Relativistic effects in electromagnetic nuclear responses in the quasi-elastic delta region

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Abstract

A new non-relativistic expansion in terms of the nucleon's momentum inside nuclear matter of the current for isobar electro-excitation from the nucleon is performed. Being exact with respect to the transferred energy and momentum, this yields new current operators which retain important aspects of relativity not taken into account in the traditional non-relativistic reductions. The transition current thus obtained differs from the leading order of the traditional expansion by simple multiplicative factors. These depend on the momentum and energy transfer and can be easily included together with relativistic kinematics in non-relativistic, many-body models of isobar electro-excitation in nuclei. The merits of the new current are tested by comparing with the unexpanded electromagnetic nuclear responses in the isobar peak computed in a relativistic Fermi gas framework. The sensitivity of the relativistic responses to the isobar's magnetic, electric and Coulomb form factors and the finite width of the isobar is analyzed.

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1   Introduction

The cross section for inclusive electron scattering (e, e') shows a pronounced peak at an energy transfer $\omega \sim \sqrt{q^2 + m_N^2} - m_N$, corresponding to the quasi-free interaction with the individual nucleons in the nucleus (here m_N = nucleon mass). For high values of the momentum transfer $q = |\mathbf{q}|$ and higher energy loss it is possible to produce real pions and the cross section shows another peak dominated by the resonant production of a $\Delta(1232)$ at $\omega \sim \sqrt{q^2 + m_\Delta^2} - m_N^2$, where m_Δ is the Δ mass [1]. The width of these peaks is related to the Fermi momentum of the nucleons inside the nucleus and, in the case of the Δ-peak, also to the decay width of the Δ in nuclear matter. Hence for a high enough value of q, these two peaks actually overlap and cannot be separated in inclusive experiments [2, 3]. Thus the response in the region above the quasi-elastic peak contains information about the nucleon's excited states and their change due to the nuclear medium.

Since the electro-excitation of the Δ requires high energy and momentum transfers, a relativistic treatment of the reaction is needed. Recently, several many-body calculations, both in nuclear matter and finite nuclei, have been performed in this region [4, 5, 6]; all of these calculations are non-relativistic in nature, although some relativistic corrections enter in two of them, including an expansion of the current to order $(p/m_N)^2$ in [4] and relativistic kinematics in [5]. Clearly some of these corrections are inadequate when one wishes to go to high momentum transfers, $q \simeq 1$ GeV/c, and there relativistic models such as the ones developed in [7, 8, 9, 10] are more appropriate. However, although these last calculations are fully relativistic, they do not include the full N-Δ vertex. For instance, in the pioneering calculation by Moniz [7] the Peccei Lagrangian was used, which is only appropriate for computing the transverse response for low momentum transfer [11, 12, 13]. In other work [8, 9] a more appropriate M1 magnetic transition current was used, although the electric E2 and Coulomb C2 excitation amplitudes of the Δ were not included. For years an important program has been pursued to determine more accurately the quadrupole $C2$ and $E2$ amplitudes in the Δ region [14], these being small compared with the dominant dipole $M1$ amplitude. Using polarized photons, the E2/M1-ratio has been measured to be around -3% at resonance [15]. The C2 amplitude however only appears in electro-production reactions $N(e, e')\Delta$. Values of the ratio $C2/M1$ around -13% have been reported in $H(e, e'\pi^0)p$ experiments at $Q^2 = 0.13$ (GeV/c)^2 [16]. However, this value differs with the findings of recent measurements of the transverse-longitudinal asymmetry and proton polarization in $H(e, e'p)\pi^0$ reactions [17]. Hence, for the $N \rightarrow \Delta$ transition our knowledge is still incomplete and has not been possible to undertake a full analysis in the sense of the work by Nozawa and Lee [18] of the effect of the C2 and E2

form factors in the nuclear Δ-peak.

In this paper we perform a new non-relativistic expansion of the electro-excitation current of the nucleon, providing an extension of our previous expansion of the electromagnetic nucleon current in powers of $\eta = p/m_N$, in which we retained the full dependence on q and ω [19]. Recently the same procedure has been also applied to meson-exchange currents (MEC) [20]. These currents can be implemented together with relativistic kinematics in standard non-relativistic models of one-particle emission near the quasi-elastic peak. In this paper we apply the same procedure to the Δ electro-excitation current, which we develop to leading order in η, again retaining the full dependence on the energy and momentum transfers. We perform this expansion for the magnetic transition current M1, which is the dominant one both in the longitudinal and transverse nuclear responses [9]. The resulting current is designed in such a way that it differs from the traditional non-relativistic limit simply by having (q, ω)-dependent factors which multiply the traditional operators. These corrections, being of leading order in η, are seen to arise mainly from the normalization factor in the Rarita-Schwinger spinor and from the lower-component spinology now included in the effective current operator.

The organization of the work is as follows. In sect. 2 we first develop the analytical expressions for the longitudinal and transverse responses in the relativistic Fermi gas (RFG) model by using a Δ-hole approach. We consider the full vertex of Jones and Scadron [12] that includes M1, E2 and C2 Δ-amplitudes. In deriving expressions for these responses we assume a stable Δ particle and later include its finite width by performing a convolution of these responses with a Lorentz distribution. In sect. 3 we perform the expansion of the magnetic current to leading order in the momentum of the bound nucleon. In sect. 4 we test the validity of the expansion by comparing the exact RFG result with a non-relativistic Fermi gas model, using the new current and relativistic kinematics. Furthermore, we compare several models of the reaction by using the RFG; in particular, we study the differences that arise upon using the Peccei and magnetic Lagrangians, and explore the effects of the E2, C2 multipoles and the finite width of the Δ on the longitudinal and transverse response functions. Finally in sect. 5 we draw our main conclusions.

2   General formalism

We start our discussion by introducing the general formalism, which is based on a relativistic treatment of the nuclear currents entering into the calculation of the response functions. It is well-known [21] that the longitudinal and transverse response functions

can be evaluated as components of the nuclear tensor $W_{\mu\nu}$, namely

$$R_L(q,\omega) = \left(\frac{q^2}{Q^2}\right)^2 \left[W_{00} - \frac{\omega}{q}\left(W_{03} + W_{30}\right) + \frac{\omega^2}{q^2}W_{33}\right] = W_{00} \tag{1}$$

$$R_T(q,\omega) = W_{11} + W_{22} , \tag{2}$$

where $Q^\mu = (\omega, \mathbf{q})$ is the space-like four-momentum carried by the virtual photon and the gauge invariance has been exploited in obtaining Eq. (1). We shall compute this nuclear tensor in the RFG framework, where nucleons are assumed to move freely inside the system with relativistic kinematics, hence being on their mass-shell.

2.1 Nuclear tensor and response functions for inelastic processes

We first consider the electro-production of a stable resonance (namely, the Δ is viewed as a particle on its mass-shell) in the RFG — in this case analytical expressions for the response functions are obtained — and later on we shall include corrections due to inclusion of the decay width of the Δ. The RFG nuclear tensor reads (see, for example, [21])

$$W_{\mu\nu} = \frac{3\pi^2 \mathcal{N} m_N^2}{k_F^3} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\theta(k_F - p)}{E(\mathbf{p})E_\Delta(\mathbf{p}+\mathbf{q})} f_{\mu\nu}(\mathbf{p}, \mathbf{p}+\mathbf{q})\delta[E_\Delta(\mathbf{p}+\mathbf{q}) - E(\mathbf{p}) - \omega] , \tag{3}$$

where $\mathcal{N}$ is the number of protons or neutrons in the nucleus. Here m_N and m_Δ are the masses of the struck nucleon and Δ, respectively, and $E(\mathbf{p}) = \sqrt{m_N^2 + p^2}$ and $E_\Delta(\mathbf{p}+\mathbf{q}) = \sqrt{m_\Delta^2 + (\mathbf{p}+\mathbf{q})^2}$ are their corresponding energies; k_F is the Fermi momentum and $f_{\mu\nu}$ the inelastic single-nucleon tensor of the $N \to \Delta$ transition.

Introducing the standard dimensionless variables

$$\boldsymbol{\eta} = \frac{\mathbf{p}}{m_N} , \quad \boldsymbol{\eta}_F = \frac{\mathbf{k}_F}{m_N} , \quad \boldsymbol{\kappa} = \frac{\mathbf{q}}{2m_N} , \tag{4}$$

$$\lambda = \frac{\omega}{2m_N} , \quad \tau = \kappa^2 - \lambda^2 , \quad \varepsilon = \sqrt{1 + \eta^2} , \quad \mu_\Delta = \frac{m_\Delta}{m_N} \tag{5}$$

and performing the angular integration in Eq. (3) we obtain

$$W_{\mu\nu}(\kappa, \lambda) = \frac{3\mathcal{N}}{8m\eta_F^3\kappa} \int_{\varepsilon_0}^{\varepsilon_F} f_{\mu\nu}(\varepsilon, \theta_0)d\varepsilon , \tag{6}$$

where $\varepsilon_F = \sqrt{1 + \eta_F^2}$ is the Fermi energy and

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$$\varepsilon_0 = \kappa \sqrt{\frac{1}{\tau} + \rho^2} - \lambda \rho \tag{7}$$

the minimum energy of the struck nucleon for fixed κ and λ, having defined the factor

$$\rho = 1 + \frac{1}{4\tau} \left(\mu_\Delta^2 - 1 \right) , \tag{8}$$

which measures the inelasticity of the elementary process.

In Eq. (6) the single-nucleon tensor $f_{\mu\nu}(\varepsilon, \theta_0)$ contains the angle θ between $\boldsymbol{\eta}$ and $\boldsymbol{\kappa}$ given via

$$\cos \theta_0 = \frac{\lambda \varepsilon - \tau \rho}{\kappa \eta} , \tag{9}$$

as required by energy conservation. The condition $|\cos \theta_0| \leq 1$ then permits the response of the Δ to occur only in the range

$$\frac{1}{2} \left[\sqrt{(2\kappa - \eta_F)^2 + \mu_\Delta^2} - \varepsilon_F \right] \leq \lambda \leq \frac{1}{2} \left[\sqrt{(2\kappa + \eta_F)^2 + \mu_\Delta^2} - \varepsilon_F \right] . \tag{10}$$

In analogy with the physics of the quasi-elastic peak [21], it is convenient to introduce a scaling variable ψ_Δ defined as follows

$$\psi_\Delta^2(\kappa, \lambda) = \frac{\varepsilon_0 - 1}{\xi_F} = \frac{1}{\xi_F} \left(\kappa \sqrt{\frac{1}{\tau} + \rho^2} - \lambda \rho - 1 \right) , \tag{11}$$

with $\xi_F = \varepsilon_F - 1$. The physical meaning of the scaling variable may be deduced from the above equation: in terms of dimensionless variables, $\xi_F \psi_\Delta^2$ is the minimum kinetic energy required to transform a nucleon inside the nucleus into a Δ when hit by a photon of energy λ and momentum κ. It is straightforward to check that, when $m_N = m_\Delta$ (hence $\rho = 1$), the ordinary quasi-elastic scaling variable ψ [21] is recovered.

