Manuscript submitted to
AIMS' Journals
Volume **XX**, Number **0X**, Month **XXXX**

doi:10.3934/dcdsb.XXXXXX
pp. **X–XX**

STOCHASTIC DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES DRIVEN BY A FRACTIONAL BROWNIAN MOTION

Ahmed Boudaoui

Laboratory of Mathematics, Univ Sidi Bel Abbes
PoBox 89, 22000 Sidi-Bel-Abbes, Algeria

Tomás Caraballo[1]

Departamento de Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla
Apdo. de Correos 1160, 41080 Sevilla, Spain

Abdelghani Ouahab

Laboratory of Mathematics, Univ Sidi Bel Abbes
PoBox 89, 22000 Sidi-Bel-Abbes, Algeria

Abstract. This paper is concerned with the existence and continuous dependence of mild solutions to stochastic differential equations with non–instantaneous impulses driven by fractional Brownian motions. Our approach is based on a Banach fixed point theorem and Krasnoselski-Schaefer type fixed point theorem.

1. Introduction.

Stochastic differential equations have many applications in science and engineering, and for this reason, these equations have been receiving much attention over the last decades (see, e.g., [4, 8, 10, 34, 15, 19, 30, 29, 1] and references therein). Also, in recent years, stochastic differential equations driven by fractional Brownian motions (fBm) have attracted much attention and there are only a few papers published in this field (see, e.g. [9, 11, 12]). Boudaoui et al. [6] discussed the existence of mild solutions to stochastic impulsive evolution equations with time delays, driven by fBm with Hurst index $H > \frac{1}{2}$.

On the other hand, impulsive effects exist in several evolution processes in which states are changed abruptly at certain moments of time, related to fields such as economics, bioengineering, chemical technology, medicine and biology etc (see [20, 33, 13, 27, 28]). Boudaoui et al. [8] obtained several new sufficient conditions to ensure the local and global existence and attractivity of mild solutions for stochastic neutral functional differential equations with instantaneous impulses, driven by fractional Brownian motions.

2010 *Mathematics Subject Classification.* Primary: 34A12, 34A37; Secondary: 35A01, 60H15, 60H20.

Key words and phrases. Fractional Brownian motion, fixed point, mild solutions, stochastic functional differential equation.

This work has been partially supported by grant MTM2015-63723-P (MINECO/FEDER, EU) and Consejería de Innovación, Ciencia y Empresa (Junta de Andalucía) under grant 2010/FQM314 and Proyecto de Excelencia P12-FQM-1492.

[1]Corresponding author

Recently, Hernández and O'Regan [18] initially study on Cauchy problems for a new type of first order evolution equations with non–instantaneous impulses. For example, impulses start abruptly at the instant t_k and their action continue on a finite time interval $(t_k, s_k]$. This type of problem motivates to study certain dynamical changes of evolution processes in pharmacotherapy [26, 14, 32]. As a motivation, we can mention a simple situation concerning the hemodynamical behavior of a person who has a decompensation of the glucose level (either high or low level). Then, this person can be prescribed some intravenous insulin to compensate the level. Since the introduction of the drug in the bloodstream and its absorption are gradually continuous processes, we can interpret the above situation as an impulsive action which remains active for a period of time, so a non-instantaneous impulse is taking place.

Hence, it is important to take into account the effect of impulses in the investigation of stochastic delay differential equations driven by fBm.

In this paper, our main objective is to establish sufficient conditions ensuring existence and continuous dependence of mild solutions to the following first order stochastic impulsive differential equation:

$$dy(t) = [Ay(t) + f(t, y_t)]dt + g(t, y_t)B_Q^H(t), \quad t \in J_k = (s_k, t_{k+1}], k = 0, \cdots, m, \tag{1}$$

$$y(t) = h_k(t, y_t) \quad t \in (t_k, s_k], k = 1, \cdots, m, \tag{2}$$

$$y(t) = \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad \text{for a.e. } t \in J_0 = (-\infty, 0], \tag{3}$$

in a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, where $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t), t \geq 0\}$ satisfying $\|S(t)\|^2 \leq M$, B_Q^H is a fractional Brownian motion on a real and separable Hilbert space $\mathcal{K}$, with Hurst parameter $H \in (1/2, 1)$, and with respect to a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ furnished with a family of right continuous and increasing σ-algebras $\{\mathcal{F}_t, t \in J = [0,T]\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. The impulse times t_k satisfy $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 < \cdots < t_m \leq s_m < t_{m+1} = T$.

As for y_t we mean the segment solution which is defined in the usual way, that is, if $y(\cdot,\cdot) : (-\infty, T] \times \Omega \to \mathcal{H}$, then for any $t \geq 0$, $y_t(\cdot,\cdot) : (-\infty, 0] \times \Omega \to \mathcal{H}$ is given by:

$$y_t(\theta, \omega) = y(t + \theta, \omega), \text{ for } \theta \in (-\infty, 0], \ \omega \in \Omega,$$

Before describing the properties fulfilled by the operators f, g, σ and h_k, we need to introduce some notation and describe some spaces.

In this work, we will use an axiomatic definition of the phase space $\mathcal{D}_{\mathcal{F}_0}$ introduced by Hale and Kato [17].

