/08704-5, and by CNPq grant 305860/2013-5, Brazil.

structure for the unperturbed linear equation and certain properties for the linear and nonlinear parts, the existence of a special bounded entire solution, as well the existence of stable and unstable manifolds of this solution are established. Moreover, it is shown that this solution and these manifolds inherit the temporal behaviour of the vector field equation. In the stable case it is shown that this special solution is the pullback attractor of the system. A class of infinite dimensional examples involving a linear operator consisting of a time independent part which generates a $\mathcal{C}_0$-semigroup plus a small time dependent part is presented and applied to systems of coupled heat and beam equations. Since in that class on cannot use Fourier Series an alternative strategy was to associate to each function of the considered class a set of sequences. Then they proved that the special solution is not more complicate than the forcing function.

In our case we study equations of the form $\dot{x} = Ax + f(t, x, \varepsilon)$ where A is hyperbolic and $f(\cdot, x, \varepsilon)$ belongs to a space L_2. First we consider the case when the forcing function is $f(t)$. When f belongs to L_2, the special bounded (in $\mathbb{R}$) solution belongs to a space of bounded functions. Each function of the considered basis of wavelets will give rise to a special solution. Therefore associated to the basis of wavelets we will have a kind of basis of bounded functions. We develop f in a series of wavelets and the special solution will be developed as a series of this basis of bounded solutions.

The multiresolution is also analyzed. We also study the stable and unstable manifolds of the special solution $x^*(t, \varepsilon)$ of $\dot{x} = Ax + f(t, x, \varepsilon)$.

What are some novelties of this paper? In the classical study of nonlinear oscillations, as treated for example in Hale [10], in the finite dimensional case, the equations are defined in terms of periodic or almost periodic functions and the Fourier Series Theory is used. In Kloeden and Rodrigues [15], in infinite dimensions, a similar problem is treated with a larger class of equations using uniformly continuous functions, instead of periodic or almost periodic functions, but Fourier Theory was not available. In this case, besides working in infinite dimensions, we allow $f(\cdot, x, \varepsilon)$ to belong to an $L_2(\mathbb{R})$ space. This has the advantage that it can be expanded in wavelets series with a convenient basis and the multiresolution analysis can be used. This treatment allows to consider local perturbations in time and also very complicate perturbations.

This paper is organized as follows. In Section 2 we consider the particular case where the perturbation is a function $f(t),\ t \in \mathbb{R}$. Besides the study of properties of the special solution, we analyze its expansion in a wavelets series and its multiresolution. We also compute the terms of the expansion in a specific example using Haar basis of wavelets and the stable and unstable manifolds of the special solution.

In Section 4 we study weakly nonlinear equations. Using Banach Fixed Point Theorem we investigate the special solution of an equation of the form $\frac{dx}{dt} = Ax + f(t, x, \varepsilon)$, where $f(\cdot, x, \varepsilon)$ to belong to an $L_2(\mathbb{R})$ space.

In Section 5, using a wavelet basis and multi resolution analysis we study approximations of this solution, the saddle point property, the existence of its stable and unstable manifolds, and approximations of these manifolds as approximations of the stable and unstable manifolds of the special solution.

Finally, in Section 6, using previous results, we discuss the existence of the pullback attractor in the case that in the linear part we have only a stable part and the multiresolution of the pullback attractor which shows that it inherits properties of the nonlinear part of the equation.

2. **Perturbation of the Equilibrium. Preliminaries.** To study a nonlinear autonomous equation one should be concerned with all its solutions, but this could be a difficult task. Then, one may try to analyze the simplest solutions, that is the equilibria. If besides the nonlinear autonomous perturbation one has a non autonomous perturbation that depends on $t \in \mathbb{R}$, then the first attempt could be to study perturbation of equilibria. This will be treated in this section and in the subsequent sections.

Let $\mathbb{X}$ be a complex Banach space with the norm $\|\cdot\|$. Let $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ be the infinitesimal generator of the $\mathcal{C}_0$ semigroup $T(t)$ for $t \geq 0$. We suppose that $P : \mathbb{X} \to \mathbb{X}$ is a projection and $f : \mathbb{R} \to \mathbb{X}$.

Consider the equation

$$\dot{x} = A\ x + f(t). \tag{1}$$

Lemma 2.1. *Suppose the following exponential dichotomy holds, that is, there exist positive constants K and α such that*

$$\|T(t)P\| \leq Ke^{-\alpha t}, \quad \textit{for } t \geq 0 \quad \textit{and} \quad \|T(t)(I-P)\| \leq Ke^{\alpha t}, \ \textit{for } t \leq 0. \tag{2}$$

Suppose that $f \in L_2(\mathbb{R}, \mathbb{X})$. Then there exists a unique solution of (1) that is bounded in $\mathbb{R}$ and it is given by:

$$x_f(t) = \int_{-\infty}^{t} T(t-s)\ P\ f(s)\ ds + \int_{\infty}^{t} T(t-s)\ (I-P)\ f(s)\ ds \tag{3}$$

This will be a solution in the mild sense as considered in Pazy [17]. or in other words it is a continuous function and will satisfy the equation a.e. If f is continuous in $\mathbb{R}$, then it will be a solution in $\mathbb{R}$ in the usual sense.

Moreover

$$\sup_{t\in\mathbb{R}} \|x_f(t)\| \leq K\sqrt{\frac{2}{\alpha}} \left[\int_{-\infty}^{\infty} \|f(s)\|^2 ds\right]^{1/2} := K^* \left[\int_{-\infty}^{\infty} \|f(s)\|^2 ds\right]^{1/2}, \tag{4}$$

where $K^* = K\sqrt{\frac{2}{\alpha}}$.

Proof: It is straightforward to check that $x_f(t)$ is solution of (1). Let us check now that the solution is bounded. The uniquenessss is obvious because the difference of two bounded soution for (1) is a bounded solution of $\dot{x} = Ax$, that is will be the zero solution.

$$
\begin{aligned}
\|x_f(t)\| &\le \int_{-\infty}^{t} \|T(t-s)\ P\|\|f(s)\|ds \\
&\quad + \int_{t}^{\infty} \|T(t-s)(I-P)\|\|f(s)\|\ ds \\
&\le Ke^{-\alpha t}\int_{-\infty}^{t} e^{\alpha s}\|f(s)\|ds \\
&\quad + Ke^{\alpha t}\int_{-\infty}^{t} e^{-\alpha s}\|f(s)\|ds \\
&\le Ke^{-\alpha t}\left[\int_{-\infty}^{t} e^{2\alpha s}\right]^{1/2}\left[\int_{-\infty}^{t}\|f(s)\|^2 ds\right]^{1/2} \\
&\quad + Ke^{\alpha t}\left[\int_{t}^{\infty} e^{-2\alpha s}\right]^{1/2}\left[\int_{t}^{\infty}\|f(s)\|^2 ds\right]^{1/2} \\
&\le K\sqrt{\frac{2}{\alpha}}\left[\int_{-\infty}^{\infty}\|f(s)\|^2 ds\right]^{1/2}.
\end{aligned}
$$

□

Let $UCB(\mathbb{R},\mathbb{X})$ be the space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{X}$ with the topology of uniform convergence.

Lemma 2.2. *Let $f_n \in UCB(\mathbb{R},\mathbb{X}),\ n \in \mathbb{N}$ be such that $f_n \ \to f$ uniformly for $t \in \mathbb{R}$. Then $f \in UCB(\mathbb{R},\mathbb{X})$.*

For a proof see Kloeden and Rodrigues [15].

For our goal, we will consider a subspace of $UCB(\mathbb{R},\mathbb{X})$. To be more precise, let $\mathcal{F}$ be the space of all uniformly continuous functions from $\mathbb{R}$ to $\mathbb{X}$ with precompact range with the topology of uniform convergence.

Lemma 2.3. *Let $(\mathbb{M}, d)$ be a complete metric space. Then a subset A of $\mathbb{M}$ is relatively compact if and only if for every $\varepsilon > 0$ there exists a relatively compact B_ε such that $A \subset V_\varepsilon(B_\varepsilon) := \{x \in \mathbb{M} :\ d(x, B_\varepsilon)\} < \varepsilon$, an ε-neighborhood of B_ε.*

See Bachman and Narici [1].

