

DYNAMICS OF SOME STOCHASTIC CHEMOSTAT MODELS WITH MULTIPLICATIVE NOISE

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(Communicated by the associate editor name)

ABSTRACT. In this paper we study two stochastic chemostat models, with and without wall growth, driven by a white noise. Specifically, we analyze the existence and uniqueness of solutions for these models, as well as the existence of the random attractor associated to the random dynamical system generated by the solution. The analysis will be carried out by means of the well-known Ornstein-Uhlenbeck process, that allows us to transform our stochastic chemostat models into random ones.

1. Introduction. Chemostat refers to a laboratory device used for growing microorganisms in a cultured environment and has been regarded as an idealization of nature to study competition modeling in mathematical biology, which is a really important and interesting problem since they can be used to study recombinant problems in genetically altered microorganisms (see e.g. [17, 18]), waste water treatment (see e.g. [13, 25]) and play an important role in theoretical ecology (see e.g. [2, 12, 16, 23, 29, 31, 32, 34]). Derivation and analysis of chemostat models are well documented in [26, 27, 33] and references therein.

Two standard assumptions for simple chemostat models are: 1) the availability of the nutrient and its supply rate are fixed and 2) the tendency of microorganisms to adhere to surfaces is not taken into account (see e.g. [7, 8]). However, these are very strong restrictions as the real world is non-autonomous and stochastic and this justifies the analysis of stochastic chemostat models, with and without wall growth.

Let us first consider the following chemostat model without wall growth

$$\frac{dS}{dt} = (S^0 - S)D - \frac{mSx}{a + S}, \quad (1)$$

$$\frac{dx}{dt} = -Dx + \frac{mSx}{a + S}, \quad (2)$$

2010 *Mathematics Subject Classification.* Primary: 34C11, 34F05; Secondary: 60H10.

Key words and phrases. Chemostat, stochastic differential equations, multiplicative noise, random dynamical systems, random attractors.

Partially supported by FEDER and Ministerio de Economía y Competitividad under grant MTM2015-63723-P and Junta de Andalucía under Proyecto de Excelencia P12-FQM-1492.

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where $S(t)$ and $x(t)$ denote concentrations of the nutrient and the microbial biomass, respectively; S^0 denotes the volumetric dilution rate, a is the half-saturation constant, D is the dilution rate and m is the maximal consumption rate of the nutrient and also the maximal specific growth rate of microorganisms. We notice that all parameters are supposed to be positive and a function Holling type-II is used as functional response of the microorganisms describing how the nutrient is consumed by the species (see e.g. [28] for more details and biological explanations about this model).

Our aim in this paper is to perturb system (1)-(2) by a noisy input such that the perturbed one becomes a more realistic model of a chemostat. Recently, in [4] the authors have analyzed system (1)-(2) by replacing the dilution rate D by $D + \alpha \dot{W}(t)$, where $W(t)$ is a Wiener process. Even though in that paper the existence and uniqueness of solutions, as well as the existence of the corresponding attractor have been stated, biologically the model does not seem completely realistic, since the substrate S in the corresponding stochastic chemostat model can take negative values. We want to overcome this biological inconsistency by considering a different kind of stochastic perturbation.

We would like to emphasize that one may consider several alternatives to model randomness and stochasticity. We will use a technique based in the one carried out by Fudenberg and Harris in [19] or by Foster and Young in [15], in which the first idea was to consider a stochastic perturbation of the payoff function in continuous-time replicator dynamics. In other words, we could write our model as

$$\begin{aligned}\frac{dS(t)}{dt} &= S(t)f_1(S(t), x(t)), \\ \frac{dx(t)}{dt} &= x(t)f_2(S(t), x(t)),\end{aligned}$$

and then we could add some stochastic perturbation $\alpha_i \dot{W}_i$ to the expected payoff $f_i(\cdot, \cdot)$, for $i \in \{1, 2\}$, instead of adding it directly to dS/dt and dx/dt , as follows

$$\begin{aligned}\frac{dS(t)}{dt} &= S(t) \left[f_1(S(t), x(t)) + \alpha_1 \dot{W}_1(t) \right], \\ \frac{dx(t)}{dt} &= x(t) \left[f_2(S(t), x(t)) + \alpha_2 \dot{W}_2(t) \right],\end{aligned}$$

or, equivalently,

$$\begin{aligned}dS(t) &= S(t)f_1(S(t), x(t))dt + \alpha_1 S(t)dW_1(t), \\ dx(t) &= x(t)f_2(S(t), x(t))dt + \alpha_2 x(t)dW_2(t).\end{aligned}$$

In this way, the populations S and x will always remain positive for any realization of the Wiener processes W_i . In fact, as explained in [19], it can be understood as the payoff to play some strategy i subjected to some external perturbations due to, for example, the *weather*.

Moreover, in the paper by Imhof and Walcher (see [22]) the authors justify mathematically that it could be reasonable to consider the following stochastic chemostat model

$$dS = \left[(S^0 - S)D - \frac{mSx}{a + S} \right] dt + \alpha_1 S dW_1(t),$$

$$dx = \left[-Dx + \frac{mSx}{a+S} \right] dt + \alpha_2 x dW_2(t),$$

where W_1 and W_2 are independent Wiener processes. To this end, a discrete Markov chain is considered for some increment δt and the convergence to the solution of the original stochastic equation is proved as δt tends to zero, whenever it exists a unique solution (see [22] for a more detailed explanation).

Motivated by this feature, in this paper we consider a noisy term in each equation (1)-(2) in the same fashion as in [22], which ensures the positivity of both the nutrient and biomass, although does not preserve the washout equilibrium from the deterministic to the stochastic model. More precisely, we consider now the following system, which is understood in the Itô sense

$$dS = \left[(S^0 - S)D - \frac{mSx}{a+S} \right] dt - \alpha S dW(t),$$

$$dx = \left[-Dx + \frac{mSx}{a+S} \right] dt - \alpha x dW(t),$$

where $W(t)$ is a standard Brownian motion, and $\alpha \geq 0$ represents the intensity of noise.

We remark that, in order to make the calculations much more tractable and clear, we consider the same noise in both equations, even though a similar analysis could be developed by using different Brownian motions in each equation. This leads to more complicated technicalities that we prefer to avoid in this first approach.

We would also like to note that there are not special reasons to consider the sign minus (-) in front of the stochastic terms, instead of the positive one used in [22], since the choice does not cause any effect over the behavior of our system.

Now, by using the well-known conversion between Itô and Stratonovich senses we obtain the following stochastic chemostat without wall growth

$$dS = \left[-\bar{D}S - \frac{mSx}{a+S} + S^0 D \right] dt - \alpha S \circ dW(t), \quad (3)$$

$$dx = \left[-\bar{D}x + \frac{mSx}{a+S} \right] dt - \alpha x \circ dW(t), \quad (4)$$

where

$$\bar{D} := D + \frac{\alpha^2}{2}. \quad (5)$$

Before analyzing the previous system, we would like to highlight some significant insights discovered throughout this work. We will only refer to the case without wall growth since similar ones hold for the other case as well.

Concerning the deterministic chemostat model (DCM) given by (1)-(2), Caraballo and Han proved in a recently published book (see [6]) the existence of a unique axial equilibrium $(S^0, 0)$ which is asymptotically stable provided $D > m$, therefore this situation corresponds to the extinction of the microorganism. However, if $D < m$ and $aD/(m - D) < S^0$ the axial equilibrium becomes unstable and a unique positive globally asymptotically stable equilibrium appears inside the positive quadrant, i.e., persistence of the microorganism can be ensured. Notice that, in this case, the global attractor exists and consists of both equilibria and the heteroclinic solutions between them. Otherwise, no more information can be deduced related to the asymptotic behavior of the system.

Regarding the stochastic chemostat model (SCM), we prove in this paper that there exists a unique global random attractor which is given by singleton components $(S^0 D \rho^*(\omega), 0)$ provided $D + \alpha^2/2 > m$ (see Section 3.1 for more details). Otherwise, the unique global random attractor is contained in a segment whose intersection with the axes $S = 0$ and $x = 0$ is reduced to two single points.

In light of the previous facts, observe that when $D < m$ and $aD/(m - D) < S^0$ we can choose α , large enough, such that $D + \alpha^2/2 > m$. This means that persistence of the microorganism holds for (DCM), while for (SCM) we have extinction since the global random attractor becomes the single random point $(S^0 D \rho^*(\omega), 0)$. This fact is closely related to the stabilizing effects that Itô's noise can produce on deterministic systems. However, if we considered a Stratonovich interpretation for our perturbation at the beginning of our study, then we would have obtained D instead of \bar{D} in (3)-(4); in other words, assumption $D + \alpha^2/2 > m$ in (SCM) would become $D > m$, the same that we had for (DCM). Consequently, no stabilizing effect is produced by the noise (see [3, 6, 21] and Remark 3.3 in [24] for a more detailed discussion on this topic). Thus, not only the type of noise but also its mathematical interpretation can provide different results, something that has to be taken into account by the modeler. A reference that could help to make the appropriate choice in a specific application is [30], where the author presents a criterion for determining which interpretation of the noise is the most useful in his work.