Definition 1.1. $\mathcal{D}_{\mathcal{F}_0}$ is a linear space of family of $\mathcal{F}_0$-measurable functions from $(-\infty, 0]$ into $\mathcal{H}$ endowed with a norm $\|\cdot\|_{\mathcal{D}_{\mathcal{F}_0}}$, which satisfies the following axioms:

- **(A-1):** If $y : (-\infty, T] \longrightarrow \mathcal{H}$, $T > 0$ is such that $y_0 \in \mathcal{D}_{\mathcal{F}_0}$, then for every $t \in [0, T)$ the following conditions hold
  - **(i):** $y_t \in \mathcal{D}_{\mathcal{F}_0}$
  - **(ii):** $\|y(t)\| \leq \mathcal{L}\|y_t\|_{\mathcal{D}_{\mathcal{F}_0}}$
  - **(iii):** $\|y_t\|_{\mathcal{D}_{\mathcal{F}_0}} \leq K(t) \sup\{\|y(s)\| : 0 \leq s \leq t\} + N(t)\|y_0\|_{\mathcal{D}_{\mathcal{F}_0}}$,

    where $\mathcal{L} > 0$ is a constant; $K, N : [0, \infty) \longrightarrow [0, \infty)$, K is continuous, N is locally bounded and K, N are independent of $y(\cdot)$.
- **(A-2):** For the function $y(\cdot)$ in (A-1), y_t is a $\mathcal{D}_{\mathcal{F}_0}$-valued function for $t \in [0, T)$.
- **(A-3):** The space $\mathcal{D}_{\mathcal{F}_0}$ is complete.

Denote

$$\widetilde{K} = \sup\{K(t) : t \in J\} \text{ and } \widetilde{N} = \sup\{N(t) : t \in J\}.$$

Now, for a given $T > 0$, we define

$$\begin{aligned}\mathcal{D}_{\mathcal{F}_T} = \big\{y\colon\ &(-\infty, T] \times \Omega \to \mathcal{H},\ y|_{J_k} \in C((t_k, t_{k+1}], \mathcal{H}), \text{for all } \omega \in \Omega, \quad \text{for } k = 0, \dots m,\\ &y_0 \in \mathcal{D}_{\mathcal{F}_0},\ \text{and there exist} \quad y(t_k^-) \text{ and } y(t_k^+)\\ &\text{with } y(t_k) = y(t_k^-),\ \ k = 1, \cdots, m, \text{ and } \sup_{t\in[0,T]} E(\|y(t)\|^2) < \infty\big\},\end{aligned}$$

endowed with the norm

$$\|y\|_{\mathcal{D}_{\mathcal{F}_T}} = \|\phi\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{0\le s\le T} (E\|y(s)\|^2)^{\frac{1}{2}},$$

where $J_k = (t_k, t_{k+1}]$, $k = 0, 1, \cdots, m$, and $J_0 = (-\infty, 0]$.

Then we will consider our initial data $\phi \in \mathcal{D}_{\mathcal{F}_0}$.

Assume that B_Q^H is a $\mathcal{K}$-valued fractional Brownian motion with increment covariance given by a non-negative trace class operator Q (see next section for more details), and let us denote by $L(\mathcal{K}, \mathcal{H})$ the space of all bounded, continuous and linear operators from $\mathcal{K}$ into $\mathcal{H}$.

Then we assume that $h_k \in C((t_k, s_k] \times \mathcal{D}_{\mathcal{F}_0}, \mathcal{H})$ for all $k = 1, \cdots, m$, $f : J \times \mathcal{D}_{\mathcal{F}_0} \to \mathcal{H}$ and $\sigma : J \times \mathcal{D}_{\mathcal{F}_0} \to L_Q^0(\mathcal{K}, \mathcal{H})$ Here, $L_Q^0(\mathcal{K}, \mathcal{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathcal{K}$ into $\mathcal{H}$, which will be also defined in the next section.

The plan of this paper is as follows. In Section 2 we introduce notations, definitions, and preliminary facts which are useful throughout the paper. In Section 3 we state and prove our main results by using the Banach fixed point theorem and Krasnoselskii-Schaefer type fixed point theorem [3]. The continuous dependence of mild solutions to problem (1)-(3) is investigated in Section 4. Finally, in Section 5, an example is exhibited to illustrate the applicability of our results.

2. **Preliminaries.**

In this section we introduce notations, definitions, and preliminary facts which will be used throughout this paper. In particular, we consider a fractional Brownian motion as well as the Wiener integral with respect to it. We also establish some important and helpful results for our analysis. Needless to say that we could omit this whole section and refer to other works already published for these preliminaries (see, e.g. [6] and the references therein), but we aim at making this paper as much self contained as possible and this is why we decided to include this material in this section.

Recall that $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a complete probability space furnished with a family of right continuous increasing σ-algebras $\{\mathcal{F}_t, t \in J\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$.

Definition 2.1. Given $H \in (0, 1)$, a continuous centered Gaussian process $\beta^H = \{\beta^H(t), t \in \mathbb{R}\}$, with the covariance function

$$R_H(t, s) = E[\beta^H(t)\beta^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), t, s \in \mathbb{R}$$

is called a two$-$sided one$-$dimensional fractional Brownian motion, and H is the Hurst parameter.

Now we aim at introducing the Wiener integral with respect to the one-dimensional β^H.

