ON THE IDEMPOTENCY OF SOME COMPOSITE FUNCTORS

RAMÓN J. FLORES

ABSTRACT. We present examples of localization functors whose composition with certain cellularization functors is not idempotent, and vice versa.

1. INTRODUCTION

The main goal of this note is to discuss some interesting questions raised by E. Dror-Farjoun, concerning the effect of the successive application of localization and cellularization functors (see definitions in next section). We deal with Conjectures 3 and 7 of [Mis08, pages 62–63], which essentially ask for any localization functor L and cellularization CW, which are homotopy idempotent by definition, whether the composite functors LCW and CWL are also homotopy idempotent. Note that the functors L and CW are respectively coaugmented and augmented, but in general neither LCW nor CWL admit coaugmentation nor augmentation.

In light of these considerations we use a previous description [FS07, Corollary 5.8] of the $B\mathbb{Z}/2$-cellularization of the smallest in the family of Suzuki simple groups, $Sz(8)$, to provide counterexamples to the questions of Farjoun. The following theorem summarizes our main results, which appear in the text as Proposition 3.3 and Proposition 3.5:

Theorem 1.1. *If CW represents cellularization with regard to a wedge $B\mathbb{Z}/2 \vee \bigvee_{p \text{ odd prime}} M(\mathbb{Z}/p, 2)$ and L denotes an n-Postnikov section for an appropriate choice of n, then the functors LCW and CWL are not homotopy idempotent.*

The paper is structured as follows: next section is devoted to recall some notions of (co)Localization Theory, and also to a little comprehensive introduction of the problem of the idempotency of the functors LCW and CWL; Section 3 contains our main results and their proofs.

---

2. Preliminaries

Recall that given a pointed map $f : A \to B$, a space X is said to be f-local if the following conditions hold: the induced function $\mathrm{map}_*(B, X) \to \mathrm{map}_*(A, X)$ is a weak equivalence; the localization associated to f is a coaugmented and idempotent endofunctor L of the category of spaces; and every map $X \to Y$ from a space X to a f-local space Y factors through LX in an unique way, up to homotopy. Analogously, given spaces A and X, X is said A-cellular if it is homotopy equivalent to an iterated homotopy colimit of copies of A. The A-cellularization of X, denoted by CW_A, is defined as an augmented idempotent endofunctor of the category of spaces such that the augmentation $CW_A X \to X$ is initial among all maps $f : Y \to X$ which induce a weak equivalence $\mathrm{map}_*(A, Y) \to \mathrm{map}_*(A, X)$. A comprehensive introduction to these functors can be found in the first two chapters of [Far95].

It is quite easy to find pairs of functors CW and L such that CWL or LCW are idempotent. For example, if L is a Postnikov section $P[n]$, or more generally localization with regard the constant map $A \to *$ for a certain space A (usually called A-nullification), and CW is the n-connected cover functor (respectively CW_A), both composites are trivial. A similar phenomenon occurs if we apply $CW_{M(\mathbb{Z}/p,2)}$, where p is a prime, and then take the Bousfield-Kan q-completion at another prime q.

It can also happen that the composite of L and CW behaves like the identity functor (up to homotopy) over certain spaces, which immediately implies idempotency. An easy example is given by taking L as the R-completion, for $R = \mathbb{Q}$ or $R = \mathbb{Z}/p$, of a simply-connected space, and CW as the universal cover functor; then LCW is the identity over LX, and CWL is idempotent over X. So the problem is to find functors L and CW such that their combined effect is not so drastic as to kill the entire space, nor so weak that the composition gives back the original space. As one can expect, for "nice" spaces homological localization functors [Bou94] combined with cellularization with regard to p-primary meaningful spaces (as $B\mathbb{Z}/p$ or Moore spaces for $\mathbb{Z}/p$) have a good behavior in this context, and idempotency can be expected at least when working with nilpotent spaces. The same happens for some well-known nullification functors ([Bou94],[Bou97]).