In terms of the scaling variable in Eq. (11) the response region given by Eq. (10) simply reduces to $-1 \leq \psi_\Delta \leq 1$. The energy position $\lambda_{\Delta P}$ of the peak of the Δ response occurs when ψ_Δ vanishes, namely for

$$\psi_\Delta = 0 \longrightarrow \lambda_{\Delta P} = \tau \rho , \kappa_{\Delta P}^2 = \tau(\tau \rho^2 + 1) . \tag{12}$$

We now turn to a consideration of the nucleonic tensor $f_{\mu\nu}$, which we will obtain in the next section for specific $N \to \Delta$ currents. For any physical process this tensor must comply with Lorentz covariance and current conservation ($Q^\mu f_{\mu\nu} = f_{\mu\nu}Q^\nu = 0$). The most general unpolarized second-rank tensor consistent with these requirements is [22]

$$f_{\mu\nu}(\mathbf{p}, \mathbf{p} + \mathbf{q}) = -w_1(\tau)\left(g_{\mu\nu} - \frac{Q_\mu Q_\nu}{Q^2}\right) + w_2(\tau)V_\mu V_\nu - \frac{i}{m_N}w_3(\tau)\varepsilon_{\mu\nu\rho\sigma}Q^\rho V^\sigma , \qquad (13)$$

where $w_1(\tau)$, $w_2(\tau)$ and $w_3(\tau)$ are *scalar* functions containing the specific dynamics of the process and

$$V_\mu = \frac{1}{m_N}\left(P_\mu - \frac{P \cdot Q}{Q^2}Q_\mu\right) , \qquad (14)$$

is a four-vector orthogonal to Q_μ, P_μ being the struck nucleon's four-momentum. Only the terms involving $w_{1,2}(\tau)$ occur in EM interactions, whereas $w_3(\tau)$ also enters for the full electroweak interaction. Energy conservation via Eq. (9) implies that $P \cdot Q/Q^2 = -\rho/2$: hence the longitudinal and transverse components of the single-nucleon tensor read

$$f_L = f_{00} = -\frac{\kappa^2}{\tau}w_1(\tau) + (\lambda\rho + \varepsilon)^2 w_2(\tau) \qquad (15)$$

and

$$f_T = f_{11} + f_{22} = 2w_1(\tau) + \left[\varepsilon^2 - 1 - \left(\frac{\lambda\varepsilon - \tau\rho}{\kappa}\right)^2\right]w_2(\tau) . \qquad (16)$$

Finally, by performing the energy integral in Eq. (6) one gets for the response functions in Eqs. (1,2) the following expressions

$$R_L(\kappa, \lambda) = \frac{3\mathcal{N}\xi_F}{8m_N\eta_F^3\kappa}\frac{\kappa^2}{\tau}\left[(1 + \tau\rho^2)w_2(\tau) - w_1(\tau) + w_2(\tau)\mathcal{D}(\kappa, \lambda)\right](1 - \psi_\Delta^2)\theta(1 - \psi_\Delta^2) \qquad (17)$$

$$R_T(\kappa, \lambda) = \frac{3\mathcal{N}\xi_F}{8m_N\eta_F^3\kappa}\left[2w_1(\tau) + w_2(\tau)\mathcal{D}(\kappa, \lambda)\right](1 - \psi_\Delta^2)\theta(1 - \psi_\Delta^2) , \qquad (18)$$

where

$$\mathcal{D}(\kappa, \lambda) \equiv \frac{\tau}{\kappa^2}\left[(\lambda\rho + 1)^2 + (\lambda\rho + 1)(1 + \psi_\Delta^2)\xi_F + \frac{1}{3}(1 + \psi_\Delta^2 + \psi_\Delta^4)\xi_F^2\right] - (1 + \tau\rho^2) \qquad (19)$$

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reflects the (modest) Fermi motion of the nucleons. In fact, at the resonance peak ($\psi_\Delta = 0$) Eq. (19) reduces to

$$\mathcal{D}(\kappa, \lambda)_{\Delta P} = \xi_F + \frac{\tau}{3\kappa^2}\xi_F^2 \;, \tag{20}$$

and, since $\xi_F \simeq 0.03$, yields a small correction.

2.2 Density dependence of the response functions

In this subsection we briefly explore the density dependence of the previously deduced responses. This is conveniently achieved by performing an expansion in the parameter ξ_F. The leading terms of the expansion of

$$R_{L,T}(\kappa, \lambda) = \frac{3\mathcal{N}\xi_F}{8m\eta_F^3\kappa}(1 - \psi_\Delta^2)\frac{\kappa^2}{\tau}\left(R_{L,T}^{(0)} + R_{L,T}^{(1)}\xi_F + R_{L,T}^{(2)}\xi_F^2\right) \tag{21}$$

are given by

$$R_L^{(0)} = -w_1(\tau) + \frac{\tau}{\kappa^2}(\lambda\rho + 1)^2 w_2(\tau) \tag{22}$$

$$R_T^{(0)} = 2w_1(\tau) + \left\{-\left(1 + \tau\rho^2\right) + \frac{\tau}{\kappa^2}(\lambda\rho + 1)^2\right\}w_2(\tau) \;, \tag{23}$$

and the next terms are found to be

$$R_L^{(1)} = R_T^{(1)} = \frac{\tau}{\kappa^2}(\lambda\rho + 1)(1 + \psi_\Delta^2)w_2(\tau) \tag{24}$$

$$R_L^{(2)} = R_T^{(2)} = \frac{1}{3}\frac{\tau}{\kappa^2}(1 + \psi_\Delta^2 + \psi_\Delta^4)w_2(\tau) \;. \tag{25}$$

In performing the $\xi_F \to 0$ limit it is of importance to realize that both responses shrink to the peak where $\kappa^2 = \tau(\tau\rho^2 + 1)$ and $\lambda = \tau\rho$. This constraint requires that $\mathcal{D} \to 0$ when $\xi_F \to 0$ (see Eq. (19)) and hence

$$R_L^{(0)}(\kappa, \lambda; \xi_F = 0) = -w_1(\tau) + (1 + \tau\rho^2)w_2(\tau) \tag{26}$$

$$R_T^{(0)}(\kappa, \lambda; \xi_F = 0) = 2w_1(\tau) \;. \tag{27}$$

Expressions for the Δ responses in this limit will be given later. In the nucleonic sector one immediately obtains $R_L^{(0)} = G_E^2$ and $R_T^{(0)} = 2\tau G_M^2$ (cf. [21]).

The expressions in Eqs. (17), (18) and (19) are valid for any process involving an initial nucleon which is converted to an on-shell resonance of mass m_Δ. In particular, the limit $\rho = 1$ yields the response functions in the quasi-elastic peak region [21]. The specific physical process gives rise to different w_1 and w_2 functions, which are evaluated in the next section for the $N \to \Delta$ transition.

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2.3 Nucleonic tensor

We now evaluate the invariant functions w_1 and w_2 relative to the $\gamma N \rightarrow \Delta$ process. These functions, being scalars, can be computed in any reference system and the most convenient one is found to be the rest system of the Δ, where the Rarita-Schwinger spinors take their simplest form. Let us denote with $Q_\mu^* = (\omega^*, \mathbf{q}^*)$ the four-momentum transfer in this system (the corresponding four-vector in the nucleus laboratory frame is Q_μ) and let $P_\mu^* = (E^*, \mathbf{p}^*)$ be the four-momentum of the struck nucleon. The Δ system is then defined by

$$\mathbf{p}_\Delta^* = 0 \Rightarrow \mathbf{p}^* = -\mathbf{q}^* = (0, 0, -q^*) \ , \tag{28}$$

having chosen the z-axis in the direction of $\mathbf{q}^*$. The energy conservation condition accordingly reads

$$E^* = m_\Delta - \omega^* \ . \tag{29}$$

Using the energy-momentum relation for the initial nucleon, namely

$$m_N^2 = E^{*2} - p^{*2} = m_\Delta^2 - 2m_\Delta \omega^* + Q^2 \ , \tag{30}$$

we find the value of ω^* in terms of Q^2

$$\omega^* = \frac{m_\Delta^2 - m_N^2 + Q^2}{2m_\Delta} \ . \tag{31}$$

In this system V_μ has its space components parallel to $\mathbf{q}^*$, while its time component reads

$$V_0^* = \frac{1}{m_N} \left(E^* - \frac{P \cdot Q}{Q^2} \omega^* \right) = -\frac{m_\Delta q^{*2}}{m_N Q^2} \ . \tag{32}$$

Then we can easily compute the 00 and 11 components of the nucleonic tensor in Eq. (13), obtaining:

$$f_{11}^* = -w_1 g_{11} = w_1 \tag{33}$$

$$f_{00}^* = -w_1 \left(1 - \frac{\omega^{*2}}{Q^2} \right) + w_2 \frac{m_\Delta^2 q^{*4}}{m_N^2 Q^4}$$

$$= w_1 \frac{q^{*2}}{Q^2} + w_2 \frac{m_\Delta^2 q^{*4}}{m_N^2 Q^4} \ . \tag{34}$$

By inverting the above equations the structure functions in the Δ-system are found to be

$$w_1 = f_{11}^* \tag{35}$$

$$w_2 = \frac{m_N^2 Q^4}{m_\Delta^2 q^{*4}} \left(f_{00}^* - \frac{q^{*2}}{Q^2} f_{11}^* \right) \ . \tag{36}$$

Hence the problem is reduced to computing just two components, namely f_{00}^* and f_{11}^*, of the nucleon tensor.

With Jones & Scadron [12] we write the transition matrix element for the Δ excitation as follows

$$\langle \Delta | j_\mu | N \rangle = \overline{u}^\beta \Gamma_{\mu\beta} u \ , \tag{37}$$

the most general form of the vertex being

$$\Gamma_{\mu\beta} = C_1 \Gamma_{\mu\beta}^1 + C_2 \Gamma_{\mu\beta}^2 + C_3 \Gamma_{\mu\beta}^3 \ . \tag{38}$$

In the above the C_a are (invariant) form factors and the three couplings $\Gamma_{\mu\beta}^a$ are given by*

$$\begin{align}
\Gamma_{\mu\beta}^1 &= (Q_\beta \gamma_\mu - \slashed{Q} g_{\beta\mu}) \gamma_5 T_3^+ \tag{39}\\
\Gamma_{\mu\beta}^2 &= (Q_\beta K_\mu - Q \cdot K g_{\beta\mu}) \gamma_5 T_3^+ \tag{40}\\
\Gamma_{\mu\beta}^3 &= (Q_\beta Q_\mu - Q^2 g_{\beta\mu}) \gamma_5 T_3^+ \ , \tag{41}
\end{align}$$

where $K_\mu = (P_\mu + P_{\Delta\mu})/2$ and $\mathbf{T}^+$ is the $N \to \Delta$ isospin transition operator [24].