Lemma 2.4. *$\mathcal{F}$ is a closed subspace of $UCB(\mathbb{R},\mathbb{X})$.*

For a proof see Kloeden and Rodrigues [15]

Lemma 2.5. *Under the assumptions of Lemma 2.1 if $f \in L_2(\mathbb{R},\mathbb{X})$ then $x_f(t) \in \mathcal{F}$.*

Proof: Let us consider the case $P = I$, the identity operator. The general case is proved in a similar way. Given $f \in L_2(\mathbb{R},\mathbb{X})$, and $\varepsilon > 0$ let $N \in \mathbb{N},\ N = N(\varepsilon)$, such that

$$\max\left\{\int_{-\infty}^{-N}\|f(t)\|^2 dt,\ \int_{N}^{\infty}\|f(t)\|^2 dt\right\} < \varepsilon.$$

Let

$$f_N(t) = \begin{cases} 0, \text{if } t \in (-\infty,-N),\ \text{ or } t \in (N,\infty), \\ f(t) \text{ if } t \in [-N,N], \end{cases}$$

and

$$x_N(t) := \int_{-\infty}^{t} T(t-s)\ f_N(s)\ ds.$$

Then $x_N(t) = 0$ if $t \in (-\infty, -N)$, $x_N(t) := \int_{-N}^{t} T(t-s)\ f_N(s)\ ds$ if $t \in [-N, N]$, $x_N(t) := \int_{-N}^{N} T(t-s)\ f_N(s)\ ds$ if $t \in [N, \infty]$.

Using the above exponential dichotomy assumptions one can show that $x_N(t) \to 0$ as $t \to\ \infty$. Therefore $x_N(t)$ is uniformly continuous in $\mathbb{R}$.

Let us consider now $x(t) - x_N(t)$. For $t \in (-\infty, -N)$,

$$\begin{aligned}
\|x(t) - x_N(t)\| &= \left\| \int_{-\infty}^{t} T(t-s)\ f(s)\ ds \right\| \\
&\leq Ke^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s}\ \|f(s)\|\ ds \\
&\leq Ke^{-\alpha t} \left(\int_{-\infty}^{t} e^{2\alpha s}\ ds \right)^{\frac{1}{2}} \left(\int_{-\infty}^{t} \|f(s)\|^2\ ds \right)^{\frac{1}{2}} \\
&\leq \frac{K}{\sqrt{2}} \varepsilon.
\end{aligned}$$

For $t \in [-N, N]$,

$$\begin{aligned}
\|x(t) - x_N(t)\| &= \left\| \int_{-\infty}^{t} T(t-s)\ [f(s) - f_N(s)]\ ds \right\| \\
&= \left\| \int_{-\infty}^{-N} T(t-s)\ [f(s) - f_N(s)]\ ds \right. \\
&\quad \left. + \int_{-N}^{t} T(t-s)\ [f(s) - f_N(s)]\ ds \right\| \\
&= \left\| \int_{-\infty}^{-N} T(t-s)\ [f(s)]\ ds \right\| \\
&\leq Ke^{-\alpha t} \int_{-\infty}^{-N} e^{\alpha s}\ \|f(s)\|\ ds \\
&\leq Ke^{-\alpha t} \left(\int_{-\infty}^{-N} e^{2\alpha s}\ ds \right)^{\frac{1}{2}} \left(\int_{-\infty}^{-N} \|f(s)\|^2\ ds \right)^{\frac{1}{2}} \\
&\leq \frac{K}{\sqrt{2}} \varepsilon e^{-\alpha t} e^{-\alpha N} \\
&= \frac{K}{\sqrt{2}} \varepsilon e^{-\alpha (t+N)} \\
&\leq \frac{K}{\sqrt{2}} \varepsilon.
\end{aligned}$$

For $t \in (N, \infty)$, using similar estimates we obtain

$$\begin{aligned}
\|x(t) - x_N(t)\| &= \left\| \int_{-\infty}^{-N} T(t-s)\ [f(s) - f_N(s)]\ ds \right. \\
&\quad + \int_{-N}^{N} T(t-s)\ [f(s) - f_N(s)]\ ds \\
&\quad \left. + \int_{N}^{t} T(t-s)\ [f(s) - f_N(s)]\ ds \right\| \\
&\leq \left\| \int_{-\infty}^{-N} T(t-s)\ f(s)\ ds \right\| + \left\| \int_{N}^{t} T(t-s)\ f(s)\ ds \right\| \\
&\leq \frac{K}{\sqrt{2}}\varepsilon + \frac{K}{\sqrt{2}}\varepsilon \\
&= K\sqrt{2}\ \varepsilon.
\end{aligned}$$

To conclude the proof we should just use Lemma 2.2 and Lemma 2.3 . □

Now we define operator $\mathcal{S} : f \in L_2(\mathbb{R}, \mathbb{X}) \mapsto \mathcal{S}f := x_f \in \mathcal{F}$ and let

$$\mathfrak{S} := \mathcal{S}(L_2(\mathbb{R}, \mathbb{X})) \subset \mathcal{F}.$$

From Lemma 2.1 it follows that operator $\mathcal{S}$ is linear and continuous. Moreover

$$\sup_{t\in\mathbb{R}} \|S(f)(t)\| \leq K^* \left(\int_{-\infty}^{-\infty} \|f(t)\|^2 dt \right)^{1/2}.$$

Now we introduce some basic concepts and notations and results about wavelets. See [9, 12, 16]. Consider the scaling function $\phi : \mathbb{R} \to \mathbb{C}$ and the wavelet function $\psi : \mathbb{R} \to \mathbb{C}$, such that

$$\phi(t) = \sqrt{2} \sum_{n\in\mathbb{N}} h_n \phi(2t - n), \quad \psi(t) = \sqrt{2} \sum_{n\in\mathbb{N}} g_n \phi(2t - n) \tag{5}$$

where $h_n \in \mathbb{C}$ and $g_n = (-1)^n \overline{h_{-n+1}}$

It is usually assumed that $\int_{-\infty}^{\infty} \phi(t)\ dt = 1$ and $\int_{-\infty}^{\infty} \psi(t)\ dt = 0$.

$$\phi_{j,k}(t) = 2^{j/2} \phi(2^j t - k), \quad \psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k) \tag{6}$$

Then $\{\phi_{j,k}(t),\ k \in \mathbb{Z}\}$ is an orthonormal basis for the space V_j. Therefore we have a chain of closed subspaces:

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots, \tag{7}$$

such that

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R}), \quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \tag{8}$$

where we indicate $L_2(\mathbb{R}) := L_2(\mathbb{R}, \mathbb{C})$. Since $V_j \subset V_{j+1}$, let W_j be the orthogonal complement of V_j in V_{j+1}, that is $V_j \oplus W_j = V_{j+1}$. An orthonormal basis of W_j is given by $\{\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k),\ k\ \in \mathbb{N}\}$.

Definition 2.6. *Let P_j be the orthogonal projection of $L_2(\mathbb{R})$ onto V_j and Q_j be the orthogonal projection of $L_2(\mathbb{R})$ onto W_j.*

It is known that P_j and Q_j are also orthogonal to each other.

With the above notation the equations in (5) can be rewritten as

$$\phi_{0,0}(t) = \sqrt{2}\sum_{n\in\mathbb{N}} h_n \phi_{1,n}(t), \quad \psi_{0,0}(t) = \sqrt{2}\sum_{n\in\mathbb{N}} g_n \phi_{1,n}(t). \tag{9}$$

Let us see now what will be the implication of the above relations on the corresponding solutions.

Lemma 2.7. *Given $x_0 \in \mathbb{X}$ let $f_1(t) := \phi_{j,n}(t)x_0$, $f_2(t) := \psi_{j,n}(t)x_0$ and let $x_{j,n}(t)$ and $y_{j,n}(t)$ be the corresponding solutions of (1) given by (3). Then $x_{j,n}(t) = x_{j,0}(t-n)$ and $y_{j,n}(t) = y_{j,0}(t-n)$. Moreover (9) implies:*

$$x_{0,0}(t) = \sqrt{2}\sum_{n\in\mathbb{N}} h_n x_{1,n}(t), \quad y_{0,0}(t) = \sqrt{2}\sum_{n\in\mathbb{N}} g_n y_{1,n}(t)$$

and so

$$x_{0,0}(t) = \sqrt{2}\sum_{n\in\mathbb{N}} h_n x_{1,0}(t-n), \quad y_{0,0}(t) = \sqrt{2}\sum_{n\in\mathbb{N}} g_n y_{1,0}(t-n)$$

Proof: Given $f \in L_2(\mathbb{R}, \mathbb{X})$, consider the function $g(t) := f(t-n)$, if we substitute in (3)

$$\begin{aligned} x_g(t) &= x_{f(\cdot - n)}(t) \\ &= \int_{-\infty}^{t} T(t-s)\ P\ f(s-n)\ ds + \int_{\infty}^{t} T(t-s)\ (I-P)\ f(s-n)\ ds. \end{aligned}$$

If we let $\tau = s - n$ in the above integrals, we obtain:

$$\begin{aligned} x_g(t) &= x_{f(\cdot - n)}(t) \\ &= \int_{-\infty}^{t-n} T(t-(\tau+n))\ P\ f(\tau)\ d\tau \\ &\quad + \int_{\infty}^{t-n} T((t-(\tau+n))\ (I-P)\ f(\tau)\ d\tau \\ &= \int_{-\infty}^{t-n} T((t-n)-\tau)\ P\ f(\tau)\ d\tau \\ &\quad + \int_{\infty}^{t-n} T((t-n)-\tau)\ (I-P)\ f(\tau)\ d\tau \\ &= x_f(t-n). \end{aligned}$$

□

Now we suppose that $\mathbb{X}$ is a separable complex Hilbert space and that $\{e_i, i \in I\}$ is an orthonormal basis, with I countable, finite or infinite.