Let $T > 0$ and denote by Λ the linear space of $\mathbb{R}-$valued step functions on $[0, T]$, that is, $\psi \in \Lambda$ if

$$\psi(t) = \sum_{i=1}^{n-1} x_i 1_{[s_i, s_{i-1})}(t),$$

where $t \in [0, T], x_i \in \mathbb{R}$ and $0 = s_1 < s_2 < \cdots < s_n = T$. For $\psi \in \Lambda$ we define its Wiener integral with respect to β^H as

$$\int_0^T \psi(\sigma) d\beta^H(\sigma) = \sum_{i=1}^{n-1} x_i(\beta^H(s_{i+1}) - \beta^H(s_i)).$$

Let $\mathcal{H}$ be the Hilbert space defined as the closure of Λ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

Then, the mapping

$$\psi = \sum_{i=1}^{n-1} x_i 1_{[s_i, s_{i+1})} \mapsto \int_0^T \psi(\sigma) d\beta^H(\sigma)$$

is an isometry between Λ and the linear space span $\{\beta^H(t), t \in [0, T]\}$, which can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fractional Brownian motion $\overline{span}^{L^2(\Omega)}\{\beta^H(t), t \in [0, T]\}$ (see [31]). The image of an element $\psi \in \mathcal{H}$ by this isometry is called the Wiener integral of ψ with respect to β^H. Our next goal is to give an explicit expression of this integral. To this end, consider the kernel

$$K_H(t, s) = c_H s^{1/2-H} \int_s^t (u - s)^{H-3/2} u^{H-1/2} du,$$

where $c_H = \left(\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}\right)^{1/2}$, with $B(\cdot, \cdot)$ denoting the Beta function, and $t \leq s$.

It is not difficult to see that

$$\frac{\partial K_H}{\partial t}(t, s) = c_H \left(\frac{t}{s}\right)^{\frac{1}{2}-H} (t - s)^{H-\frac{3}{2}}.$$

Consider the linear operator $K_H^* : \Lambda \longrightarrow L^2([0, T])$ given by

$$(K_H^* \Phi)(s) = \int_s^t \Phi(t) \frac{\partial K_H}{\partial t}(t, s) dt.$$

Then

$$(K_H^* 1_{[0,t]})(s) = K_H(t, s) 1_{[0,t]}(s),$$

and K_H^* is an isometry between Λ and $L^2([0, T])$ that can be extended to Λ (see [2]). Considering $W = \{W(t), t \in [0, T]\}$ defined by

$$W(t) = \beta^H((K_H^*)^{-1} 1_{[0,t]}),$$

it turns out that W is a Wiener process and β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s) dW(s).$$

In addition, for any $\Phi \in \Lambda$,

$$\int_0^T \Phi(s) \beta^H(s) dW(s) = \int_0^T (K_H^* \Phi)(t) dW(t)$$

if and only if $K_H^*\Phi \in L^2([0,T])$.

Also denoting

$$L^2_{\mathcal{H}}([0,T]) = \{\Phi \in \Lambda, K_H^*\Phi \in L^2([0,T])\},$$

since $H > 1/2$, we have

$$L^{1/H}([0,T]) \subset L^2_{\mathcal{H}}([0,T]), \tag{4}$$

see [23]. Moreover, the following useful result holds:

Lemma 2.2. [24]. *For* $\Phi \in L^{1/H}([0,T])$,

$$H(2H-1)\int_0^T\int_0^T |\Phi(r)\|\Phi(u)\|r-u|^{2H-2}drdu \le c_H\|\Phi\|^2_{L^{1/H}([0,T])}.$$

Next we are interested in considering a fractional Brownian motion with values in a Hilbert space and giving the definition of the corresponding stochastic integral.

Let $Q \in L(\mathcal{K},\mathcal{H})$ be a non-negative self$-$adjoint operator. Denote by $L^0_Q(\mathcal{K},\mathcal{H})$ the space of all $\xi \in L(\mathcal{K},\mathcal{H})$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$|\xi|^2_{L^0_Q(\mathcal{K},\mathcal{H})} = tr(\xi Q \xi^*).$$

Then ξ is called a Q-Hilbert-Schmidt operator from $\mathcal{K}$ to $\mathcal{H}$.

Let $\{\beta_n^H(t)\}_{n\in N}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega,\mathcal{F},P)$. When one considers the following series

$$\sum_{n=1}^{\infty}\beta_n^H(t)e_n, \quad t \ge 0,$$

where $\{e_n\}_{n\in N}$ is a complete orthonormal basis in $\mathcal{K}$, this series does not necessarily converge in the space $\mathcal{K}$. Thus we consider a $\mathcal{K}-$valued stochastic process $B_Q^H(t)$ given formally by the following series:

$$B_Q^H(t) = \sum_{n=1}^{\infty}\beta_n^H(t)Q^{\frac{1}{2}}e_n, \quad t \ge 0,$$

which is well-defined as a $\mathcal{K}$-valued Q-cylindrical fractional Brownian motion.

Let $\varphi : [0,T] \mapsto L^0_Q(\mathcal{K},\mathcal{H})$ such that

$$\sum_{n=1}^{\infty}\|K_H^*(\varphi Q^{\frac{1}{2}}e_n)\|_{L^{1/H}([0,T];\mathcal{H})} < \infty \tag{5}$$

Definition 2.3. Let $\varphi : [0,T] \longrightarrow L^0_Q(\mathcal{K},\mathcal{H})$ satisfy (5). Then, its stochastic integral with respect to the fractional Brownian motion B_Q^H is defined, for $t \ge 0$, as follows

$$\int_0^t \varphi(s)dB_Q^H(s) := \sum_{n=1}^{\infty}\int_0^t \varphi(s)Q^{1/2}e_n d\beta_n^H(s) = \sum_{n=1}^{\infty}\int_0^t (K_H^*(\varphi Q^{1/2}e_n))(s)dW(s).$$

Notice that if

$$\sum_{n=1}^{\infty}\|\varphi Q^{1/2}e_n\|_{L^{1/H}([0,T];\mathcal{H})} < \infty, \tag{6}$$

then in particular (5) holds, which follows immediately from (4).