The $N \to \Delta$ tensor $f_{\mu\nu}^*$ then reads

$$f_{\mu\nu}^* = \frac{4}{3} \frac{m_\Delta}{m_N} \sum_{ss_\Delta} (\overline{u}_\Delta^\lambda \Gamma_{\mu\lambda} u)^* (\overline{u}_\Delta^\beta \Gamma_{\nu\beta} u) \ , \tag{42}$$

the factor $4/3$ arising from the isospin trace

$$\sum_{tt_\Delta} < t|T_3|t_\Delta >< t_\Delta|T_3^+|t >= \frac{4}{3} \tag{43}$$

and u_Δ^β being the Rarita-Schwinger spinor describing a spin $3/2$ particle. In the Δ rest frame the latter has the simple form

$$\begin{align}
u_\Delta^0(\mathbf{0}, s_\Delta) &= 0 \tag{44}\\
u_\Delta^i(\mathbf{0}, s_\Delta) &= \sum_{\lambda s'} \langle \tfrac{1}{2} s' 1 \lambda | \tfrac{3}{2} s_\Delta \rangle e_\lambda^i u_\Delta(\mathbf{0}, s') \ , \tag{45}
\end{align}$$

where the $\mathbf{e}_\lambda$ ($\lambda = -1, 0, +1$) are spherical vectors and

$$u_\Delta(\mathbf{0}, s') = \begin{pmatrix} \chi_{s'}' \\ 0 \end{pmatrix} \tag{46}$$

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*Note that the expressions for the transition matrix element of Dufner and Tsai [11] and of Devenish et al. [23] differ from the ones of Jones and Scadron because the latter employ the set of basis vectors (K, Q), whereas Dufner and Devenish use (P_Δ, P) and (P_Δ, Q) respectively. Hence both Γ_2 and Γ_3 and the form factors C_2 and C_3 of these three sets of authors will be different.

is a zero momentum 1/2-spinor.

Since the initial nucleon spinor is

$$u(\mathbf{p}^*, s) = \sqrt{\frac{1 + \varepsilon^*}{2}} \begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\eta}^*}{1 + \varepsilon^*} \chi_s \end{pmatrix} \ , \tag{47}$$

the matrix element of the four-dimensional gamma-matrix

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \tag{48}$$

in the Δ-system will be related to the corresponding bispinor matrix element according to

$$\overline{u}_\Delta \Gamma u = \sqrt{\frac{1 + \varepsilon^*}{2}} \chi'^\dagger \left[\Gamma_{11} + \Gamma_{12} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\eta}^*}{1 + \varepsilon^*} \right] \chi \equiv \chi'^\dagger \overline{\Gamma} \chi \ , \tag{49}$$

which defines $\overline{\Gamma}$. The general matrix element between 1/2 and 3/2 spinors can be written as

$$\overline{u}_\Delta^i \Gamma u = \langle \tfrac{3}{2} s_\Delta | S^{i\dagger} \overline{\Gamma} | \tfrac{1}{2} s \rangle \ , \tag{50}$$

where the spin transition operator for the Δ

$$\langle \tfrac{3}{2} s_\Delta | (S^\dagger)_\lambda | \tfrac{1}{2} s' \rangle = \langle \tfrac{1}{2} s' 1 \lambda | \tfrac{3}{2} s_\Delta \rangle \tag{51}$$

has been introduced.

The tensor $f_{\mu\nu}^*$ evaluated in the Δ rest frame then turns out to be

$$\begin{aligned}
f_{\mu\nu}^* &= \frac{4}{3} \frac{m_N}{m_\Delta} \sum_{ss_\Delta} \langle \tfrac{3}{2} s_\Delta | S^{i\dagger} \overline{\Gamma}_{\mu i} | \tfrac{1}{2} s \rangle^* \langle \tfrac{3}{2} s_\Delta | S^{j\dagger} \overline{\Gamma}_{\nu j} | \tfrac{1}{2} s \rangle \\
&= \frac{4}{3} \frac{m_N}{m_\Delta} \mathrm{Tr} \left[\overline{\Gamma}_{\mu i}^\dagger \left(\frac{2}{3} \delta_{ij} - \frac{i}{3} \epsilon_{ijk} \sigma_k \right) \overline{\Gamma}_{\nu j} \right] \\
&= \frac{4}{9} \frac{m_N}{m_\Delta} \mathrm{Tr} \left[2 \overline{\Gamma}_{\mu i}^\dagger \overline{\Gamma}_{\nu i} - i \epsilon_{ijk} \overline{\Gamma}_{\mu i}^\dagger \sigma_k \overline{\Gamma}_{\nu j} \right] \ ,
\end{aligned} \tag{52}$$

where the trace is meant to be performed in a two-dimensional non-relativistic spin space.

By inserting Eq. (38) into the above expression and by performing the traces we get

$$f_{00}^* = \frac{8 m_\Delta}{9 m_N^2} q^{*2} (E^* - m_N) \left[C_1 + \frac{E^* + m_\Delta}{2} C_2 + \omega^* C_3 \right]^2 \tag{53}$$

$$f_{11}^* = \frac{8 m_\Delta}{9 m_N^2} (E^* - m_N) \left[C_1^2 (E^* + m_N)^2 + C^2 (m_N + m_\Delta)^2 - C_1 C (m_N + m_\Delta)(E^* + m_N) \right] \ , \tag{54}$$

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where a form factor

$$C = C_1 + \frac{1}{2}C_2(m_\Delta - m_N) + C_3\frac{Q^2}{m_\Delta + m_N} \tag{55}$$

has been defined.

It thus appears from Eq. (53) that f_{00}^* only depends upon a particular, quadratic combination of the three form factors C_i, namely

$$G_C = \frac{4m_\Delta m_N}{3(m_\Delta + m_N)}\left[C_1 + \frac{1}{2}(E^* + m_\Delta)C_2 + \omega^* C_3\right], \tag{56}$$

commonly referred to as the Coulomb form factor. On the other hand, the expression f_{11}^*, in addition to the terms C_1^2 and C^2, also contains their cross product CC_1. In the spirit of the familiar Sachs form factors, new form factors can however be defined in such a way that only squares will appear: indeed by diagonalizing the quadratic form in Eq. (54) one obtains

$$f_{11}^* = \frac{8m_\Delta}{9m_N^2}(E^* - m_\Delta)\left[\frac{3(m_N + m_\Delta)}{4m_N}\right]^2 [G_M^2 + 3G_E^2], \tag{57}$$

where

$$G_E = \frac{2}{3}m_N\left(C - C_1\frac{E^* + m_N}{m_\Delta + m_N}\right) \tag{58}$$

$$G_M = \frac{2}{3}m_N\left(C + C_1\frac{E^* + m_N}{m_\Delta + m_N}\right) \tag{59}$$

are usually referred to as electric and magnetic form factors. It is easily verified that the expressions in Eqs. (56), (58) and (59) coincide with the definitions of the form factors given in [12]. Furthermore the combination $G_M^2 + 3G_E^2$ entering in Eq. (57) is the one which usually appears in the generalized Rosenbluth formula for the transverse cross section in terms of magnetic and electric form factors [12].

Finally, by means of Eqs. (35,36) we obtain the two structure functions w_1 and w_2:

$$w_1 = \frac{8m_\Delta}{9m_N^2}(E^* - m_N)\left[\frac{3(m_N + m_\Delta)}{4m_N}\right]^2\left(G_M^2 + 3G_E^2\right) \tag{60}$$

$$w_2 = \frac{8Q^2}{9m_\Delta q^{*2}}(E^* - m_N)\left[\frac{3(m_N + m_\Delta)}{4m_N}\right]^2\left(\frac{Q^2}{m_\Delta^2}G_C^2 - G_M^2 - 3G_E^2\right). \tag{61}$$

To finish this section it is useful to write these structure functions in an arbitrary system, a result which will be also used in the next section to compare with the non-relativistic result. Here we only consider the structure functions coming from the magnetic

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$M1$ nucleon multipole (proportional to G_M^2), which is the leading contribution. Later in the calculations we will study the effect of including the electric and Coulomb form factors. In order to relate these expressions to the ones obtained using the so-called magnetic form of the operator, we introduce a new dimensionless form factor G defined by

$$G = -\frac{m_N + m_\Delta}{2m_N}\frac{3m_N^2 G_M}{(m_N + m_\Delta)^2 - Q^2} . \tag{62}$$

Using this form factor, the magnetic invariant functions can be written

$$w_1 = \frac{8m_\Delta^3}{9m_N^6}q^{*2}(E^* + m_N)G^2 \tag{63}$$

$$w_2 = -\frac{m_N^2 Q^2}{m_\Delta^2 q^{*2}}w_1 , \tag{64}$$

where we have used the fact that in the Δ-system

$$(m_N + m_\Delta)^2 - Q^2 = 2m_\Delta(E^* + m_N) . \tag{65}$$

In the Δ system we have the following expressions for the quantities involved, written in terms of the invariant τ:

$$q^{*2} = 4m_N^2\tau\frac{1 + \tau\rho^2}{\mu_\Delta^2} \tag{66}$$

$$m_N + E^* = \frac{m_N}{\mu_\Delta}(1 + \mu_\Delta + 2\tau\rho) . \tag{67}$$

Using these relations, the structure functions can be written in a general system as functions of τ

$$w_1 = \frac{32}{9}\tau(1 + \tau\rho^2)(1 + \mu_\Delta + 2\tau\rho)G^2 \tag{68}$$

$$w_2 = \frac{w_1}{1 + \tau\rho^2} = \frac{32}{9}\tau(1 + \mu_\Delta + 2\tau\rho)G^2 . \tag{69}$$

From the above equations we note that for a pure $M1$ transition $(1 + \tau\rho^2)w_2 - w_1 = 0$, hence only the w_2 structure function contributes to the longitudinal response, i.e.,

$$R_L^{M1}(\kappa, \lambda) = \frac{3\mathcal{N}\xi_F}{8m_N\eta_F^3\kappa}\frac{\kappa^2}{\tau}w_2(\tau)\mathcal{D}(\kappa, \lambda)(1 - \psi_\Delta^2)\theta(1 - \psi_\Delta^2) . \tag{70}$$

