RANDOM PULLBACK EXPONENTIAL ATTRACTORS: GENERAL EXISTENCE RESULTS FOR RANDOM DYNAMICAL SYSTEMS IN BANACH SPACES

TOMÁS CARABALLO, STEFANIE SONNER

Abstract. We derive general existence theorems for random pullback exponential attractors and deduce explicit bounds for their fractal dimension. The results are formulated for asymptotically compact random dynamical systems in Banach spaces.

1. Introduction

Studying the longtime behavior of infinite dimensional dynamical systems can often be reduced to analyzing the dynamics on the *global attractor*. Global attractors are compact subsets of the phase space that are strictly invariant under the time evolution of the system and attract all bounded subsets as time tends to infinity. If the global attractor exists, it is unique, and in most cases of finite fractal dimension. The rate of convergence to the attractor, however, is typically unknown. It can be arbitrarily slow and hence, global attractors are generally not stable under perturbations. To overcome these drawbacks the notion of an *exponential attractor* was introduced in [7]. Exponential attractors are compact, semi-invariant sets of finite fractal dimension that contain the global attractor and attract all bounded subsets at an exponential rate. Due to the exponential rate of convergence they are more stable under perturbations. However, since exponential attractors are only semi-invariant under the time evolution of the system, they are not unique.

Different methods have been developed to show the existence of exponential attractors for infinite dimensional dynamical systems. The first existence proof was established for semigroups acting in Hilbert spaces, cf. [7]. It is non-constructive, based on the so-called *squeezing property* of the semigroup and essentially uses the Hilbert space structure of the phase space. In [9] an alternative method and explicit construction of exponential attractors for discrete time semigroups in Banach spaces was proposed. The approach relies on the compact embedding of the phase space into an auxiliary normed space, the existence of a bounded absorbing set and the so-called *smoothing property* of the semigroup. This property implies that the semigroup is eventually compact and is mainly satisfied by parabolic problems.

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More recently, in [6, 11] and [2, 3] the construction of exponential attractors [9] has been extended to non-autonomous dynamical systems using the notion of *pullback attraction*. In [6] and [11] evolution processes satisfying the smoothing property were considered and the existence of a fixed bounded pullback absorbing set was assumed. Moreover, rather strong regularity assumptions were imposed that mainly restrict applications to parabolic problems. In [2, 3] the existence of *pullback exponential attractors* was shown under significantly weaker hypotheses and the construction was generalized for asymptotically compact evolution processes. In particular, the Hölder continuity of the process, as hypothesized in [6, 11], could be omitted and the fixed bounded pullback absorbing set was replaced by a time-dependent family of absorbing sets. As a consequence, the results allow for more general non-autonomous terms in the equations and are also applicable, e.g. to hyperbolic problems. Moreover, unlike in [6, 11] the pullback attractors are allowed to be unbounded in the past, a property that is inherent to random pullback attractors in most applications.

The aim of our paper is to extend the construction [2, 3] to the setting of random dynamical systems. We formulate general existence results for *random pullback exponential attractors* and derive explicit estimates for their fractal dimension. The generalizations developed in [2, 3] are hereby essential, since the absorbing sets as well as the constants in the estimates depend on the random parameter, and hence, are time-dependent. Moreover, random attractors of PDEs perturbed by additive or multiplicative noise are typically unbounded in the past.

Exponential attractors for random dynamical systems have previously been considered in [12], however, under restrictive assumptions that are difficult to verify in applications. The construction was carried out in the setting of Hilbert spaces and the attraction universe was the family of deterministic sets. The random dynamical system was assumed to satisfy the smoothing property and to be uniformly Hölder continuous in time. Moreover, certain stability assumptions and the compactness of the absorbing set were imposed. We improve this result in various directions and show that several of these hypotheses are not required. We consider asymptotically compact random dynamical systems in Banach spaces, i.e. the cocycle can be represented as a sum of operators satisfying the smoothing property and a family of contractions, and the attraction universe is the family of tempered random sets. Our proof yields the measurability of the random exponential attractor without the technical auxiliary results needed and established in [12], and does not require a stability assumption or the compactness of the absorbing set. Moreover, we derive explicit estimates for the fractal dimension of the attractors. For continuous time random dynamical systems we propose to weaken the notion of positive invariance. This allows to simplify the construction of random pullback exponential attractors such that the assumption of Hölder continuity in time of the cocycle can be omitted.

In a forthcoming paper [1] we will apply the theoretical results to a stochastic semilinear damped wave equation with multiplicative noise,

$$
\begin{aligned}
du_t + \big(\beta u_t - \Delta u\big)dt &= f(u)dt + \sigma u \circ dW && t > \tau, \\
u|_{\partial D} &= 0 && t \geq \tau, \\
u|_{t=\tau} = u_0, \quad u_t|_{t=\tau} &= v_0 && \tau \in \mathbb{R},
\end{aligned}
\tag{1}
$$

where $D \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain and $W : \Omega \to C_0(\mathbb{R})$ a standard scalar Wiener process. Assuming that the nonlinearity f is subcritical and dissipative and the noise $|\sigma|$ is small w.r.t. β we prove the existence of a random pullback exponential attractor and derive estimates for its fractal dimension. The previous existence result for random exponential attractors [12] is not applicable to problem (1), since it is based on the smoothing property and Hölder continuity in time of the generated random dynamical system, i.e. on properties that are not satisfied in this situation.

The outline of our paper is as follows: In Section 2 we collect several notions from the theory of random dynamical systems and recall results about entropy properties of embeddings that we will need in the sequel. General existence results for random pullback exponential attractors are derived in Section 3, where the construction is first carried out for discrete time random dynamical systems and subsequently extended to the continuous time setting.

2. Preliminaries

2.1. **Random dynamical systems.**

We recall basic notions from the theory of random dynamical systems that we will need in the subsequent sections and introduce the concept of random exponential attractors. Here and in the sequel, we assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(V, \|\cdot\|_V)$ a Banach space. Moreover, let $\mathbb{T}$ denote $\mathbb{R}$ or $\mathbb{Z}$, and $\mathbb{T}_+$ be the non-negative real numbers, or integers respectively.

Definition 1. A *random dynamical system* (θ, φ) on V consists of a measurable and measure-preserving dynamical system $\{\theta_t\}_{t\in\mathbb{T}}$, $\theta_t : \Omega \to \Omega$, on $(\Omega, \mathcal{F}, \mathbb{P})$, i.e.

$$\begin{aligned}
\theta_0 &= \mathrm{Id}, && \\
\theta_t \circ \theta_s(\omega) &= \theta_{t+s}(\omega) && \forall t, s \in \mathbb{T}, \omega \in \Omega, \\
(t, \omega) &\mapsto \theta_t(\omega) && \text{is measurable}, \\
\theta_t \mathbb{P} &= \mathbb{P} && \forall t \in \mathbb{T},
\end{aligned}$$

where Id denotes the identity operator in Ω, and a *cocycle mapping* $\varphi : \mathbb{T}_+ \times \Omega \times V \to V$, i.e.

$$\begin{aligned}
\varphi(0, \omega, v) &= v && \forall \omega \in \Omega, v \in V, \\
\varphi(s + t, \omega, v) &= \varphi(s, \theta_t(\omega), \varphi(t, \omega, v)) && \forall s, t \in \mathbb{T}_+, \omega \in \Omega, v \in V, \\
(t, \omega, v) &\mapsto \varphi(t, \omega, v) && \text{is measurable}, \\
v &\mapsto \varphi(t, \omega, v) && \text{is continuous } \forall t \in \mathbb{T}_+, \omega \in \Omega.
\end{aligned}$$

Definition 2. A *random set* B is a subset of $\Omega \times V$ that is measurable with respect to the product σ-algebra $\mathcal{F} \otimes \mathcal{B}_V$, where $\mathcal{B}_V$ denotes the Borel σ-algebra of V. Moreover, the *ω-section* of a random set B is defined by

$$B(\omega) = \{v \in V : (\omega, v) \in B\}.$$

A random set B is called *tempered*, if there exists a random variable $r_B(\omega) \geq 0$ such that $B(\omega)$ is contained in a ball with center zero and radius $r_B(\omega)$ and

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \log^+ \big(r_B(\theta_t(\omega))\big) = 0.$$

We will denote a general family of random sets by $\mathcal{D}$. It is usually called *universe* and can represent, for instance, the family of bounded deterministic sets, or the family of tempered random sets.

In the remainder of this subsection, when stating properties involving a random parameter we assume that they hold a.s., unless otherwise specified (i.e., there exists a subset $\bar{\Omega} \subset \Omega$ of full measure such that the property is satisfied for all $\omega \in \bar{\Omega}$).

Definition 3. A random set $\mathcal{A}$ with compact sections $\mathcal{A}(\omega) \neq \emptyset$ is a *random pullback $\mathcal{D}$-attractor* for the random dynamical system (θ, φ) on V, if $\mathcal{A}$ is *φ-invariant*, i.e.

$$\varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t(\omega)) \qquad \forall t \in \mathbb{T}_+,$$

and it *pullback attracts* the family $\mathcal{D}$, i.e.

$$\lim_{t\to\infty} \mathrm{dist}_{\mathrm{H}}(\varphi(t, \theta_{-t}(\omega), D(\theta_{-t}(\omega))), \mathcal{A}(\omega)) = 0 \qquad \forall D \in \mathcal{D}.$$

Here, $\mathrm{dist}_{\mathrm{H}}(\cdot,\cdot)$ denotes the Hausdorff semidistance in V, i.e.

$$\mathrm{dist}_{\mathrm{H}}(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_V.$$

There exist several criteria ensuring the existence of random pullback attractors. The simplest one states that if a random dynamical system in V possesses a compact random pullback $\mathcal{D}$-attracting set, then a random pullback $\mathcal{D}$-attractor exists (see Theorem 4 in [5]).

Theorem 1. *Let (θ, φ) be a random dynamical system on a separable Banach space V. There exists a random pullback $\mathcal{D}$-attractor for (θ, φ) if and only if there exists a compact random pullback $\mathcal{D}$-attracting set K, i.e., the sections $K(\omega)$ are compact and*

$$\lim_{t\to\infty} \mathrm{dist}_{\mathrm{H}}(\varphi(t, \theta_{-t}(\omega), D(\theta_{-t}(\omega))), K(\omega)) = 0 \qquad \forall D \in \mathcal{D}.$$

Remark 1. If a random dynamical system possesses a random pullback $\mathcal{D}$-attractor $\mathcal{A}$ and the universe $\mathcal{D}$ contains the family of compact deterministic sets, then $\mathcal{A}$ is unique a.s. (see Corollary 1 in [5]).

We now use the concept of random pullback attractors to introduce exponential attractors for random dynamical systems.