So, when searching for counterexamples there are two different but complementary paths: to seek for exotic localizations/cellularizations, or to apply the aforementioned functors to non-nilpotent spaces. The first strategy seems rather unpromising, as precise descriptions of the effect of L, or even more so of CW, are in general not available for functors that are outside the context of p-primary homotopy; and even less is known of the structure of $LCWX$ or $CWLX$. If the second path is taken, one quickly realizes that two features of L and CW are needed: their effect on the target space X should be very precisely known, and both must change X in a "moderately opposite" way, so $CWLX$ or $LCWX$ should be non-trivial but not very complicated spaces. This is the approach we take in next section, where our main results are proven.

3. Idempotency and classifying spaces

We will always consider cellularization with regard to the classifying space $B\mathbb{Z}/2$, and we will denote this functor simply by CW. In turn, if M is the wedge of all Moore spaces $M(\mathbb{Z}/q, 2)$ for q odd, L will be nullification with regard to a wedge of M with a sphere S^{n+1} whose dimension will be determined in due course.

To keep this exposition brief we refer the reader to [Gor80, 16.4] for details on the 2-local structure of $Sz(8)$. As this group is our main object of study, we henceforth denote it simply by G.

Before applying the functors, we need some technical remarks concerning to the primary structure of BG. Bousfield-Kan q-completion of a space X (see [BK72] for a main reference) is denoted by $X_q^\wedge$.

Lemma 3.1. *The space $BG_q^\wedge$ is 2-connected for q odd.*

Proof. As G is a simple group, the q-completion of BG is simply-connected, and according to [BK72, VII.4.3], $\pi_2 BG_q^\wedge$ must be a finite q-group for every q. Moreover, the Schur multiplier of G is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, so $H_2(BG; \mathbb{F}_q) = 0$ by universal coefficients. By Hurewicz's Theorem, we have $\pi_2 BG_q^\wedge = H_2(BG_q^\wedge; \mathbb{F}_q)$, which is in turn isomorphic to $H_2(BG; \mathbb{F}_q)$ because BG is q-good [BK72, VII.5]. This completes the proof.

$\square$

Note that according to [RS01, Theorem 6.2], the space $BG_q^\wedge$ should be $M(\mathbb{Z}/q, 2)$-cellular for every q odd, and then it is killed by $M(\mathbb{Z}/q, 2)$-nullification, i.e., $\mathbf{P}_{M(\mathbb{Z}/q,2)} BG_q^\wedge$ is contractible.

Recall now from [Gor80] that a Sylow 2-subgroup, S, of G satisfies: $Z(S) = S' = \Omega_1(S) \cong (\mathbb{Z}/2)^3$ and $S/Z(S) \cong (\mathbb{Z}/2)^3$. Thus $Z(S) = Cl(S)$ is the minimal strongly closed subgroup that contains all elements of order 2 in S. The normalizer, N, of S in G acts transitively on the nonidentity elements of both $Z(S)$ and $S/Z(S)$. Following the notation of [FS07], let $\Gamma = N/Z(S) \cong (\mathbb{Z}/2)^3 \rtimes \mathbb{Z}/7$. The following result describes the low-dimensional homology of Γ:

Lemma 3.2. *The group Γ is 2-superperfect, i.e. $H_i(\Gamma, \mathbb{Z}/2) = 0$ for $i = 1, 2$.*

Proof. The group Γ is 2-perfect by [FS07, page 51], and moreover it is not hard to check that $H_1(\Gamma; \mathbb{Z}) = \mathbb{Z}/7$. Using now Gaschütz's Theorem and the Fitting Lemma [Gor80], it can also be seen that the Schur multiplier of Γ is trivial. Then $H_2(\Gamma; \mathbb{F}_2)$ is likewise trivial by universal coefficients, and we are done. $\square$

We can now give the precise definition of L. The group Γ is 2-perfect and moreover has torsion in odd primes, so according to [Lev95, 1.1.4] $B\Gamma_2^\wedge$ has an infinite number of nonzero homotopy groups. If n is the smallest natural number such that $\pi_n B\Gamma_2^\wedge$ is nontrivial, then our localization L will be nullification with regard to the wedge $S^{n+1} \vee M$, where M is the wedge of Moore spaces defined above. Observe that according to the previous lemma and 3.1, $n > 2$.