Lemma 2.4. [11] *if* $\varphi : [0,T] \longrightarrow L_Q^0(\mathcal{K},\mathcal{H})$ *satisfies*

$$\int_0^T \|\varphi(s)\|^2_{L_Q^0(\mathcal{K},\mathcal{H})} ds < \infty,$$

then the series in (6) is well defined as a $\mathcal{H}$*-valued random variable and we have*

$$E\left\|\int_0^t \varphi(s) dB_Q^H(s)\right\|^2 \leq 2Ht^{2H-1}\int_0^t \|\varphi(s)\|^2_{L_Q^0(\mathcal{K},\mathcal{H})} ds.$$

Definition 2.5. The map $f : J \times \mathcal{D}_{\mathcal{F}_0} \to \mathcal{H}$ is said to be L^2-Carathéodory if

(i): $t \mapsto f(t,v)$ is measurable for each $v \in \mathcal{D}_{\mathcal{F}_0}$;
(ii): $v \mapsto f(t,v)$ is continuous for almost all $t \in J$;
(iii): for each $q > 0$, there exists $\alpha_q \in L^1(J,\mathbb{R}^+)$ such that

$$E\|f(t,v)\|^2 \leq \alpha_q(t), \text{for all } \|v\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq q \text{ and for a.e. } t \in J.$$

Definition 2.6. The map $g : J \times \mathcal{D}_{\mathcal{F}_0} \to L_Q^0(\mathcal{K},\mathcal{H})$ is said to be L^2-Carathéodory if

(i): $t \mapsto g(t,v)$ is measurable for each $v \in \mathcal{D}_{\mathcal{F}_0}$;
(ii): $v \mapsto g(t,v)$ is continuous for almost all $t \in J$;
(iii): for each $q > 0$, there exists $\alpha_{1q} \in L^1(J,\mathbb{R}^+)$ such that

$$E\|g(t,v)\|^2_{L_Q^0(\mathcal{K},\mathcal{H})} \leq \alpha_{1q}(t), \text{for all } \|v\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq q \text{ and for a.e. } t \in J.$$

The following result is known as Gronwal-Bihari Theorem.

Lemma 2.7. [5] *Let* u, g: $J \to \mathbb{R}$ *be positive real continuous functions. Assume there exist* $c > 0$ *and a continuous nondecreasing function* h: $\mathbb{R} \to (0,+\infty)$ *such that*

$$u(t) \leq c + \int_a^t g(s)h(u(s))\, ds, \quad \forall\, t \in J.$$

Then

$$u(t) \leq H^{-1}\left(\int_a^t g(s)\, ds\right), \quad \forall\, t \in J$$

provided

$$\int_c^{+\infty} \frac{dy}{h(y)} > \int_a^b g(s)\, ds.$$

Here H^{-1} *refers to the inverse of the function* $H(u) = \int_c^u \frac{dy}{h(y)}$ *for* $u \geq c$.

One of the key tools in our approach is the following form of Burton-Kirk's fixed point theorem

Theorem 2.8 (Burton-Kirk's fixed point theorem [3])**.** *Let* E *be a Banach space, and* $G_1, G_2 : E \to E$ *be two operators satisfying:*

1. G_1 *is a contraction, and*
2. G_2 *is completely continuous*

Then, either the operator equation $y = G_1(y) + G_2(y)$ *possesses a solution, or the set* $\Xi = \left\{y \in E : \lambda G_1(\frac{y}{\lambda}) + \lambda G_2(y) = y, \text{ for some } \lambda \in (0,1)\right\}$ *is unbounded.*

3. **Existence of mild solution.** In this section, we first establish the existence of mill solutions to stochastic differential equations with non-instantaneous impulses driven by fractional Brownian motions (1)-(3). More precisely, we will formulate and prove sufficient conditions for the existence of solutions to (1)-(3) with infinite delay. In order to establish the results, we will need to impose some of the following conditions.