Definition 4. A random set $\mathcal{M}$ is a *random pullback exponential $\mathcal{D}$-attractor* for the random dynamical system (θ, φ) on V, if the sections $\mathcal{M}(\omega) \neq \emptyset$ are compact and $\mathcal{M}$ is *positively φ-invariant*, i.e.

$$\varphi(t, \omega, \mathcal{M}(\omega)) \subset \mathcal{M}(\theta_t(\omega)) \qquad \forall t \in \mathbb{T}_+.$$

Moreover, the fractal dimension of $\mathcal{M}(\omega)$ is finite, i.e. there exists a random variable $k(\omega) \geq 0$ such that

$$\dim_{\mathrm{f}}(\mathcal{M}(\omega)) \leq k(\omega) < \infty,$$

and $\mathcal{M}$ is pullback $\mathcal{D}$-attracting at an exponential rate, i.e. there exists $\alpha > 0$ such that

$$\lim_{t\to\infty} e^{\alpha t} \mathrm{dist}_{\mathrm{H}} \big(\varphi\big(t, \theta_{-t}(\omega), D(\theta_{-t}(\omega))\big), \mathcal{M}(\omega)\big) = 0 \qquad \forall D \in \mathcal{D}.$$

Here, $\dim_f(\cdot)$ denotes the *fractal dimension*, i.e. if $A \subset V$ is precompact, then

$$\dim_f(A) = \limsup_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}} \big(N_\varepsilon^V(A)\big),$$

where $N_\varepsilon^V(A)$ denotes the minimal number of ε-balls in V with centers in A needed to cover A.

2.2. **(Kolmogorov) ε-entropy and entropy numbers.** Our construction of random pullback exponential attractors is based on the embedding of the phase space into an auxiliary normed space, and the entropy properties of this embedding will play a crucial role. In this subsection we recall the corresponding notions and results that we will need in the sequel.

The *(Kolmogorov) ε-entropy* of a precompact subset A of a Banach space V is defined as

$$\mathcal{H}_\varepsilon^V(A) = \log_2(N_\varepsilon^V(A)),$$

where $N_\varepsilon^V(A)$ denotes the minimal number of ε-balls in V with centers in A needed to cover A. It was first introduced by Kolmogorov and Tihomirov in [10]. The order of growth of $\mathcal{H}_\varepsilon^V$ as ε tends to zero is a measure for the massiveness of the set A in V, even if its fractal dimension is infinite.

If V and U are Banach spaces such that the embedding $V \hookrightarrow\hookrightarrow U$ is compact we use the notation

$$\mathcal{H}_\varepsilon(V;U) = \mathcal{H}_\varepsilon^U(B_1^V(0)),$$

where $B_1^V(0)$ denotes the closed unit ball in V.

Remark 2. The ε-entropy is related to the *entropy numbers* e_k for the embedding $V \hookrightarrow U$, which are defined by

$$e_k = \inf\Big\{\varepsilon > 0 : B_1^V(0) \subset \bigcup_{j=1}^{2^{k-1}} B_\varepsilon^U(w_j),\ w_j \in U,\ j = 1, \dots, 2^{k-1}\Big\},$$

$k \in \mathbb{N}$. If the embedding is compact, then e_k is finite for all $k \in \mathbb{N}$. For certain function spaces the entropy numbers can explicitly be estimated (see [8]). For instance, if $D \subset \mathbb{R}^n$ is a smooth bounded domain, then the embedding of the Sobolev spaces

$$W^{l_1,p_1}(D) \hookrightarrow W^{l_2,p_2}(D), \qquad l_1, l_2 \in \mathbb{R},\ p_1, p_2 \in (1, \infty),$$

is compact if $l_1 > l_2$ and $\frac{l_1}{n} - \frac{1}{p_1} > \frac{l_2}{n} - \frac{1}{p_2}$. Moreover, the entropy numbers grow polynomially, namely,

$$e_k \simeq k^{-\frac{l_1-l_2}{n}}$$

(see Theorem 2, Section 3.3.3 in [8]), and consequently,

$$\mathcal{H}_\varepsilon(W^{l_1,p_1}(D); W^{l_2,p_2}(D)) \leq c\varepsilon^{-\frac{n}{l_1-l_2}},$$

for some constant $c > 0$.

Here and in the sequel, we write $f \simeq g$, if there exist positive constants c_1 and c_2 such that

$$c_1 f \leq g \leq c_2 f.$$

3. Construction of random pullback exponential attractors

Let $(V, \|\cdot\|_V)$ be a separable Banach space, (θ, φ) be a random dynamical system on V and $\mathcal{D}$ denote the universe of tempered random sets. Our construction of random exponential attractors is based on the compact embedding of the phase space into an auxiliary normed space, the decomposition of the cocycle as a sum of operators satisfying the smoothing property and a family of contractions, and the existence of a tempered pullback $\mathcal{D}$-absorbing random set. We assume that the following properties are satisfied on a subset of full measure $\overline{\Omega} \subset \Omega$ that, for simplicity, we will denote again by Ω.

(H_0) We assume $(U, \|\cdot\|_U)$ is another separable Banach space such that the embedding $V \hookrightarrow\hookrightarrow U$ is dense and compact,

$$\|v\|_U \leq \mu \|v\|_V \qquad \forall v \in V,$$

for some constant $\mu > 0$, and the ε-entropy $\mathcal{H}_\varepsilon(V;U)$ grows polynomially, i.e.

$$\mathcal{H}_\varepsilon(V;U) \leq c\varepsilon^{-\gamma},$$

for some positive constants c and γ.

(H_1) There exists a random closed set $B \in \mathcal{D}$ that is *pullback $\mathcal{D}$-absorbing*, i.e. for every $D \in \mathcal{D}$ and $\omega \in \Omega$ there exists $T_{D,\omega} \geq 0$ such that

$$\varphi(t, \theta_{-t}(\omega), D(\theta_{-t}(\omega))) \subset B(\omega) \qquad \forall t \geq T_{D,\omega},$$

and we assume that $T_{D,\theta_{-\tau}(\omega)} \leq T_{D,\omega}$ for all $\tau \in \mathbb{T}_+$.

Moreover, the cocycle φ can be represented as sum $\varphi = \phi + \psi$, where $\phi : \mathbb{T}_+ \times \Omega \times V \to V$, and $\psi : \mathbb{T}_+ \times \Omega \times V \to V$, are families of operators satisfying the following hypotheses:

(H_2) There exists a positive $\tilde{t} \geq T_{B,\omega}$ such that the family ϕ satisfies the *smoothing property* within B, i.e. there exists a random variable $\kappa(\omega)$ such that

$$\|\phi(\tilde{t}, \omega, u) - \phi(\tilde{t}, \omega, v)\|_V \leq \kappa(\omega)\|u - v\|_U \qquad \forall u, v \in B(\omega),$$

and κ satisfies

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \big(\kappa(\theta_{-k}(\omega))\big)^\gamma < \infty, \tag{2}$$

where γ is the growth exponent of the ε-entropy in (H_0).

(H_3) The family of operators ψ is a *contraction* within B, i.e.

$$\|\psi(\tilde{t}, \omega, u) - \psi(\tilde{t}, \omega, v)\|_V \leq \lambda \|u - v\|_V \qquad \forall u, v \in B(\omega),$$

where $0 \leq \lambda < \frac{1}{2}$.

Remark 3. If the random variable κ satisfies $\kappa^\gamma \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, Birkhoff's ergodic theorem implies property (2).

Our main result is the following existence result for discrete time random dynamical systems that we later extend for continuous time random dynamical systems.

Theorem 2. *Let (θ, φ) be a discrete time random dynamical system on V, i.e. $\mathbb{T} = \mathbb{Z}$ and the assumptions (H_0)–(H_3) be satisfied. Then, for any $\nu \in (0, \frac{1}{2} - \lambda)$*

there exists a random pullback exponential attractor $\mathcal{M}^\nu$ *for* (θ,φ) *in* V*, and the fractal dimension of its sections is bounded by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}^\nu(\omega)) \le \frac{d(\omega)}{-\log_2(2(\nu+\lambda))} \qquad \forall \omega\in\Omega,$$

where

$$d(\omega)=\frac{c}{\nu^\gamma}\limsup_{m\to\infty}\frac{1}{m-1}\sum_{k=1}^{m}\big(\kappa(\theta_{-k}(\omega))\big)^\gamma,$$

and c *and* γ *are the constants determined by the entropy properties in* (H_0)*.*

In the particular case that the constant κ in (H_2) can be chosen uniformly w.r.t. ω we recover the bound for the fractal dimension of deterministic pullback exponential attractors in [2, 3], namely,

$$\dim_{\mathrm{f}}^V(\mathcal{M}^\nu(\omega)) \le \frac{\left(\frac{\kappa}{\nu}\right)^\gamma}{-\log_2(2(\nu+\lambda))}.$$

Remark 4. Our results improve the previous existence result for random pullback exponential attractors by Shirikyan & Zelik [12]. The hypotheses are significantly weaker and easier to verify in applications. In particular, we generalize the construction for asymptotically compact random dynamical systems, i.e. for cocycles that can be represented as sum of operators ϕ satisfying the smoothing property and a family of contractions ψ. Moreover, we formulate the setting in Banach spaces instead of Hilbert spaces and replace the attraction universe of bounded deterministic sets by tempered random sets. The hypotheses on the random pullback absorbing set B are essentially weaker, since we do not suppose its compactness nor impose growth conditions for the ε-entropy of its sections. The measurability of the exponential attractor is achieved by a modified construction that does not require the technical auxiliary results in [12]. In Section 3.2 we extend the construction for continuous time random dynamical systems. In order to apply the approach in [2] developed for non-autonomous deterministic problems, that does not require the Hölder continuity in time of the cocycle, we propose to weaken the notion of positive invariance for continuous time random exponential attractors. It essentially simplifies the construction and leads to better, explicit estimates for the fractal dimension.

During the revision of our article S. Zhou published an existence result for exponential attractors for non-autonomous random dynamical systems [13] and applied the results to stochastic lattice systems. The setting is different, since he considers non-autonomous random problems, and the hypotheses are more difficult to verify in applications.

We first prove Theorem 2 and subsequently construct random exponential attractors for continuous time random dynamical systems in Subsection 3.2.

3.1. **Discrete random dynamical systems.** In this subsection, we consider discrete time dynamical systems (θ,φ), i.e. $\mathbb{T}=\mathbb{Z}$.

Proof of Theorem 2. Without loss of generality we assume that $\tilde{t}=1$ in assumptions (H_2) and (H_3).

We follow and extend the method used in [2] to construct pullback exponential attractors for nonautonomous evolution processes. Different from the deterministic setting, the constants now depend on the random parameter ω, and the construction has to be done in such a way that the random pullback exponential attractor is measurable.