Up to now, we have just mentioned the chemostat model without wall growth. Nevertheless, we are also interested in studying the equivalent model with wall growth since it will allow us to work in a more realistic situation and we will also be able to obtain more useful results from the biological point of view. Then, let us now introduce the simplest chemostat model with wall growth

$$\frac{dS}{dt} = D(S^0 - S) - \frac{mS}{a + S}x_1 - \frac{mS}{a + S}x_2 + b\nu x_1, \quad (6)$$

$$\frac{dx_1}{dt} = -(\nu + D)x_1 + c\frac{S}{a + S}x_1 - r_1x_1 + r_2x_2, \quad (7)$$

$$\frac{dx_2}{dt} = -\nu x_2 + c\frac{S}{a + S}x_2 + r_1x_1 - r_2x_2, \quad (8)$$

where $S(t)$, $x_1(t)$ and $x_2(t)$ denote concentrations of the nutrient and the two different microorganisms, respectively; $b \in (0, 1)$ describes the fraction of dead biomass which is recycled, $\nu > 0$ is the collective death rate coefficient, $r_1 > 0$ and $r_2 > 0$ represent the rates at which the species stick on to and shear off from the walls, respectively, and $0 < c \leq m$ is the growth rate coefficient of the consumer species.

By introducing again a white noise in each equation of (6)-(8) and using the conversion between Itô and Stratonovich interpretations, we finally obtain the following stochastic system with wall growth

$$dS = \left[-\bar{D}S + b\nu x_1 - \frac{mS}{a + S}x_1 - \frac{mS}{a + S}x_2 + DS^0 \right] dt - \alpha S \circ dW(t), \quad (9)$$

$$dx_1 = \left[-(\nu + \bar{D} + r_1)x_1 + c\frac{S}{a + S}x_1 + r_2x_2 \right] dt - \alpha x_1 \circ dW(t), \quad (10)$$

$$dx_2 = \left[r_1x_1 - \left(\nu + r_2 + \frac{\alpha^2}{2} \right) x_2 + c\frac{S}{a + S}x_2 \right] dt - \alpha x_2 \circ dW(t). \quad (11)$$

The paper is organized as follows: in Section 2 we recall some basic results on random dynamical systems. Then, in Section 3 we analyze both random chemostat models, with and without wall growth, and we provide some results regarding existence and uniqueness of global solution just like the generation of a random dynamical system and existence of random pullback attractor, describing its internal structure explicitly. Moreover, in Section 4 we use a conjugation result in order to explain how the global attractor behaves in the stochastic model. Finally, we state some numerical simulations to illustrate our study and some final comments in Section 5.

2. Random dynamical systems. Although there are very good references (see e.g. [1]) in the literature which provide a very detailed information about random dynamical systems (RDSs), we prefer to recall very briefly here some definitions and results to make our presentation as much self-contained as possible.

Let $(X, \|\cdot\|_X)$ be a separable Banach space.

Definition 2.1. A RDS on X consists of two ingredients: (a) a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and the family of mappings $\theta_t : \Omega \rightarrow \Omega$ satisfies

- (1) $\theta_0 = \text{Id}_\Omega$,
- (2) $\theta_s \circ \theta_t = \theta_{s+t}$ for all $s, t \in \mathbb{R}$,
- (3) the mapping $(t, \omega) \mapsto \theta_t \omega$ is measurable,
- (4) the probability measure \mathbb{P} is preserved by θ_t , i.e., $\theta_t \mathbb{P} = \mathbb{P}$

and (b) a mapping $\varphi : [0, \infty) \times \Omega \times X \rightarrow X$ which is $(\mathcal{B}[0, \infty) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$ -measurable, such that for each $\omega \in \Omega$,

- (i) the mapping $\varphi(t, \omega) : X \rightarrow X$, $x \mapsto \varphi(t, \omega)x$ is continuous for every $t \geq 0$,
- (ii) $\varphi(0, \omega)$ is the identity operator on X ,
- (iii) (cocycle property) $\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \varphi(s, \omega)$ for all $s, t \geq 0$.

Definition 2.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random set K is a measurable subset of $X \times \Omega$ with respect to the product σ -algebra $\mathcal{B}(X) \times \mathcal{F}$. Moreover K will be said a closed or a compact random set if $K(\omega) = \{x : (x, \omega) \in K\}$, $\omega \in \Omega$, is closed or compact for \mathbb{P} -almost all $\omega \in \Omega$, respectively.

Definition 2.3. A bounded random set $K(\omega) \subset X$ is said to be tempered with respect to $\{\theta_t\}_{t \in \mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} e^{-\beta t} \sup_{x \in K(\theta_{-t} \omega)} \|x\|_X = 0, \quad \text{for all } \beta > 0;$$

a random variable $\omega \mapsto r(\omega) \in \mathbb{R}$ is said to be tempered with respect to $\{\theta_t\}_{t \in \mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} e^{-\beta t} \sup_{t \in \mathbb{R}} |r(\theta_{-t} \omega)| = 0, \quad \text{for all } \beta > 0.$$

In what follows we use $\mathcal{E}(X)$ to denote the set of all tempered random sets of X .

Definition 2.4. A random set $B(\omega) \subset X$ is called a random absorbing set in $\mathcal{E}(X)$ if for any $E \in \mathcal{E}(X)$ and a.e. $\omega \in \Omega$, there exists $T_E(\omega) > 0$ such that

$$\varphi(t, \theta_{-t} \omega) E(\theta_{-t} \omega) \subset B(\omega), \quad \text{for all } t \geq T_E(\omega).$$

Definition 2.5. Let $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ be an RDS over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ with state space X and let $A(\omega) (\subset X)$ be a random set. Then $\mathcal{A} = \{A(\omega)\}_{\omega \in \Omega}$ is called a global random \mathcal{E} -attractor (or pullback \mathcal{E} -attractor) for $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ if

- (i) (compactness) $A(\omega)$ is a compact set of X for any $\omega \in \Omega$;
- (ii) (invariance) for any $\omega \in \Omega$ and all $t \geq 0$, it holds

$$\varphi(t, \omega)A(\omega) = A(\theta_t \omega);$$

- (iii) (attracting property) for any $E \in \mathcal{E}(X)$ and a.e. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} \text{dist}_X(\varphi(t, \theta_{-t} \omega)E(\theta_{-t} \omega), A(\omega)) = 0,$$

where $\text{dist}_X(G, H) = \sup_{g \in G} \inf_{h \in H} \|g - h\|_X$ is the Hausdorff semi-metric for $G, H \subseteq X$.

Proposition 2.1. [See [11, 14]] Let $B \in \mathcal{E}(X)$ be a closed absorbing set for the continuous random dynamical system $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ that satisfies the asymptotic compactness condition for a.e. $\omega \in \Omega$, i.e., each sequence $x_n \in \varphi(t_n, \theta_{-t_n} \omega)B(\theta_{-t_n} \omega)$ has a convergent subsequence in X when $t_n \rightarrow \infty$. Then φ has a unique global random attractor $\mathcal{A} = \{A(\omega)\}_{\omega \in \Omega}$ with component subsets

$$A(\omega) = \bigcap_{\tau \geq T_B(\omega)} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t} \omega)B(\theta_{-t} \omega)}.$$

Remark 1. When the state space $X = \mathbb{R}^d$ as in this paper, the asymptotic compactness follows trivially.

The next result ensures when two RDSs are conjugated (see [5, 9]).

Lemma 2.6. Let φ_u be an RDS on X . Suppose that the mapping $\mathcal{T} : \Omega \times X \rightarrow X$ possesses the following properties: for fixed $\omega \in \Omega$, $\mathcal{T}(\omega, \cdot)$ is a homeomorphism on X , and for $x \in X$, the mappings $\mathcal{T}(\cdot, x)$, $\mathcal{T}^{-1}(\cdot, x)$ are measurable. Then the mapping

$$(t, \omega, x) \rightarrow \varphi_v(t, \omega)x := \mathcal{T}^{-1}(\theta_t \omega, \varphi_u(t, \omega)\mathcal{T}(\omega, x))$$

is a (conjugated) RDS.

3. Random chemostat. In this section we will study the stochastic systems (3)-(4) and (9)-(11) by transforming them into differential equations with random coefficients.