We are ready to explore the behavior of BG under the recursive action of the functors L and CW. As $n > 1$, it is clear that BG is L-local, and therefore by [FS07, Corollary 5.8], $CW BG \simeq CW LBG$ fits in a fibre sequence:

$$CW BG \longrightarrow BG \longrightarrow B\Gamma_2^\wedge \times \prod_{q \neq 2} BG_q^\wedge$$

where the product on the right is extended to odd primes. Here the map on the right is given by the composition

$$BG \longrightarrow BG_2^\wedge \longrightarrow BN_2^\wedge \longrightarrow B\Gamma_2^\wedge,$$

where the second map is an equivalence because the normalizer of S controls G-fusion in S (see details in [FS07, Example 5.2]).

Now let us describe $LCW BG$. Looping the base of the previous fibre sequencee we obtain another one, which is principal:

$$(3.1) \qquad \Omega(B\Gamma_2^\wedge) \times \Omega(\prod_{q \neq 2} BG_q^\wedge) \longrightarrow CW BG \longrightarrow BG.$$

The space BG is clearly $S^{n+1} \vee M$-null, so by [Far95, Proposition 3.D.3] the fibration is preserved by L. On the other hand, Lemma 3.1 implies that $\Omega(\prod_{q \neq 2} BG_q^\wedge)$ is simply connected, so it is M-cellular by [RS01, Theorem 6.2] and its L-localization is contractible. Then, by definition of L we have that $L\Omega(B\Gamma_2^\wedge)$ has the homotopy type of an Eilenberg-Mac Lane space $K(H, n-1)$, where $H = \pi_n B\Gamma_2^\wedge$. Thus $LCW BG(= LCW LBG)$ is homotopy equivalent to a Postnikov piece X whose only nonzero homotopy groups are $\pi_1 X = G$ and $\pi_{n-1} X = H$.

We are now ready to give a solution (in the negative) to the first part of Problem 3 and Problem 7 in [Mis08, pages. 62-63].

Proposition 3.3. *The functor CWL is not idempotent.*

Proof. Consider a non-trivial map $\Sigma^{n-2} B\mathbb{Z}/2 \to K(H, n-1)$, which exists because H is a finite 2-group, and observe that it remains essential when composing with the covering map $K(H, n-1) \to X$ (otherwise it would lift non-trivially to the discrete space $\Omega BG \simeq G$). On the other hand, as BG is aspherical and $n > 2$, there are no non-trivial maps from $\Sigma^{n-2} B\mathbb{Z}/2$ to BG. Hence, $\mathrm{map}_*(B\mathbb{Z}/2, BG)$ is not equivalent to $\mathrm{map}_*(B\mathbb{Z}/2, X)$, and so $CWX \not\simeq CW BG$. As $X = LCW LBG$ and $BG \simeq LBG$, we are done. $\square$

So our goal now is to check that LCW is not idempotent either. To undertake this task, we will need to explicitly compute CWX. A theorem of Chachólski [Cha96, 20.10] establishes that the A-cellularization of any space Y is the fibre of the ΣA-nullification of the cofibre of the map $\bigvee_{[A,Y]_*} A \to Y$, where the wedge is extended to all the homotopy classes of maps $A \to Y$. In our case then we consider the cofibre sequence

$$\bigvee_{[B\mathbb{Z}/2, X]_*} B\mathbb{Z}/2 \longrightarrow X \longrightarrow C,$$

and we need to describe its $\Sigma B\mathbb{Z}/2$-nullification.

To accomplish this we first show that C is simply connected. Because G is generated by elements of order two (it is simple), by the Mayer-Vietoris sequence this amounts to checking that every element of order two in $\pi_1 X = G$ is in the image of the

homomorphism induced at the level of fundamental groups by a certain map $B\mathbb{Z}/2 \to X$. So, let f be such a map, and x be an element of G of order two which is in the image of $\pi_1 f$. As the fibre sequence 3.1 is principal and $S^{n+1} \vee M$ is equivalent to a suspension, the fibration $K(H, n-1) \to X \to BG$ which defines X is principal by [DF09, Theorem 3.1]. Moreover, it is classified by the cohomology class defined by the composition $BG \to B\Gamma_2^\wedge \to K(H, n)$, where the second map is $(n+1)$-Postnikov section. According to the definition of Γ, the composition $B\mathbb{Z}/2 \to BG \to K(H, n)$ is null-homotopic for every map $B\mathbb{Z}/2 \to BG$. This implies in particular that f lifts to X, so C is simply connected.