Let us see the implications of the previous relations on the coordinates.

Definition 2.8. *Let $f = (f^i e_i)_{i\in I} \in L_2(\mathbb{R}, \mathbb{X})$, where $f^i \in L_2(\mathbb{R})$. For each $j \in \mathbb{Z}$ we define a projection $\mathbb{P}_j : L_2(\mathbb{R}, \mathbb{X}) \to L_2(\mathbb{R}, \mathbb{X})$ as $\mathbb{P}_j f := ((P_j f^i)e_i)_{i\in I}$ and let $\mathbb{V}_j := \mathbb{P}_j(L_2(\mathbb{R}, \mathbb{X}))$. Similarly we define $\mathbb{Q}_j : L_2(\mathbb{R}, \mathbb{X}) \to L_2(\mathbb{R}, \mathbb{X})$ as $\mathbb{Q}_j f := ((Q_j f^i)e_i)_{i\in I}$ and let $\mathbb{W}_j := \mathbb{Q}_j(L_2(\mathbb{R}, \mathbb{X}))$.*

From (7) and (8) it follows that

$$\cdots \mathbb{V}_{-2} \subset \mathbb{V}_{-1} \subset \mathbb{V}_0 \subset \mathbb{V}_1 \subset \mathbb{V}_2 \cdots, \tag{10}$$

such that

$$\overline{\bigcup_{j\in\mathbb{Z}} \mathbb{V}_j} = L_2(\mathbb{R},\mathbb{X}), \quad \bigcap_{j\in\mathbb{Z}} \mathbb{V}_j = \{0\}, \tag{11}$$

It also follows that $\mathbb{V}_{j+1} = \mathbb{V}_j \oplus \mathbb{W}_j$, $\mathbb{V}_j$ and $\mathbb{W}_j$ orthogonal.

Recalling that $\mathfrak{S} := \mathcal{S}(L_2(\mathbb{R},\mathbb{X})) \subset \mathcal{F}$, for each $j \in \mathbb{Z}$ we define the subspace $\mathcal{V}_j := \mathcal{S}(\mathbb{V}_j)$ and $\mathcal{W}_j := \mathcal{S}(\mathbb{W}_j)$. Then $\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j$ and

$$\cdots \mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \cdots, \tag{12}$$

For each $x = x(\cdot) \in \mathfrak{S}$ let $f \in L_2(\mathbb{R},\mathbb{X})$ such that $\mathcal{S}f = x$. For such a $x \in \mathfrak{S}$ and for each $j \in \mathbb{Z}$ we define the projection $\mathcal{P}_j x := \mathcal{S}(\mathbb{P}_j f)$. Therefore $\mathcal{P}_j(\mathfrak{S}) = \mathcal{V}_j$.

In a similar way we can define for each $y = y(\cdot) \in \mathfrak{S}$ let $f \in L_2(\mathbb{R},\mathbb{X})$ such that $\mathcal{S}f = y$. For such a $y \in \mathfrak{S}$ and for each $j \in \mathbb{Z}$ we define the projection $\mathcal{Q}_j x := \mathcal{S}(\mathbb{Q}_j f)$. Therefore $\mathcal{Q}_j(\mathfrak{S}) = \mathcal{W}_j$.

Now we introduce the following notation. Given $f \in L_2(\mathbb{R},\mathbb{X})$ we write $f = \sum_{i\in I} f^i e_i$ and for each $i \in I$ we consider the special solutions $x^i := x_{f^i e_i}$, $y^i := y_{f^i e_i}$. Therefore $x_f = \sum_{i\in I} x^i$, $y_f = \sum_{i\in I} y^i$.

Lemma 2.9. *Let A be the generator of a $\mathcal{C}_0$-semigroup, such that A^{-1} is bounded and that the above exponential dichotomy (2) is satisfied. Consider the equations*

$$\dot{x} = Ax + 2^{j/2}\phi(2^j t)e_i = Ax + \phi_{j,0}(t)\ e_i, \quad \dot{y} = Ay + 2^{j/2}\psi(2^j t)e_i = Ax + \psi_{j,0}(t)\ e_i, \tag{13}$$

where ϕ, ψ are given as above. Let $x^i_j(t)$ (respectively $y^i_j(t)$) be the unique bounded solution on $\mathbb{R}$ of the above equation. Let P the projection defined by the exponential dichotomy. Then $x^i_j(t)$ can be written as

$$x^i_j(t) = x^{i-}_j(t) + x^{i+}_j(t),$$

where,

$$x^{i-}_j(t) = 2^{-j/2} e^{At} \int_{-\infty}^{2^j t} e^{-A(2^{-j}s)} P\phi(s) ds\ e_i,$$

$$x^{i+}_j(t) = -2^{-j/2} e^{-At} \int_{2^j t}^{\infty} e^{A(2^{-j}s)} (I-P)\phi(s) ds\ e_i.$$

Similar result for $y^i_j(t)$.

$$y^i_j(t) = y^{i-}_j(t) + y^{i+}_j(t),$$

where,

$$y^{i-}_j(t) = 2^{-j/2} e^{At} \int_{-\infty}^{2^j t} e^{-A(2^{-j}s)} P\psi(s) ds\ e_i,$$

$$y^{i+}_j(t) = -2^{-j/2} e^{-At} \int_{2^j t}^{\infty} e^{A(2^{-j}s)} (I-P)\psi(s) ds\ e_i.$$

Proof: Let $a = 2^{-j}$. From (13) we obtain the equations:

$$\dot{x} = Ax + a^{-1/2}\phi(a^{-1}t)e_i, \quad \dot{y} = Ay + a^{-1/2}\psi(a^{-1}t)e_i, \tag{14}$$

Let $z(t) := x(at)$. Then $z(t)$ satisfies the following equation:

$$\dot{z} = aAz + a^{1/2}\phi(t)e_i.$$

The unique bounded solution on $\mathbb{R}$ is given by:

$$z(t) = a^{1/2}\left[\int_{-\infty}^{t} e^{Aa(t-s)} Pa^{1/2}\phi(s)\ ds\ e_i - \int_{t}^{\infty} e^{Aa(s-t)}(I-P)\phi(s)\ ds\ e_i\right].$$

Returning to the original variable we obtain

$$\begin{aligned} x(t) &= z(a^{-1}t) \\ &= a^{1/2}\left[\int_{-\infty}^{ta^{-1}} e^{Aa(a^{-1}t-s)} P\phi(s)\ ds\ e_i - \int_{ta^{-1}}^{\infty} e^{Aa(s-a^{-1}t)}(I-P)\phi(s)\ ds\ e_i\right] \\ &= a^{1/2}\left[\int_{-\infty}^{ta^{-1}} e^{A(t-as)} P\phi(s)\ ds\ e_i - \int_{ta^{-1}}^{\infty} e^{A(as-t)}(I-P)\phi(s)\ ds\ e_i\right]. \end{aligned}$$

Now we let $a = 2^{-j}$ again to obtain

$$\begin{aligned} x(t) &= 2^{-j/2}\left[\int_{-\infty}^{2^j t} e^{A(t-2^{-j}s)} P\phi(s)\ ds\ e_i \right.\\ &\qquad \left. - \int_{2^j t}^{\infty} e^{A(2^{-j}s-t)}(I-P)\phi(s)\ ds\ e_i\right] \\ &= x^-_{j,i}(t) + x^+_{j,i}(t). \end{aligned}$$

□

2.1. **The Multiresolution Analysis.** At this point we should do the multiresolution for functions $f \in L_2(\mathbb{R})$.