- (H1) There exist constants $L_f > 0$, such that
$$E\|f(t,\phi_1) - f(t,\phi_2)\|^2 \le L_f\|\phi_1 - \phi_2\|^2_{\mathcal{D}_{\mathcal{F}_0}}$$
for all $\phi_1, \phi_2 \in \mathcal{D}_{\mathcal{F}_0}$,$t \in (s_k, t_{k+1}]$ and $k = 0, \dots, m$.
- (H2) There exist constants $L_g > 0$ such that
$$E\|g(t,\phi_1) - g(t,\phi_2)\|^2_{L^0_Q(\mathcal{K},\mathcal{H})} \le L_g\|\phi_1 - \phi_2\|^2_{\mathcal{D}_{\mathcal{F}_0}}$$
for all $\phi_1, \phi_2 \in \mathcal{D}_{\mathcal{F}_0}$, $t \in (s_k, t_{k+1}]$ and $k = 0, \dots, m$.
- (H3) There exist constants $L_{h_k} > 0$, for all $\phi_1, \phi_2 \in \mathcal{D}_{\mathcal{F}_0}$, $t \in (t_k, s_k]$ and $k = 1, \dots, m$ such that
$$E\|h_k(t,\phi_1) - h_k(t,\phi_2)\|^2 \le L_{h_k}\|\phi_1 - \phi_2\|^2_{\mathcal{D}_{\mathcal{F}_0}}$$
and
$$h_k \in C((t_k, s_k] \times \mathcal{D}_{\mathcal{F}_0}, \mathcal{H}), \text{ for all } k = 1, \cdots, m.$$
- (H4) f and g are a L^2-Caratheodory map and for every $t \in [0,T]$ the function $t \to f(t, y_t)$ and $t \to g(t, y_t)$, $y_t \in \mathcal{D}_{\mathcal{F}_0}$ are mesurable
- (H5) For the initial value $\phi \in \mathcal{D}_{\mathcal{F}_0}$, there exists a continuous nondecreasing functions $\psi, \psi_1 : [0,\infty) \to [0,\infty)$ and $p, p_1 \in L^1(J, \mathbb{R}_+)$ such that
$$E\|f(t,y)\|^2 \le p(t)\psi(\|y\|^2_{\mathcal{D}_{\mathcal{F}_0}})$$
and
$$E\|g(t,y)\|^2_{L^0_Q(\mathcal{K},\mathcal{H})} \le p_1(t)\psi_1(\|y\|^2_{\mathcal{D}_{\mathcal{F}_0}})$$
for a.e. $t \in J$ and $y \in \mathcal{D}_{\mathcal{F}_0}$ with
$$K_1\int_{s_k}^{t_{k+1}} p(s)ds + K_2\int_{s_k}^{t_{k+1}} p_1(s)ds < \int_{K_0}^{\infty}\frac{ds}{\psi(s)+\psi_1(s)} \quad k = 0, \cdots, m,$$
where
$$K_0 = 4\widetilde{K}^2ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2\|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}},\ K_1 = 8\widetilde{K}^2Mt_1,\ K_2 = 16\widetilde{K}^2MHt_1^{2H-1},$$
for $t \in [0, t_1]$, and
$$K_0 = \frac{4\widetilde{K}^2ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2\|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}}{1 - 12\widetilde{K}^2ML_h}, K_1 = \frac{12\widetilde{K}^2M(t_{k+1} - s_k)}{1 - 12\widetilde{K}^2ML_h},$$
for $t \in [s_k, t_{k+1}]$, $k = 1, \cdots, m$.
$$K_2 = \frac{24\widetilde{K}^2MHt_1^{2H-1}}{1 - 12\widetilde{K}^2ML_h}, \text{for } t \in [s_k, t_{k+1}],\ k = 1, \cdots, m.$$
- (H6) Operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t)\}$, $t \in J$ which is compact for $t > 0$ in $\mathcal{H}$.

Now, we present the definition of mild solutions to our problem.

Definition 3.1. Given $\phi \in \mathcal{D}_{\mathcal{F}_0}$, a $\mathcal{H}-$valued stochastic process $\{y(t), t \in (-\infty, T]\}$ is called a mild solution of the problem (1)-(3) if $y(t)$ is measurable and $\mathcal{F}_t$-adapted, for each $t > 0$, $y(t) = \phi(t)$ on $(-\infty, 0]$, for each $t > 0$, $y(t) = h_k(t, y_t)$ for all $t \in (t_k, s_k]$ and each $k = 1, \cdots, m$, and

$$y(t) = S(t)\phi(0) + \int_0^t S(t-s)f(s, y_s)ds + \int_0^t g(s, y_s)dB_Q^H d(s) \text{ for all } t \in [0, t_1]$$

and

$$y(t) = S(t - s_k)h_k(s_k, y_{s_k}) + \int_{s_k}^t S(t-s)f(s, y_s)ds + \int_{s_k}^t g(s, y_s)dB_Q^H d(s)$$

for all $t \in [s_k, t_{k+1}]$ and every $k = 1, \cdots, m$.

For our main consideration of Problem (1)-(3), a Banach fixed point is used to investigate the existence and uniqueness of solutions for impulsive stochastic differential equations.

Theorem 3.2. *Assume conditions (H1)-(H3) are satisfied and*

$$L_0 = \max\{\mu_1, \mu_2, \mu_3\} < 1,$$

where $\mu_1 = 4M\widetilde{K}^2(t_1^2 L_f + 2Ht_1^{2H} l_g)$, $\mu_2 = 2\widetilde{K}^2 L_{h_k}$ *and* $\mu_3 = 6M\widetilde{K}^2(L_{h_k} + L_f(t_{k+1} - s_k)^2 + 2H(t_{k+1} - s_k)^{2H} L_g)$. *Then, for every initial function* $\phi \in \mathcal{D}_{\mathcal{F}_0}$ *there exists a unique associated mild solution* $y \in \mathcal{D}_{\mathcal{F}_T}$ *of the problem (1)-(3).*

Remark 1. It is worth mentioning that the assumption $L_0 < 1$ in Theorem 3.2 implies some kind of smallness of the functions f, g and h_k in comparison with the periods of time when the impulses are active or viceversa.

Proof. We first transform problem (1)-(3) into a fixed point one. Consider operator $\Phi : \mathcal{D}_{\mathcal{F}_T} \to \mathcal{D}_{\mathcal{F}_T}$ defined by

$$\Phi(y)(t) = \begin{cases} \phi(t), t \in (-\infty, 0], \\ S(t)\phi(0) + \int_0^t S(t-s)f(s, y_s)ds + \int_0^t S(t-s)g(s, y_s)dB_Q^H(s), \text{ if } t \in [0, t_1], \\ h_k(t, y_t), \text{ if } t \in (t_k, s_k], \\ S(t - s_k)h_k(s_k, y_{s_k}) + \int_{s_k}^t S(t-s)f(s, y_s)ds \\ + \int_{s_k}^t S(t-s)g(s, y_s)dB_Q^H(s), \text{ if } t \in [s_k, t_{k+1}], k = 1, 2, \cdots, m. \end{cases}$$

For $\phi \in \mathcal{D}_{\mathcal{F}_0}$, we define $\widehat{\phi}$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ S(t)\phi(0), & t \in [0, T]; \end{cases}$$

then $\widehat{\phi} \in \mathcal{D}_{\mathcal{F}_T}$.