Now we can describe C more precisely: As $\bigvee B\mathbb{Z}/2$ and X are rationally trivial, C is too by Mayer-Vietoris sequence in rational homology. Moreover, as the universal covering fibration of X is principal, it is preserved by p-completion ([BK72, II.5.2]), and hence the "formal completions" $(\prod_p X_p^\wedge)_\mathbb{Q}$ and $(\prod_p C_p^\wedge)_\mathbb{Q}$ are also trivial. Now, Sullivan's arithmetic square [BK72, VI.8.1] implies that the cofibre splits as a product $\prod C_q^\wedge$ over all primes q. Clearly $\mathrm{map}_*(\Sigma B\mathbb{Z}/2, C_q^\wedge)$ is weakly trivial for every q odd; moreover, as $\bigvee_{[B\mathbb{Z}/2, X]_*} B\mathbb{Z}/2$ is q-acyclic, the q-completion of the cofibre sequence that defines C gives an equivalence $BG_q^\wedge \simeq C_q^\wedge$. Then by Chachólski's Theorem CWX is equivalent to the homotopy fibre of $X \to \mathbf{P}_{\Sigma B\mathbb{Z}/2}(C_2^\wedge) \times \prod_{q \neq 2} BG_q^\wedge$, and we only need to describe $\mathbf{P}_{\Sigma B\mathbb{Z}/2}(C_2^\wedge)$ to get a complete characterization of CWX.

Proposition 3.4. *The space* $\mathbf{P}_{\Sigma B\mathbb{Z}/2}(C_2^\wedge)$ *has the homotopy type of* $B\Gamma_2^\wedge$.

Proof. We follow the same ideas as in the proof of [FS07, Proposition 5.5], see also [FF09, Section 3] for a comprehensive introduction to Zabrodsky Lemma in this context. We denote the $\Sigma B\mathbb{Z}/2$-null space $\mathbf{P}_{\Sigma B\mathbb{Z}/2}(C_2^\wedge)$ by P. We need to define maps between P and $B\Gamma_2^\wedge$; and we start by constructing the map $P \to B\Gamma_2^\wedge$ by

considering the diagram

$$(3.2)$$

$$\bigvee_{[B\mathbb{Z}/2,X]_*} B\mathbb{Z}/2 \longrightarrow X \xrightarrow{i} C \xrightarrow{(-)_2^\wedge} C_2^\wedge \xrightarrow{\eta} P$$

Here η denotes the coaugmentation of the $\Sigma B\mathbb{Z}/2$-nullification, and $B\pi_2^\wedge$ denotes the composition of the isomorphism $BG_2^\wedge \simeq BN_2^\wedge$ with the map induced in 2-completed classifying spaces by the projection $N \to \Gamma$. (Recall that N stands for the normalizer of the Sylow 2-subgroup S of G, which controls fusion in S.)

Observe that the composite $B\mathbb{Z}/2 \to X \to BG$ is trivial for every map $B\mathbb{Z}/2 \to X$. Therefore, the composite of the vertical maps extends to C (all the extensions are considered up to homotopy). As $B\Gamma_2^\wedge$ is $\Sigma B\mathbb{Z}/2$-null (see [Mil84, 9.9]) and 2-complete the corresponding map $C \to B\Gamma_2^\wedge$ also extends to $C_2^\wedge$ and then to P. So we have defined g, one of the desired maps.