Finally, we end this section by obtaining the relativistic structure functions w_1 and w_2 in the limit $\eta_F = 0$. We begin with the expressions in Eqs. (60,61) written in the Δ-system which, in terms of the dimensionless variables introduced previously, read

$$w_1(\tau) = \frac{1}{2}\mu_\Delta(\mu_\Delta + 1)^2\xi^*\left[G_M^2(\tau) + 3G_E^2(\tau)\right] \tag{71}$$

$$w_2(\tau) = \frac{1}{2\mu_\Delta}(\mu_\Delta + 1)^2\xi^*\frac{\tau}{\kappa^{*2}}\left[4\frac{\tau}{\mu_\Delta^2}G_C^2(\tau) - G_M^2(\tau) - 3G_E^2(\tau)\right] , \tag{72}$$

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where $\xi^* = (E^* - m_N)/m_N$ is the nucleon kinetic energy in the Δ rest frame. In the $\xi_F = 0$ limit the following kinematical relations hold:

$$\kappa^{*2} = \frac{\tau}{\mu_\Delta^2}(1 + \tau\rho^2) \tag{73}$$

$$\xi^* = \frac{1}{\mu_\Delta}(2\tau\rho + 1 - \mu_\Delta) . \tag{74}$$

As a consequence the single-nucleon structure functions reduce to

$$w_1(\tau; \xi_F = 0) = \frac{1}{2}(\mu_\Delta + 1)^2(2\tau\rho + 1 - \mu_\Delta)\left(G_M^2 + 3G_E^2\right) \tag{75}$$

$$w_2(\tau; \xi_F = 0) = \frac{1}{2}(\mu_\Delta + 1)^2\frac{2\tau\rho + 1 - \mu_\Delta}{1 + \tau\rho^2}\left(4\frac{\tau}{\mu_\Delta^2}G_C^2 - G_M^2 - 3G_E^2\right) \tag{76}$$

and the response functions in Eqs. (26,27) read

$$R_L^{(0)}(\xi_F = 0) = \frac{2\tau}{\mu_\Delta^2}(\mu_\Delta + 1)^2(2\tau\rho + 1 - \mu_\Delta)G_C^2 \tag{77}$$

$$R_T^{(0)}(\xi_F = 0) = (\mu_\Delta + 1)^2(2\tau\rho + 1 - \mu_\Delta)\left(G_M^2 + 3G_E^2\right) . \tag{78}$$

3 Non-relativistic reduction of the delta current

In this section we present a new "relativized" expression for the delta current that can be implemented very easily in standard non-relativistic models. Contrary to previous work on the $N \to \Delta$ transition, where an expansion in the transferred momentum (q) and transferred energy (ω) was performed, here we only consider expansions in powers of the bound nucleon momentum $\eta = p/m_N$, keeping the *exact* dependence in both $\kappa = q/2m_N$ and $\lambda = \omega/2m_N$. Thus, we follow closely our previous work [19, 20] where new relativized expressions were obtained for the electroweak nucleon and meson-exchange currents. It is important to realize that for high-energy conditions the traditional current operators — obtained assuming that also $\kappa \ll 1$ and $\lambda \ll 1$ — are bound to fail, whereas our past work provides a way to include relativistic aspects into improved, effective operators for use with the same non-relativistic wave functions.

Following [12] the full $\gamma N\Delta$ vertex in Eq. (38) can be decomposed into magnetic dipole, electric quadrupole and Coulomb (longitudinal) quadrupole contributions. In what follows we focus on the contribution of the magnetic term: due to the smallness of the E2 and C2 form factors, the magnetic term is the dominant one for both the

longitudinal and transverse nuclear response functions. Thus, in the particular case of the magnetic contribution, the $\gamma N\Delta$ vertex current can be written in the form [12]

$$\Gamma^M_{\mu\beta} = \frac{G}{m_N^2}\epsilon_{\beta\mu}(KQ) , \tag{79}$$

where we use the notation $\epsilon_{\beta\mu}(KQ) = \epsilon_{\beta\mu\alpha\gamma}K^\alpha Q^\gamma$ with $K^\alpha = (P^\alpha + P^\alpha_\Delta)/2$ and Q^α the four-momentum transfer. The form factor G in Eq. (62) is proportional to the magnetic form factor G_M. It can be proven that the relativistic structure functions obtained with the current in Eq. (79) coincide with the general expressions given by Eqs. (60,61) neglecting the electric and Coulomb form factors, i.e., $G_E = G_C = 0$.

In order to proceed with the non-relativistic reduction of the $\gamma N \to \Delta$ current, let us rewrite the transition matrix element for the Δ excitation

$$\langle\Delta|j_\mu(P_\Delta, P)|N\rangle = \overline{u}^\beta_\Delta(P_\Delta, s_\Delta)\Gamma^M_{\mu\beta}u(P, s) , \tag{80}$$

where $u^\beta_\Delta(P_\Delta, s_\Delta)$ is the relativistic Rarita-Schwinger spinor describing a spin-3/2 particle whose general expression is given by [24]

$$u^\beta_\Delta(P_\Delta, s_\Delta) = \sum_{\lambda s'} \langle\tfrac{1}{2}s'1\lambda|\tfrac{3}{2}s_\Delta\rangle e^\beta_\Delta(P_\Delta, \lambda)u_\Delta(P_\Delta, s') . \tag{81}$$

Here $e^\beta_\Delta(P_\Delta, \lambda)$ are the spherical vectors boosted to momentum $\mathbf{p}_\Delta$ and $u_\Delta(P_\Delta, s')$ are spin-1/2 Dirac spinors. Introducing the dimensionless variables $\boldsymbol{\eta}_\Delta = \frac{\mathbf{p}_\Delta}{m_\Delta}$ and $\varepsilon_\Delta = \frac{E_\Delta}{m_\Delta}$, we can simply write

$$e^\beta_\Delta(P_\Delta, \lambda) = \left(\mathbf{e}_\lambda\cdot\boldsymbol{\eta}_\Delta, \mathbf{e}_\lambda + \frac{\mathbf{e}_\lambda\cdot\boldsymbol{\eta}_\Delta}{1+\varepsilon_\Delta}\boldsymbol{\eta}_\Delta\right) \tag{82}$$

$$u_\Delta(P_\Delta, s') = \sqrt{\frac{1+\varepsilon_\Delta}{2}}\left[\begin{array}{c}1\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\eta}_\Delta}{1+\varepsilon_\Delta}\end{array}\right]\chi_{s'} . \tag{83}$$

Thus, following the same procedure developed in [19, 20], the matrix element for a general four-dimensional gamma matrix Γ can be related to the corresponding bi-spinor matrix element according to

$$\overline{u}_\Delta\Gamma u \equiv \chi^\dagger_{s'}\overline{\Gamma}\chi_s = \chi^\dagger_{s'}\frac{1}{\sqrt{2(1+\varepsilon_\Delta)}}\left[(1+\varepsilon_\Delta)\Gamma_{11} - \boldsymbol{\sigma}\cdot\boldsymbol{\eta}_\Delta\Gamma_{21}\right]\chi_s + O(\eta) , \tag{84}$$

where the expansion only to leading order in η has been considered.

To make the discussion that follows easier we introduce new dimensionless variables which are analogous to the ones already used in the case of the nucleon quasi-elastic peak [19, 20]. In particular, let us define the variable λ' given by the relation: $\varepsilon_\Delta = 1 + 2\lambda'$. It is straightforward to derive the following relations (valid up to first order in η)

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$$\lambda' = \frac{1}{\mu_\Delta}\left(\lambda - \frac{\mu_\Delta - 1}{2}\right) = \tau' + \boldsymbol{\eta} \cdot \boldsymbol{\kappa}' + O(\eta^2) \tag{85}$$

$$\tau' = \frac{1}{\mu_\Delta}\left[\tau + \left(\frac{\mu_\Delta - 1}{2}\right)^2\right] \tag{86}$$

$$\boldsymbol{\kappa}' = \frac{\boldsymbol{\kappa}}{\mu_\Delta}; \qquad \boldsymbol{\eta}_\Delta = 2\boldsymbol{\kappa}' + \frac{\boldsymbol{\eta}}{\mu_\Delta} \tag{87}$$

with μ_Δ as defined in Eq. (5). The current operator $\overline{\Gamma}$ in Eq. (84) can be then written up to leading order in η in the form

$$\overline{\Gamma} = \frac{1}{\sqrt{1+\tau'}}\left[(1+\tau')\Gamma_{11} - \boldsymbol{\sigma} \cdot \boldsymbol{\kappa}'\Gamma_{21}\right] + O(\eta) . \tag{88}$$

Moreover, in terms of the spin transition operator for the Δ in Eq. (51), the general matrix elements between $1/2$ and $3/2$ spinors result

$$\overline{u}_\Delta^0 \Gamma u = \langle \tfrac{3}{2}s_\Delta|2\mathbf{S}^\dagger \cdot \boldsymbol{\kappa}'\overline{\Gamma}|\tfrac{1}{2}s\rangle + O(\eta) \tag{89}$$

$$\overline{u}_\Delta^i \Gamma u = \langle \tfrac{3}{2}s_\Delta|\left[\mathbf{S}^\dagger + \frac{\mathbf{S}^\dagger \cdot \boldsymbol{\kappa}'}{1+\tau'}2\boldsymbol{\kappa}'\right]^i \overline{\Gamma}|\tfrac{1}{2}s\rangle + O(\eta) . \tag{90}$$

The non-relativistic reduction (valid up to leading order in η) for the magnetic $N\Delta$ current operator, J_μ, is defined through the relation

$$\langle \tfrac{3}{2}s_\Delta|J_\mu|\tfrac{1}{2}s\rangle = \langle \Delta|j_\mu(P_\Delta, P)|N\rangle \tag{91}$$

and can be obtained by using the general relations given by Eqs. (85-87) and taking into account the fact that the magnetic current operator $\Gamma_{\mu\beta}^M$ is diagonal in spin space. The time and space components are given by

$$J_0 \simeq \frac{G}{m_N^2}\sqrt{1+\tau'}\left[\mathbf{S}^\dagger + \frac{\mathbf{S}^\dagger \cdot \boldsymbol{\kappa}'}{1+\tau'}2\boldsymbol{\kappa}'\right]^i \epsilon_{i0}(KQ) \tag{92}$$

$$J_i \simeq \frac{G}{m_N^2}\sqrt{1+\tau'}\left\{2\mathbf{S}^\dagger \cdot \boldsymbol{\kappa}'\epsilon_{0i}(KQ) + \left[\mathbf{S}^\dagger + \frac{\mathbf{S}^\dagger \cdot \boldsymbol{\kappa}'}{1+\tau'}2\boldsymbol{\kappa}'\right]^j \epsilon_{ji}(KQ)\right\} . \tag{93}$$

Using the relations

$$\epsilon_{i0}(KQ) = \epsilon_{i0jk}K^jQ^k \simeq -2m_N^2(\boldsymbol{\eta} \times \boldsymbol{\kappa})_i \tag{94}$$

$$\epsilon_{ji}(KQ) = \epsilon_{ji\alpha\beta}K^\alpha Q^\beta \simeq 2m_N^2\epsilon_{jik}\kappa^k \tag{95}$$

and taking into account the fact that $\boldsymbol{\kappa}'$ and $\boldsymbol{\kappa}$ are parallel vectors (see Eq. (87)), we finally get

$$J_0 = -2G\sqrt{1+\tau'}\mathbf{S}^\dagger \cdot (\boldsymbol{\eta} \times \boldsymbol{\kappa}) + O(\eta^2) \tag{96}$$

$$\mathbf{J} = 2G\sqrt{1+\tau'}(\mathbf{S}^\dagger \times \boldsymbol{\kappa}) + O(\eta) . \tag{97}$$

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Note that the current $\mathbf{J}$ is of order $O(1)$, whereas the charge J_0 is of order $O(\eta)$. Thus, one expects the contribution of the $N\Delta$ current to be considerably more important for the transverse response.