Coverings of $\varphi\big(n,\theta_{-n}(\omega),B(\theta_{-n}(\omega))\big)$**.** By Proposition 1.3.2 in [4] there exists a sequence of measurable functions $v_j:\Omega\to V$, $j\in\mathbb{N}$, such that $v_j(\omega)\in B(\omega)$ for all $j\in\mathbb{N}$, and

$$B(\omega)=\overline{\{v_j(\omega),j\in\mathbb{N}\}}^V \qquad \forall\omega\in\Omega,$$

where $\overline{\cdot}^V$ is the closure in V. We denote the countable dense random subset by

$$\mathcal{V}(\omega)=\bigcup_{j\in\mathbb{N}}v_j(\omega),\qquad \omega\in\Omega. \tag{3}$$

We remark that by the assumption $1=\tilde{t}\geq T_{B,\omega}$ in (H_2) and property (H_1) it follows that

$$\varphi(n,\theta_{-n-\tau}(\omega),B(\theta_{-n-\tau}(\omega)))\subset B(\theta_{-\tau}(\omega))\qquad \forall\tau\geq 0,\ n\in\mathbb{N}. \tag{4}$$

Let $\nu\in(0,\frac{1}{2}-\lambda)$ be arbitrary and $\beta=2(\nu+\lambda)$. By property (3) and since B is tempered, there exist $v^\omega\in\mathcal{V}(\omega)$ and a random variable $R(\omega)\geq 0$ such that $B(\omega)$ is contained in a ball with center zero and radius $R(\omega)$,

$$\lim_{t\to\pm\infty}\frac{1}{|t|}\log^+\big(R(\theta_t\omega)\big)=0,$$

and $B(\omega)\subset B^V_{R(\omega)}(v^\omega)$. Moreover, we choose elements $p_1^\omega,\ldots,p^\omega_{N(\omega)}\in V$ such that

$$B_1^V(0)\subset\bigcup_{i=1}^{N(\omega)}B^U_{\frac{\nu}{\kappa(\omega)}}(p_i^\omega),$$

where $N(\omega)=N^U_{\frac{\nu}{\kappa(\omega)}}(B_1^V(0))$. We define $U^0(\omega)=\{v^\omega\}$ for all $\omega\in\Omega$ and construct by induction in $n\in\mathbb{N}$ sets $U^n(\omega),n\in\mathbb{N}$, such that

$$U^n(\omega)\subset\varphi\big(n,\theta_{-n}(\omega),\mathcal{V}(\theta_{-n}(\omega))\big)\subset B(\omega) \tag{U_1}$$

$$\sharp U^n(\omega)\leq\prod_{i=1}^n N(\theta_{-i}(\omega)) \tag{U_2}$$

$$\varphi\big(n,\theta_{-n}(\omega),\mathcal{V}(\theta_{-n}(\omega))\big)\subset\bigcup_{u\in U^n(\omega)}B^V_{\beta^nR(\theta_{-n}(\omega))}(u), \tag{U_3}$$

where $\sharp$ denotes the cardinality of a set. Certainly, property (U_3) implies that

$$\begin{aligned}\varphi\big(n,\theta_{-n}(\omega),B(\theta_{-n}(\omega))\big)&=\varphi\big(n,\theta_{-n}(\omega),\overline{\mathcal{V}(\theta_{-n}(\omega))}^V\big)\\&\subset\bigcup_{u\in U^n(\omega)}\overline{B^V_{\beta^nR(\theta_{-n}(\omega))}(u)}^V.\end{aligned}$$

First, we build a suitable covering of the image $\varphi\big(1,\theta_{-1}(\omega),\mathcal{V}(\theta_{-1}(\omega))\big)$. For any $v\in B^V_{R(\theta_{-1}(\omega))}(v^{\theta_{-1}(\omega)})$ we have

$$\frac{1}{R(\theta_{-1}(\omega))}(v-v^{\theta_{-1}(\omega)})\in B_1^V(0)\subset\bigcup_{i=1}^{N(\theta_{-1}(\omega))}B^U_{\frac{\nu}{\kappa(\theta_{-1}(\omega))}}(p_i^{\theta_{-1}(\omega)})$$

and consequently,

$$B^V_{R(\theta_{-1}(\omega))}(v^{\theta_{-1}(\omega)}) \subset \bigcup_{i=1}^{N(\theta_{-1}(\omega))} B^U_{\frac{\nu R(\theta_{-1}(\omega))}{\kappa(\theta_{-1}(\omega))}}\big(R(\theta_{-1}(\omega))p_i^{\theta_{-1}(\omega)} + v^{\theta_{-1}(\omega)}\big).$$

Due to the smoothing property (H_2) we obtain

$$\|\phi(1,\theta_{-1}(\omega),\tilde{u}) - \phi(1,\theta_{-1}(\omega),\tilde{v})\|_V \leq \kappa(\theta_{-1}(\omega))\|\tilde{u}-\tilde{v}\|_U < 2\nu R(\theta_{-1}(\omega))$$

for all $\tilde{u},\tilde{v} \in B^U_{\frac{\nu R(\theta_{-1}(\omega))}{\kappa(\theta_{-1}(\omega))}}\big(R(\theta_{-1}(\omega))p_i^{\theta_{-1}(\omega)} + v^{\theta_{-1}(\omega)}\big) \cap \mathcal{V}(\theta_{-1}(\omega)) \subset B(\theta_{-1}(\omega))$, which yields the covering

$$\phi\big(1,\theta_{-1}(\omega), B^V_{R(\theta_{-1}(\omega))}(v^{\theta_{-1}(\omega)}) \cap \mathcal{V}(\theta_{-1}(\omega))\big) \subset \bigcup_{i=1}^{N(\theta_{-1}(\omega))} B^V_{2\nu R(\theta_{-1}(\omega))}(z_i),$$

for some $z_1,\ldots,z_{N(\theta_{-1}(\omega))} \in \phi\big(1,\theta_{-1}(\omega),\mathcal{V}(\theta_{-1}(\omega))\big)$. In particular, we can choose elements $y_1,\ldots,y_{N(\theta_{-1}(\omega))} \in \mathcal{V}(\theta_{-1}(\omega))$ such that $z_i = \phi(1,\theta_{-1}(\omega),y_i)$ for $i = 1,\ldots,N(\theta_{-1}(\omega))$.

Moreover, if $u \in \mathcal{V}(\theta_{-1}(\omega)) \subset B(\theta_{-1}(\omega))$ the contraction property (H_3) implies that

$$\|\psi(1,\theta_{-1}(\omega),u) - \psi(1,\theta_{-1}(\omega),y_i)\|_V \leq \lambda\|u-y_i\|_V < 2\lambda R(\theta_{-1}(\omega)),$$

for all $i = 1,\ldots,N(\theta_{-1}(\omega))$, and we conclude that

$$\psi\big(1,\theta_{-1}(\omega),\mathcal{V}(\theta_{-1}(\omega))\big) \subset B^V_{2\lambda R(\theta_{-1}(\omega))}\big(\psi(1,\theta_{-1}(\omega),y_i)\big).$$

Finally, we obtain the covering

$$\begin{aligned}
&\varphi\big(1,\theta_{-1}(\omega),\mathcal{V}(\theta_{-1}(\omega))\big) = (\phi+\psi)\big(1,\theta_{-1}(\omega),\mathcal{V}(\theta_{-1}(\omega))\big)\\
&\subset \bigcup_{i=1}^{N(\theta_{-1}(\omega))} \Big(B^V_{2\nu R(\theta_{-1}(\omega))}(\phi(1,\theta_{-1}(\omega),y_i)) + B^V_{2\lambda R(\theta_{-1}(\omega))}(\psi(1,\theta_{-1}(\omega),y_i))\Big)\\
&\subset \bigcup_{i=1}^{N(\theta_{-1}(\omega))} B^V_{\beta R(\theta_{-1}(\omega))}\big(\varphi(1,\theta_{-1}(\omega),y_i)\big),
\end{aligned}$$

with centers $\varphi(1,\theta_{-1}(\omega),y_i) \in \varphi\big(1,\theta_{-1}(\omega),\mathcal{V}(\theta_{-1}(\omega))\big)$, $i = 1,\ldots,N(\theta_{-1}(\omega))$, where $\beta = 2(\nu+\lambda)$. Denoting the new set of centers by $U^1(\omega)$ it follows that

$$\begin{aligned}
&\varphi\big(1,\theta_{-1}(\omega),\mathcal{V}(\theta_{-1}(\omega))\big) \subset \bigcup_{u\in U^1(\omega)} B^V_{\beta R(\theta_{-1}(\omega))}(u),\\
&U^1(\omega) \subset \varphi\big(1,\theta_{-1}(\omega),\mathcal{V}(\theta_{-1}(\omega))\big) \subset B(\omega),\\
&\sharp U^1(\omega) \leq N(\theta_{-1}(\omega)),
\end{aligned}$$

which proves properties (U_1)–(U_3) for $n = 1$.

We now assume that the sets $U^k(\omega)$ satisfying (U_1)–(U_3) have been constructed for all $k \leq n$. In order to construct a covering of the set

$$\begin{aligned}
&\varphi\big(n+1,\theta_{-(n+1)}(\omega),\mathcal{V}(\theta_{-(n+1)}(\omega))\big)\\
=&\varphi\Big(1,\theta_{-1}(\omega),\varphi\big(n,\theta_{-n}\circ\theta_{-1}(\omega),\mathcal{V}(\theta_{-n}\circ\theta_{-1}(\omega))\big)\Big)\\
\subset&\bigcup_{u\in U^n(\theta_{-1}(\omega))} \varphi\big(1,\theta_{-1}(\omega), B^V_{\beta^n R(\theta_{-(n+1)}(\omega))}(u)\big),
\end{aligned}$$

let $u \in U^n(\theta_{-1}(\omega))$. We proceed as before using a covering of the unit ball $B_1^V(0)$ by $\frac{\nu}{\kappa(\theta_{-(n+1)}(\omega))}$-balls in U to conclude

$$\begin{aligned} &B^V_{\beta^n R(\theta_{-(n+1)}(\omega))}(u) \\ &\subset \bigcup_{i=1}^{N(\theta_{-(n+1)}(\omega))} B^U_{\frac{\nu(\beta^n R(\theta_{-(n+1)}(\omega))}{\kappa(\theta_{-1}(\omega))}}\big(\beta^n R(\theta_{-(n+1)}(\omega)) p_i^{\theta_{-(n+1)}(\omega)} + u\big). \end{aligned}$$