Again for clarity, we summarize the information of the second part of the proof in a new diagram, which we prove to be commutative up to unpointed homotopy:
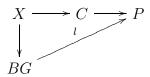
$$(3.3)$$

$$\bigvee_{[B\mathbb{Z}/2,X]_*} B\mathbb{Z}/2 \longrightarrow X \xrightarrow{i} C \xrightarrow{(-)_2^\wedge} C_2^\wedge \xrightarrow{\eta} P$$

To define the second map we consider the covering fibre sequence:

$$K(H, n-1) \longrightarrow X \longrightarrow BG.$$

As H is a 2-torsion finite group and $n > 2$, the mapping space $\mathrm{map}_*(K(H, n-1), \Omega P)$ is weakly trivial. But the composition $K(H, n-1) \to X \to C \to P$ is inessential, so Dwyer's version of the Zabrodsky Lemma ([Dwy96, Proposition 3.5], see also [CCS, Lemma 2.3]) establishes the existence of an extension (up to unpointed homotopy)

$$
\begin{array}{ccc}
X & \longrightarrow C \longrightarrow & P \\
\downarrow & \overset{l}{\nearrow} & \\
BG & &
\end{array}
$$

which, in particular, is null-homotopic if and only if the upper composition is. As P is 2-complete, the extension $BG \to P$ factors through $BG_2^\wedge$, which has the homotopy type of $BN_2^\wedge$. Now for $Z(S) = Cl(S) < S$ as discussed just before Lemma 3.2, there is another fibre sequence $BCl(S) \to BN \to B\Gamma$. By the definition of Γ and P, the composition

$$
BCl(S) \longrightarrow BN \longrightarrow BN_2^\wedge \simeq BG_2^\wedge \longrightarrow P
$$

is trivial. Then as P is $\Sigma B\mathbb{Z}/2$-null, Zabrodsky's Lemma again shows the existence of an extension from $BN_2^\wedge \to P$ to $B\Gamma_2^\wedge$. This is the other map we needed, and we denote it by f.

Our task now is to establish that $f \circ g$ and $g \simeq f$ are (pointed) homotopic to the respective identities. As P and $B\Gamma_2^\wedge$ are simply connected, it is enough to check this in the unpointed category. Let us denote by F the composition of the horizontal maps in the previous diagram, and by G the composition of the vertical maps.

Consider first the composition $f \circ g : P \to P$. The universal property of localization guarantees that $f \circ g \simeq Id_P$ if and only if $f \circ g \circ \eta \simeq \eta$. Moreover, as P is 2-complete, this is equivalent to establishing that $f \circ g \circ \eta \circ (-)_2^\wedge \simeq \eta \circ (-)_2^\wedge$. Now the Puppe sequence of the cofibration $\bigvee_{[B\mathbb{Z}/2, X]_*} B\mathbb{Z}/2 \to X$ implies that the latter is true if and only if the maps are homotopic when precomposing with i. By the definition of F, this is the same as checking that $f \circ g \circ F$ is homotopic to F. According to Diagram 1.1, $g \circ F \simeq G$, and then Diagram 1.2 implies $f \circ G \simeq F$.

On the other hand, consider $g \circ f : B\Gamma_2^\wedge \to B\Gamma_2^\wedge$. As this space is 2-complete and $\Sigma B\mathbb{Z}/2$-null, Zabrodsky's Lemma again applied to the fibration $BCl(S) \to BG \to B\Gamma$ implies that $g \circ f \simeq Id_{B\Gamma_2^\wedge}$ if and only if $g \circ f \circ B\pi_2^\wedge \circ (-)_2^\wedge$ is homotopic to $B\pi_2^\wedge \circ (-)_2^\wedge$. Applying the Zabrodsky Lemma again to the universal covering fibration of X, we obtain that the latter is equivalent to $g \circ f \circ B\pi_2^\wedge \circ (-)_2^\wedge \circ p \simeq (-)_2^\wedge \circ p$, or, in other

words, $g \circ f \circ G \simeq G$. By Diagram 1.2 now, $f \circ G \simeq F$, and then by Diagram 1.1, $g \circ F \simeq G$. So we are done.

$\square$

Proposition 3.5. *The functor LCW is not homotopy idempotent.*

Proof. The previous proposition gives a fibre sequence

$$CWX \longrightarrow X \longrightarrow B\Gamma_2^\wedge \times \prod_{q \text{ odd}} BG_q^\wedge$$

that characterizes the cellularization of X. Recall at this point that X is, by definition, $LCW\,LBG \simeq LCW\,BG$. We show how this space and the previous description of CWX give a counterexample for the second part of Problem 7 of Farjoun in [Mis08].