It is important to note that in order to be consistent one should treat the charge and current at the same level and perform an expansion of the current to order $O(\eta)$. Such program can be carried out by using the techniques developed in [19, 20]. However, for the present case an additional simplification can be made by using the expression of the charge operator to order $O(\eta)$ in Eq. (96), and taking into account the fact that, before performing the expansion, the original current was gauge-invariant. This invariance property should be valid for all the orders in the expansion (we are not expanding in $\mathbf{q}$ or ω). Hence we can relate the longitudinal component of the current with the density, i.e.,

$$\mathbf{J} \cdot \boldsymbol{\kappa} = \lambda J_0 = -2G\lambda\sqrt{1+\tau'}(\mathbf{S}^\dagger \times \boldsymbol{\eta}) \cdot \boldsymbol{\kappa} \ . \tag{98}$$

Hence we can obtain an improved and gauge-invariant current by adding to the expression in Eq. (97) a new piece of order $O(\eta)$ given by

$$\mathbf{J}_{\text{gauge}} = -2G\lambda\sqrt{1+\tau'}(\mathbf{S}^\dagger \times \boldsymbol{\eta}) \ .$$

Of course there could be additional corrections of order $O(\eta)$ in the transverse current that cannot be fixed by just using the continuity equation. However, as we will show below, the transverse response function computed with this improved current differs from the exact relativistic results by terms only of order $O(\eta^2)$, proving the high quality of this expansion for most applications in nuclear physics. Therefore the new expression of the current that we will consider below is

$$\mathbf{J} \ = \ 2G\sqrt{1+\tau'}\,\mathbf{S}^\dagger \times (\boldsymbol{\kappa} - \lambda\boldsymbol{\eta}) \ . \tag{99}$$

In order to test the quality of the above expansion of the current we proceed to compute the response functions in a non-relativistic Fermi gas using relativistic kinematics and the new currents in Eqs. (96,99). We begin by computing the analytical expressions and compare with the exact relativistic answer. Results are shown and discussed in the next section.

The calculation of the non-relativistic nucleon tensor needed to evaluate the nuclear response functions can be done by performing the following traces

$$f_{00}^{nr} \ = \ \frac{4}{3}\frac{m_\Delta}{m_N} \, \text{Tr}\,[J_0^\dagger J_0] \equiv w_{nr}^L \boldsymbol{\eta}_T^2 \tag{100}$$

$$f_{11}^{nr} + f_{22}^{nr} \ = \ \frac{4}{3}\frac{m_\Delta}{m_N} \, \text{Tr}\,[\mathbf{J}_T^\dagger \cdot \mathbf{J}_T] \equiv 2w_{nr}^T \ , \tag{101}$$

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where we have introduced the longitudinal and transverse, non-relativistic structure functions w_{nr}^L and w_{nr}^T

$$w_{nr}^L = \frac{4}{3}\frac{16m_\Delta}{3m_N}G^2(1+\tau')\kappa^2 \tag{102}$$

$$w_{nr}^T = \frac{4}{3}\frac{16m_\Delta}{3m_N}G^2(1+\tau')(\kappa^2 - 2\lambda\boldsymbol{\kappa}\cdot\boldsymbol{\eta}) + O(\eta^2) . \tag{103}$$

In order to compare the relativistic and non-relativistic response functions it is convenient to use the kinematical relations:

$$1 + \tau' = \frac{1}{2\mu_\Delta}(1 + \mu_\Delta + 2\tau\rho) \tag{104}$$

$$\kappa^2 - 2\lambda\boldsymbol{\kappa}\cdot\boldsymbol{\eta} = \tau(1 + \tau\rho^2) + O(\eta^2) . \tag{105}$$

We can then write, up to first order, the relations

$$w_{nr}^T = w_1 + O(\eta^2) \tag{106}$$

$$w_{nr}^L = \frac{\kappa^2}{\tau}\frac{w_1}{1+\tau\rho^2} + O(\eta^2) = \frac{\kappa^2}{\tau}w_2 + O(\eta^2) , \tag{107}$$

where w_1 and w_2 are the magnetic relativistic functions given in Eqs. (68,69).

The nuclear response functions are given finally by

$$\begin{aligned} R_L^{nr} &= \frac{3\mathcal{N}\xi_F}{8m_N\eta_F^3\kappa}\theta(1-\psi_\Delta^2)(1-\psi_\Delta^2)w_{nr}^L\mathcal{D} \tag{108} \\ &= \frac{3\mathcal{N}\xi_F}{8m_N\eta_F^3\kappa}\theta(1-\psi_\Delta^2)(1-\psi_\Delta^2)\frac{\kappa^2}{\tau}\left[w_2 + O(\eta^2)\right]\mathcal{D} \\ R_T^{nr} &= \frac{3\mathcal{N}\xi_F}{8m_N\eta_F^3\kappa}\theta(1-\psi_\Delta^2)(1-\psi_\Delta^2)2w_{nr}^T \tag{109} \\ &= \frac{3\mathcal{N}\xi_F}{8m_N\eta_F^3\kappa}\theta(1-\psi_\Delta^2)(1-\psi_\Delta^2)2\left[w_1 + O(\eta^2)\right] \end{aligned}$$

with $\mathcal{D}$ as given by Eq. (19). Although we call these functions "non-relativistic", actually they contain enough relativistic ingredients to be high-quality approximations to the exact RFG result. In fact, first we use relativistic kinematics, so that the phase space and momentum integrals are done exactly. Second, we consider the new currents expanded to include effects up to order $O(\eta)$, so that the dynamics of the problem are correct to that order. Comparing these expressions with the exact relativistic responses in Eqs. (18,70), we see that the relative differences between R_L^{nr} and R_L, and between R_T^{nr} and R_T are of order $O(\eta^2)$. In the next section we test numerically the quality of the non-relativistic responses of Eqs. (108,109).

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4 Results

In this section we present numerical results for the relativistic response functions in the region of the Δ-peak and check numerically the quality of our new approximation to the $N \to \Delta$ electromagnetic current. We also investigate the importance of different contributions in the relativistic responses. We shall present results for medium and high momentum transfers, ranging from $q = 0.5$ to 2 GeV/c, and thus will also be able to analyze the validity of the traditional non-relativistic calculations performed for different values of q.

Before starting our analysis it is convenient to check our model with some of the available experimental data [25]. In this way we can fix some of the ingredients that enter in the model, in particular, the magnetic form factor G_M of the Δ and the modification of the response due to the finite Δ width as a consequence of its later decay into the N-π channel, which we incorporate by performing a convolution with the responses for stable particles.

In fig. 1 we show the transverse cross section σ_T for the inclusive reaction $H(e, e')$ from the nucleon. The transverse nuclear cross section is defined by

$$\sigma_T = \hbar c \frac{2\pi\alpha^2}{\omega + \frac{Q^2}{2m_N}} r_T ~, \tag{110}$$

where $r_T = \frac{m_\Delta}{E_\Delta} 2w_1$ is the transverse response of a single nucleon. In this calculation we have included the Δ width by substituting for the energy-conserving delta function a Lorentzian shape

$$\delta(\omega + m_N - E_\Delta) \longrightarrow \frac{E_\Delta}{m_\Delta} \frac{1}{\pi} \frac{\Gamma(s)/2}{(\sqrt{s} - m_\Delta)^2 + \Gamma(s)^2/4} ~, \tag{111}$$

where $s = (m_N + \omega)^2 - q^2$ is the invariant mass of the initial photon and nucleon. Results that include a width $\Gamma(s)$ (which is zero at threshold and equal to the width $\Gamma_0 = 120$ MeV at resonance) are represented by solid lines in fig. 1. The dependence of $\Gamma(s)$ on the invariant mass is given by [4]

$$\Gamma(s) = \Gamma_0 \frac{m_\Delta}{\sqrt{s}} \left(\frac{p_\pi^*}{p_\pi^{res}} \right)^3 ~, \tag{112}$$

where p_π^* is the momentum of the final pion resulting from the Δ decay (in the Δ-system) given by

$$p_\pi^* = \frac{1}{\sqrt{s}} \left[\frac{(s - m_N^2 - m_\pi^2)^2}{4} - m_N^2 m_\pi^2 \right]^{1/2} \tag{113}$$

and p_π^{res} is its value at resonance, obtained from the above expression for $\sqrt{s} = m_\Delta$. For comparison, we show in fig. 1 with dashed lines results obtained by considering a constant width $\Gamma = \Gamma_0$.

Other ingredients that enter in the cross section are the Δ form factors. We use the parameterization

$$G_M(Q^2) = G_M(0)f(Q^2) \tag{114}$$

$$G_E(Q^2) = G_E(0)f(Q^2) \, , \tag{115}$$

where we assume that the same dependence in Q^2 is valid for the electric and magnetic form factors, given by the function [8]

$$f(Q^2) = G_E^P(Q^2)\left(1 - \frac{Q^2}{3.5\,(\mathrm{GeV/c})^2}\right)^{-1/2} \tag{116}$$

with G_E^P the electric form factor of the proton, for which we use the Galster parameterization $(1 + 4.97\tau)^{-2}$ [26]. The above equation reflects the fact that the isobar form factor falls off faster than the proton form factor. Unless otherwise indicated, we take $G_C = 0$ and use the following values [12] of the form factors at the origin

$$G_M(0) = 2.97, \qquad G_E(0) = -0.03. \tag{117}$$

Later on we show the effect on the longitudinal response function introduced by considering a C2 form factor that is different from zero.