By the smoothing property (H_2) it then follows that

$$\begin{aligned} &\phi\Big(1, \theta_{-1}(\omega), \varphi\big(n, \theta_{-n} \circ \theta_{-1}(\omega), \mathcal{V}(\theta_{-n} \circ \theta_{-1}(\omega))\big) \cap B^V_{\beta^n R(\theta_{-(n+1)}(\omega))}(u)\Big) \\ &\subset \bigcup_{i=1}^{N(\theta_{-1}(\omega))} B^V_{\beta^n 2\nu R(\theta_{-(n+1)}(\omega))}(\phi(1, \theta_{-1}(\omega), y_i^u)), \end{aligned}$$

for some $y_1^u, \ldots, y^u_{N(\theta_{-1}(\omega))} \in \varphi\big(n, \theta_{-n} \circ \theta_{-1}(\omega), \mathcal{V}(\theta_{-n} \circ \theta_{-1}(\omega))\big)$. Furthermore, the contraction property (H_3) implies that

$$\begin{aligned} &\psi\Big(1, \theta_{-1}(\omega), \varphi\big(n, \theta_{-n} \circ \theta_{-1}(\omega), \mathcal{V}(\theta_{-n} \circ \theta_{-1}(\omega))\big) \cap B^V_{\beta^n R(\theta_{-(n+1)}(\omega))}(u)\Big) \\ &\subset B^V_{\beta^n 2\lambda R(\theta_{-(n+1)}(\omega))}(\psi(1, \theta_{-1}(\omega), y_i^u)), \end{aligned}$$

for all $i = 1, \ldots, N(\theta_{-1}(\omega))$. Consequently, we obtain the covering

$$\begin{aligned} &\varphi\Big(1, \theta_{-1}(\omega), \varphi\big(n, \theta_{-n} \circ \theta_{-1}(\omega), \mathcal{V}(\theta_{-n} \circ \theta_{-1}(\omega))\big) \cap B^V_{\beta^n R(\theta_{-(n+1)}(\omega))}(u)\Big) \\ =&(\phi + \psi)\Big(1, \theta_{-1}(\omega), \varphi\big(n, \theta_{-n} \circ \theta_{-1}(\omega), \mathcal{V}(\theta_{-n} \circ \theta_{-1}(\omega))\big) \cap B^V_{\beta^n R(\theta_{-(n+1)}(\omega))}(u)\Big) \\ \subset& \bigcup_{i=1}^{N(\theta_{-1}(\omega))} B^V_{\beta^n 2\nu R(\theta_{-(n+1)}(\omega))}(\phi(1, \theta_{-1}(\omega), y_i^u)) \\ &\qquad + B^V_{\beta^n 2\lambda R(\theta_{-(n+1)}(\omega))}(\psi(1, \theta_{-1}(\omega), y_i^u)) \\ \subset& \bigcup_{i=1}^{N(\theta_{-1}(\omega))} B^V_{\beta^{n+1} R(\theta_{-(n+1)}(\omega))}\big((\phi + \psi)(1, \theta_{-1}(\omega), y_i^u)\big) \\ =& \bigcup_{i=1}^{N(\theta_{-1}(\omega))} B^V_{\beta^{n+1} R(\theta_{-(n+1)}(\omega))}\big(\varphi(1, \theta_{-1}(\omega), y_i^u)\big) \end{aligned}$$

with centers $\varphi\big(1, \theta_{-1}(\omega), y_i^u\big) \in \varphi\big(n+1, \theta_{-(n+1)}(\omega), \mathcal{V}(\theta_{-(n+1)}(\omega))\big)$, for $i = 1, \ldots, N(\theta_{-1}(\omega))$. Constructing for every $u \in U^n(\theta_{-1}(\omega))$ such a covering by balls with radius $\beta^{n+1} R(\theta_{-(n+1)}(\omega))$ in V we obtain a covering of the set

$$\varphi\big(n+1, \theta_{-(n+1)}(\omega), \mathcal{V}(\theta_{-(n+1)}(\omega))\big)$$

and denote the new set of centers by $U^{n+1}(\omega)$. This yields

$$\sharp U^{n+1}(\omega) \leq N(\theta_{-1}(\omega)) \sharp U^n(\theta_{-1}(\omega)) \leq \prod_{k=1}^{n+1} N(\theta_{-k}(\omega)),$$

by construction the set of centers $U^{n+1}(\omega) \subset \varphi\big(n+1, \theta_{-(n+1)}(\omega), \mathcal{V}(\theta_{-(n+1)}(\omega))\big)$, and

$$\varphi\big(n+1, \theta_{-(n+1)}(\omega), \mathcal{V}(\theta_{-(n+1)}(\omega))\big) \qquad \subset \bigcup_{u\in U^{n+1}(\omega)} B^V_{\beta^{n+1}R(\theta_{-(n+1)}(\omega))}(u),$$

which concludes the proof of the properties (U_1)-(U_3).

Construction of measurable sets of centres. Let $n \in \mathbb{N}$ and $\delta(\omega) = \beta^n R(\theta_{-n}(\omega))$. We recall that $\mathcal{V}(\omega) = \bigcup_{j\in\mathbb{N}} v_j(\omega)$ is a measurable selection for $B(\omega)$, the sets $U^n(\omega) \subset \varphi\big(n, \theta_{-n}(\omega), \mathcal{V}(\theta_{-n}(\omega))\big)$ and

$$\varphi\big(n, \theta_{-n}(\omega), \mathcal{V}(\theta_{-n}(\omega))\big) \subset \bigcup_{u\in U^n(\omega)} B^V_{\delta(\omega)}(u).$$

For an l-tuple $\mathbf{k} = (k_1, \dots, k_l) \in \mathbb{N}^l$, we define the random variable $G_{\mathbf{k}} : \Omega \to \{0, 1\}$,

$$G_{\mathbf{k}}(\omega) = \begin{cases} 1 & \text{if } \varphi\big(n, \theta_{-n}(\omega), \mathcal{V}(\theta_{-n}(\omega))\big) \subset \bigcup_{j=1}^{l} B^V_{\delta(\omega)}\big(\varphi\big(n, \theta_{-n}(\omega), v_{k_j}(\theta_{-n}(\omega))\big)\big), \\ 0 & \text{otherwise,} \end{cases}$$

and denote by $\Omega_l \subset \Omega$ those $\omega \in \Omega$ for which there exists an l-tuple $\mathbf{k}$ such that $G_{\mathbf{k}}(\omega) = 1$ and $G_{\tilde{\mathbf{k}}}(\omega) = 0$ for any $\tilde{\mathbf{k}}$ containing less than l elements. The sets Ω_l are intersections of measurable sets,

$$\Omega_l = \bigcap_{\sharp\mathbf{k}=l-1} \{\omega \in \Omega : G_{\mathbf{k}}(\omega) = 0\} \cap \bigcup_{\sharp\mathbf{k}=l} \{\omega \in \Omega : G_{\mathbf{k}}(\omega) = 1\},$$

and hence, are measurable. Since $\Omega = \bigcup_{l\in\mathbb{N}} \Omega_l$, it suffices to construct the set of measurable centres $\widetilde{U}^n$ on each subset Ω_l. For $l \in \mathbb{N}$ let $\sigma : \mathbb{N} \to \mathbb{N}^l$, $\sigma(i) = \mathbf{k} = (k_1, \dots, k_l)$ be an indexing of all l-tuples in $\mathbb{N}^l$. We define random variables $F_i : \Omega_l \to \{0, 1\}$ such that $F_i(\omega) = 1$, if

$$i = \min_{\tilde{i}\in\mathbb{N}} \Big\{ \varphi\big(n, \theta_{-n}(\omega), \mathcal{V}(\theta_{-n}(\omega))\big) \subset \bigcup_{j\in\sigma(\tilde{i})(\omega)} B^V_{\delta(\omega)}\big(\varphi\big(n, \theta_{-n}(\omega), v_j(\theta_{-n}(\omega))\big)\big) \Big\},$$

and $F_i(\omega) = 0$, otherwise. Finally, we set

$$\widehat{U}^n(\omega) = \big\{ v_j(\theta_{-n}(\omega)) : F_i(\omega) = 1,\ j \in \sigma(i)(\omega) \big\},$$

and $\widetilde{U}^n(\omega) = \varphi\big(n, \theta_{-n}(\omega), \widehat{U}^n(\omega)\big)$. The set $\widehat{U}^n$ is a finite random set, since

$$d(v, \widehat{U}^n) = \min \big\{ \|v - v_j(\theta_{-n}(\omega))\|_V : F_i(\omega) = 1,\ j \in \sigma(i)(\omega) \big\}$$

for all $v \in V$, which by the continuity and measurability of the cocycle φ implies that also $\widetilde{U}^n$ is a finite random set. Constructing for all $l \in \mathbb{N}$ the sets $\widetilde{U}^n$ on Ω_l we obtain the random finite set $\widetilde{U}^n(\omega), \omega \in \Omega$.

If necessary, we now replace for all $n \in \mathbb{N}$ and $\omega \in \Omega$ the sets $U^n(\omega)$ by the sets $\widetilde{U}^n(\omega)$, and obtain a family of random finite sets $\widetilde{U}^n, n \in \mathbb{N}$, which by construction, satisfies the properties (U_1)–(U_3).

Definition of the random pullback exponential attractor. We define $E^0(\omega) = \widetilde{U}^0(\omega)$, for all $\omega \in \Omega$, and set

$$E^n(\omega) = \widetilde{U}^n(\omega) \cup \varphi\big(1, \theta_{-1}(\omega), E^{n-1}(\theta_{-1}(\omega))\big), \qquad n \in \mathbb{N}.$$

Then, the family $E^n(\omega)$, $n \in \mathbb{N}_0$, satisfies

$$(E_1)\qquad \varphi\big(1,\theta_{-1}(\omega),E^n(\theta_{-1}(\omega))\big)\subset E^{n+1}(\omega),$$
$$E^n(\omega)\subset\varphi\big(n,\theta_{-n}(\omega),B(\theta_{-n}(\omega))\big)\subset B(\omega),$$
$$(E_2)\qquad E^n(\omega)=\bigcup_{k=0}^{n}\varphi\big(k,\theta_{-k}(\omega),\widetilde{U}^{n-k}(\theta_{-k}(\omega))\big),$$
$$\sharp E^n(\omega)\le\sum_{k=0}^{n}\prod_{l=1}^{k}N(\theta_{-n-l+k}(\omega)),$$
$$(E_3)\qquad \varphi\big(n,\theta_{-n}(\omega),B(\theta_{-n}(\omega))\big)\subset\bigcup_{u\in\widetilde{U}^n(\omega)}\overline{B^V_{\beta^n R(\theta_{-n}(\omega))}(u)}^V.$$

These relations can be proved by induction, and are immediate consequences of the definition of the sets $E^n(\omega)$, the properties (U_1)–(U_3) of the family $\widetilde{U}^n(\omega), n\in\mathbb{N}_0$, and property (4). Using the sets $E^n(\omega)$ we define $\widetilde{\mathcal{M}}^\nu(\omega)=\bigcup_{n\in\mathbb{N}_0}E^n(\omega)$ and show that its closure

$$\mathcal{M}^\nu=\{\mathcal{M}^\nu(\omega):\omega\in\Omega\}=\big\{\overline{\widetilde{\mathcal{M}}^\nu(\omega)}^V:\omega\in\Omega\big\}$$

is a random pullback exponential attractor for (θ,φ).