As the map $X \to B\Gamma_2^\wedge \times \prod_{q \text{ odd}} BG_q^\wedge$ factors through BG, its composition with $K(H, n-1) \to X$ is homotopically trivial, and thus the long exact homotopy sequence of the previous fibration defines $\pi_{n-1}CWX$ as an extension

$$\pi_n(B\Gamma_2^\wedge \times \prod_{q \text{ odd}} BG_q^\wedge) \longrightarrow \pi_{n-1}CWX \longrightarrow H.$$

Observe that L does not affect the 2-torsion subgroup of $\pi_{n-1}CWX$, which is therefore isomorphic to the 2-torsion subgroup of $\pi_{n-1}LCWX$. In particular it is *not* isomorphic to the 2-torsion subgroup of $\pi_{n-1}X = H$ because $\pi_n(B\Gamma_2^\wedge \times \prod_{q \text{ odd}} BG_q^\wedge)$, by definition of n, is nontrivial. Hence $LCW\,BG$ is not homotopy equivalent to $LCWX$, which is by definition $LCW\,LCW\,BG$. So LCW is not homotopy idempotent either, and the argument is complete.

$\square$

References

[Bou94]   A.K. Bousfield. The localization of spaces with respect to homology. *Topology*, 14: 133–150, 1975.

[Bou94]   A.K. Bousfield. Localization and periodicity in unstable homotopy theory. *J. Amer. Math. Soc.*, 7(4): 831–873, 1994.

[Bou97]   A.K. Bousfield. Homotopical localization of spaces. *Amer. J. Math.*, 119(6): 1321–1354, 1997.

[BK72]   A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations.* Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.

[CCS]   N. Castellana, J. A. Crespo, and J. Scherer. Postnikov pieces and $B\mathbb{Z}/p$-homotopy theory. *Trans. Amer. Math. Soc.*, 359(4):1791–1816, 2007.

[Cha96]   W. Chachólski. On the functors CW_A and P_A. *Duke Math. J.*, 84(3):599–631, 1996.

[DF09]   W.G.   Dwyer   and   E.Dror-Farjoun   Localization   and   cellu-
            larization   of   principal   fibrations.   ,   Preprint,   available   at
            http://www.nd.edu/∼wgd/Dvi/LocalizationCellularizationPrincipalFibrations.pdf

[Dwy96]   W. G. Dwyer. The centralizer decomposition of BG. In *Algebraic topology: new trends
            in localization and periodicity (Sant Feliu de Guíxols, 1994)*, volume 136 of *Progr.
            Math.*, pages 167–184. Birkhäuser, Basel, 1996.

[Far95]   E. Dror-Farjoun. *Cellular spaces, null spaces and homotopy localization*, volume 1622
            of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

[FF09]   R.J. Flores and R. Foote. The cellular structure of the classifying spaces of finite
            groups. Preprint, available at http://arxiv.org/pdf/0809.4116.

[FS07]   R.J. Flores and J. Scherer. Cellularization of classifying spaces and fusion properties
            of finite groups. *J. Lond. Math. Soc.(2)* 76(1):41–56, 2007.

[Gor80]   D. Gorenstein. *Finite groups*. Chelsea Publishing Co., New York, second edition, 1980.

[Lev95]   R. Levi. On finite groups and homotopy theory. *Mem. Amer. Math. Soc.*,
            118(567):xiv+100, 1995.

[Mil84]   H. Miller. The Sullivan conjecture on maps from classifying spaces. *Ann. of Math.
            (2)*, 120(1):39–87, 1984.

[Mis08]   Guido's book of conjectures. A gift to Guido Mislin on the occasion of his retirement
            from ETHZ, June 2006. Collected by Indira Chatterji.. *Enseign. Math.*,54(2): vols.
            1–3, 2008.

[RS01]   J. L. Rodríguez and J. Scherer. Cellular approximations using Moore spaces. In *Coho-
            mological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*,
            pages 357–374. Birkhäuser, Basel, 2001.

Ramón J. Flores

Departamento de Estadística, Universidad Carlos III de Madrid, C/ Madrid 126

E – 28903 Getafe

e-mail: `rflores@est-econ.uc3m.es`