Let W be a two sided Wiener process. Kolmogorov's theorem ensures that W has a continuous version, that we will denote by ω , whose canonical interpretation is as follows: let Ω be defined by

$$\Omega = \{\omega \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\} = \mathcal{C}_0(\mathbb{R}, \mathbb{R}),$$

\mathcal{F} the Borel σ -algebra on Ω generated by the compact open topology (see [1] for details) and \mathbb{P} the corresponding Wiener measure on \mathcal{F} . We consider the Wiener shift flow given by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system. Now let us introduce the following Ornstein-Uhlenbeck process on $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$

$$z^*(\theta_t \omega) = - \int_{-\infty}^0 e^s \theta_t \omega(s) ds, \quad t \in \mathbb{R}, \quad \omega \in \Omega,$$

which solves the following Langevin equation (see [1, 10])

$$dz + zdt = d\omega(t), \quad t \in \mathbb{R}. \quad (12)$$

Proposition 3.1. [See [1, 10]] *There exists a θ_t -invariant set $\tilde{\Omega} \in \mathcal{F}$ of Ω of full \mathbb{P} measure such that for $\omega \in \tilde{\Omega}$, we have*

- (i) *the random variable $|z^*(\omega)|$ is tempered.*
- (ii) *the mapping*

$$(t, \omega) \rightarrow z^*(\theta_t \omega) = - \int_{-\infty}^0 e^s \omega(t+s) ds + \omega(t)$$

is a stationary solution of (12) with continuous trajectories;

- (iii) *in addition, for any $\omega \in \tilde{\Omega}$:*

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} \frac{|z^*(\theta_t \omega)|}{t} &= 0; \\ \lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t z^*(\theta_s \omega) ds &= 0; \\ \lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t |z^*(\theta_s \omega)| ds &= \mathbb{E}[z^*] < \infty. \end{aligned}$$

In what follows we will consider the restriction of the Wiener shift θ to the set $\tilde{\Omega}$, and we restrict accordingly the metric dynamical system to this set, that is also a metric dynamical system, see [5]. For simplicity, we will still denote the restricted metric dynamical system by the old symbols $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$.

From now on, we denote $\mathcal{X} := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$.

3.1. Random chemostat without wall growth. In what follows we use the Ornstein-Uhlenbeck process to transform (3)-(4) into a random system. To this end, we first define two new variables σ and κ as follows

$$\sigma(t) = S(t)e^{\alpha z^*(\theta_t \omega)} \quad \text{and} \quad \kappa(t) = x(t)e^{\alpha z^*(\theta_t \omega)}. \quad (13)$$

For the sake of simplicity, and when no confusion is possible, we will write z^* instead of $z^*(\theta_t \omega)$, and σ and κ instead of $\sigma(t)$ and $\kappa(t)$.

Hence, it is straightforward that

$$\frac{d\sigma}{dt} = -(\bar{D} + \alpha z^*)\sigma - \frac{m\sigma e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*}}\kappa + S^0 D e^{\alpha z^*}, \quad (14)$$

$$\frac{d\kappa}{dt} = -(\bar{D} + \alpha z^*)\kappa + \frac{m\sigma e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*}}\kappa. \quad (15)$$

Next we prove that the random chemostat (14)-(15) generates an RDS.

Theorem 3.1. *For any $\omega \in \Omega$ and any initial value $u_0 := (\sigma_0, \kappa_0) \in \mathcal{X}$, where σ_0 and κ_0 denote $\sigma(0)$ and $\kappa(0)$ respectively, the system (14)-(15) possesses a unique global solution $u(\cdot; \omega, u_0) := (\sigma(\cdot; \omega, u_0), \kappa(\cdot; \omega, u_0)) \in \mathcal{C}^1([0, \infty), \mathcal{X})$ with $u(0; \omega, u_0) = u_0$. Moreover the solution mapping generates a RDS $\varphi_u : \mathbb{R}^+ \times \Omega \times \mathcal{X} \rightarrow \mathcal{X}$ defined as*

$$\varphi_u(t, \omega)u_0 = u(t; \omega, u_0), \quad \text{for all } t \in \mathbb{R}^+, u_0 \in \mathcal{X}, \omega \in \Omega.$$

Proof. Observe that we can rewrite one of the terms in the previous equations as

$$\frac{m\sigma e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*}} \kappa = \frac{m\sigma e^{-\alpha z^*} + ma - ma}{a + \sigma e^{-\alpha z^*}} \kappa = m\kappa - \frac{ma}{a + \sigma e^{-\alpha z^*}} \kappa$$

and therefore system (14)-(15) turns into

$$\frac{d\sigma}{dt} = -(\bar{D} + \alpha z^*)\sigma - m\kappa + \frac{ma}{a + \sigma e^{-\alpha z^*}} \kappa + S^0 D e^{\alpha z^*}, \quad (16)$$

$$\frac{d\kappa}{dt} = -(\bar{D} + \alpha z^*)\kappa + m\kappa - \frac{ma}{a + \sigma e^{-\alpha z^*}} \kappa. \quad (17)$$

Denoting $u(\cdot; \omega, u_0) := (\sigma(\cdot; \omega, u_0), \kappa(\cdot; \omega, u_0))$, system (16)-(17) can be rewritten as

$$\frac{du}{dt} = L(\theta_t \omega) \cdot u + F(u, \theta_t \omega),$$

where

$$L(\theta_t \omega) = \begin{pmatrix} -(\bar{D} + \alpha z^*) & -m \\ 0 & -(\bar{D} + \alpha z^*) + m \end{pmatrix}$$

and $F : \mathcal{X} \times \Omega \rightarrow \mathbb{R}^2$ is given by

$$F(\eta, \omega) = \begin{pmatrix} \frac{ma}{a + \eta_1 e^{-\alpha z^*(\omega)}} \eta_2 + DS^0 e^{\alpha z^*(\omega)} \\ \frac{-ma}{a + \eta_1 e^{-\alpha z^*(\omega)}} \eta_2 \end{pmatrix},$$

where $\eta = (\eta_1, \eta_2) \in \mathcal{X}$.

Since $t \mapsto z^*(\theta_t \omega)$ is continuous, L generates an evolution system on \mathbb{R}^2 . Moreover, we notice that

$$\frac{\partial}{\partial \eta_1} \left[\pm \frac{ma}{a + \eta_1 e^{-\alpha z^*}} \eta_2 + \tilde{C} \right] = \mp \frac{ma e^{-\alpha z^*}}{(a + \eta_1 e^{-\alpha z^*})^2} \eta_2,$$

and

$$\frac{\partial}{\partial \eta_2} \left[\pm \frac{ma}{a + \eta_1 e^{-\alpha z^*}} \eta_2 + \tilde{C} \right] = \pm \frac{ma}{a + \eta_1 e^{-\alpha z^*}},$$

where \tilde{C} is a constant which does not depend on $(\eta_1, \eta_2) \in \mathcal{X}$, therefore $F(\cdot, \theta_t \omega) \in \mathcal{C}(\mathcal{X} \times [0, \infty); \mathbb{R}^2)$ and is continuously differentiable with respect to the variables (η_1, η_2) , which implies that it is locally Lipschitz with respect to $(\eta_1, \eta_2) \in \mathcal{X}$.

Therefore, thanks to classical results from the theory of ordinary differential equations, system (16)-(17) possesses a unique local solution. Let us check now that in fact this solution is a global one.

We define $V(t) := \sigma(t) + \kappa(t)$ and thanks to (16)-(17) we have

$$\frac{dV}{dt} = -(\bar{D} + \alpha z^*)V + S^0 D e^{\alpha z^*}.$$

By solving the previous differential equation we obtain

$$V(t) = V(0)e^{-\bar{D}t - \alpha \int_0^t z^* ds} + S^0 D \int_0^t e^{\alpha z^*} e^{-\bar{D}(t-s) - \alpha \int_s^t z^* dr} ds, \quad (18)$$

hence V is clearly bounded by above by an expression which does not blow up.

On the other hand, from (14) we obtain

$$\frac{d\sigma}{dt} \leq -(\bar{D} + \alpha z^*)\sigma + S^0 D e^{\alpha z^*},$$

hence, similarly to previous calculations, we obtain

$$\sigma(t) \leq \sigma(0)e^{-\bar{D}t - \alpha \int_0^t z^* ds} + S^0 D \int_0^t e^{\alpha z^*} e^{-\bar{D}(t-s) - \alpha \int_s^t z^* dr} ds,$$

thus σ does not blow up either.

Summing up, we have proved that $V(t)$ and $\sigma(t)$ do not blow up and the same happens to $\kappa(t) = V(t) - \sigma(t)$. Therefore, the unique local solution to system (16)-(17) can be extended to a unique global one.