Let $f_J \in L_2(\mathbb{R})$ an approximation of f, $f_J \in V_J$, for J sufficiently large. Then f_J can be written as $f_J = f_{J-1} + g_{J-1}$, where $f_{J-1} \in V_{J-1}$ and $g_{J-1} \in W_{J-1}$. Then $f_J = f_{J-2} + g_{J-2} + g_{J-1}$, where $f_{J-2} \in V_{J-2}$ and $g_{J-2} \in W_{J-2}$ and so on, until $f_J = f_0 + \cdots$. In each step we can consider f_{J-1} as a lower resolution of f_J, f_{J-2} a more lower resolution of f_J untilf_0 the lowest resolution of f_J, in this sequence. Then $f_J(t) = \sum_{n\in\mathbb{Z}} a_{J,n}\phi_{J,n}(t)$.

The multiresolution implies

$$f_J(t) = \sum_{n\in\mathbb{Z}} c_{J,n}\phi_{0,n}(t) + \sum_{0\le j\le J-1, n\in\mathbb{Z}} d_{j,n}\psi_{j,n}(t) \tag{15}$$

The first sum above will be the lowest resolution in this series. The coefficients above can be obtained as follows:

$$c_{J,n} = \int_{-\infty}^{+\infty} f_J(s)\phi_{0,n}(s)ds, \quad d_{j,n} = \int_{-\infty}^{+\infty} f_J(s)\psi_{j,n}(s)ds.$$

Remark 2.10. *When in (15) J is very large the coefficients $d_{j,n}$ will be associated to the details. Some of them may be zero or very small. The last ones are considered as zeroes in the reconstruction process.*

We are going to show that the multiresolution analysis that is done for functions in $L_2(\mathbb{R}, \mathbb{X})$ induces a similar resolution for the special solutions defined by Lemma 2.1.

Let $f \in L_2(\mathbb{R}, \mathbb{X})$. We first obtain an approximation of f in a space $\mathbb{V}_J$, say $f_J(t) = \sum_{i\in I}\sum_{n\in\mathbb{Z}} a^i_{J,n}\phi_{J,n}(t)e_i$, by taking the orthogonal projections of f on the space $\mathbb{V}_J$.

Using the multiresolution analysis in $L_2(\mathbb{R}, \mathbb{X})$ we can write

$$f_J(t) = \sum_{i\in I}\sum_{n\in\mathbb{Z}} c^i_{J,n}\phi_{0,n}(t)e_i + \sum_{i\in I}\sum_{0\le j\le J-1, n\in\mathbb{Z}} d^i_{j,n}\psi_{j,n}(t)e_i. \tag{16}$$

For each $f \in L_2(\mathbb{R},\mathbb{X})$ we have the special solution $x_f \in \mathfrak{S}$. Then we have the corresponding approximated solution in the space $\mathcal{V}_J$:

$$x_J(t) = \sum_{i\in I}\sum_{n\in\mathbb{Z}} a^i_{J,n}x_{J,n}(t)\,ds\,e_i.$$

The corresponding multiresolution analysis for this solution becomes

$$x_J(t) = \sum_{i\in I}\sum_{n\in\mathbb{Z}} c^i_{J,n}x_{0,n}(t)e_i + \sum_{i\in I}\sum_{0\le j\le J-1, n\in\mathbb{Z}} d^i_{j,n}y_{j,n}(t)e_i. \tag{17}$$

We remark that coefficients in (16) and in (17) are the same.

Remark 2.11. *From the previous remark and as it was stated in (2.10) the reconstruction of the solution $x_J(t)$ could be done directly from (17).*

2.2. Some Special Cases.

In some cases, the functions ϕ, ψ are zero outside the bounded unit interval. This is the case of Haar wavelets, which takes the value 1 inside the bounded interval. In such situation the relations obtained in Lemma 2.9 can be improved.

Lemma 2.12. *Under assumptions of Lemma 2.9, if $\phi(s) = 0,\ s < 0$ and $\phi(s) = 0,\ s > 1$ we have that*

$$x^{i-}_j(t) = 0,\ t < 0$$

$$x^{i-}_j(t) = 2^{-j/2}\int_0^{2^j t} e^{A(t-2^{-j}s)}P\phi(s)\,ds\,e_i,\ 0 \le t \le 2^{-j},$$

$$x^{i-}_j(t) = 2^{-j/2}e^{At}\int_0^1 e^{-A(2^{-j}s)}P\phi(s)\,ds\,e_i,\ t \ge 2^{-j}.$$

Similar results hold for $x^{i+}_j(t)$:

$$x^{i+}_j(t) = 0,\ t > 2^{-j},$$

$$x^{i+}_j(t) = -2^{-j/2}e^{-At}\int_{2^j t}^1 e^{A(2^{-j}s)}(I-P)\phi(s)\,ds\,e_i,\ 0 \le t \le 2^{-j},$$

$$x^{i+}_j(t) = -2^{-j/2}e^{-At}\int_0^1 e^{A(2^{-j}s)}(I-P)\phi(s)\,ds\,e_i,\ t < 0.$$

Similar results hold for $y^i_j(t)$.

$$y^{i-}_j(t) = 0,\ t < 0$$

$$y^{i-}_j(t) = 2^{-j/2}e^{At}\int_0^{2^j t} e^{-A(2^{-j}s)}P\psi(s)\,ds\,e_i,\ 0 \le t \le 2^{-j},$$

$$y^{i-}_j(t) = 2^{-j/2}e^{At}\int_0^1 e^{-A(2^{-j}s)}P\psi(s)\,ds\,e_i,\ t \ge 2^{-j}.$$

Similar results hold for $y^{i+}_j(t)$.

$$y_j^{i+}(t) = 0,\ t > 2^{-j}.$$

$$y_j^{i+}(t) = -2^{-j/2}e^{-At}\int_{2^j t}^{1} e^{A(2^{-j}s)}(I-P)\psi(s)\,ds\,e_i,\ 0 \le t \le 2^{-j}$$

$$y_j^{i+}(t) = -2^{-j/2}e^{-At}\int_{0}^{1} e^{A(2^{-j}s)}(I-P)\psi(s)\,ds\,e_i,\ t < 0$$

Proof: The proof is elementary. □

Lemma 2.13. *Let $f \in L_2(\mathbb{R}, \mathbb{X})$, and $f(t) = \sum_{i\in J} f_i(t)e_i$, where*

$$f_i(t) = \sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} a_{j,n}^i 2^{j/2}\phi(2^j t - n).$$

Then the unique bounded solution on $\mathbb{R}$ of

$$\dot{x} = Ax + f(t)$$

is given by

$$x(t) = \sum_{i\in J}\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} a_{j,n}^i x_j^i(t-n),$$

where $x_j^i(t)$ is given in Lemma 2.9.

A similar result holds true for $y(t)$.

Proof: The proof follows from Lemma 2.9 and by linearity. □

3. Applications with the Haar basis.

As a first example we consider the Haar basis:

$$\phi = \mathcal{X}_{[0,1]}, \quad \psi = \mathcal{X}_{[0,1/2)} - \mathcal{X}_{[[1/2,1)]}.$$

In this case $h_0 = h_1 = \frac{1}{\sqrt{2}}$, $g_0 = \frac{1}{\sqrt{2}}$, $g_1 = -\frac{1}{\sqrt{2}}$.

$$\phi(t) = \sqrt{2}[\frac{1}{\sqrt{2}}\phi(2t) + \frac{1}{\sqrt{2}}\phi(2t-1)] = \sqrt{2}[h_0\phi(2t) + h_1\phi(2t-1)]$$

$$\psi(t) = \sqrt{2}[\frac{1}{\sqrt{2}}\phi(2t) - \frac{1}{\sqrt{2}}\phi(2t-1)] = \sqrt{2}[g_0\phi(2t) + g_1\phi(2t-1)]$$