Positive φ-invariance. The set $\widetilde{\mathcal{M}}^\nu$ is positively φ-invariant: Indeed, for all $k\in\mathbb{N}$ and $\omega\in\Omega$ property (E_1) implies that

$$\varphi\big(k,\theta_{-k}(\omega),\widetilde{\mathcal{M}}^\nu(\theta_{-k}(\omega))\big)=\bigcup_{n\in\mathbb{N}_0}\varphi\big(k,\theta_{-k}(\omega),E^n(\theta_{-k}(\omega))\big)$$
$$\subset\bigcup_{n\in\mathbb{N}_0}E^{n+k}(\omega)\subset\bigcup_{n\in\mathbb{N}_0}E^n(\omega)=\widetilde{\mathcal{M}}^\nu(\omega).$$

Since φ is continuous, it follows the positive φ-invariance of $\mathcal{M}^\nu$,

$$\varphi\big(k,\theta_{-k}(\omega),\mathcal{M}^\nu(\theta_{-k}(\omega))\big)=\varphi\big(k,\theta_{-k}(\omega),\overline{\widetilde{\mathcal{M}}^\nu(\theta_{-k}(\omega))}^V\big)$$
$$\subset\overline{\varphi\big(k,\theta_{-k}(\omega),\widetilde{\mathcal{M}}^\nu(\theta_{-k}(\omega))\big)}^V\subset\overline{\widetilde{\mathcal{M}}^\nu(\omega)}^V=\mathcal{M}^\nu(\omega),$$

for all $k\in\mathbb{N},\ \omega\in\Omega$.

Compactness and finite fractal dimension. We first prove that the sections $\widetilde{\mathcal{M}}^\nu(\omega)$ are precompact and of finite fractal dimension in V. For any $m\in\mathbb{N}$ and $n\ge m$ the cocycle property implies that

$$\begin{aligned}E^n(\omega)&\subset\varphi\big(n,\theta_{-n}(\omega),B(\theta_{-n}(\omega))\big)\\&=\varphi\Big(m,\theta_{-m}(\omega),\varphi\big(n-m,\theta_{-(n-m)}\circ\theta_{-m}(\omega),B(\theta_{-(n-m)}\circ\theta_{-m}(\omega))\big)\Big)\\&\subset\varphi\big(m,\theta_{-m}(\omega),B(\theta_{-m}(\omega))\big),\end{aligned}$$

where property (4) was used in the last inclusion. Consequently, for all $m\in\mathbb{N}$ we obtain

$$\widetilde{\mathcal{M}}^\nu(\omega)=\bigcup_{n=0}^{m}E^n(\omega)\cup\bigcup_{n=m+1}^{\infty}E^n(\omega)\subset\bigcup_{n=0}^{m}E^n(\omega)\cup\varphi\big(m,\theta_{-m}(\omega),B(\theta_{-m}(\omega))\big).$$

Let $\varepsilon_m>0, m\in\mathbb{N}$, be a sequence converging to 0 as $m\to\infty$. Since B is tempered and $\beta\in(0,1)$, there exists a subsequence $m_j, j\in\mathbb{N}$, such that $m_j\to\infty$ as $j\to\infty$

and

$$\beta^{m_j} R(\theta_{-m_j}(\omega)) < \varepsilon_m \leq \beta^{m_j-1} R(\theta_{-m_j+1}(\omega))$$

holds. Hence, it follows that

$$\varphi\big(m_j, \theta_{-m_j}(\omega), B(\theta_{-m_j}(\omega))\big) \subset \bigcup_{u\in \widetilde{U}^{m_j}(\omega)} B^V_\varepsilon(u),$$

and we can estimate the number of ε_m-balls in V needed to cover $\widetilde{\mathcal{M}}(\omega)$ by

$$\begin{aligned}
N^V_{\varepsilon_m}(\widetilde{\mathcal{M}}^\nu(\omega)) &\leq \sharp\Big(\bigcup_{n=0}^{m_j} E^n(\omega)\Big) + \sharp\widetilde{U}^{m_j}(\omega)\\
&\leq \Big(\sum_{n=0}^{m_j}\sum_{k=0}^{n}\prod_{l=1}^{k} N(\theta_{-n-l+k}(\omega))\Big) + \prod_{k=1}^{m_j} N(\theta_{-k}(\omega))\\
&\leq \Big(\sum_{n=0}^{m_j}(n+1)\prod_{l=1}^{n} N(\theta_{-l}(\omega))\Big) + \prod_{k=1}^{m_j} N(\theta_{-k}(\omega))\\
&\leq (m_j+1)^2\prod_{l=1}^{m_j} N(\theta_{-l}(\omega)) + \prod_{k=1}^{m_j} N(\theta_{-k}(\omega))\\
&\leq 2(m_j+1)^2\prod_{k=1}^{m_j} N(\theta_{-k}(\omega)),
\end{aligned}$$

where we used properties (U_2) and (E_2). This proves the precompactness of $\widetilde{\mathcal{M}}^\nu(\omega)$ in V, and taking the closure $\mathcal{M}^\nu(\omega) = \overline{\widetilde{\mathcal{M}}^\nu(\omega)}^{\|\cdot\|_V}, \omega \in \Omega$, we obtain compact subsets in V.

For the fractal dimension of $\widetilde{\mathcal{M}}^\nu(\omega)$ we obtain

$$\begin{aligned}
\dim^V_{\mathrm{f}}(\widetilde{\mathcal{M}}^\nu(\omega)) &= \limsup_{m\to\infty} \frac{\ln\big(N^V_{\varepsilon_m}(\widetilde{\mathcal{M}}^\nu(\omega))\big)}{-\ln\varepsilon_m}\\
&\leq \limsup_{j\to\infty} \frac{\ln(2) + 2\ln(m_j+1) + \sum_{k=1}^{m_j}\ln\big(N(\theta_{-k}(\omega))\big)}{-\ln(\beta^{m_j-1}R(\theta_{-m_j+1}(\omega)))}\\
&= \limsup_{j\to\infty} \frac{\sum_{k=1}^{m_j}\ln(N(\theta_{-k}(\omega)))}{-\ln\big(\beta^{m_j-1}R(\theta_{-m_j+1}(\omega))\big)}.
\end{aligned}$$

Let now $\alpha \in (\beta, 1)$ be arbitrary and $\delta = \ln(\frac{\alpha}{\beta})$. Since B is tempered there exists $n_0 \in \mathbb{N}$ such that

$$\beta^n R(\theta_{-n}(\omega)) < \alpha^n \qquad \forall n \geq n_0.$$

Consequently, it follows that

$$\begin{aligned}
\dim^V_{\mathrm{f}}(\widetilde{\mathcal{M}}^\nu(\omega)) &\leq \limsup_{j\to\infty} \frac{\sum_{k=1}^{m_j}\ln(N(\theta_{-k}(\omega)))}{-\ln\big(\alpha^{m_j-1}\big)}\\
&= \frac{1}{-\log_2(\alpha)}\limsup_{j\to\infty}\frac{1}{(m_j-1)}\sum_{k=1}^{m_j} c\Big(\frac{\kappa(\theta_{-k}(\omega))}{\nu}\Big)^\gamma = \frac{d(\omega)}{-\log_2(\alpha)},
\end{aligned}$$

where we used the growth of the ε-entropy $\mathcal{H}_\varepsilon(V;U)$ in (H_0) and assumption (H_2). This estimates holds for all $\alpha \in (\beta, 1)$, which implies the bound stated in the

Theorem. Finally, since

$$\dim_{\mathrm{f}}^V(\mathcal{M}^\nu(\omega)) = \dim_{\mathrm{f}}^V(\overline{\widetilde{\mathcal{M}}^\nu(\omega)}^V) = \dim_{\mathrm{f}}^V(\widetilde{\mathcal{M}}^\nu(\omega)),$$

the fractal dimension of the sections $\mathcal{M}^\nu(\omega)$ is bounded by the same value.

Pullback exponential attraction. It remains to show that $\mathcal{M}^\nu$ pullback attracts all tempered random sets at an exponential rate. By assumption (H_1) for any $D \in \mathcal{D}$ and $\omega \in \Omega$ there exists $N_{D,\omega} \in \mathbb{N}$ such that

$$\varphi\big(m, \theta_{-(m+k)}(\omega), D(\theta_{-(m+k)}(\omega))\big) \subset B(\theta_{-k}\omega)$$

for all $m \geq N_{D,\omega}$ and $k \in \mathbb{N}$. If $n \geq N_{D,\omega} + 1$, i.e. $n = N_{D,\omega} + n_0$ for some $n_0 \in \mathbb{N}$, then

$$\begin{aligned}
&\mathrm{dist}_{\mathrm{H}}^V\Big(\varphi\big(n, \theta_{-n}(\omega), D(\theta_{-n}(\omega))\big), \widetilde{\mathcal{M}}^\nu(\omega)\Big)\\
\leq&\mathrm{dist}_{\mathrm{H}}^V\Big(\varphi\big(n_0, \theta_{-n_0}(\omega), \varphi(N_{D,\omega}, \theta_{-(N_{D,\omega}+n_0)}(\omega), D(\theta_{-(N_{D,\omega}+n_0)}(\omega)))\big), \bigcup_{n=0}^{\infty} E^n(\omega)\Big)\\
\leq&\mathrm{dist}_{\mathrm{H}}^V\Big(\varphi\big(n_0, \theta_{-n_0}(\omega), B(\theta_{-n_0}(\omega))\big), \bigcup_{n=0}^{\infty} E^n(\omega)\Big)\\
\leq&\mathrm{dist}_{\mathrm{H}}^V\Big(\varphi\big(n_0, \theta_{-n_0}(\omega), B(\theta_{-n_0}(\omega))\big), E^{n_0}(\omega)\Big)\\
\leq&(2(\nu+\lambda))^{n_0} R(\theta_{-n_0}(\omega)) \leq Ce^{-\alpha n},
\end{aligned}$$

for some constants $C \geq 0$ and $\alpha > 0$, where we used that B is tempered in the last inequality. Finally, the sections $\mathcal{M}^\nu(\omega) = \overline{\widetilde{\mathcal{M}}^\nu(\omega)}^V$ are certainly pullback $\mathcal{D}$-attracting at an exponential rate, since

$$\begin{aligned}
&\mathrm{dist}_{\mathrm{H}}^V\Big(\varphi\big(n, \theta_{-n}(\omega), D(\theta_{-n}(\omega))\big), \mathcal{M}^\nu(\omega)\Big)\\
=&\mathrm{dist}_{\mathrm{H}}^V\Big(\varphi\big(n, \theta_{-n}(\omega), D(\theta_{-n}(\omega))\big), \overline{\widetilde{\mathcal{M}}^\nu(\omega)}^V\Big)\\
\leq&\mathrm{dist}_{\mathrm{H}}^V\Big(\varphi\big(n, \theta_{-n}(\omega), D(\theta_{-n}(\omega))\big), \widetilde{\mathcal{M}}^\nu(\omega)\Big).
\end{aligned}$$

Measurability. By Proposition 1.3.1 in [4] the pullback exponential attractor $\mathcal{M}^\nu = \overline{\widetilde{\mathcal{M}}^\nu}^V$ is a random set if and only if $\widetilde{\mathcal{M}}^\nu$ is a random set. Moreover, $\widetilde{\mathcal{M}}^\nu$ is the countable union of the sets E^n, $n \in \mathbb{N}$, and hence, it suffices to show that each set E^n is a random set. However, E^n is the union of $\widetilde{U}^n$ and images of the sets $\widetilde{U}^{n_0}$, $n_0 < n$, under the continuous and measurable cocycle φ. Since $\widetilde{U}^n = \{\widetilde{U}^n(\omega), \omega \in \Omega\}$ is a finite random set for all $n \in \mathbb{N}$, it follows the measurability of the sets E^n, $n \in \mathbb{N}$, and therefore the measurability of $\mathcal{M}^\nu$.