Now we would like to check that the global solution of (16)-(17) belongs to the set \mathcal{X} for any $t \in \mathbb{R}^+$. From (14)-(15), if $\sigma(t) = 0$ for some $t^* \in \mathbb{R}^+$, we have

$$\frac{d\sigma}{dt}(t^*) = \left[-(\bar{D} + \alpha z^*)\sigma - \frac{m\sigma e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*}}\kappa + S^0 D e^{\alpha z^*} \right](t^*) = S^0 D e^{\alpha z^*} > 0.$$

Besides, given $(\sigma_0, 0)$ with $\sigma_0 > 0$, there exists a unique solution of system (14)-(15) satisfying $\sigma(t_0) = \sigma_0$ and $\kappa(t_0) = 0$ for some initial time $t_0 \geq 0$. Imposing $\kappa \equiv 0$ we deduce that $\sigma(t)$ is given by

$$\sigma(t) = \sigma(t_0)e^{-\bar{D}(t-t_0) - \alpha \int_{t_0}^t z^* ds} + S^0 D \int_{t_0}^t e^{\alpha z^*} e^{-\bar{D}(t-s) - \alpha \int_s^t z^* dr} ds. \quad (19)$$

Now, let us pick $(\sigma_0, \kappa_0) \in \mathcal{X}$. Thus, there exists a unique solution $(\sigma(t), \kappa(t))$ such that $\sigma(0) = \sigma_0$ and $\kappa(0) = \kappa_0$. If there is some first $t^* > 0$ verifying $\kappa(t^*) = 0$, then we have that $(\sigma(\cdot), \kappa(\cdot))$ is the unique solution of system (14)-(15) with $\sigma(t^*) = \sigma^*$ and $\kappa(t^*) = 0$. Moreover $\kappa(t) > 0$ for all $0 \leq t < t^*$; however, we already have another solution $(\sigma(t), 0)$ for all $t \geq t^* - \delta$ (for any $\delta > 0$ small enough) for this problem, so we obtain a contradiction. As a result, we deduce that for any initial data $u_0 \in \mathcal{X}$ the solution $u(t)$ remains in \mathcal{X} .

Now we can define the mapping $\varphi_u : \mathbb{R}^+ \times \Omega \times \mathcal{X} \rightarrow \mathcal{X}$ given by

$$\varphi_u(t, \omega)u_0 := u(t; \omega, u_0), \quad \text{for all } t \geq 0, u_0 \in \mathcal{X}, \omega \in \Omega.$$

Since the function F is continuous in (u, t) , and is measurable in ω , we obtain the $(\mathcal{B}[0, \infty) \times \mathcal{F} \times \mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{X}))$ -measurability of the previous mapping. It then follows that (16)-(17) generate the continuous RDS $\varphi_u(t, \omega)(\cdot)$. \square

Now we study the existence of a random attractor, describing its internal structure if possible.

Proposition 3.2. *There exists a tempered compact random absorbing set $B(\omega) \in \mathcal{E}(\mathcal{X})$ of the random dynamical system $\{\varphi_u(t, \omega)\}_{t \geq 0, \omega \in \Omega}$.*

Proof. Recall that $\varphi_u(t, \theta_{-t}\omega)u_0 = u(t; \theta_{-t}\omega, u_0)$ denotes the solution of system (14)-(15), satisfying $u(0; \theta_{-t}\omega, u_0) = u_0$, where $u_0 := u_0(\theta_{-t}\omega) \in E(\theta_{-t}\omega)$.

First we define $\|\cdot\|$ as

$$\begin{aligned} \|\varphi_u(t, \theta_{-t}\omega)u_0\| &= \|u(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega))\| \\ &:= \sigma(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega)) + \kappa(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega)). \end{aligned}$$

By replacing ω by $\theta_{-t}\omega$ in (18), we have

$$\begin{aligned} &\sigma(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega)) + \kappa(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega)) \\ &= (\sigma + \kappa)(0)e^{-\bar{D}t - \alpha \int_{-t}^0 z^*(\theta_s\omega) ds} \\ &\quad + S^0 D \int_0^t e^{\tau \left[-\bar{D} + \frac{\alpha z^*(\theta_{-\tau}\omega)}{\tau} - \frac{\alpha}{\tau} \int_{-\tau}^0 z^*(\theta_r\omega) dr \right]} d\tau, \end{aligned} \quad (20)$$

and therefore

$$\lim_{t \rightarrow \infty} \{\sigma(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega)) + \kappa(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega))\} = S^0 D \rho^*(\omega)$$

since \bar{D} given by (5) is always positive, where $\rho^*(\omega)$ is defined by

$$\rho^*(\omega) := \int_0^\infty e^{\tau \left[-\bar{D} + \frac{\alpha z^*(\theta_{-\tau}\omega)}{\tau} - \frac{\alpha}{\tau} \int_{-\tau}^0 z^*(\theta_r\omega) dr \right]} d\tau.$$

Note that the above integrand converges to zero when τ goes to infinity, but not the integral. Moreover, $\rho^*(\omega)$ has sub-exponential growth.

Therefore, for any given $\varepsilon > 0$, there exists $T_E(\omega, \varepsilon) > 0$ such that

$$S^0 D \rho^*(\omega) - \varepsilon \leq \|u(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega))\| \leq S^0 D \rho^*(\omega) + \varepsilon$$

for all $u_0 \in E(\theta_{-t}\omega)$ and $t \geq T_E(\omega, \varepsilon)$.

We now define

$$B_\varepsilon(\omega) := \{(\sigma, \kappa) \in \mathcal{X} : S^0 D \rho^*(\omega) - \varepsilon \leq \sigma + \kappa \leq S^0 D \rho^*(\omega) + \varepsilon\},$$

thus $B_\varepsilon(\omega) \in \mathcal{E}(\omega)$ is absorbing in \mathcal{X} for any $\varepsilon > 0$. \square

Hence, from Proposition 2.1, the RDS generated by the system (16)-(17) possesses a unique random attractor given by $\mathcal{A} = \{A(\omega)\}_{\omega \in \Omega} \subset B_\varepsilon(\omega)$ for any $\varepsilon > 0$. Thus $\mathcal{A} = \{A(\omega)\}_{\omega \in \Omega} \subset B_0(\omega)$, i.e., we have the following expression for each component of our attractor

$$A(\omega) := (S^0 D \rho^*(\omega) - \kappa(\omega), \kappa(\omega)).$$

Proposition 3.3. *For \bar{D} defined by (5) assume that $\bar{D} > m$. Then, the random attractor \mathcal{A} associated to the RDS φ_u has the following structure:*

$$\mathcal{A} = \{A(\omega)\}_{\omega \in \Omega}, \quad \text{where } A(\omega) = (S^0 D \rho^*(\omega), 0).$$

Proof. Thanks to (15) we know that

$$\frac{d\kappa}{dt} \leq -(\bar{D} - m + \alpha z^*)\kappa,$$

whose solution, after replacing ω by $\theta_{-t}\omega$ and making t go to infinity, tends to zero provided $\bar{D} > m$, thus the internal structure of the attractor in this case consists of singleton subsets $A(\omega) = (S^0 D \rho^*(\omega), 0)$ which means that there is not persistence of the microorganism (see Figure 2 in Section 5). \square

However, we cannot ensure the persistence of the microorganism in case $\bar{D} < m$ by using mathematical arguments even though our simulations show that the random attractor in this case is totally contained in \mathcal{X} , in other words, our model seems to guarantee the persistence of the microorganism (see Figure 1 in Section 5).

3.2. Random chemostat with wall growth. In what follows we use the Ornstein-Uhlenbeck process to transform (9)-(11) into a random system. Similarly to Section 3.1, we first define three new variables σ , κ_1 and κ_2 as follows

$$\sigma(t) = S(t)e^{\alpha z^*(\theta_t \omega)}, \quad \kappa_1(t) = x_1(t)e^{\alpha z^*(\theta_t \omega)} \quad \text{and} \quad \kappa_2(t) = x_2(t)e^{\alpha z^*(\theta_t \omega)}. \quad (21)$$

By differentiation, we obtain the following random system

$$\frac{d\sigma}{dt} = -(\bar{D} + \alpha z^*)\sigma + b\nu\kappa_1 - \frac{m\sigma e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*}}(\kappa_1 + \kappa_2) + DS^0 e^{\alpha z^*}, \quad (22)$$

$$\frac{d\kappa_1}{dt} = -(\nu + \bar{D} + r_1 + \alpha z^*)\kappa_1 + c \frac{\sigma e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*}}\kappa_1 + r_2\kappa_2, \quad (23)$$

$$\frac{d\kappa_2}{dt} = r_1\kappa_1 - \left(\nu + r_2 + \frac{\alpha^2}{2} + \alpha z^*\right)\kappa_2 + c \frac{\sigma e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*}}\kappa_2. \quad (24)$$

Now, we define two new variables

$$\kappa(t) = \kappa_1(t) + \kappa_2(t) \quad \text{and} \quad \xi(t) = \frac{\kappa_1(t)}{\kappa_1(t) + \kappa_2(t)} = \frac{x_1(t)}{x_1(t) + x_2(t)} \quad (25)$$

in order to transform our random system (22)-(24) into another system which will be more useful to understand the dynamics of the model. For the sake of simplicity we will write κ and ξ instead of $\kappa(t)$ and $\xi(t)$.

Taking into account (25), the system (22)-(24) is equivalent to the following random one

$$\frac{d\sigma}{dt} = -(\bar{D} + \alpha z^*)\sigma + b\nu\xi\kappa - \frac{m\sigma e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*}}\kappa + DS^0 e^{\alpha z^*}, \quad (26)$$

$$\frac{d\kappa}{dt} = -\left(\nu + \alpha z^* + \frac{\alpha^2}{2}\right)\kappa + c \frac{\sigma e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*}}\kappa - D\xi\kappa, \quad (27)$$

$$\frac{d\xi}{dt} = -D\xi(1 - \xi) - r_1\xi + r_2(1 - \xi). \quad (28)$$

We first study the Riccati equation held by $\xi(t)$ since the dynamics of $\xi(t) = \xi(t; \omega, \xi_0)$ is uncoupled with $\sigma(t)$ and $\kappa(t)$.