Lemma 3.1. *Under the assumptions of Lemma 2.12 using now the Haar basis with $\phi = \mathcal{X}_{[0,1]}$, we obtain the following results:*

$$x_j^{i-}(t) = 0,\ t < 0,$$

$$x_j^{i-}(t) = -2^{j/2}A^{-1}e^{At}(e^{-At} - I)Pe_i,\ 0 \le t \le 2^{-j},$$

$$x_j^{i-}(t) = -2^{j/2}A^{-1}e^{At}(e^{-2^{-j}A} - I)Pe_i,\ t \ge 2^{-j}.$$

$$x_j^{i+}(t) = 0,\ t > 2^{-j}.$$

$$x_j^{i+}(t) = -2^{j/2}A^{-1}e^{-At}(e^{2^{-j}A} - e^{At})(I-P)e_i,\ 0 \le t \le 2^{-j},$$

$$x_j^{i+}(t) = -2^{j/2}A^{-1}e^{-At}(e^{2^{-j}A} - I)(I-P)e_i,\ t<0.$$

The corresponding results for $y(t)$, using ψ instead of ϕ, are given by:

$$y_j^{i-}(t) = 0,\ t<0,$$

$$y_j^{i-}(t) = -2^{j/2}A^{-1}e^{At}(e^{-At} - I)Pe_i,\ 0 \le t \le 2^{-(j+1)},$$

$$y_j^{i-}(t) = 2^{j/2}A^{-1}e^{At}[I - 2e^{-A2^{-(j+1)}} + e^{-At}]Pe_i,\ 2^{-(j+1)} \le t \le 2^{-j},$$

$$y_j^{i-}((t) = 2^{j/2}A^{-1}e^{At}[I - 2e^{-A2^{-(j+1)}} + e^{-A2^{-j}}]Pe_i,\ 2^{-j} \le t.$$

$$y_j^{i+}(t) = 0,\ 2^{-j} \le t$$

$$y_j^{i+}(t) = -2^{j/2}A^{-1}e^{-At}(e^{2^{-j}A} - e^{At})(I-P)e_i,\ 2^{-(j+1)} \le t \le 2^{-j},$$

$$y_j^{i+}(t) = -2^{j/2}A^{-1}e^{-At}(2e^{A2^{-(j+1)}} - e^{2^{-j}A} - e^{At})(I-P)e_i,\ 0 \le t \le 2^{-(j+1)},$$

$$y_j^{i+}(t) = -2^{j/2}A^{-1}e^{-At}(2e^{A2^{-(j+1)}} - I - e^{2^{-j}A})(I-P)e_i,\ t \le 0.$$

Now we consider a more specific example.

$$\dot{x} = -x + 2^{j/2}\phi(2^j t)$$

From the expressions,

$$x_j^{i-}(t) = -2^{j/2}A^{-1}e^{At}(e^{-At} - I)Pe_i,\ 0 \le t \le 2^{-j},$$

$$x_j^{i-}(t) = -2^{j/2}A^{-1}e^{At}(e^{-2^{-j}A} - I)Pe_i,\ t \ge 2^{-j},$$

taking $e_i = 1,\ A = -1\ P = I$, we obtain:

$$x_j(t) = 0,\ t \le 0,$$

$$x_j(t) = 2^{j/2}(1 - e^{-t}),\ 0 \le t \le 2^{-j},$$

$$x_j(t) = 2^{j/2}e^{-t}(e^{2^{-j}} - 1),\ t \ge 2^{-j}.$$

Corresponding results can be obtained for $y(t)$.

$$\dot{y} = -y + 2^{j/2}\psi(2^j t)e_i,$$

$$y_j(t) = 0,\ t<0,$$

$$y_j(t) = 2^{j/2}e^{-t}(e^t - 1),\ 0 \le t \le 2^{-(j+1)},$$

$$y_j(t) = -2^{j/2}e^{-t}[1 - 2e^{2^{-(j+1)}} + e^t],\ 2^{-(j+1)} \le t \le 2^{-j},$$

$$y_j(t) = -2^{j/2}e^{-t}[1 - 2e^{2^{-(j+1)}} + e^{2^{-j}}],\ 2^{-j} \le t.$$

Proof: To obtain the results above the calculations are elementary. □

Below in Fig. 1 we show scaling solutions for $j = 2$ and $j = -2$. In Fig. 2 we show scaling solutions for $j = 1$ and $j = -1$

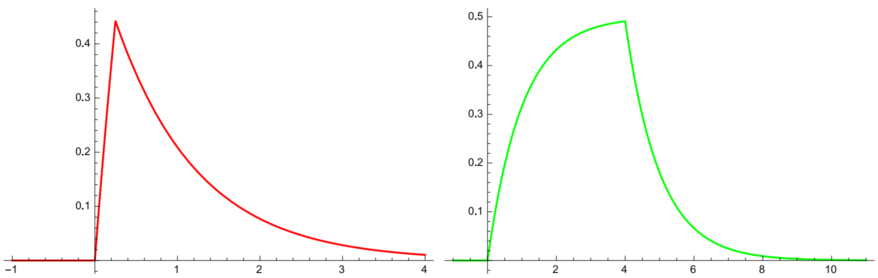

FIGURE 1. Scaling solution $x_j(t)$ for $j = 2$ (red), and $j = -2$ (green).

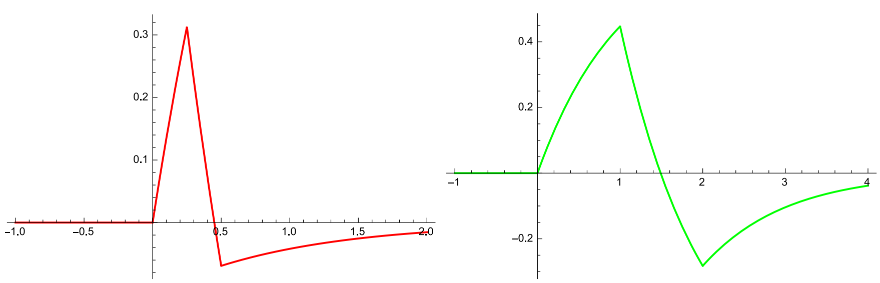

FIGURE 2. Wavelet solution $y_j(t)$ for $j = 1$ (red), and $j = -1$ (green).

3.1. **The Stable and the Unstable Manifolds.** Under the above assumptions let $f \in L_2(\mathbb{R}, \mathbb{X})$ and $x_f(t)$ be the special solution given in (3). The object of this section is to describe the stable and the unstable manifolds associated to it.

Following the assumptions and the notations of Lemma 2.1, let $S := P\mathbb{X}$ and $U := (I-P)\mathbb{X}$, respectively, the stable space and the unstable space associated to the hyperbolic linear map A.

The section of the stable manifold associated to the special solution $x_f(t)$ of the equation given in Lemma 2.1 will be given by $\mathcal{S}(t) := S + x_f(t)$. Similarly $\mathcal{U}(t) := U + x_f(t)$.

Considering the multiresolution, we obtain an approximation of f in a space V_J, say $f_J(t) = \sum_{n\in\mathbb{Z}} a_{J,n}\phi_{J,n}(t)$, by taking the orthogonal projections of f on the space V_J. Then we have the corresponding approximated special solution:

$$x_J(t) = \sum_{n\in\mathbb{Z}} a_{J,n} x_{J,n}(t),$$

where the special solution $x_J(t)$ is given in Lemma 2.9.

Using the estimate

$$\sup_{t\in\mathbb{R}} \|S(f)(t)\| \leq K^* \int_{-\infty}^{-\infty} \|f(t)\|^2 dt,$$

it follows that

$$\sup_{t\in\mathbb{R}} \|x_J(t)\| \leq K^* \sum_{n\in\mathbb{Z}} |c_{J,n}|^2.$$

The approximated section of stable and unstable manifolds in the time t will be given by,

$$x_J(t) + S = \sum_{n\in\mathbb{Z}} a_{J,n} x_{J,n}(t) + S, \quad x_J(t) + U = \sum_{n\in\mathbb{Z}} a_{J,n} x_{J,n}(t) + U.$$

Using the wavelet multiresolution analysis we can decompose it in the form:

$$x_J(t) = \sum_{n\in\mathbb{Z}} c_{J,n} x_{0,n}(t) + \sum_{0\leq j\leq J-1, n\in\mathbb{Z}} d_{j,n} y_{j,n}(t).$$

In this case, the approximated section of stable and unstable manifolds will be given by

$$\mathcal{S}^J(t) := x_J(t) + S = \sum_{n\in\mathbb{Z}} c_{J,n} x_{0,n}(t) + S + \sum_{0\leq j\leq J-1, n\in\mathbb{Z}} d_{j,n} y_{j,n}(t).$$

$$\mathcal{U}^J(t) := x_J(t) + U = \sum_{n\in\mathbb{Z}} c_{J,n} x_{0,n}(t) + U + \sum_{0\leq j\leq J-1, n\in\mathbb{Z}} d_{j,n} y_{j,n}(t).$$

4. **Weakly Nonlinear Equations.** To give a motivation for what will be treated in the next section we present a simple example.