Let $y(t) = z(t) + \widehat{\phi}(t), -\infty < t \leq T$. It is evident that z satisfies $z_0 = 0$, for all $t \in (-\infty, 0]$,

$$z(t) = \int_0^t S(t-s)f(s, z_s + \widehat{\phi}_s)ds + \int_0^t S(t-s)g(s, z_s + \widehat{\phi}_s)dB_Q^H(s), t \in [0, t_1],$$

$z(t) = h_k(t, z_t + \widehat{\phi}_t), t \in (t_k, s_k]$
and

$$z(t) = S(t - s_k)h_k(s_k, z_{s_k} + \widehat{\phi}_{s_k}) + \int_{s_k}^{t} S(t-s)f(s, z_s + \widehat{\phi}_s)ds$$
$$+ \int_{s_k}^{t} S(t-s)g(s, z_s + \widehat{\phi}_s)dB_Q^H(s), \text{ if } t \in [s_k, t_{k+1}], k = 1, 2, \cdots, m.$$

Set $\mathcal{D}'_{\mathcal{F}_T} = \{z \in \mathcal{D}_{\mathcal{F}_T}, \quad \text{such that} \quad z_0 = 0 \in \mathcal{D}_{\mathcal{F}_0}\}$ and for any $z \in \mathcal{D}_{\mathcal{F}_0}$ we have

$$\|z\|_{\mathcal{F}_T} = \|z_0\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{t \in J}(E\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t \in J}(E\|z(t)\|^2)^{\frac{1}{2}},$$

then $(\mathcal{D}'_{\mathcal{F}_T}, \|\cdot\|_{\mathcal{F}_T})$ is a Banach space.

Let the operator $\widehat{\Phi} : \mathcal{D}'_{\mathcal{F}_T} \to \mathcal{D}'_{\mathcal{F}_T}$ defined by

$$\widehat{\Phi}(z) = \begin{cases} 0, \text{ if } t \in (-\infty, 0], \\ \int_0^t S(t-s)f(s, z_s + \widehat{\phi}_s)ds + \int_0^t S(t-s)g(s, z_s + \widehat{\phi}_s)dB_Q^H(s), t \in [0, t_1], \\ h_k(t, z_t + \widehat{\phi}_t), t \in (t_k, s_k], \\ S(t - t_k)h_k(s_k, z_{s_k} + \widehat{\phi}_{s_k}) + \int_{s_k}^t S(t-s)f(s, z_s + \widehat{\phi}_s)ds \\ + \int_{s_k}^t S(t-s)g(s, z_s + \widehat{\phi}_s)dB_Q^H(s), \text{ if } t \in [s_k, t_{k+1}]. \end{cases}$$

From the assumption it is easy to see that $\widehat{\Phi}$ is well defined. Now we only need to show that $\widehat{\Phi}$ is a contraction mapping.

Case 1: For u, $v \in \mathcal{D}'_{\mathcal{F}_T}$ and for $t \in [0, t_1]$, we have

$$\begin{aligned} &E\|\widehat{\Phi}(u)(t) - \widehat{\Phi}(v)(t)\|^2 \\ &\leq 2E\left\|\int_0^t S(t-s)(f(s, u_s + \widehat{\phi}_s) - f(s, v_s + \widehat{\phi}_s))ds\right\|^2 \\ &\quad +2E\left\|\int_0^t S(t-s)(g(s, u_s + \widehat{\phi}_s) - g(s, v_s + \widehat{\phi}_s))dB_Q^H(s)\right\|^2 \\ &\leq 2M(t_1^2 L_f + 2Ht_1^{2H} L_g)\|u - v\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ &\leq 2M(t_1^2 L_f + 2Ht_1^{2H} L_g) \times \left[2\widetilde{K}^2 \sup_{0 \leq s \leq T} E\|u(s) - v(s)\|^2 \right. \\ &\quad \left. +2\widetilde{N}\|u_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 2\widetilde{N}\|v_0\|^2_{\mathcal{D}_{\mathcal{F}_0}}\right], \\ &\leq 4M\widetilde{K}^2(t_1^2 L_f + 2Ht_1^{2H} L_g) \sup_{0 \leq s \leq t_1} E\|u(s) - v(s)\|^2 \end{aligned}$$

Since $\|u_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} = 0$, $\|v_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} = 0$. Taking the supremum over t, we obtain

$$\|\widehat{\Phi}(u) - \widehat{\Phi}(v)\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq 4M\widetilde{K}^2(t_1^2 L_f + 2Ht_1^{2H} L_g)\|v - u\|^2_{\mathcal{D}'_{\mathcal{F}_T}},$$

Case 2: For u, $v \in \mathcal{D}'_{\mathcal{F}_T}$ and for $t \in (t_k, s_k]$, $k = 1, 2, \cdots, m$, we have

$$\begin{aligned} E\|\widehat{\Phi}(u)(t) - \widehat{\Phi}(v)(t)\|^2 &\leq E\left\|h_k(t, u_t + \widehat{\phi}_t) - h_k(t, v_t + \widehat{\phi}_t)\right\|^2 \\ &\leq L_{h_k}\|u - v\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ &\leq 2\widetilde{K}^2 L_{h_k}\|u - v\|_{\mathcal{D}'_{\mathcal{F}_T}} \end{aligned}$$