Defining $F_\xi : [0, \infty) \times [0, 1] \rightarrow \mathbb{R}$ as

$$F_\xi(t, \xi) = -D\xi(1 - \xi) - r_1\xi + r_2(1 - \xi) = -D\xi + D\xi^2 - r_1\xi + r_2 - r_2\xi,$$

it is straightforward to check that F_ξ is continuous (it is a polynomial function) and locally Lipschitz respect to ξ , hence there exists a unique local solution of (28) which can be extended to a global one since ξ is bounded.

Moreover, by solving explicitly (28) we obtain

$$\xi^*(t) := \hat{\xi} + \frac{1}{\left[\frac{1}{\xi(0) - \hat{\xi}} + \frac{D}{D + r_1 + r_2 - 2D\hat{\xi}}\right] e^{(D + r_1 + r_2 - 2D\hat{\xi})t} - \frac{D}{D + r_1 + r_2 - 2D\hat{\xi}}}, \quad (29)$$

where $\hat{\xi} := \frac{D + r_1 + r_2 - \sqrt{(D + r_1 + r_2)^2 - 4Dr_2}}{2D}$, so that $D + r_1 + r_2 - 2D\hat{\xi} > 0$.

Now we can define $\varphi_\xi : \mathbb{R}^+ \times \Omega \times (0, 1) \rightarrow (0, 1)$ as

$$\varphi_\xi(t, \omega)\xi_0 := \xi(t; \omega, \xi_0), \quad \text{for all } t \geq 0, \xi_0 \in (0, 1), \omega \in \Omega.$$

Since the function F_ξ is continuous in (ξ, t) , and is measurable in ω , we obtain the $(\mathcal{B}[0, \infty) \times \mathcal{F} \times \mathcal{B}(0, 1), \mathcal{B}(0, 1))$ -measurability of the previous mapping. Hence (28) generates the continuous RDS $\varphi_\xi(t, \omega)(\cdot)$.

By replacing ω by $\theta_{-t}\omega$ in (29) we have

$$\begin{aligned} \varphi_\xi(t, \theta_{-t}\omega)\xi_0 &= \xi(t; \theta_{-t}\omega, \xi_0(\theta_{-t}\omega)) \\ &= \widehat{\xi} + \frac{1}{\left[\frac{1}{\xi_0 - \widehat{\xi}} + \frac{D}{D+r_1+r_2-2D\widehat{\xi}} \right] e^{(D+r_1+r_2-2D\widehat{\xi})t} - \frac{D}{D+r_1+r_2-2D\widehat{\xi}}}. \end{aligned}$$

Hence, since $D + r_1 + r_2 - 2D\widehat{\xi} > 0$, we obtain

$$\lim_{t \rightarrow \infty} \xi(t; \theta_{-t}\omega, \xi_0(\theta_{-t}\omega)) = \widehat{\xi}.$$

Therefore, given $\xi_0 \in E(\theta_{-t}\omega)$, there exists $T_E(\omega) > 0$ such that

$$\xi(t; \theta_{-t}\omega, \xi_0(\theta_{-t}\omega)) = \widehat{\xi}$$

for all $t \geq T_E(\omega)$. Hence, it follows directly from Proposition 2.1 that the RDS generated by (28) possesses a unique random attractor given by

$$\mathcal{A}_\xi = \{A_\xi(\omega)\}_{\omega \in \Omega} = \{\widehat{\xi}\}.$$

Now, we are interested in studying the system

$$\frac{d\sigma}{dt} = -(\bar{D} + \alpha z^*)\sigma + b\nu\xi^*\kappa - \frac{m\sigma e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*}}\kappa + DS^0 e^{\alpha z^*}, \quad (30)$$

$$\frac{d\kappa}{dt} = -\left(\nu + \alpha z^* + \frac{\alpha^2}{2}\right)\kappa + c \frac{\sigma e^{-\alpha z^*}}{a + \sigma e^{-\alpha z^*}}\kappa - D\xi^*\kappa. \quad (31)$$

Theorem 3.2. *For any $\omega \in \Omega$, any $(\sigma_0, \kappa_{01}, \kappa_{02}) \in \mathbb{R}_+^3$ and any initial value $u_0 := (\sigma_0, \kappa_0) \in \mathcal{X}$, where $\sigma_0, \kappa_0, \kappa_{01}$ and κ_{02} denote $\sigma(0), \kappa(0), \kappa_1(0)$ and $\kappa_2(0)$, respectively, the system (30)-(31) possesses a unique global solution $u(\cdot; \omega, u_0) := (\sigma(\cdot; \omega, u_0), \kappa(\cdot; \omega, u_0)) \in \mathcal{C}^1([0, \infty), \mathcal{X})$ with $u(0; \omega, u_0) = u_0$. Moreover the solution mapping generates a RDS $\varphi_u : \mathbb{R}^+ \times \Omega \times \mathcal{X} \rightarrow \mathcal{X}$ defined as*

$$\varphi_u(t, \omega)u_0 = u(t; \omega, u_0), \quad \text{for all } t \in \mathbb{R}^+, u_0 \in \mathcal{X}, \omega \in \Omega.$$

Proof. Arguing in the same way as in the proof of Theorem 3.1, the classical results from the theory of ordinary differential equations ensures that system (30)-(31) possesses a unique local solution. Let us check now that in fact this solution is a global one. In order to do that, we define $V(t) := \sigma(t) + \kappa(t)$, which satisfies the following differential equation

$$\frac{dV}{dt} \leq -\left(\frac{\alpha^2}{2} + \alpha z^*\right)(\sigma + \kappa) + DS^0 e^{\alpha z^*}, \quad (32)$$

since $c \leq m$, $\xi^* \leq 0$ and $b \in (0, 1)$. Hence

$$V(t) \leq V(0)e^{-\frac{\alpha^2}{2}t - \alpha \int_0^t z^* ds} + DS^0 \int_0^t e^{\alpha z^*} e^{\frac{\alpha^2}{2}s + \alpha \int_0^s z^* dr} e^{-\frac{\alpha^2}{2}t - \alpha \int_0^t z^* dr} ds, \quad (33)$$

thus V is clearly bounded by above by an expression which does not blow up at any finite time.

On the other hand,

$$\frac{d\kappa}{dt} \leq -\left(\nu + \alpha z^* + \frac{\alpha^2}{2} - c\right)\kappa,$$

thus we have

$$\kappa(t) \leq \kappa(0)e^{-\left(\nu + \frac{\alpha^2}{2} - c\right)t - \alpha \int_0^t z^* ds}, \quad (34)$$

hence κ does not blow up at any finite time either. As a result, $\sigma(t) = V(t) - \kappa(t)$ does not blow up. Therefore, the unique local solution to system (30)-(31) can be extended to a unique global one.

Furthermore,

$$\frac{dV}{dt} \geq -(\bar{D} + \alpha z^* + m + \nu)V + DS^0 e^{\alpha z^*},$$

therefore we obtain the following inequality, which will be further very useful

$$\begin{aligned} V(t) &\geq V(0)e^{-(\bar{D}+m+\nu)t - \alpha \int_0^t z^* ds} \\ &\quad + DS^0 \int_0^t e^{\alpha z^*} e^{(\bar{D}+m+\nu)s + \alpha \int_0^s z^* dr} e^{-(\bar{D}+m+\nu)t - \alpha \int_0^t z^* dr} ds. \end{aligned} \quad (35)$$

It is straightforward to verify, similarly to the case without wall growth, that the global solution $u(t)$ of (30)-(31) belongs to \mathcal{X} for any initial data $u_0 \in \mathcal{X}$ and $t \in \mathbb{R}^+$.

Now we can define the mapping $\varphi_u : \mathbb{R}^+ \times \Omega \times \mathcal{X} \rightarrow \mathcal{X}$ given by

$$\varphi_u(t, \omega)u_0 := u(t; \omega, u_0), \quad \text{for all } t \geq 0, u_0 \in \mathcal{X}, \omega \in \Omega.$$

Analogously to the case without wall growth, we obtain the $(\mathcal{B}[0, \infty) \times \mathcal{F} \times \mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{X}))$ -measurability of the previous mapping. Hence $\varphi_u(t, \omega)(\cdot)$ is an RDS. \square

Now we study the existence of a random attractor, describing it explicitly whenever it is possible.