Example 4.1. *Let us consider the following differential equation*

$$\dot{x} = -x + h(t)x^3 + \varepsilon f(t),$$

where h and f belong to $L_2(\mathbb{R}, \mathbb{R})$.

When we project h and f on V_J we obtain the approximated equation:

$$\dot{x} = -x + h_J(t)x^3 + \varepsilon f_J(t).$$

4.1. **The special solution.** Following the ideas of Hale [10] and of Kloeden and Rodrigues [15] we now consider nonlinear differential equations of the form

$$\frac{dx}{dt} = Ax + f(t,x,\varepsilon), \tag{18}$$

where A is as before and the nonlinear term belongs to the class

$$\mathcal{L}ip(\eta, M) := \Big\{ f : \mathbb{R}\times\Omega(\rho_0,\sigma_0) \to \mathbb{X}, \Big(\textstyle\int_{-\infty}^{+\infty} \|f(t,0,\varepsilon)\|^2 dt\Big)^{1/2} \le M(\varepsilon),$$
$$\Big(\textstyle\int_{-\infty}^{+\infty} \|f(t,x(t),\varepsilon) - f(t,y(t),\varepsilon)\|^2 dt\Big)^{1/2} \le \eta(\rho,\sigma)\sup_{t\in\mathbb{R}}\|x(t)-y(t)\| \Big\}$$

for all $(t,x(t),\varepsilon)$, $(t,y(t),\varepsilon) \in \mathbb{R}\times\Omega(\rho,\sigma)$, $\rho \le \rho_0$ and $0 < \sigma \le \sigma_0$, where

$$\Omega(\rho,\sigma) := \left\{ (x(t),\varepsilon) \in \mathbb{X}\times\mathbb{R}^m \ : \ \sup_{t\in\mathbb{R}}\|x(t)\| \le \rho, \ |\varepsilon| \le \sigma \right\},$$

$$B_\rho := \left\{ x(t) \in \mathbb{X} \ : \ \sup_{t\in\mathbb{R}}\|x(t)\| \le \rho \right\},$$

and let $\eta(\rho,\sigma)$, $M(\sigma)$ for $\rho \ge 0$, $\sigma \ge 0$, be continuous functions which are nondecreasing in both variables with $\eta(0,0) = 0$ and $M(0) = 0$.

Theorem 4.2. *Suppose that assumptions in Lemma 2.1 hold. Suppose also that the function $f \in \mathcal{L}ip(\eta, M)$. Then, there are constants $\rho_1 > 0$, $\sigma_1 > 0$ and a function $x^*(t,\varepsilon)$ such that $x^*(\cdot,\varepsilon) \in \mathcal{F}$, $x^*(t,0) = 0$, $|x^*(\cdot,\varepsilon)| \le \rho_1$, $0 \le |\varepsilon| \le \sigma_1$, such that $x^*(\cdot,\varepsilon)$ is the unique solution of (18) with norm $\le \rho_1$ and $x^*(\cdot,\varepsilon) \in \mathcal{F}$.*

Proof: Let $B_{\rho_1} := \{x(\cdot) \in \mathcal{F} \ : \ |x(t)| \le \rho_1, \ \forall\, t \in \mathbb{R}\ \}$, where $0 < \rho_1 \le \rho_0$, and consider the operator

$$(\mathcal{T}x(\cdot))(t) = (\mathcal{T}_\varepsilon x(\cdot))(t) =$$
$$\int_{-\infty}^{t} e^{A(t-s)} P f(s,x(s),\varepsilon)\, ds + \int_{t}^{\infty} e^{A(t-s)} (I-P) f(s,x(s),\varepsilon)\, ds,$$

which is motivated by the solution operator $\mathcal{S} : f \in L_2(\mathbb{R},\mathbb{X}) \mapsto \mathcal{S}f := x_f \in \mathcal{F}$ defined by (3) for the nonhomogeneous linear differential equation (1).

We will show that $\mathcal{T}$ has a unique fixed point in B_{ρ_1}. Following Hale [10] and Kloeden and Rodrigues[15], using (4) we obtain

$$\begin{aligned}
\sup_{t\in\mathbb{R}}\|(\mathcal{T}x(\cdot))(t)\| &\le K^*\left[\int_{-\infty}^{\infty}\|f(s,x(s),\varepsilon)\|^2 ds\right]^{1/2} \\
&\le K^*\ \Big(\eta(\rho_1,\sigma_1)\sup_{t\in\mathbb{R}}\|x(t)\| + M(\sigma_1)\Big) \\
&< \rho_1,
\end{aligned}$$

if ρ_1 and σ_1 are taken sufficiently small.

$$\begin{aligned}
\sup_{t\in\mathbb{R}}|\mathcal{T}x(\cdot))(t) - \mathcal{T}y(\cdot)(t)| &\le K^*\eta(\rho_1,\sigma_1)\|x(t)-y(t)\| \\
&\le \theta\ \sup_{t\in\mathbb{R}}\|x(t)-y(t)\|,
\end{aligned}$$

where θ can be taken sufficiently small, say $\theta < \frac{1}{2}$.

This shows that $\mathcal{T}$ is a uniform contraction on B_{ρ_1} for $|\varepsilon| \leq \sigma_1$ and, thus, has a unique fixed point $x^*(\cdot,\varepsilon)$ in B_{ρ_1}. Moreover, $x^*(t,\varepsilon)$ is a solution of (18) and $x^*(t,0)=0$. □

Remark 4.3. *Now we will consider the multiresolution analysis on the special solution. We recall that*

$$x^*(t,\varepsilon) = \int_{-\infty}^{t} e^{A(t-s)} P f(s,x^*(s,\varepsilon),\varepsilon)\,ds + \int_{t}^{\infty} e^{A(t-s)}(I-P) f(s,x^*(s,\varepsilon),\varepsilon)\,ds, \tag{19}$$

Since $f(\cdot,x^*(\cdot,\varepsilon),\varepsilon) \in L_2(\mathbb{R},\mathbb{X})$, now $f(\cdot,x^*(\cdot,\varepsilon),\varepsilon)$ will play the role of f in (3). Let $x_J^*(\cdot,\varepsilon)$ the projection of $x^*(\cdot,\varepsilon)$ in the space $\mathcal{V}_J$ and $f_J(\cdot,x^*(\cdot,\varepsilon),\varepsilon)$ the projection of $f(\cdot,x^*(\cdot,\varepsilon),\varepsilon)$ in the space $\mathbb{V}_J$.

Therefore from (19) we obtain the following approximation of $x^*(\cdot,\varepsilon)$,

$$x_J^*(t,\varepsilon) = \int_{-\infty}^{t} e^{A(t-s)} P f_J(s,x^*(s,\varepsilon),\varepsilon)\,ds + \int_{t}^{\infty} e^{A(t-s)}(I-P) f_J(s,x^*(s,\varepsilon),\varepsilon)\,ds, \tag{20}$$

As in Lemma 2.9, using the Hilbert space structure, we write $f(s,x^*(s,\varepsilon),\varepsilon) = \sum_{i\in I} f^i(s,,x^*(s,\varepsilon),\varepsilon)e_i$.

Next we consider the orthogonal projection of $f^i(\cdot,x^*(\cdot,\varepsilon),\varepsilon)$ in the space V_J, obtaining $f_J^i(s,x^*(s,\varepsilon),\varepsilon) = \sum_{n\in\mathbb{Z}} a_{J,n}^i(\varepsilon)\phi_{J,n}(s)$, where

$$a_{J,n}^i(\varepsilon) = \int_{-\infty}^{+\infty} f^i(s,x^*(s,\varepsilon),\varepsilon)\phi_{J,n}(s)ds.$$

Now we define $f_J(s,x^*(s,\varepsilon),\varepsilon) = \sum_{i\in I} f_J^i(s,x^*(s,\varepsilon),\varepsilon)e_i$.

From now on the multiresolution analysis follows the same steps as in (16).

$$x_J^*(t,\varepsilon) = \sum_{i\in I}\sum_{n\in\mathbb{Z}} a_{J,n}^i(\varepsilon)x_{J,n}(t)e_i,$$

$$x_J^*(t,\varepsilon) = \sum_{i\in I}\sum_{n\in\mathbb{Z}} c_{J,n}^i(\varepsilon)x_{0,n}(t)e_i + \sum_{i\in I}\sum_{0\leq j\leq J-1,n\in\mathbb{Z}} d_{j,n}^i(\varepsilon)y_{j,n}(t)e_i. \tag{21}$$

5. **Another approximation of the perturbed equilibrium.** Under the above assumptions, for each $x\in\mathbb{X}$ the function $\|f(\cdot,x,\varepsilon)\|$ is square integrable in $\mathbb{R}$. Next we will find an approximation $f_J(\cdot,x,\varepsilon)$ for the function $f(\cdot,x,\varepsilon)$.