Case 3: For $u,\, v \in \mathcal{D}^{'}_{\mathcal{F}_T}$ and for $t \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, m$, we have

$$\begin{aligned}
& E\|\widehat{\Phi}(u)(t) - \widehat{\Phi}(v)(t)\|^2 \\
& \quad \leq 3E\|S(t-t_k)(h(s_k, u_{s_k} + \widehat{\phi}_{s_k}) - h(s_k, v_{s_k} + \widehat{\phi}_{s_k}))\|^2 \\
& \qquad + 3E\left\|\int_{s_k}^{t} S(t-s)(f(s, u_s + \widehat{\phi}_s) - f(s, v_s + \widehat{\phi}_s))ds\right\|^2 \\
& \qquad + 3E\left\|\int_{0}^{t} S(t-s)(g(s, u_s + \widehat{\phi}_s)) - g(s, v_s + \widehat{\phi}_s)))dB^H_Q(s)\right\|^2 \\
& \quad \leq 3ML_{h_k}\|u-v\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 3ML_f(t_{k+1}-s_k)^2\|x-y\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\
& \qquad + 6H(t_{k+1}-s_k)^{2H}L_g)\|x-y\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\
& \quad \leq 6M\widetilde{K}^2(L_{h_k} + L_f(t_{k+1}-s_k)^2 + 2H(t_{k+1}-s_k)^{2H}L_g)\|x-y\|^2_{\mathcal{D}'_{\mathcal{F}_T}}.
\end{aligned}$$

From above, we obtain

$$\|\widehat{\Phi}u - \widehat{\Phi}v\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq L_0\|u-v\|^2_{\mathcal{D}'_{\mathcal{F}_T}},$$

which implies that $\widehat{\Phi}$ is a contraction and therefore has a unique fixed point $z \in \mathcal{D}^{'}_{\mathcal{F}_T}$. Since $y(t) = z(t) + \widehat{\phi}(t)$, $t \in (-\infty, T]$, then y is a fixed point of the operator Φ which is a mild solution of the problem (1)-(3). This completes the proof.

□

The second result is established using a Krasnoselskii-Schaefer type fixed point theorem. As we can easily see, we will weaken the assumption $L_0 < 1$ in Theorem 3.2, but at the same time we need to impose some Carathéodory and Nagumo type of assumptions as well as an additional smallness hypothesis. The counterpart is that we can prove existence of at least one bounded solution for any initial value satisfying appropriate assumptions.

Theorem 3.3. *Assume that $h_k(t,0) = 0, t \geq 0$, $k \in \mathbb{N}^*$, and hypotheses $(H3)-(H6)$ hold. If $L_1 = \max\{2\widetilde{K}^2 L_{h_k}, 2M\widetilde{K}^2 L_{h_k}\} < 1$ then, problem (1)-(3) possesses at least one mild solution on $(-\infty, T]$.*

Proof. As in the proof of Theorem 3.2, we first transform our problem (1)-(3) into the same fixed point formulation, and keep the same notation for the operators $\Phi, \widehat{\Phi}$ and the spaces $\mathcal{D}_{\mathcal{F}_T}$ and $\mathcal{D}^{'}_{\mathcal{F}_T}$.

Now we split our operator $\widehat{\Phi}$ into two parts in the following way:

$$\widehat{\Phi}_1(z) = \begin{cases} 0, \text{ if } t \in (-\infty, t_1], \\ \\ h_k(t, z_t + \widehat{\phi}_t), t \in (t_k, s_k], k = 1, \ldots, m, \\ \\ S(t-s_k)h_k(s_k, z_{s_k} + \widehat{\phi}_{s_k}), \text{ if } t \in [s_k, t_{k+1}], \quad k = 1, \ldots, m, \end{cases}$$

and

$$\widehat{\Phi}_2(z)(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0], \\ 0, & t \in (t_k, s_k], k = 1, \ldots, m, \\ \displaystyle\int_{s_k}^{t} S(t-s)f(s, z_s + \widehat{\phi}_s)ds & \\ \displaystyle + \int_{s_k}^{t} S(t-s)g(s, z_s + \widehat{\phi}_s)dB^H_Q(s), & \text{if } t \in [s_k, t_{k+1}], k = 0, \ldots, m \end{cases}$$

To use Theorem 2.8 we will verify that $\widehat{\Phi}_1$ is a contraction while $\widehat{\Phi}_2$ is a completely continuous operator. For the sake of convenience, we split the proof into several steps.

Step 1 : $\widehat{\Phi}_1$ is a contraction.

Case 1: For $u,\, v \in \mathcal{D}'_{\mathcal{F}_T}$ and for $t \in (t_k, s_k]$, $k = 1, 2, \cdots, m$, we have

$$\begin{aligned} E\|(\widehat{\Phi}_1 u)(t) - (\widehat{\Phi}_1 v)(t)\|^2 &\leq E\left\| h_k(t, u_t + \widehat{\phi}_t) - h_k(t, v_t + \widehat{\phi}_t)\right\|^2 \\ &\leq 2\widetilde{K}^2 L_{h_k} \|u - v\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \end{aligned}$$

Case 2: For $u,\, v \in \mathcal{D}'_{\mathcal{F}_T}$ and for $t \in [s_k, t_{k+1}]$, $k = 1, 2, \cdots, m$, we have

$$\begin{aligned} E\|(\widehat{\Phi}_1 u)(t) - (\widehat{\Phi}_1 v)(t)\|^2 &\leq E\|S(t - s_k)[h(s_k, u_{s_k} + \widehat{\phi}_{s_k}) - h(s_k, v_{s_k} + \widehat{\phi}_{s_k})]\|^2 \\ &\leq 2M\widetilde{K}^2 L_{h_k} \|u - v\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \end{aligned}$$

Taking the supremum over t, we obtain

$$\|\widehat{\Phi}_1(u) - \widehat{\Phi}_1(v)\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq L_1 \|u - v\|^2_{\mathcal{D}'_{\mathcal{F}_T}}.$$

Thus $\widehat{\Phi}_1$ is a contraction.