Proposition 3.4. *There exists a tempered compact random absorbing set $B(\omega) \in \mathcal{E}(\mathcal{X})$ of the RDS $\{\varphi_u(t, \omega)\}_{t \geq 0, \omega \in \Omega}$.*

Proof. Remember that

$$\begin{aligned} \|\varphi_u(t, \theta_{-t}\omega)u_0\| &= \|u(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega))\| \\ &:= \sigma(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega)) + \kappa(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega)). \end{aligned}$$

By replacing ω by $\theta_{-t}\omega$ in (33), we have

$$\begin{aligned} &\sigma(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega)) + \kappa(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega)) \\ &\leq (\sigma + \kappa)(0)e^{-\frac{\alpha^2}{2}t - \alpha \int_{-t}^0 z^*(\theta_r\omega)dr} + DS^0 \int_0^t e^{-\tau} \left[\frac{\alpha^2}{2} - \frac{\alpha z^*(\theta_{-\tau}\omega)}{\tau} + \frac{\alpha}{\tau} \int_{-\tau}^0 z^*(\theta_r\omega)dr \right] d\tau, \end{aligned}$$

which tends to $DS^0 \rho_u^*(\omega)$ when t goes to infinity, where

$$\rho_u^*(\omega) := \int_0^\infty e^{-\tau} \left[\frac{\alpha^2}{2} - \frac{\alpha z^*(\theta_{-\tau}\omega)}{\tau} + \frac{\alpha}{\tau} \int_{-\tau}^0 z^*(\theta_r\omega)dr \right] d\tau.$$

Now we replace ω by $\theta_{-t}\omega$ in (35) thus we obtain

$$\begin{aligned} &\sigma(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega)) + \kappa(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega)) \\ &= (\sigma + \kappa)(0)e^{-(\bar{D}+m+\nu)t - \alpha \int_{-t}^0 z^*(\theta_r\omega)dr} \end{aligned}$$

$$+DS^0 \int_0^t e^{-\tau} \left[(\bar{D}+m+\nu) - \frac{\alpha z^*(\theta_{-\tau}\omega)}{\tau} + \frac{\alpha}{\tau} \int_{-\tau}^0 z^*(\theta_r\omega) dr \right] d\tau. \quad (36)$$

The first term tends to zero since $\bar{D}+m+\nu > 0$. The second one tends to $DS^0 \rho_l^*(\omega)$, where

$$\rho_l^*(\omega) := \int_0^\infty e^{-\tau} \left[(\bar{D}+m+\nu) - \frac{\alpha z^*(\theta_{-\tau}\omega)}{\tau} + \frac{\alpha}{\tau} \int_{-\tau}^0 z^*(\theta_r\omega) dr \right] d\tau.$$

Note that the integrands defining $\rho_u^*(\omega)$ and $\rho_l^*(\omega)$ converge to zero when τ goes to infinity, but not the integrals. Moreover, $\rho_u^*(\omega)$ and $\rho_l^*(\omega)$ have sub-exponential growth.

Therefore, for any given $\varepsilon > 0$, there exists $T_E(\omega, \varepsilon) > 0$ such that

$$DS^0 \rho_l^*(\omega) - \varepsilon \leq \|u(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega))\| \leq DS^0 \rho_u^*(\omega) + \varepsilon$$

for all $u_0 \in E(\theta_{-t}\omega)$, when $t \geq T_E(\omega, \varepsilon)$.

We define

$$\widehat{B}_\varepsilon(\omega) := \{(\sigma, \kappa) \in \mathcal{X} : DS^0 \rho_l^*(\omega) - \varepsilon \leq \sigma + \kappa \leq DS^0 \rho_u^*(\omega) + \varepsilon\},$$

thus $\widehat{B}_\varepsilon(\omega) \in \mathcal{E}(\omega)$ is absorbing in \mathcal{X} . \square

Hence, it follows directly from Proposition 2.1 that the RDS generated by the system (30)-(31) possesses a unique random attractor given by $\widehat{\mathcal{A}} = \{\widehat{A}(\omega)\}_{\omega \in \Omega} \subset \widehat{B}_\varepsilon(\omega)$, for all $\varepsilon > 0$. Thus, $\widehat{\mathcal{A}} = \{\widehat{A}(\omega)\}_{\omega \in \Omega} \subset \widehat{B}_0(\omega)$.

Now we would like to go deeper into the equations of our model with wall growth in order to know the internal structure of the random attractor in more detail.

On the one hand, thanks to (34), after replacing ω by $\theta_{-t}\omega$ we know that $\lim_{t \rightarrow \infty} \kappa(t) \leq \varepsilon$, for any $\varepsilon > 0$, provided $\nu + \frac{\alpha^2}{2} > c$. However, when $\nu + \frac{\alpha^2}{2} < c$, (34) does not give any extra information about the long-time behaviour of κ .

On the other hand, from (27) we obtain the following inequalities

$$-\left(\nu + \frac{\alpha^2}{2} + D + \alpha z^*\right) \kappa \leq \frac{d\kappa}{dt} \leq -\left(\nu + \frac{\alpha^2}{2} - c + \alpha z^*\right) \kappa. \quad (37)$$

Moreover, we can easily obtain the next lower bound from (26)

$$\frac{d\sigma}{dt} \geq -(\bar{D} + \alpha z^*)\sigma + (b\nu c_\xi - m)\kappa + DS^0 e^{\alpha z^*}, \quad (38)$$

where c_ξ is defined as

$$c_\xi = \begin{cases} \hat{\xi}, & \text{if } \xi^*(0) \geq \hat{\xi}, \\ \xi^*(0), & \text{if } \xi^*(0) < \hat{\xi}, \end{cases}$$

where ξ^* is given by (29). By using (37) we are able to solve (38) whichever the sign of $b\nu c_\xi - m$, so that we split our analysis into two different cases.

- **Case A:** If $b\nu c_\xi - m \geq 0$ holds, we have

$$\lim_{t \rightarrow \infty} \sigma(t) \geq S^0 D \rho_\sigma^*(\omega) - \varepsilon \quad \text{for any } \varepsilon > 0,$$

where

$$\rho_\sigma^*(\omega) := \int_0^\infty e^{-\tau} \left[\bar{D} - \frac{\alpha z^*(\theta_{-\tau}\omega)}{\tau} + \frac{\alpha}{\tau} \int_{-\tau}^0 z^*(\theta_r\omega) dr \right] d\tau.$$

We note that $\rho_\sigma^*(\omega)$ is well-defined and has sub-exponential growth. Hence, we analyze the following cases

- **(A-1)** If $\nu + \frac{\alpha^2}{2} > c$, we obtain

$$\lim_{t \rightarrow \infty} \sigma(t) \geq S^0 D\rho_\sigma^*(\omega) - \varepsilon \quad \text{and} \quad \lim_{t \rightarrow \infty} \kappa(t) \leq \varepsilon$$

for any $\varepsilon > 0$. In this case the random attractor satisfies

$$\widehat{\mathcal{A}} \subset \{(\sigma, 0) \in \mathcal{X} : S^0 D\rho_\sigma^*(\omega) \leq \sigma \leq S^0 D\rho_u^*(\omega)\},$$

which means that there is not persistence of the microorganisms (see Figures 5-6 in Section 5).

- **(A-2)** If $\nu + \frac{\alpha^2}{2} < c$, we obtain

$$\lim_{t \rightarrow \infty} \sigma(t) \geq S^0 D\rho_\sigma^*(\omega) - \varepsilon \quad \text{and} \quad \lim_{t \rightarrow \infty} \kappa(t) \leq \varepsilon$$

for any $\varepsilon > 0$. In this case the random attractor satisfies

$$\widehat{\mathcal{A}} \subset \{(\sigma, \kappa) \in \mathcal{X} : \sigma + \kappa \leq S^0 D\rho_u^*(\omega), \sigma \geq S^0 D\rho_\sigma^*(\omega)\}.$$

In that case we are not able to establish conditions to ensure the persistence of both microorganisms. However, the numerical simulations show that we can obtain persistence in the current case (see Figures 3-4 in Section 5).

- **Case B:** If $b\nu c_\xi - m < 0$ holds, we distinguish two cases again:

- **(B-1)** If $\nu + \frac{\alpha^*}{2} > c$, we have

$$\lim_{t \rightarrow \infty} \sigma(t) \geq S^0 D\rho_\sigma^*(\omega) - \varepsilon \quad \text{and} \quad \lim_{t \rightarrow \infty} \kappa(t) \leq \varepsilon$$

for any $\varepsilon > 0$. In this case the random attractor satisfies

$$\widehat{\mathcal{A}} \subset \{(\sigma, 0) \in \mathcal{X} : S^0 D\rho_\sigma^*(\omega) \leq \sigma \leq S^0 D\rho_u^*(\omega)\},$$

which means that there is not persistence of the microorganisms (see Figures 5-6 in Section 5).

- **(B-2)** If $\nu + \frac{\alpha^2}{2} < c$, we have

$$\lim_{t \rightarrow \infty} \sigma(t) \geq -\infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \kappa(t) \leq \infty.$$

In this case the global attractor satisfies $\widehat{\mathcal{A}} \subset \widehat{B}_0(\omega)$. We are not able to guarantee the persistence of the microorganisms even though the numerical simulations show that we can obtain it (see Figures 3-4 in Section 5).

Finally, we state Table 1 to summarize the results of the previous study.