As in Lemma 2.9, taking into account the Hilbert space structure, we write $f(s,x,\varepsilon) = \sum_{i\in I} f^i(s,x,\varepsilon)e_i$. Next we consider the orthogonal projection of $f^i(\cdot,x,\varepsilon)$ in the space V_J, that we denote by $f_J^i(s,x,\varepsilon)$. Then we denote $f_J(\cdot,x,\varepsilon) := \sum_{i\in I} f_J^i(s,x,\varepsilon)$.

The function $f_J(s,x,\varepsilon)$ satisfies similar conditions as $f(s,x,\varepsilon)$ in the above theorem.

Therefore there exists a unique small solution $x^*(t,\varepsilon,J)$, which is the fixed point of

$$\begin{aligned} x^*(t,\varepsilon,J) &= \int_{-\infty}^{t} e^{A(t-s)} P f_J(s,x^*(s,\varepsilon,J),\varepsilon)\,ds \\ &\quad + \int_{t}^{\infty} e^{A(t-s)}(I-P) f_J(s,x^*(s,\varepsilon,J),\varepsilon)\,ds. \end{aligned}$$

Remark 5.1. *At this point we can also provide a multiresolution analysis. In order to do this we have to project the function $f_J(\cdot, x^*(\cdot,\varepsilon,J),\varepsilon)$ in the space $\mathbb{V}_J$ and proceed as in Remark 4.3.*

5.1. **A motivation example for the Stable and the Unstable Manifolds.** In this section we follow the ideas of Hale [10] and of Kloeden and Rodrigues [15]. As a motivation we will consider the example:

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = -x_2 + h(t)x_1^3 + \varepsilon\phi(t), \end{cases} \tag{22}$$

where $\phi = \mathcal{X}_{[0,1]}$, $h(t) = e^t,\ t \le 0$ and $h(t) = e^{-t},\ t \ge 0$. We have already computed the unique small bounded solution in $\mathbb{R}$, $x^0(t)$ and it is given by

$$\begin{pmatrix} x_1^0(t) \\ x_2^0(t) \end{pmatrix},$$

where $x_1^0(t) = 0,\ \forall\, t \in \mathbb{R}$, $x_2^0(t) = 0,\ t \in (-\infty, 0]$, $x_2^0(t) = \varepsilon(1 - e^{-t}),\ t \in [0,1]$ and $x_2^0(t) = \varepsilon(e^{-t}(e-1)),\ t \in [1,\infty)$.

In order to compute the stable and unstable manifolds of this special solution, we consider the change of variables $y_1 = x_1,\ y_2 = x_2 - x_2^0(t)$ and obtain the new system:

$$\begin{cases} \dot{y}_1 = y_1 \\ \dot{y}_2 = -y_2 + h(t)y_1^3. \end{cases} \tag{23}$$

Now we will compute the stable and unstable manifolds of the equilibrium solution.

The stable manifold is given by $\mathcal{S} = \begin{pmatrix} 0 \\ y_2 \end{pmatrix}$.

Now we will calculate the unstable manifold for $t \le 0$. If we solve the first equation we obtain the solution $y_1(t) = ae^t$. If we substitute in the second equation we obtain the equation $\dot{y}_1(t) = -y_2 + a^3e^{4t}$, for $t \le 0$. If we look for a bounded solution for $t \le 0$ we solve this equation and obtain $y_2(t) = \frac{a^3}{5}e^{4t}$.

Therefore the section of the unstable manifold at the time $t \le 0$ is given by

$$\mathcal{U}(t) = \begin{pmatrix} ae^t \\ \frac{a^3}{5}e^{4t} \end{pmatrix}.$$

The section of the stable and unstable manifolds of the solution $x^0(t)$ are given, respectively, by: $x^0(t) + \mathcal{S}$ and $x^0(t) + \mathcal{U}(t)$.

5.2. **The saddle property.** In this section we will establish the existence of stable and unstable manifolds of the special solution $x^*(t,\varepsilon)$ and also prove that they inherit some properties of the original equation (18). Under the set up and assumptions of the last section we consider the change of variables on the equation (18)

$$x = x^*(t,\varepsilon) + y,$$

and for the equation on y we obtain

$$\frac{dy}{dt} = Ay + F(t,y,\varepsilon) \tag{24}$$

where $F(t,y,\varepsilon) := f(t, x^*(t,\varepsilon)+y,\varepsilon) - f(t,x^*(t,\varepsilon),\varepsilon)$. The function $F(t,y,\varepsilon)$ plays the role for (24) that the function $f(t,y,\varepsilon)$ played for (18). Since $F(t,0,\varepsilon) = 0$ and $M(\varepsilon) = 0$, the unique entire solution in B_{ρ_1} is the zero solution.

For any $t_0 \in \mathbb{R}$, let $y(t, t_0, y^{t_0}, \varepsilon)$ denote the solution of (24) with the initial value $y(t_0, t_0, y^{t_0}, \varepsilon) = y^{t_0}$ and let K^* be as in Lemma 2.1. For each $\delta > 0$ we define the local stable and unstable manifolds as

$$S(t_0, \delta, \varepsilon) := \left\{ y^{t_0} \in \mathbb{X} \ : \ |Py^{t_0}| < \frac{\delta}{2K^*}, \ |y(t, t_0, y^{t_0}, \varepsilon)| < \delta, \ t \geq t_0 \right\}, \tag{25}$$

and

$$U(t_0, \delta, \varepsilon) := \left\{ y^{t_0} \in \mathbb{X} \ : \ |(I-P)y^{t_0}| < \frac{\delta}{2K^*}, \ |y(t, t_0, y^{t_0}, \varepsilon)| < \delta, \ t \leq t_0 \right\}, \tag{26}$$

respectively. Their existence and other properties are given by the next theorem.

Theorem 5.2. *Suppose that assumptions of Theorem 4.2 hold with F in the place of f. If $F \in \mathcal{L}ip(\eta, 0)$, then there are constants $\delta > 0$, $\varepsilon_1 > 0$, $\beta > 0$ such that, for any $t_0 \in \mathbb{R}$, $|\varepsilon| \leq \varepsilon_1$, the mapping P is a homeomorphism of $S(t_0, \delta, \varepsilon)$ onto $P\mathbb{X} \cap B_{\delta/2K}$, $S(t_0, \delta, 0)$ which is tangent to $P\mathbb{X}$ at zero and*

$$\left|y(t, t_0, y^{t_0}, \varepsilon)\right| \leq 2K^* |Py^{t_0}| e^{-\beta(t-t_0)}, \ t \geq t_0,$$

for any $y^{t_0} \in S(t_0, \delta, \varepsilon)$ and the mapping $I - P$ is a homeomorphism of $U(t_0, \delta, \varepsilon)$ onto $(I-P)\mathbb{X} \cap B_{\delta/2K}$, $U(t_0, \delta, 0)$ which is tangent to $(I-P)\mathbb{X}$ at zero and

$$|y(t, t_0, y^{t_0}, \varepsilon)| \leq 2K^* |(I-P)y^{t_0}| e^{\beta(t-t_0)}, \ t \leq t_0,$$

for any $y^{t_0} \in U(t_0, \delta, \varepsilon)$.

Moreover, if $g(\cdot, t_0, \varepsilon) : P\mathbb{X} \cap B_{\delta/2K} \to S(t_0, \delta, \varepsilon)$ is the inverse of the homeomorphism P, then $g(y_-, t_0, \varepsilon)$ is Lipschitz in y_- with Lipschitz constant $2K^*$.

The same conclusions hold for the inverse of the homeomorphism $I - P$ of $U(t_0, \delta, \varepsilon)$ onto $(I-P)\mathbb{X} \cap B_{\delta/2K^*}$.

Proof: The proof is similar to the proof of Theorem 3.1, page 159, in Hale [10] or in Kloeden and Rodrigues [15]. Since we are working in a different but similar class of nonlinear problems, we will present just some key steps to adapt the proof there to our notation and assumptions.