Our calculation (solid lines) displayed in fig. 1 is slightly below the data for the three values of the momentum transfer $Q^2 = -0.2, -0.3, -0.4$ $(\mathrm{GeV/c})^2$, reflecting the fact that we have not included the background contributions of non-resonant pion production, which produce an additional increase of the cross section.

In fig. 2 we show results for the nuclear inclusive cross section per nucleon from ^{12}C compared with the experimental data taken from [2, 27]. Dotted lines correspond to the RFG for a stable Δ. A more realistic model of the Δ peak requires the inclusion of the Δ width in the cross section, which we show with dashed lines. The nuclear responses including the width, $R_\Gamma(q, \omega)$, are computed from the responses $R(q, \omega, W)$ for a stable Δ with mass W by a convolution

$$R_\Gamma(q, \omega) = \int_{m_N + m_\pi}^{W_{max}} \frac{1}{\pi} \frac{\Gamma(W)/2}{(W - m_\Delta)^2 + \Gamma(W)^2/4} R(q, \omega, W)dW \, , \tag{118}$$

where the integration interval goes from threshold to the maximum value allowed in the Fermi gas model, $W_{max}^2 = (E_F + \omega)^2 - (q - k_F)^2$. The inclusion of the Δ width produces a broadening of the Δ peak and correspondingly a decrease of the strength.

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As an illustration of how one could improve the model in the quasi-elastic-peak region, we also show with dot-dashed lines the quasi-elastic cross section computed with the PWIA model of ref. [28]. Here the mean difference with the RFG model is the inclusion of the momentum distribution of the finite-sized nucleus, which produces the "tails" of the cross section, and the binding energy of the nucleons in the nucleus, which produces a shift to higher energies (in the direction of data). Here for the PWIA calculation we use relativistic kinematics, final states are described as plane waves and the electromagnetic current used contains relativistic corrections to order η. Finally, we show with solid lines the results computed with a hybrid model in which we add the PWIA cross section for the quasi-elastic contribution to the RFG result for the Δ contribution.

As we can see in fig. 2, our results are below the data in the dip and Δ region. This was expected because other contributions coming mainly from two-nucleon emission and non-resonant pion production (not included in our model) also enter here [1, 4, 10, 29]. However, our intention in this work is not to reproduce the experimental data nor to present a complete model including all of the physical contributions in this energy region, but instead to discuss the effect of different ingredients in the calculation and present a new set of improved currents specifically for excitation of the Δ peak that now include the relevant relativistic content — these could now straightforwardly be used in standard non-relativistic many-body models with relativistic kinematics.

The quality of the new approximation to the relativistic, magnetic Δ current is shown in fig. 3, where the exact RFG longitudinal and transverse responses using magnetic and electric form factors are displayed with solid lines. Here we show just the Δ contribution to the responses. In addition we show with dashed lines the responses computed in the non-relativistic Fermi gas model with relativistic kinematics and the new currents in Eqs. (96,99). For comparison we also show with dot-dashed lines results for the non-relativistic Fermi gas model and relativistic kinematics, but using the traditional non-relativistic current. The improvement of the description of the relativistic results using our currents is clear from this figure — the solid and dashed lines almost coincide. This proves that our expansion to order $O(\eta)$ is precise enough to describe the Δ excitation in nuclei with negligible error for high momentum transfers.

In fig. 4 several relativistic effects and ingredients of the calculation are analyzed. Therein we show the longitudinal and transverse responses for $q = 0.5, 1$ and 2 GeV/c. With solid lines we show the Δ peak computed within our model, while with dashed lines we show the Δ peak computed using the Peccei Lagrangian [30]. The Peccei Lagrangian only includes the first coupling $\Gamma^1_{\mu\nu}$ given in Eq. (39) with coupling constant $C_1(0) = 2.5$ GeV^{-1}. This value has been chosen so that the transverse response is equal to the one

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computed with the full Lagrangian in the Δ peak for $q = 500$ MeV/c. The importance of using the full vertex in Eq. (38) is clear from this figure. First, although the coupling constants can be chosen so that the transverse responses computed with both Lagrangians are similar for moderate $q = 500$ MeV/c, they begin to fail for higher q-values. These differences are seen to be most important in the longitudinal response, where the Peccei Lagrangian clearly gives an extremely large result. The reason for this unphysical behavior of the Peccei Lagrangian is that in Eq. (53) there are important cancelations among the C_1, C_2 and C_3 pieces in the longitudinal channel [11]. Second, as the Peccei Lagrangian is the one usually employed to compute the MEC contribution involving virtual Δ excitation [1], it is mandatory to use the full Lagrangian — or at least the magnetic piece — if one wants to compute the longitudinal contribution of the MEC in this channel.

As reference, in fig. 4 we also show with dot-dashed lines the quasi-elastic peak responses. For high q-values the two peaks overlap and the importance of the Δ in both responses also increases with q. This is better seen in fig. 5 where we also show the effect on the response functions produced by incorporating the finite Δ width (solid lines). Dashed lines correspond to results without including the isobar finite width, while dot-dashed lines correspond to the quasi-elastic peak. In the above results no Coulomb form factor has been included. In this case the contribution of the Δ in the longitudinal response is due to the Fermi motion of the nucleons inside the nucleus [9], the main contribution here coming from the magnetic Δ excitation, which is zero only for nucleons at rest. This is better seen in Eq. (100) where the longitudinal, magnetic single-nucleon response is seen to be proportional to η_T^2 and to the function w_L^{nr} which is proportional to κ^2 (see Eq. (102)), explaining the increase with q of the longitudinal response observed in fig. 5. In the static limit, the longitudinal response becomes proportional to the Coulomb $C2$ form factor, as shown at the end of sect. 2.

In fig. 6 we show the dependence of the longitudinal response on the Coulomb form factor of the Δ. With solid lines we show the Δ peak without C2 multipoles, while the dashed lines include a Coulomb contribution with form factor

$$G_C(Q^2) = -0.15 G_M(Q^2) \ .$$

We can see that the longitudinal response is quite sensitive to this form factor, especially for high q. This can be easily understood from the analytical expression of the structure functions in Eqs. (60,61). Only the structure function w_2 depends on G_C and its dependence on the form factors is carried by the quadratic combination $\frac{Q^2}{m_\Delta^2} G_C^2 - G_M^2 - 3G_E^2$, which is not very sensitive to small values of G_C. However looking at R_L in Eq. (17), we see that the correction due to the term $w_2 \mathcal{D}$ is small; hence the main contribution

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comes from the combination $(1 + \tau\rho^2)w_2 - w_1$. The important point is that, if no C2 term is present, that combination is exactly zero, i.e., $(1 + \tau\rho^2)w_2 - w_1 = 0$. Thus, the only contribution to the longitudinal response comes from the higher-order term $w_2\mathcal{D}$. This explains why the longitudinal isobar response is so small. On the other hand, if the C2 form factor is nonzero, there will be a contribution from the leading-order term $(1 + \tau\rho^2)w_2 - w_1$ which is proportional to G_C^2. This term, although small, is of the same order of magnitude as the $w_2\mathcal{D}$ piece. Therefore, we can conclude (at least for values of G_C such as those assumed here) that a correct treatment of the isobar longitudinal response requires the inclusion of the Coulomb form factor.

To finish this section, we have also explored the sensitivity of the response functions to inclusion of the electric E2 form factor, finding that both responses are quite insensitive. The reason is that the E2 contribution always adds incoherently to the M1 form factor in the combination $G_M^2 + 3G_E^2$, as seen in Eqs. (60,61). Therefore, the inclusion of a small E2 form factor does not significantly modify any of the responses. In order to obtain appreciable effects due to the electric form factor, one could explore other observables where interferences can occur; for instance, one could analyze the angular distribution of the Δ emission in the transverse channel [18].

5   Conclusions

In summary, in this paper we have obtained a new expansion of the relativistic Δ electro-excitation current to first order in $\eta = p/m_N$, maintaining the exact dependence on the momentum and energy transfers. We have tested this new expansion by performing a calculation using a non-relativistic Fermi gas model together with relativistic kinematics. The resulting longitudinal and transverse responses are found to be very close to the exact result computed with the RFG model. Therefore it is expected that the use of this current will provide a significant improvement when used in more realistic models of inclusive electron scattering from nuclei.

We have also performed a comparison between different Lagrangians in the treatment of the Δ excitation, finding that the Peccei Lagrangian is inappropriate in the longitudinal channel for all of the q-values analyzed. We have studied the contribution of the isobar emission to the longitudinal response, in general finding a small contribution for medium q-values, although increasing significance as q increases. In our calculations we include the finite width of the Δ assuming a Lorentzian shape. With regards to the quadrupole amplitudes of the Δ, we have found a large sensitivity of the longitudinal response to inclusion of the Coulomb form factor of the isobar, especially for high q, a fact that could

be of importance for the present investigations of the longitudinal nuclear response. On the other hand, both L and T responses are found to be insensitive to the quadrupole E2 form factor.

Finally, it is of interest to extend the present studies to the two-particle emission channels which provide important contributions in this energy region. In particular, one expects these ideas to be relevant in the analysis of non-resonant pion production $(e, e'p\pi)$ and two-nucleon emission $(e, e'2N)$ reactions, where there is the hope of extracting detailed information on short-range correlations in nuclei. It is clear that, in order to extract ground-state properties from these reactions, high-q values are needed, since one wishes to reach a reasonably quasi-free regime in which final-state interaction effects are expected to be minimal. However, for such kinematical conditions, as our present studies indicate, relativistic effects play a role and it is important to have them theoretically under control. At this point in time we have explored ways to relativize the electroweak currents for selected situations — our intent is to continue along the same path for other related problems such as those mentioned above.

Acknowledgments

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Figures

Figure 1: Transverse cross section σ_T for the inclusive reaction $H(e, e')$ in the Δ peak as a function of the Δ invariant mass $\sqrt{s}$. The solid lines include a width $\Gamma(s)$ which is zero at threshold. The dashed lines use a constant width $\Gamma = 120$ MeV. Experimental data are from [25].

Figure 2: Inclusive cross section per nucleon from ^{12}C, for two beam energies and two scattering angles. Dotted lines: RFG without Δ width; dashed: RFG including the finite Δ width; dot-dashed: PWIA for the quasi-elastic peak; solid: hybrid model obtained by adding the PWIA cross section in the quasi-elastic peak and RFG cross section in the Δ-peak. Experimental data are from [2, 27].

Figure 3: Electromagnetic responses in the Δ-peak for ^{12}C with $k_F = 225$ MeV/c, without a Δ-width. Solid: exact results within the RFG model; dashed: new expansion of the electromagnetic current to first order in η; dot-dashed: traditional non-relativistic current. We use relativistic kinematics in all cases.