		ASYMPTOTIC BOUNDS	ATTRACTOR INTERNAL STRUCTURE
Case A: $b\nu c_\xi - m \geq 0$	(A-1) $\nu + \frac{\alpha^2}{2} > c$	$\lim_{t \rightarrow \infty} \sigma(t) \geq S^0 D\rho_\sigma^*(\omega) - \varepsilon$	
		$\lim_{t \rightarrow \infty} \kappa(t) \leq \varepsilon$	
Case A: $b\nu c_\xi - m \geq 0$	(A-2) $\nu + \frac{\alpha^2}{2} < c$	$\lim_{t \rightarrow \infty} \sigma(t) \geq S^0 D\rho_\sigma^*(\omega) - \varepsilon$	
		$\kappa(t)$ does not provide any extra information	
Case B: $b\nu c_\xi - m < 0$	(B-1) $\nu + \frac{\alpha^2}{2} > c$	$\lim_{t \rightarrow \infty} \sigma(t) \geq S^0 D\rho_\sigma^*(\omega) - \varepsilon$	
		$\lim_{t \rightarrow \infty} \kappa(t) \leq \varepsilon$	
Case B: $b\nu c_\xi - m < 0$	(B-2) $\nu + \frac{\alpha^2}{2} < c$	$\sigma(t)$ does not provide any extra information	
		$\kappa(t)$ does not provide any extra information	

TABLE 1. Internal structure of the random attractor - Random chemostat model with wall growth

4. Existence of the random attractor for the stochastic system.

4.1. Stochastic model without wall growth. We have proved that the system (14)-(15) has a unique global solution $u(t; \omega, u_0)$ which remains in \mathcal{X} for all $u_0 \in \mathcal{X}$ and generates the RDS φ_u .

Now, we define a mapping $\mathcal{T} : \Omega \times \mathcal{X} \rightarrow \mathcal{X}$ as $\mathcal{T}(\omega, \zeta) = (\zeta_1 e^{\alpha z^*(\omega)}, \zeta_2 e^{\alpha z^*(\omega)})$ whose inverse is given by $\mathcal{T}^{-1}(\omega, \zeta) = (\zeta_1 e^{-\alpha z^*(\omega)}, \zeta_2 e^{-\alpha z^*(\omega)})$.

We know that $v(t) = (S(t), x(t))$ and $u(t) = (\sigma(t), \kappa(t))$ are related by (13). Since \mathcal{T} is a homeomorphism, thanks to Lemma 2.6 we obtain a conjugated RDS given by

$$\varphi_v(t, \omega)v_0 := \mathcal{T}^{-1}(\theta_t \omega, \varphi_u(t, \omega)\mathcal{T}(\omega, v_0))$$

$$\begin{aligned}
&= \mathcal{T}^{-1}(\theta_t \omega, \varphi_u(t, \omega) u_0) \\
&= \mathcal{T}^{-1}(\theta_t \omega, u(t; \omega, u_0)) \\
&= v(t; \omega, v_0)
\end{aligned}$$

which means that φ_v is an RDS for our original stochastic system (3)-(4).

Moreover, the global random attractor of the random system without wall growth (14)-(15), $\mathcal{A} = \{A(\omega)\}_{\omega \in \Omega} \subset B_0(\omega)$, becomes into $\mathcal{A}^{\mathcal{T}} = \{A^{\mathcal{T}}(\omega)\}_{\omega \in \Omega} \subset B_0^{\mathcal{T}}(\omega)$, the global random attractor of the system (3)-(4), where

$$B_0^{\mathcal{T}}(\omega) := \left\{ (S, x) \in \mathcal{X} : S + x = DS^0 \rho^*(\omega) e^{-\alpha z^*(\omega)} \right\}.$$

In other words, each component $A^{\mathcal{T}}(\omega)$, $\omega \in \Omega$, of our attractor can be written as

$$A^{\mathcal{T}}(\omega) := \left(S^0 D \rho^*(\omega) - S e^{-\alpha z^*(\omega)}, S e^{-\alpha z^*(\omega)} \right).$$

Moreover, we know that the internal structure of the attractor consists of singleton subsets $A^{\mathcal{T}}(\omega) = (S^0 D \rho^*(\omega) e^{-\alpha z^*(\omega)}, 0)$ as long as $\bar{D} > m$ and we cannot ensure the persistence of the microorganism otherwise. However, our simulations show that we can get the persistence for several values of the parameters (see Figures 1-2 in Section 5).

4.2. Stochastic model with wall growth. We have also proved that the system (30)-(31) has a unique global solution $u(t; \omega, u_0)$ which remains in \mathcal{X} for all $u_0 \in \mathcal{X}$ and generates the RDS φ_u .

Now, we define a mapping $\mathcal{T} : \Omega \times \mathcal{X} \rightarrow \mathcal{X}$ as in Section 4.1. Since $v(t) = (S(t), x(t))$ and $u(t) = (\sigma(t), \kappa(t))$, where $x(t) := x_1(t) + x_2(t)$ and $\kappa(t) := \kappa_1(t) + \kappa_2(t)$, are related by (21) and \mathcal{T} is a homeomorphism, thanks to Lemma 2.6 we obtain again a conjugated RDS given by

$$\varphi_v(t, \omega) v_0 = v(t; \omega, v_0)$$

which means that φ_v is an RDS for our original stochastic system (9)-(11).

Moreover, the global random attractor of the random system with wall growth (30)-(31), $\hat{\mathcal{A}} = \{\hat{A}(\omega)\}_{\omega \in \Omega} \subset \hat{B}_0(\omega)$, becomes into $\hat{\mathcal{A}}^{\mathcal{T}} = \{\hat{A}^{\mathcal{T}}(\omega)\}_{\omega \in \Omega} \subset \hat{B}_0^{\mathcal{T}}(\omega)$, the global random attractor of the system (9)-(11), where

$$\hat{B}_0^{\mathcal{T}}(\omega) = \left\{ (S, x_1, x_2) \in \mathbb{R}_3^+ : DS^0 \rho_l^*(\omega) e^{-\alpha z^*(\omega)} \leq S + x_1 + x_2 \leq DS^0 \rho_u^*(\omega) e^{-\alpha z^*(\omega)} \right\}.$$

Table 2 in the next page shows information on the random attractor $\hat{\mathcal{A}}^{\mathcal{T}} = \{\hat{A}^{\mathcal{T}}(\omega)\}_{\omega \in \Omega}$, taking into account the analysis carried out at the end of Section 3.2.

		ASYMPTOTIC BOUNDS	ATTRACTOR INTERNAL STRUCTURE
Case A: $b\nu c\xi - m \geq 0$	(A-1) $\nu + \frac{\sigma^2}{2} > c$	$\lim_{t \rightarrow \infty} S(t) \geq S^0 D\rho_\sigma^*(\omega) e^{-\alpha z^*(\omega)} - \varepsilon$	
		$\lim_{t \rightarrow \infty} [x_1(t) + x_2(t)] \leq \varepsilon$	
	(A-2) $\nu + \frac{\sigma^2}{2} < c$	$\lim_{t \rightarrow \infty} S(t) \geq S^0 D\rho_\sigma^*(\omega) e^{-\alpha z^*(\omega)} - \varepsilon$	
		$x_1 + x_2$ does not provide any extra information	
Case B: $b\nu c\xi - m < 0$	(B-1) $\nu + \frac{\sigma^2}{2} > c$	$\lim_{t \rightarrow \infty} S(t) \geq S^0 D\rho_\sigma^*(\omega) e^{-\alpha z^*(\omega)} - \varepsilon$	
		$\lim_{t \rightarrow \infty} [x_1(t) + x_2(t)] \leq \varepsilon$	
	(B-2) $\nu + \frac{\sigma^2}{2} < c$	S does not provide any extra information	
		$x_1 + x_2$ does not provide any extra information	

TABLE 2. Internal structure of the random attractor - Stochastic chemostat model with wall growth

5. Numerical simulations and final comments. In this section we will show some numerical simulations which support clearly the results obtained throughout this paper.

We firstly consider the following system of stochastic differential equations

$$dX(t) = f(X(t))dt + g(X(t)) \circ dW(t), \quad X(0) = X_0, \quad (39)$$

where $X \in \mathbb{R}^d$. Here $d = 2$ and $d = 3$ correspond to the stochastic chemostat models without and with wall growth, respectively.

Now we define a partition $\Delta := \{0 = \tau_0 < \tau_1 < \dots < \tau_N = T\}$ by dividing a time interval $[0, T] \subset \mathbb{R}$, $T > 0$, into N subintervals and setting $\delta t = T/N$ and $\tau_j = j \cdot \delta t$, $j = 0, \dots, N$. We want to approximate $X(\tau_j) \approx X_j$ by using the Euler-Maruyama method (see [20]).

In this way we integrate the equation in (39) on $\tau_{j-1} \leq t \leq \tau_j$ for some arbitrary $j \in \{0, \dots, N\}$ and we use the following approximations of the integrals

$$\int_{\tau_{j-1}}^{\tau_j} f(X(s))ds \approx f(X_{j-1})\delta t \quad \text{and} \quad \int_{\tau_{j-1}}^{\tau_j} g(X(s))dW(s) \approx g(X_{j-1})\delta W_j,$$

where $\delta W_j := W(\tau_j) - W(\tau_{j-1}) \sim \mathcal{N}(0, \delta t)$ are independent normally distributed random variables.