Any solution of (24) which is bounded in $[t_0, \infty)$ will have the form:

$$\begin{aligned} x(t) \ &= \ e^{A(t-t_0)} x_- + \int_{t_0}^{t} e^{A(t-s)} PF(s, x(s), \varepsilon)\, ds \\ &+ \ \int_{\infty}^{t} e^{A(s-t)} (I-P) F(s, x(s), \varepsilon)\, ds \end{aligned} \tag{27}$$

for $t \geq t_0$, where $x_- \in P\mathbb{X}$.

In addition, any solution of (24) which is bounded in $(-\infty, t_0]$ has the form

$$\begin{aligned} x(t) \ &= \ e^{A(t_0-t)} x_+ + \int_{t_0}^{t} e^{A(s-t)} (I-P) F(s, x(s), \varepsilon)\, ds \\ &+ \int_{-\infty}^{t} e^{A(t-s)} PF(s, x(s), \varepsilon)\, ds \end{aligned} \tag{28}$$

for $t \leq t_0$, where $x_+ \in (I-P)\mathbb{X}$.

It follows from (2) that $|P| \leq K$ and $|I-P| \leq K$. If K and α are as in (2) and η is the Lipschitz constant of $f(t,x,\varepsilon)$ with respect to x we can choose $\delta > 0$ and $\varepsilon_1 > 0$ such that

$$4K^*K\eta(\delta,\varepsilon_1) < 1\ .$$

For each $x_- \in P\mathbb{X}$ with $|x_-| \leq \delta/(2K)$ consider the set

$$\mathcal{G}(t_0,x_-,\delta) := \big\{x : [t_0,\infty) \to \mathbb{X} \text{ continuous, } |x| := \sup_{t\in[t_0,\infty)} |x(t)| \leq \delta,\ P(t_0)x = x_-\big\},$$

which is a complete metric space with the topology of the uniform convergence. For $x \in \mathcal{G}(t_0,x_-,\delta)$ define $\mathcal{T}x$ by

$$(\mathcal{T}x)(t) = e^{A(t-t_0)}x_- + \int_{t_0}^{t} e^{A(t-s)}PF(s,x(s),\varepsilon)\,ds \tag{29}$$

$$+\int_{\infty}^{t} e^{A(s-t)}(I-P)F(s,x(s),\varepsilon)\,dst \geq t_0.$$

As in Hale [10], page 160, it follows from the contraction principle that $\mathcal{T}$ has a unique fixed point $x^*(\cdot,t_0,x_-,\varepsilon)$ for $|\varepsilon| \leq \varepsilon_1$, which depends continuously upon t, t_0, x_- and $x^*(\cdot,t_0,0,\varepsilon) = 0$ and satisfies the estimate

$$\big|x^*(t,t_0,x_-,\varepsilon) - x^*(t,t_0,x'_-,\varepsilon)\big| \leq 2KK^*e^{\dfrac{-\alpha(t-t_0)}{2}}\big|x_- - x'_-\big| \tag{30}$$

for $t \geq t_0$. In view of its definition, here

$$S(t_0,\delta,\varepsilon) = \big\{x\ :\ x = x^*(t_0,t_0,x_-,\varepsilon)\ \in (P\mathbb{X}) \cap B_{\delta/2K}\big\} \tag{31}$$

for $|\varepsilon| \leq \varepsilon_1$. Since $x^*(\cdot,t_0,0,\varepsilon) = 0$, from (30) and (31) we obtain

$$\big|x^*(\cdot,t_0,x^{t_0},\varepsilon)\big| \leq 2K|P(t_0)|\,e^{-\beta(t-t_0)}, \qquad t \geq t_0,$$

with $\beta := \alpha/2$.

Remark 5.3. *From 5, as before we can approximate the function $f(t,x,\varepsilon)$ by $f_J(t,x,\varepsilon)$ and consider the equation*

$$\frac{dx}{dt} = Ax + f_J(t,x,\varepsilon) \tag{32}$$

obtaining the special solution $x^*(t,\varepsilon,J)$.

If we let $y = x - x^*(t,\varepsilon,J)$ and

$$F_J(t,y,\varepsilon) := f_J(t,x^*(t,\varepsilon,J)+y,\varepsilon) - f_J(t,x^*(t,\varepsilon),\varepsilon,J),$$

we obtain the equation

$$\frac{dy}{dt} = Ay + F_J(t,y,\varepsilon), \tag{33}$$

where $F_J(t,0,\varepsilon) = 0$.

Therefore we can obtain the stable and unstable manifolds associated to the zero solution of the above equation. These manifolds can be interpreted as approximations of the previous ones.

6. **The Pullback Attractor.** The theory of pullback attractors has recently proven very useful in analyzing the dynamics of nonautonomous dynamical systems appearing in the applied sciences, and has experienced a significant development over the last two decades (see, for instance, [3, 4, 5, 6, 13, 14, 15] and the references therein). We would like to emphasize now that this theory can be also applied to provide interesting information about the problem we have considered in this paper. Indeed, let us consider the nonautonomous equation

$$\dot{x} = Ax + f(t, x, \varepsilon), \tag{34}$$

and assume that there exists positive constants $K, \ \alpha$ such that

$$\|e^{At}\| \leq Ke^{-\alpha t}$$

for $t \geq 0$.

Then, is we also suppose that $f(t, x, \varepsilon)$ satisfies the conditions of Theorem 4.2, we can follow the ideas of Kloeden and Rodrigues in [15] to conclude that the special solution $x^*(t, \varepsilon)$ of (34) gives rise to the pullback attractor for equation (34). Therefore, as we did before, we can perform a similar analysis to construct the multiresolution analysis of the pullback attractor.

REFERENCES

[1] G. Bachman and L. Narici, *Functional Analysis,* Academic Press, New York, (1966).

[2] T. Gnana Bhaskar, S. Hariharan, N. Nataraj, Heatlet approach to diffusion equation on unbounded domains, *Applied Mathematics and Computation*, **197** (2008), 891–903.

[3] T. Caraballo, R. Colucci, X. Han, Non-autonomous dynamics of a semi-Kolmogorov population model with periodic forcing, *Nonlinear Anal. Real World Appl.*, **31** (2016), 661–680.

[4] T. Caraballo, X. Han, *Applied Nonautonomous and Random Dynamical Systems*, SpringerBriefs in Mathematics, Springer International Publishing, (2016). DOI 10.1007/978-3-319-49247-6

[5] T. Caraballo, X. Han, P. E. Kloeden, Nonautonomous chemostats with variable delays, *SIAM J. Math. Anal.*, **47**(3) (2015), 2178–2199.

[6] T. Caraballo, G. Łukaszewicz, J. Real, Pullback attractors for asymptotically compact nonautonomous dynamical systems, *Nonlinear Anal.*, **64**(3) (2006), 484–498.

[7] W.A. Coppel, Almost periodic properties of ordinary differential equations, *Anal. Mat. Pura Appl.*, **76** (4) (1967), 27–49.

[8] W.A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, D-.C. Heath & Co., Boston, (1965).

[9] I. C. Daubechies and A. C. Gilbert, Harmonic Analysis, Wavelets and Applications, *Proceedings of the IEEE.* **84** (4) (1996).

[10] J.K. Hale, *Ordinary Differential Equations,* Second Edition, Krieger Publishing Co., Huntington, New York, (1980).

[11] D. Henry, *Geometric Theory of Semilinear Parabolic Equations,* Springer Lecture Notes Math., Vol. 840, Springer-Verlag, Berlin, (1981).

[12] E. Hernández and G. Weiss, A First Course on Wavelets, *CRC Press LLC* (1996).

[13] P.E. Kloeden, Pullback attractors of nonautonomous semidynamical systems, *Stochastics & Dynamics* **3** , No. 1, (2003), 101-112.

[14] P. E. Kloeden, M. Rasmussen, *Nonautonomous dynamical systems*, volume 176 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2011.

[15] P. E. Kloeden, H. M. Rodrigues, Dynamics of a class of ODEs more general than almost periodic, *Nonlinear Analysis*, **74** (2011), 2695–2719.

[16] S. Mallat, A Wavelet Tour of Signal Processing, *Academic Press* (1998).

[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations* Springer-Verlag, New York Berlin Heidelberg Tokyo (1983).

[18] J. Shen and G. Strang, On Wavelets Fundamental Solutions to the Heat Equation, *J. of Differential Equations* **161** (2000), 403-421.

[19] B. Vidakovic and P. Mueller, Wavelets for Kids. A Tutorial Introduction Duke University. http://www2.isye.gatech.edu/ brani/wp/kidsA.pdf

E-mail address: hmr@icmc.usp.br
E-mail address: caraball@us.es
E-mail address: gameiro@icmc.usp.br