Figure 4: Electromagnetic responses for ^{12}C in the RFG model with $k_F = 225$ MeV/c, with Δ-width. Solid: Δ-peak Using the Jones & Scadron Lagrangian; dashed: Δ-peak using the Peccei Lagrangian; dot-dashed: quasi-elastic peak.

Figure 5: Electromagnetic responses for ^{12}C in the RFG model with $k_F = 225$ MeV/c. Solid: Δ-peak including the Δ-width; dashed: Δ-peak for stable isobar; dot-dashed: quasi-elastic peak.

Figure 6: Longitudinal response for ^{12}C. Solid: Δ-peak without Coulomb form factor; dashed: Δ-peak with Coulomb form factor $G_C(0) = -0.15 G_M(0)$.

Figure 1
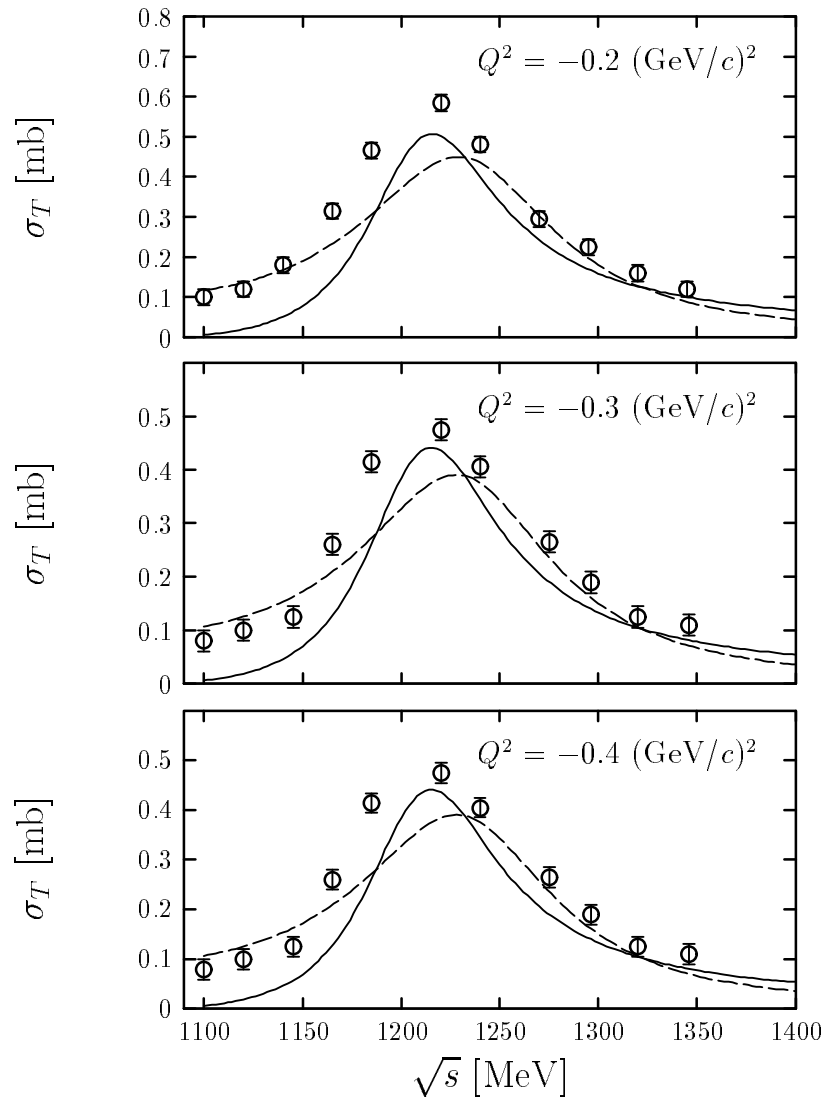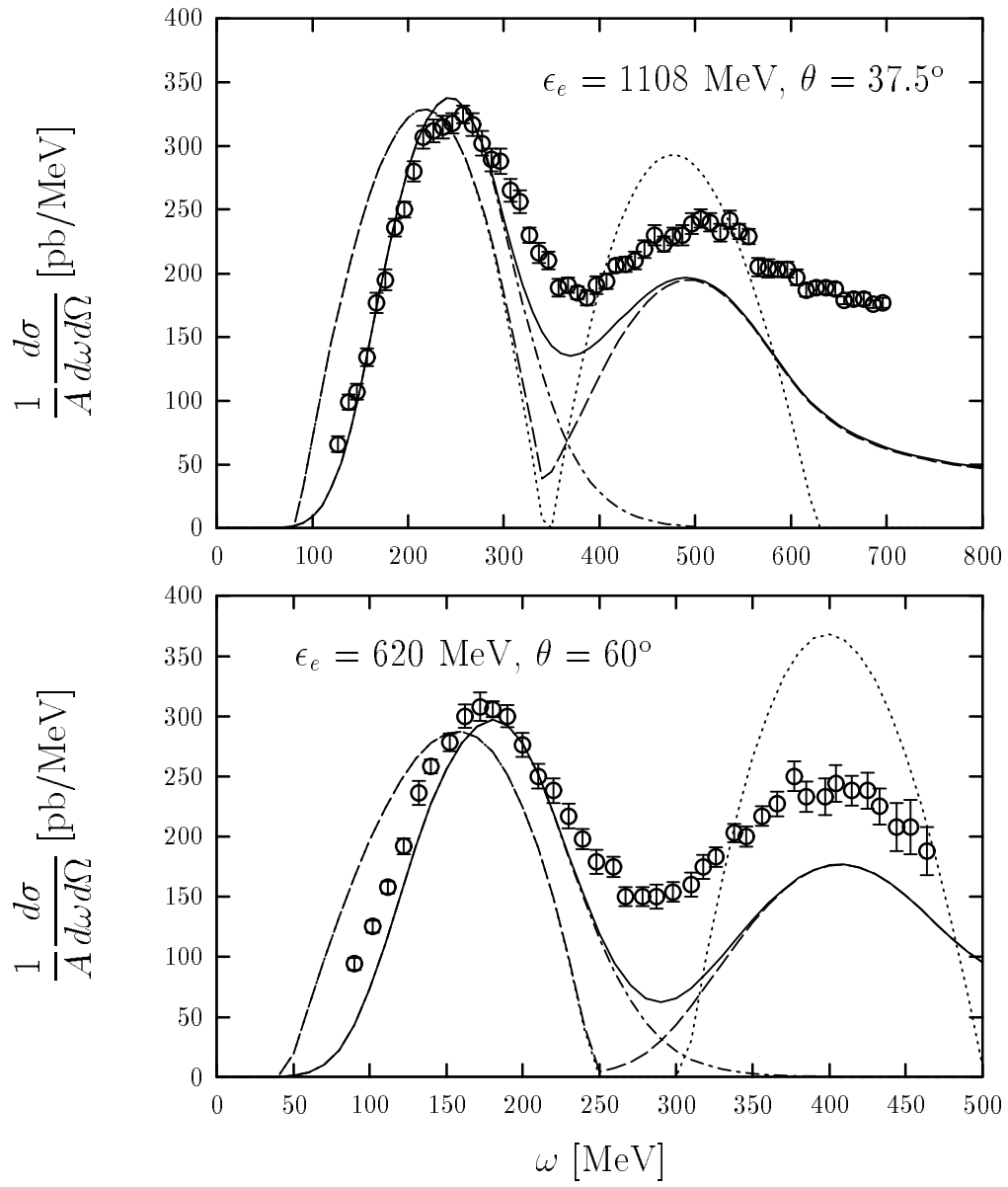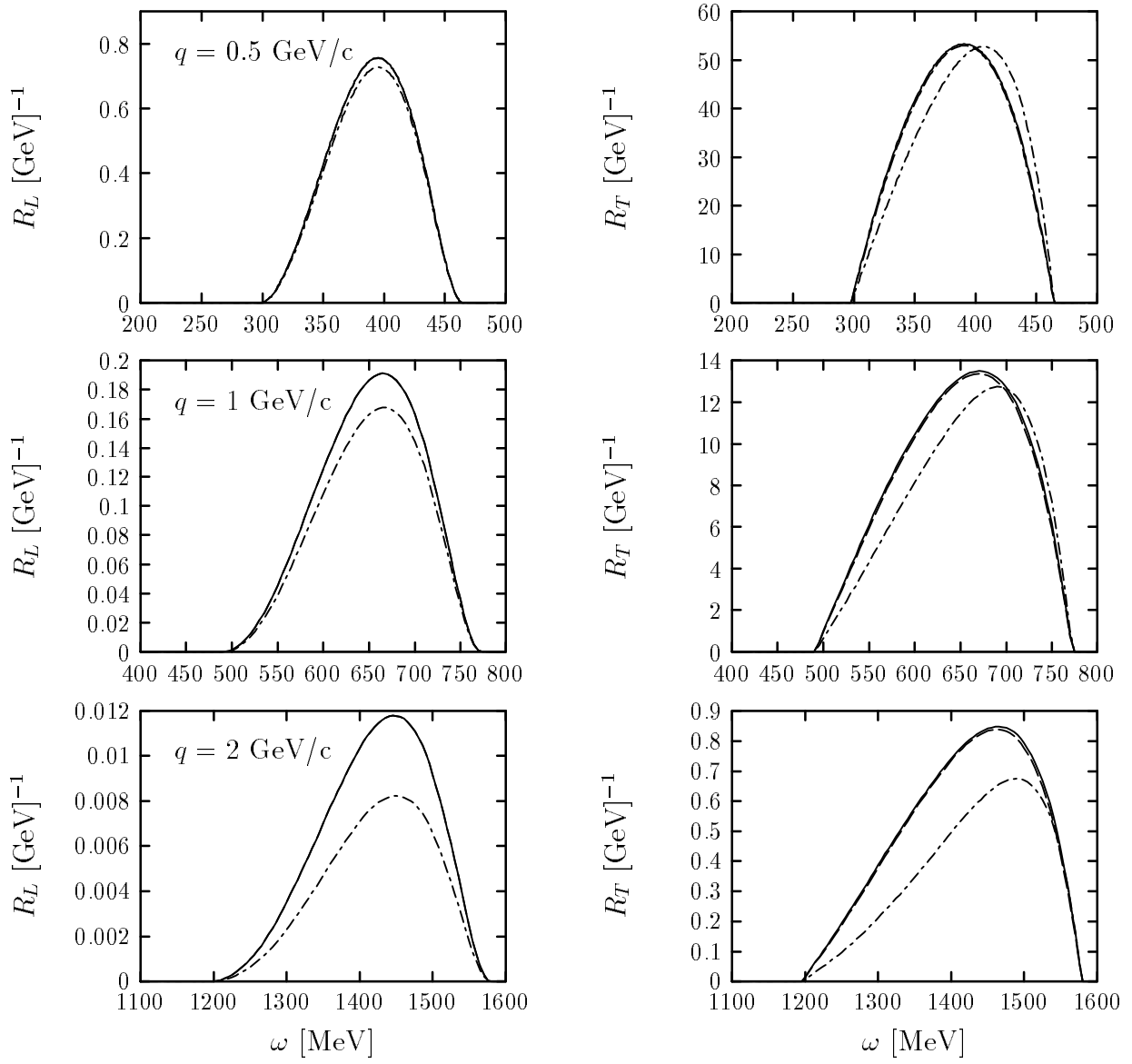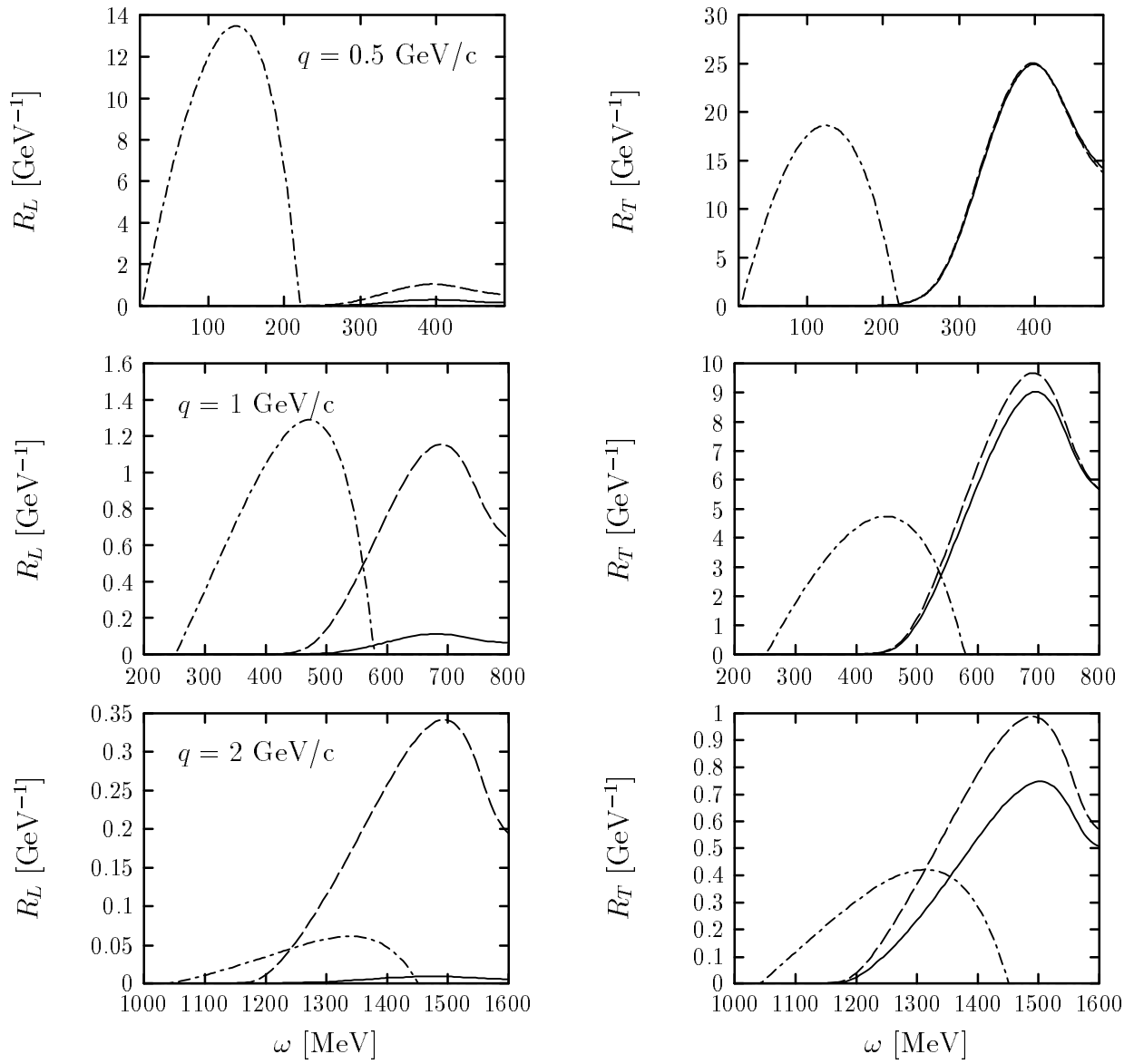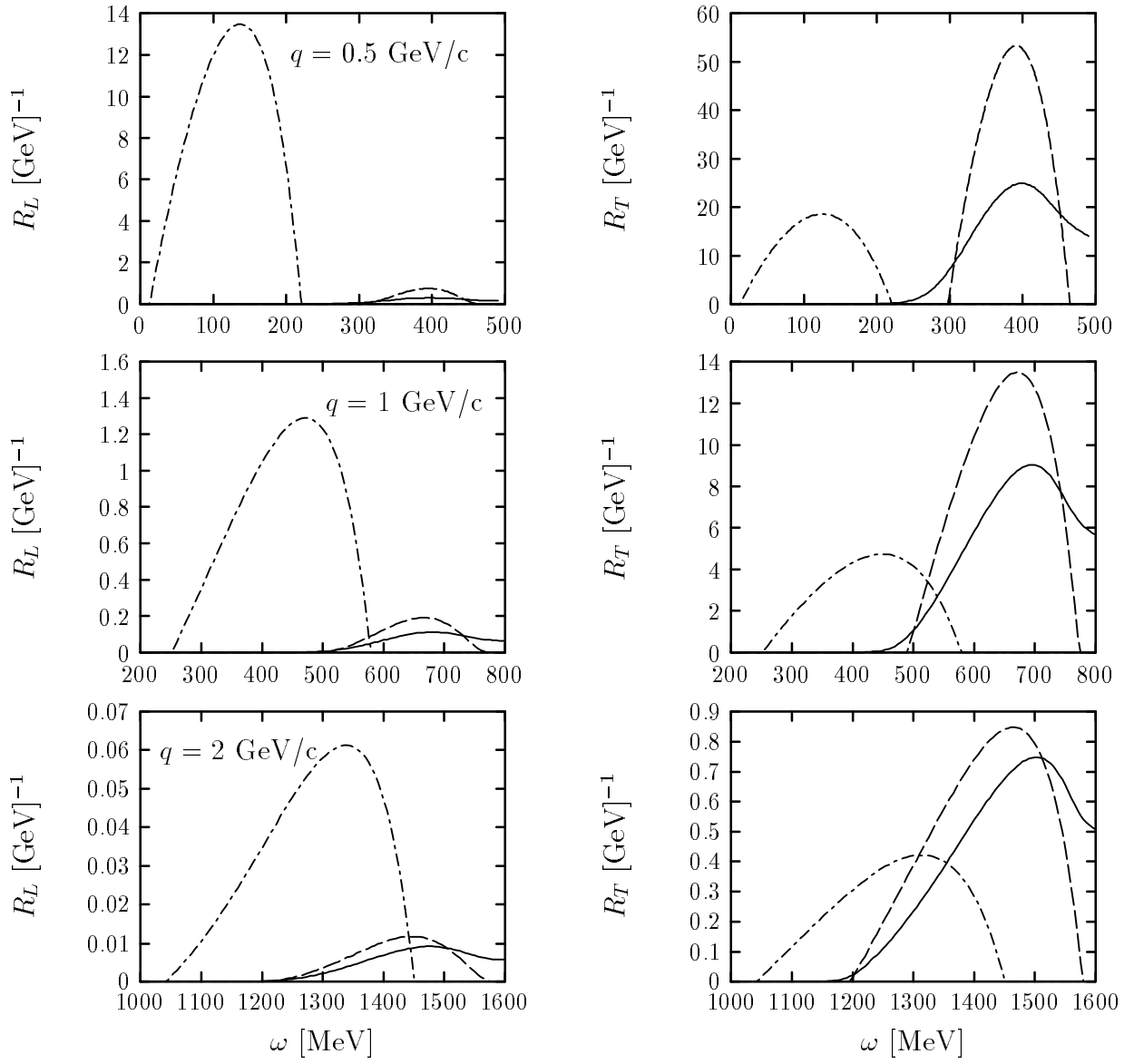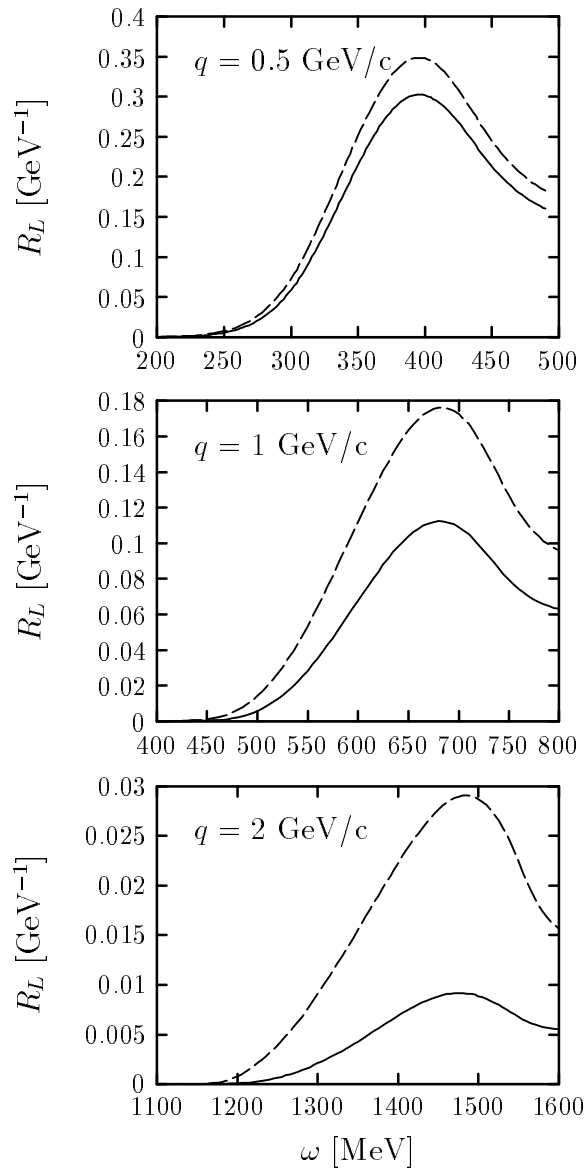
Figure 2

Figure 3

Figure 4

Figure 5

Figure 6