Hence, we can already define the following numerical scheme given by

$$X_j = X_{j-1} + f(X_{j-1})\delta t + g(X_{j-1})\delta W_j$$

for $j = 1, \dots, N$ and we obtain the simulations below for different values in the parameters of both models without and with wall growth, where the dashed line corresponds to the deterministic solutions and the other lines to the stochastic ones.

Firstly, we will show some simulations of the stochastic chemostat model without wall growth.

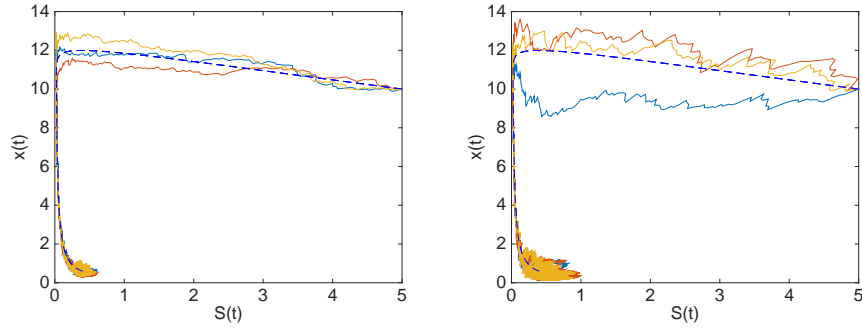


FIGURE 1. Stochastic chemostat without wall growth. Values of parameters: $S_0 = 5$, $x_0 = 10$, $S^0 = 1$, $D = 2$, $a = 0.6$, $m = 5$, $\alpha = 0.2$ (left) and $\alpha = 0.5$ (right)

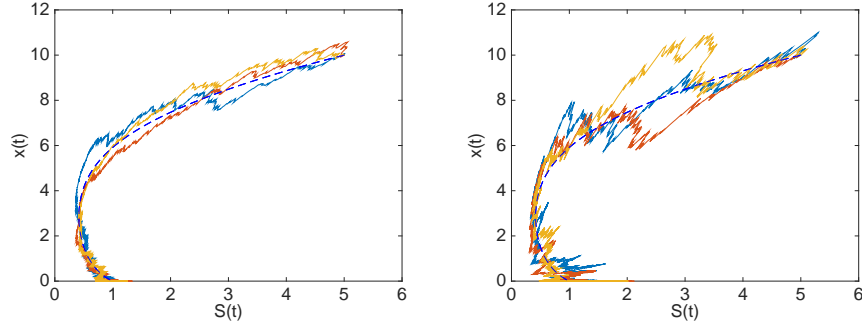


FIGURE 2. Stochastic chemostat without wall growth. Values of parameters: $S_0 = 5$, $x_0 = 10$, $S^0 = 1$, $D = 2$, $a = 0.6$, $m = 1$, $\alpha = 0.2$ (left) and $\alpha = 0.5$ (right)

Now, we will show some simulations for the stochastic chemostat model with wall growth by displaying two different panels in each figure: on the left we will show a general point of view of the dynamics; on the right, the viewer is supposed to be looking at the dynamics from point $(S_0, x_{01}, 0)$ in order to make the reader easier check whether the populations involved in our model remain strictly positive or not. Moreover, the thick black asterisk denotes the initial value (S_0, x_{01}, x_{02}) .

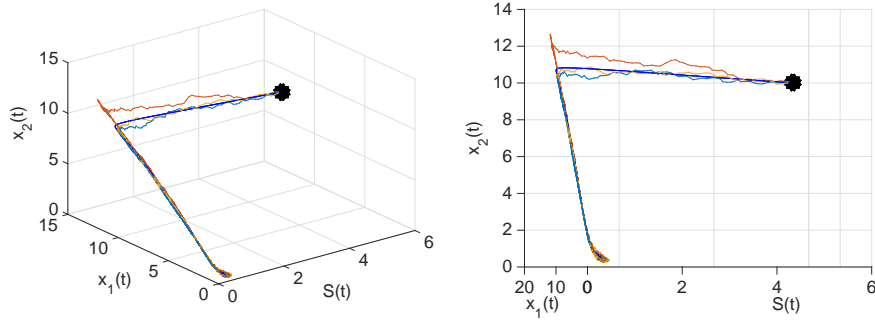


FIGURE 3. Stochastic chemostat with wall growth. Values of parameters: $S_0 = 5$, $x_{01} = 10$, $x_{02} = 10$, $S^0 = 1$, $D = 2$, $a = 0.6$, $m = 5$, $b = 0.5$, $r_1 = 0.2$, $r_2 = 0.8$, $\nu = 0.3$, $c = 3$, $\alpha = 0.2$

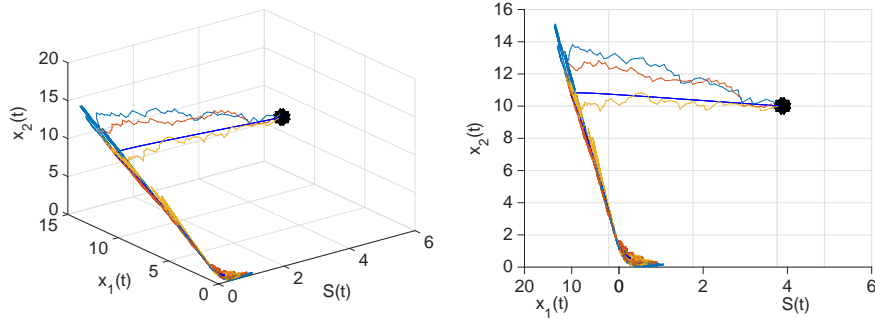


FIGURE 4. Stochastic chemostat with wall growth. Values of parameters: $S_0 = 5$, $x_{01} = 10$, $x_{02} = 10$, $S^0 = 1$, $D = 2$, $a = 0.6$, $m = 5$, $b = 0.5$, $r_1 = 0.2$, $r_2 = 0.8$, $\nu = 0.3$, $c = 3$, $\alpha = 0.5$

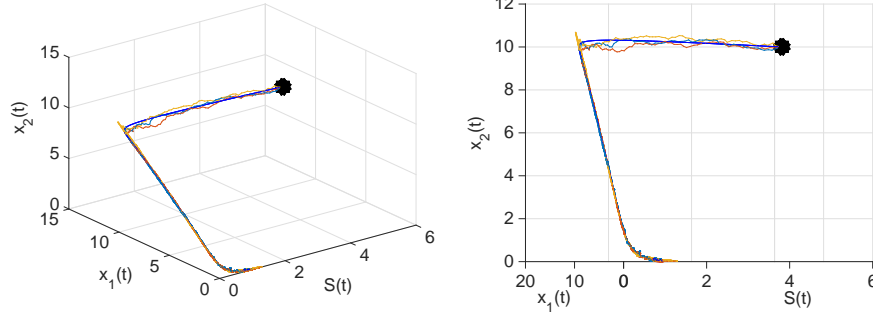


FIGURE 5. Stochastic chemostat with wall growth. Values of parameters: $S_0 = 5$, $x_{01} = 10$, $x_{02} = 10$, $S^0 = 1$, $D = 2$, $a = 0.6$, $m = 5$, $b = 0.5$, $r_1 = 0.2$, $r_2 = 0.8$, $\nu = 1.2$, $c = 1$, $\alpha = 0.2$

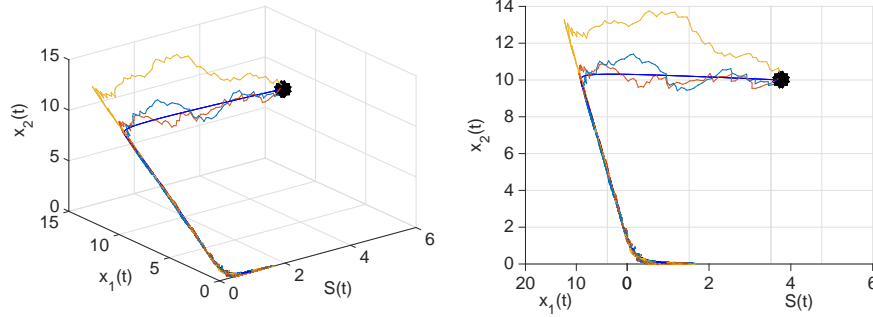


FIGURE 6. Stochastic chemostat with wall growth. Values of parameters: $S_0 = 5$, $x_{01} = 10$, $x_{02} = 10$, $S^0 = 1$, $D = 2$, $a = 0.6$, $m = 5$, $b = 0.5$, $r_1 = 0.2$, $r_2 = 0.8$, $\nu = 1.2$, $c = 1$, $\alpha = 0.5$

Acknowledgments. We would like to thank the referee for the interesting and helpful suggestions which allowed us to improve the presentation of our paper.

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Received xxxx 20xx; revised xxxx 20xx.

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