This shows that $\mathcal{M}^\nu$ is a random pullback exponential attractor for the discrete random dynamical system (θ, φ) in V. □

3.2. **Continuous time random dynamical systems.** We now consider the continuous time setting, i.e. $\mathbb{T} = \mathbb{R}$. If (θ, φ) is a continuous time random dynamical system satisfying the hypotheses (H_0)–(H_3), we can construct as in the previous subsection a random set satisfying all the properties of a random pullback exponential attractor, except for the positive φ-invariance. To obtain a positively φ-invariant attractor requires additional assumptions, namely, the Hölder continuity in time of the cocycle.

(H_4) The cocycle φ is Hölder continuous in time within B, in particular, there exist constants $\delta_\omega \in (0,1]$ and $K_\omega > 0$ such that

$$\mathrm{dist}^V_{\mathrm{H,\ symm}}\Big(\varphi\big(s,\theta_{-s}(\omega),E^n(\theta_{-s}(\omega))\big),\varphi\big(t,\theta_{-t}(\omega),E^n(\theta_{-t}(\omega))\big)\Big) \le K_\omega |t-s|^{\delta_\omega},$$

for all $s,t\in[0,\tilde{t}]$ and $n\in\mathbb{N}_0$, where $E^n(\omega)$ are the sets of centers constructed in the proof of Theorem 2 and $\mathrm{dist}^V_{\mathrm{H,\ symm}}(\cdot,\cdot)$ denotes the symmetric Hausdorff distance in V.

Theorem 3. *Let (θ,φ) be a continuous time random dynamical system on V, and the assumptions (H_0)–(H_4) be satisfied. Then, for any $\nu\in(0,\frac{1}{2}-\lambda)$ there exists a random pullback exponential attractor $\mathcal{M}^\nu$ for (θ,φ) in V, and the fractal dimension of its sections is bounded by*

$$\dim^V_{\mathrm{f}}(\mathcal{M}^\nu(\omega)) \le \frac{d(\omega)}{-\log_2(2(\nu+\lambda))} + \frac{1}{\delta_\omega} \qquad \forall \omega\in\Omega.$$

Proof. Let $(\widetilde{\theta},\widetilde{\varphi})$ be the discrete random dynamical system defined by $\widetilde{\theta}_n=\theta_{n\tilde{t}}$, $n\in\mathbb{Z}$, and $\widetilde{\varphi}(n,\omega,v)=\varphi(n\tilde{t},\omega,v)$, $n\in\mathbb{N}_0$, $\omega\in\Omega, v\in V$. Then, $(\widetilde{\theta},\widetilde{\varphi})$ satisfies the hypotheses of Theorem 2, which implies the existence of a random pullback exponential attractor $\mathcal{M}^\nu_d$ for $(\widetilde{\theta},\widetilde{\varphi})$, where $\mathcal{M}^\nu_d(\omega)=\overline{\widetilde{\mathcal{M}}^\nu_d(\omega)}^V$, $\omega\in\Omega$, and the sets $\widetilde{\mathcal{M}}^\nu_d(\omega)=\bigcup_{n\in\mathbb{N}_0}E^n(\omega)$ are as constructed in the proof of Theorem 2. To obtain a random pullback exponential attractor for the continuous time dynamical system (θ,φ) we set

$$\begin{aligned}\widetilde{\mathcal{M}}^\nu(\omega) &= \bigcup_{t\in[0,\tilde{t}]}\varphi\big(t,\theta_{-t}(\omega),\widetilde{\mathcal{M}}^\nu_d(\theta_{-t}(\omega))\big),\\ \mathcal{M}^\nu(\omega) &= \overline{\widetilde{\mathcal{M}}^\nu(\omega)}^V.\end{aligned}$$

Positive φ-invariance. Let $\tau\in[0,\tilde{t}]$ and $t\ge 0$. Then, $t+\tau=k\tilde{t}+s$ for some $k\in\mathbb{N}_0$ and $s\in[0,\tilde{t}]$. By the cocycle property and the positive φ-invariance of the discrete attractor $\widetilde{\mathcal{M}}^\nu_d$ we obtain

$$\begin{aligned}&\varphi\Big(t,\omega,\varphi\big(\tau,\theta_{-\tau}(\omega),\widetilde{\mathcal{M}}^\nu_d(\theta_{-\tau}(\omega))\big)\Big) = \varphi\Big(t+\tau,\theta_{-\tau}(\omega),\widetilde{\mathcal{M}}^\nu_d(\theta_{-\tau}(\omega))\Big)\\ &= \varphi\Big(k\tilde{t}+s,\theta_{-\tau}(\omega),\widetilde{\mathcal{M}}^\nu_d(\theta_{-\tau}(\omega))\Big) = \varphi\Big(s,\theta_{k\tilde{t}-\tau}(\omega),\varphi\big(k\tilde{t},\theta_{-\tau}(\omega),\widetilde{\mathcal{M}}^\nu_d(\theta_{-\tau}(\omega))\big)\Big)\\ &\subset \varphi\Big(s,\theta_{k\tilde{t}-\tau}(\omega),\widetilde{\mathcal{M}}^\nu_d(\theta_{k\tilde{t}-\tau}(\omega))\Big) = \varphi\Big(s,\theta_{-s}\circ\theta_t(\omega),\widetilde{\mathcal{M}}^\nu_d(\theta_{-s}\circ\theta_t(\omega)\Big)\\ &\subset \bigcup_{\tau\in[0,\tilde{t}]}\varphi\Big(\tau,\theta_{-\tau}\circ\theta_t(\omega),\widetilde{\mathcal{M}}^\nu_d\big(\theta_{-t}\circ\theta_t(\omega)\big)\Big) = \widetilde{\mathcal{M}}^\nu(\theta_t(\omega)).\end{aligned}$$

Since $\tau\in[0,\tilde{t}]$ was arbitrary and φ is continuous, it follows that $\mathcal{M}^\nu$ is positively φ-invariant.

Compactness and finite fractal dimension. First, we observe that

$$\widetilde{\mathcal{M}}^\nu(\omega) = \bigcup_{t\in[0,\tilde t]} \varphi\big(t,\theta_{-t}(\omega),\widetilde{\mathcal{M}}^\nu_d(\theta_{-t}(\omega))\big) = \bigcup_{t\in[0,\tilde t]} \varphi\Big(t,\theta_{-t}(\omega),\bigcup_{n\in\mathbb{N}_0} E^n(\theta_{-t}(\omega))\Big)$$
$$= \bigcup_{n\in\mathbb{N}_0}\bigcup_{t\in[0,\tilde t]} \varphi\Big(t,\theta_{-t}(\omega),E^n(\theta_{-t}(\omega))\Big).$$

Moreover, by (H_1) and the cocycle property we have

$$\varphi\big(t,\theta_{-t}(\omega),E^n(\theta_{-t}(\omega))\big) \subset \varphi\Big(t,\theta_{-t}(\omega),\varphi(n\tilde t,\theta_{-n\tilde t-t}(\omega),B(\theta_{-n\tilde t-t}(\omega))\Big)$$
$$=\varphi\big(n\tilde t+t,\theta_{-t-n\tilde t}(\omega),B(\theta_{-n\tilde t-t}(\omega))\big)$$
$$=\varphi\Big(m\tilde t,\theta_{-m\tilde t}(\omega),\varphi\big(t+(n-m)\tilde t,\theta_{-(n-m)\tilde t-t}\circ\theta_{-m\tilde t}(\omega),B(\theta_{-(n-m)\tilde t-t}\circ\theta_{-m\tilde t}(\omega))\big)\Big)$$
$$\subset\varphi\big(m\tilde t,\theta_{-m\tilde t}(\omega),B(\theta_{-m\tilde t}(\omega))\big)$$

for all $n>m$ and $t\in[0,\tilde t]$. Let $\varepsilon_m>0, m\in\mathbb{N}$, be a sequence converging to 0 as $m\to\infty$. Since B is tempered and $\beta\in(0,1)$, there exists a subsequence $m_j, j\in\mathbb{N}$, such that $m_j\to\infty$ as $j\to\infty$ and

$$\beta^{m_j}R(\theta_{-m_j}(\omega)) < \varepsilon_m \le \beta^{m_j-1}R(\theta_{-m_j+1}(\omega))$$

holds. Hence, for all $n>m_j$ it follows that

$$\varphi\big(t,\theta_{-t}(\omega),E^n(\theta_{-t}(\omega))\big)\subset\varphi\big(m_j,\theta_{-m_j}(\omega),B(\theta_{-m_j}(\omega))\big)\subset\bigcup_{u\in\widetilde U^{m_j}(\omega)} B^V_{\varepsilon_m}(u),$$

and we can estimate the number of ε_m-balls in V needed to cover $\widetilde{\mathcal{M}}(\omega)$ by

$$N^V_{\varepsilon_m}(\widetilde{\mathcal{M}}^\nu(\omega)) \le \sharp\Big(\bigcup_{n=0}^{m_j}\bigcup_{t\in[0,\tilde t]}\varphi\big(t,\theta_{-t}(\omega),E^n(\theta_{-t}(\omega))\big)\Big)+\sharp\widetilde U^{m_j}(\omega)$$
$$\le \sharp\Big(\bigcup_{t\in[0,\tilde t]}\varphi\big(t,\theta_{-t}(\omega),\bigcup_{n=0}^{m_j}E^n(\theta_{-t}(\omega))\big)\Big)+\prod_{k=1}^{m_j}N(\theta_{-k}(\omega)).$$

As in the proof of Theorem 2 it follows that

$$\sharp\Big(\bigcup_{n=0}^{m_j}E^n(\theta_{-t}(\omega))\Big)\le(m_j+1)^2\prod_{k=1}^{m_j}N(\theta_{-k}(\theta_{-t}(\omega))).$$

Let $n\in\{0,\dots,m_j\}$. We now construct a covering of

$$\bigcup_{t\in[0,\tilde t]}\varphi\big(t,\theta_{-t}(\omega),E^n(\theta_{-t}(\omega))\big).$$

We subdivide the interval $[0,\tilde t]$ in at most $p_\omega=\Big\lfloor\tilde t\left(\frac{K_\omega}{\varepsilon_m}\right)^{\frac{1}{\delta_\omega}}\Big\rfloor+1$ intervals I_i of length $\left(\frac{\varepsilon_m}{K_\omega}\right)^{\frac{1}{\delta_\omega}}$, where $\lfloor x\rfloor$ denotes the largest integer less than or equal to x. For each $i=1,\dots,p_\omega$ let $s_i\in I_i$ be an arbitrary point in the subinterval. The hypothesis (H_4) then implies that

$$\mathrm{dist}^V_{\mathrm{H,symm}}\Big(\varphi\big(t,\theta_{-t}(\omega),E^n(\theta_{-t}(\omega))\big),\varphi\big(s_i,\theta_{-s_i}(\omega),E^n(\theta_{-s_i}(\omega))\big)\Big)$$
$$\le K_\omega|t-s_i|^{\delta_\omega}<\varepsilon_m$$

for all $t \in I_i$, and hence,

$$\bigcup_{t\in[0,\tilde t]} \varphi\big(t,\theta_{-t}(\omega),E^n(\theta_{-t}(\omega))\big) \subset \bigcup_{i=1}^{p_\omega} \bigcup_{u\in E^n(\theta_{-s_i}(\omega))} B^V_{\varepsilon_m}\big(\varphi(s_i,\theta_{-s_i}(\omega),u)\big).$$

Constructing for every $n \in \{0,\dots,m_j\}$ such a covering we conclude that

$$\begin{aligned} N^V_{\varepsilon_m}(\widetilde{\mathcal{M}}^\nu(\omega)) &\le p_\omega(m_j+1)^2 \prod_{k=1}^{m_j} N(\theta_{-k}(\omega)) + \prod_{k=1}^{m_j} N(\theta_{-k}(\omega)) \\ &\le 2p_\omega(m_j+1)^2 \prod_{k=1}^{m_j} N(\theta_{-k}(\omega)), \end{aligned}$$

which proves the precompactness of $\widetilde{\mathcal{M}}^\nu$. For the fractal dimension of $\widetilde{\mathcal{M}}^\nu(\omega)$ we obtain, similarly as in the proof of Theorem 2,

$$\begin{aligned} &\dim^V_{\mathrm{f}}(\widetilde{\mathcal{M}}^\nu(\omega)) = \limsup_{m\to\infty} \frac{\ln\big(N^V_{\varepsilon_m}(\widetilde{\mathcal{M}}^\nu(\omega))\big)}{-\ln\varepsilon_m} \\ &\le \limsup_{j\to\infty} \frac{\ln(2)+2\ln(m_j+1)+\sum_{k=1}^{m_j}\ln\big(N(\theta_{-k}(\omega))\big)+\ln(p_\omega)}{-\ln(\beta^{m_j-1}R(\theta_{-m_j+1}(\omega)))} \\ &\le \frac{d(\omega)}{-\log_2(2(\nu+\lambda))} + \limsup_{j\to\infty} \frac{\ln\Big(\tilde t\big(\frac{K_\omega}{\beta^{m_j}R(\theta_{-m_j}(\omega))}\big)^{\frac{1}{\delta_\omega}}+1\Big)}{-\ln(\beta^{m_j-1}R(\theta_{-m_j+1}(\omega)))} \\ &\le \frac{d(\omega)}{-\log_2(2(\nu+\lambda))} + \frac{1}{\delta_\omega}. \end{aligned}$$

Finally, since

$$\dim^V_{\mathrm{f}}(\mathcal{M}^\nu(\omega)) = \dim^V_{\mathrm{f}}(\overline{\widetilde{\mathcal{M}}^\nu(\omega)}^V) = \dim^V_{\mathrm{f}}(\widetilde{\mathcal{M}}^\nu(\omega)),$$

the fractal dimension of the sections $\mathcal{M}^\nu(\omega)$ is bounded by the same value.

Exponential attraction. Since $\widetilde{\mathcal{M}}^\nu_d \subset \widetilde{\mathcal{M}}^\nu$, and the discrete attractor $\widetilde{\mathcal{M}}^\nu_d$ is pullback exponentially attracting all tempered sets, the property of pullback exponential attraction for $\widetilde{\mathcal{M}}^\nu$ and hence, for $\mathcal{M}^\nu$ follows immediately.

Measurability. By Proposition 1.3.1 in [4] the pullback exponential attractor $\mathcal{M}^\nu = \overline{\widetilde{\mathcal{M}}^\nu}^V$ is a random set if and only if $\widetilde{\mathcal{M}}^\nu$ is a random set. Moreover, since $\widetilde{\mathcal{M}}^\nu(\omega)$ is the countable union of the sets

$$M^\nu_n(\omega) = \bigcup_{t\in[0,\tilde t]} \varphi\big(t,\theta_{-t}(\omega),E^n(\theta_{-t}(\omega))\big), \qquad n \in \mathbb{N}_0,$$

it suffices to show that M^ν_n is a random set for every $n \in \mathbb{N}_0$. To this end let $n \in \mathbb{N}_0$, the set $\widehat{M}^\nu_n \subset \Omega \times \big([0,\tilde t] \times V\big)$ be defined by

$$\widehat{M}^\nu_n(\omega) = \big\{(\tau,v) \in [0,\tilde t] \times V : v \in E^n(\theta_{-\tau}(\omega))\big\},$$

and the mapping $\chi_\omega : [0,\tilde t] \times V \to V$ be defined by

$$(\tau,v) \mapsto \varphi\big(\tau,\theta_{-\tau}(\omega),v\big).$$

First, we observe that $\widehat{M}_n^\nu$ is a compact random set, since for every $(s,u) \in [0,\tilde{t}] \times V$ the mapping

$$\omega \mapsto \inf_{(\tau,v)\in\widehat{M}_n^\nu(\omega)} \big\{|s-\tau| + \|u-v\|_V\big\} = \inf_{\tau\in\mathbb{Q}\cap[0,\tilde{t}]} \Big\{ \inf_{v\in E^n(\theta_{-\tau}(\omega))} \big\{\|u-v\|_V\big\}\Big\},$$

is measurable. Moreover, for every fixed $\omega \in \Omega$ the mapping $\chi_\omega : [0,\tilde{t}] \times V \to V$ is continuous, and for every $(\tau, v) \in [0,\tilde{t}] \times V$ the mapping

$$\omega \mapsto \chi_\omega(\tau, v) = \varphi\big(\tau, \theta_{-\tau}(\omega), v\big)$$

is measurable. Since $M_n^\nu(\omega) = \chi_\omega(\widehat{M}_n^\nu(\omega))$ the measurability of M_n^ν now follows from Proposition 5.6 in [12], which concludes the proof. ☐

The Hölder continuity in time (H_4) is a limiting assumption in applications. We therefore propose to weaken the invariance property of random pullback exponential attractors, and to consider *non-autonomous* random pullback exponential attractors instead. This allows to apply the construction developed for non-autonomous deterministic systems in [2]. It requires only the Lipschitz continuity in space of the cocycle and not its Hölder continuity in time.

(H_4') The cocycle φ is Lipschitz continuous in B, i.e. for all $s \in]0,\tilde{t}]$ there exists a constant $L_{\omega,s} > 0$ such that

$$\|\varphi(s,\omega,u) - \varphi(s,\omega,v)\|_V \le L_{\omega,s}\|u-v\|_V \qquad \forall u, v \in B(\omega).$$

Definition 5. The non-autonomous random set $\mathcal{M} \subset \mathbb{R} \times \Omega \times V$ is called a *non-autonomous random pullback exponential attractor* for (θ, φ) on V, if there exists $\hat{t} > 0$ such that $\mathcal{M}(t+\hat{t}, \omega) = \mathcal{M}(t,\omega)$ for all $t \in \mathbb{R}$, the sections $\mathcal{M}(t,\omega) \neq 0$ are compact, and $\mathcal{M}$ is positively φ-invariant in the non-autonomous sense, i.e.

$$\varphi(s,\omega,\mathcal{M}(t,\omega)) \subset \mathcal{M}(t+s,\theta_s(\omega)) \qquad \forall s \ge 0, t \in \mathbb{R}.$$

Moreover, there exists a random variable $\kappa(\omega)$ such that

$$\sup_{t\in\mathbb{R}} \dim_f^V(\mathcal{M}(t,\omega)) \le \kappa(\omega) < \infty,$$

and $\mathcal{M}$ is pullback $\mathcal{D}$-attracting at an exponential rate, i.e. there exists $\alpha > 0$ such that

$$\lim_{s\to\infty} e^{\alpha s}\mathrm{dist}_{\mathrm{H}}\left(\varphi\big(s,\theta_{-s}(\omega), D(\theta_{-s}(\omega))\big), \mathcal{M}(t,\omega)\right) = 0 \qquad \forall D \in \mathcal{D}, t \in \mathbb{R}.$$

Remark 5. We emphasize that for every $t \in \mathbb{R}$, the random set $\mathcal{M}(t,\cdot)$ contains the global random attractor and satisfies all properties of a random pullback exponential attractor, except for the positive φ-invariance. Considering positive φ-invariance in the non-autonomous sense instead allows to weaken the invariance property of random pullback exponential attractors in continuous time settings. The construction of the exponential attractor then essentially simplifies, it does not require the Hölder continuity assumption (H_4), and leads to better bounds for the fractal dimension.

Theorem 4. *Let (θ,φ) be a continuous time random dynamical system on V, and the assumptions (H_0)–(H_3), (H_4') be satisfied. Then, for any $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a non-autonomous random pullback exponential attractor $\mathcal{M}^\nu$ for (θ,φ) in V, and the fractal dimension of its sections is bounded by*

$$\sup_{t\in\mathbb{R}} \dim_{\mathrm{f}}^V(\mathcal{M}^\nu(t,\omega)) \le \frac{d(\omega)}{-\log_2(2(\nu+\lambda))} \qquad \forall \omega \in \Omega.$$

Proof. Let $(\widetilde{\theta},\widetilde{\varphi})$ be the discrete random dynamical system defined by $\widetilde{\theta}_n = \theta_{n\tilde{t}}$, $n \in \mathbb{Z}$, and $\widetilde{\varphi}(n,\omega,v) = \varphi(n\widetilde{t},\omega,v)$, $n \in \mathbb{N}_0$, $\omega \in \Omega, v \in V$. Then, $(\widetilde{\theta},\widetilde{\varphi})$ satisfies the hypotheses of Theorem 2, which implies the existence of a random pullback exponential attractor $\mathcal{M}_d^\nu$ for $(\widetilde{\theta},\widetilde{\varphi})$, where $\mathcal{M}_d^\nu(\omega) = \overline{\widetilde{\mathcal{M}}_d^\nu(\omega)}^V$, $\omega \in \Omega$, and the sets $\widetilde{\mathcal{M}}_d^\nu(\omega)$ are as constructed in the proof of Theorem 2. To obtain a non-autonomous random pullback exponential attractor for the continuous time dynamical system (θ,φ) we set

$$
\begin{aligned}
\widetilde{\mathcal{M}}^\nu(t,\omega) &= \varphi\big(t,\theta_{-t}(\omega),\widetilde{\mathcal{M}}_d^\nu(\theta_{-t}(\omega))\big),\\
\mathcal{M}^\nu(t,\omega) &= \overline{\widetilde{\mathcal{M}}^\nu(t,\omega)}^V,
\end{aligned}
$$

for all $0 \le t < \tilde{t}$, and $\mathcal{M}^\nu(k\tilde{t}+t,\omega) = \mathcal{M}^\nu(t,\omega)$ for all $k \in \mathbb{Z}$ and $0 \le t < \tilde{t}$.

Compactness and finite fractal dimension. Due to the continuity of φ, the sections $\widetilde{\mathcal{M}}^\nu(\omega)$, $\omega \in \Omega$, are precompact in V. Furthermore, the Lipschitz-continuity (H_4) implies the following estimate,

$$
\begin{aligned}
&\dim_{\mathrm{f}}^V\big(\mathcal{M}^\nu(t,\omega)\big) = \dim_{\mathrm{f}}^V\big(\widetilde{\mathcal{M}}^\nu(t,\omega)\big)\\
=&\dim_{\mathrm{f}}^V\big(\varphi(t,\theta_{-t}(\omega),\widetilde{\mathcal{M}}_d^\nu(\theta_{-t}(\omega)))\big) \le \dim_{\mathrm{f}}^V\big(\widetilde{\mathcal{M}}_d^\nu(\theta_{-t}(\omega))\big)
\end{aligned}
$$

for $0 \le t < \tilde{t}$. The bound for the fractal dimension now follows from Theorem 2.

Positive φ-invariance. Let $t \ge 0$ and $s \in \mathbb{R}$. Then, $s = l\tilde{t}+s_0$ and $t+s_0 = k\tilde{t}+t_0$ for some $l \in \mathbb{Z}, k \in \mathbb{N}_0$ and $s_0, t_0 \in [0,\tilde{t}[$. The definition of $\widetilde{\mathcal{M}}^\nu(t,\omega)$ and the cocycle property imply that

$$
\begin{aligned}
&\varphi\big(t,\omega,\widetilde{\mathcal{M}}^\nu(s,\omega)\big) = \varphi\big(t,\omega,\widetilde{\mathcal{M}}^\nu(s_0,\omega)\big)\\
=\ &\varphi\big(t,\omega,\varphi(s_0,\theta_{-s_0}(\omega),\widetilde{\mathcal{M}}_d^\nu(\theta_{-s_0}(\omega)))\big) = \varphi\big(t+s_0,\theta_{-s_0}(\omega),\widetilde{\mathcal{M}}_d^\nu(\theta_{-s_0}(\omega))\big)\\
=\ &\varphi\big(k\tilde{t}+t_0,\theta_{-s_0}(\omega),\widetilde{\mathcal{M}}_d^\nu(\theta_{-s_0}(\omega))\big)\\
=\ &\varphi\Big(t_0,\theta_{k\tilde{t}-s_0}(\omega),\varphi\big(k\tilde{t},\theta_{-s_0}(\omega),\widetilde{\mathcal{M}}_d^\nu(\theta_{-s_0}(\omega))\big)\Big)\\
\subset\ &\varphi\Big(t_0,\theta_{k\tilde{t}-s_0}(\omega),\widetilde{\mathcal{M}}_d^\nu(\theta_{k\tilde{t}-s_0}(\omega))\Big) = \varphi\Big(t_0,\theta_{-t_0+t}(\omega),\widetilde{\mathcal{M}}_d^\nu(\theta_{-t_0+t}(\omega))\Big)\\
=\ &\widetilde{\mathcal{M}}^\nu(t_0,\theta_t(\omega)) = \widetilde{\mathcal{M}}^\nu((k+l)\tilde{t}+t_0,\theta_t(\omega)) = \widetilde{\mathcal{M}}^\nu(t+s,\theta_t(\omega)),
\end{aligned}
$$

where we used the positive φ-invariance of the discrete random pullback attractor $\mathcal{M}_d^\nu$. By the continuity of φ now follows that $\mathcal{M}^\nu$ is positively φ-invariant.

Pullback exponential attraction. This is a straightforward consequence of the pullback exponential attracting property of the discrete random attractor $\mathcal{M}_d^\nu$.

Measurability. By Proposition 1.3.1 in [4] the pullback exponential attractor $\mathcal{M}^\nu = \overline{\widetilde{\mathcal{M}}^\nu}^V$ is a random set if and only if $\widetilde{\mathcal{M}}^\nu$ is a random set. By construction, the sections of the sets $\widetilde{\mathcal{M}}^\nu$ are images of the countable random sets $\widetilde{\mathcal{M}}_d^\nu$ under the measurable and continuous cocycle φ, which implies the measurability of $\mathcal{M}^\nu$. □

An immediate consequence of our results is the existence and finite dimensionality of the global random pullback attractor. We remark that even in the continuous time case neither the Hölder continuity (H_4) nor the Lipschitz continuity (H_4') are needed. In fact, (H_0)-(H_3) are sufficient conditions for the existence and finite dimensionality of the global random pullback attractor.

Corollary 1. *Let (θ, φ) be a random dynamical system on V and $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$. If the assumptions (H_0)-(H_3) are satisfied, the global random pullback attractor $\mathcal{A}$ exists, it is contained in the random pullback exponential attractor constructed in Theorem 2, and the fractal dimension of its sections is bounded by*

$$\dim_{\mathrm{f}}^{V}(\mathcal{A}(\omega)) \leq \inf_{\nu \in (0, \frac{1}{2} - \lambda)} \frac{d(\omega)}{-\log_2(2(\nu + \lambda))} \qquad \forall \omega \in \Omega.$$

Proof. For discrete time random dynamical systems it is an immediate consequence of Theorem 2 and the characterization of cocycles possessing a random pullback attractor in Theorem 1.

Let now $\mathbb{T} = \mathbb{R}$ and the discrete random dynamical system $(\widetilde{\theta}, \widetilde{\varphi})$ be defined by $\widetilde{\theta}_n = \theta_{n\tilde{t}}$, $n \in \mathbb{Z}$, and $\widetilde{\varphi}(n, \omega, v) = \varphi(n\widetilde{t}, \omega, v)$, $n \in \mathbb{N}_0$, $\omega \in \Omega, v \in V$. Then, $(\widetilde{\theta}, \widetilde{\varphi})$ satisfies the hypotheses of Theorem 2. Hence, for every $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a random pullback exponential attractor $\mathcal{M}_d^{\nu}$ for $(\widetilde{\theta}, \widetilde{\varphi})$. It remains to show that the compact random set $\mathcal{M}_d^{\nu}$ is pullback $\mathcal{D}$-attracting for (θ, φ), the corollary then follows from Theorem 1. Let $D \in \mathcal{D}$, $\omega \in \Omega$ and $t \geq N_{D,\omega} + 1$, where $N_{D,\omega}$ denotes the absorbing time corresponding to $(\widetilde{\theta}, \widetilde{\varphi})$. Then, $t = (N_{D,\omega} + n_0)\tilde{t} + t_0$ for some $n_0 \in \mathbb{N}$ and $t_0 \in [0, \tilde{t}]$, and

$$\begin{aligned}
&\operatorname{dist}_{\mathrm{H}}^{V}\Big(\varphi\big(t, \theta_{-t}(\omega), D(\theta_{-t}(\omega))\big), \mathcal{M}_d^{\nu}(\omega)\Big) \\
=&\operatorname{dist}_{\mathrm{H}}^{V}\Big(\varphi\big(n_0\tilde{t}, \theta_{-n_0\tilde{t}}(\omega), \varphi(N_{D,\omega}\tilde{t} + t_0, \theta_{-(N_{D,\omega}\tilde{t}+t_0)}(\omega), D(\theta_{-t}(\omega)))\big), \mathcal{M}_d^{\nu}(\omega)\Big) \\
\leq&\operatorname{dist}_{\mathrm{H}}^{V}\Big(\varphi\big(n_0, \theta_{-n_0}(\omega), B(\theta_{-n_0}(\omega))\big), E^{n_0}(\omega)\Big).
\end{aligned}$$

As in the proof of Theorem 2 we conclude, that $\mathcal{M}_d^{\nu}$ pullback attracts every tempered set $D \in \mathcal{D}$ at an exponential rate. □

Finally, we consider the special case $\lambda = 0$ and a slightly modified setting. We formulate here only the results corresponding to Theorem 2, i.e., for discrete time dynamical systems. The statements of Theorem 3, Theorem 4 and Corollary 1 hold accordingly.

In the special case that $\lambda = 0$ Theorem 2 yields the result for random dynamical systems satisfying the smoothing property with respect to the spaces V and U. This situation was considered in [12]. However, the existence of random exponential attractors has been proved under essentially stronger assumptions that are difficult to verify in applications.

Corollary 2. *Let (θ, ϕ) be a discrete time random dynamical system on V and the assumptions (H_0) and (H_2) be satisfied. Moreover, we assume that (H_1) holds with φ replaced by ϕ, where it suffices that the absorbing set is tempered with respect to the norm in U. Then, for any $\nu \in (0, \frac{1}{2})$ there exists a random pullback exponential attractor $\mathcal{M}^{\nu}$ for (θ, ϕ), and the fractal dimension of its sections is bounded by*

$$\dim_{\mathrm{f}}^{V}(\mathcal{M}^{\nu}(\omega)) \leq \frac{d(\omega)}{-\log_2(2\nu)} \qquad \forall \omega \in \Omega.$$

We could also consider random dynamical systems and exponential attractors in the weaker phase space U as analyzed, e.g. in [9] for autonomous deterministic systems. Such attractors are also called *bi-space attractors* or (V, U)-*attractors.*

Theorem 5. *Let (θ, φ) be a a discrete time random dynamical system in U and the assumptions (H_0) and (H_2) be satisfied. Moreover, we assume that (H_1) holds for an absorbing set B that is tempered with respect to the metric in U, and property (H_3) is satisfied with V replaced by U. Then, for any $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a random pullback exponential attractor $\mathcal{M}^\nu$ for (θ, φ) in U, and the fractal dimension of its sections is bounded by*

$$\dim_f^U(\mathcal{M}^\nu(\omega)) \leq \frac{d(\omega)}{-\log_2(2(\nu + \lambda))} \qquad \forall \omega \in \Omega.$$

Proof. The statement follows by slightly modifying the construction of the random pullback exponential attractor in the proof of Theorem 2. □

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Departamento Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080 Sevilla, Spain

E-mail address: `caraball@us.es`

Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität Graz, Heinrichstr. 36, 8010 Graz, Austria

E-mail address: `stefanie.sonner@uni-graz.at`