Next, we aim to prove that the operator $\widehat{\Phi}_2$ is completely continuous.

Step 2 : $\widehat{\Phi}_2$ is continuous .

Let z^n be a sequence such that $z^n \to z$ in $\mathcal{D}'_{\mathcal{F}_T}$. Then, for $t \in (s_k, t_{k+1}]$, $k = 0, 1, \cdots, m$, and thanks to $(H4)$,
we have by the dominated convergence theorem

$$\begin{aligned} & E\|(\widehat{\Phi}_2 z^n)(t) - \widehat{\Phi}_2(z)(t)\|^2 \\ &\leq 2E\left\|\int_{s_k}^t S(t-s)[f(s, z^n_s + \widehat{\phi}_s) - f(s, z_s + \widehat{\phi}_s)]ds\right\|^2 \\ &\quad + 2E\left\|\int_{s_k}^t S(t-s)[g(s, z^n_s + \widehat{\phi}_s) - g(s, z_s + \widehat{\phi}_s)]dB^H_Q(s)\right\|^2 \\ &\leq 2M(t_{k+1} - s_k)\int_{s_k}^t E\|f(s, z^n_s + \widehat{\phi}_s) - f(s, z_s + \widehat{\phi}_s)\|^2 ds \\ &\quad + 4MH(t_{k+1} - s_k)^{2H-1}\int_{s_k}^t E\|g(s, z^n_s + \widehat{\phi}_s) - g(s, z_s + \widehat{\phi}_s)\|^2_{L^0_Q(\mathcal{K},\mathcal{H})} ds \to 0. \end{aligned}$$

as $n \to +\infty$. Thus $\widehat{\Phi}_2$ is continuous on $\mathcal{D}'_{\mathcal{F}_T}$.

Step 3. $\widehat{\Phi}_2$ maps bounded sets into bounded sets in $\mathcal{D}'_{\mathcal{F}_T}$.

Let us first define $\mathcal{B}_q = \left\{ z \in \mathcal{D}'_{\mathcal{F}_T}, \quad \|z\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq q, \right\}$ where $q > 0$ is a given number.

The set $\mathcal{B}_q$ is clearly a bounded closed convex set in $\mathcal{D}'_{\mathcal{F}_T}$. We then have

$$\begin{aligned} \|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq\ & 2(\|z_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\ &\leq 4(\widetilde{N}^2\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \widetilde{K}^2(q + ME\|\phi(0)\|^2)) \\ &= q'. \end{aligned}$$

Now, to prove that $\widehat{\Phi}_2$ maps bounded sets into bounded sets in $\mathcal{D}^{'}_{\mathcal{F}_T}$, it is enough to show that for any $q > 0$, there exists a positive constant l such that for each $z \in \mathcal{B}_q = \{z \in \mathcal{D}^{'}_{\mathcal{F}_T} : \|z\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq q\}$, one has $||\widehat{\Phi}_2(z)||^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq l$.
Let $z \in \mathcal{B}_q$, then for each $t \in [s_k, t_{k+1}]$, $k = 0, 1, \cdots, m$, we have

$$\begin{aligned}
&E\|(\widehat{\Phi}_2 z)(t)\|^2 \\
&= E\left\|\int_{s_k}^{t_{k+1}} S(t-s)f(s, z_s + \widehat{\phi}_s)ds + \int_{s_k}^{t_{k+1}} S(t-s)g(s, z_s + \widehat{\phi}_s)dB^H_Q(s)\right\|^2 \\
&\leq 2ME\left\|\int_{s_k}^{t_{k+1}} f(s, z_s + \widehat{\phi}_s)ds\right\|^2 + 2ME\left\|\int_{s_k}^{t_{k+1}} g(s, z_s + \widehat{\phi}_s)dB^H_Q(s)\right\|^2 \\
&\leq 2M(t_{k+1} - s_k)\int_{s_k}^{t_{k+1}} p(s)\psi(\|z_s + \widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}})ds \\
&\leq 4MH(t_{k+1} - s_k)^{2H-1}\int_{s_k}^{t_{k+1}} p_1(s)\psi_1(\|z_s + \widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}})ds \\
&\leq 2M(t_{k+1} - s_k)\psi(q^{'})\|p\|_{L^1} + 4MH(t_{k+1} - s_k)^{2H-1}\psi_1(q^{'})\|p_1\|_{L^1} \\
&= l,
\end{aligned}$$

where

$$l = \max_{0\leq k\leq m}\left\{2M(t_{k+1} - s_k)\psi(q^{'})\|p\|_{L^1} + 4MH(t_{k+1} - s_k)^{2H-1}\psi_1(q^{'})\|p_1\|_{L^1}\right\},$$

and therefore

$$\|\widehat{\Phi}_2(z)\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq l.$$

Step 4 : The map $\widehat{\Phi}_2$ is equicontinuous .

Let $\tau_1, \tau_2 \in (s_k, t_{k+1}], k = 0, \cdots, m$, $\tau_1 < \tau_2$ and $z \in \mathcal{B}_q$, we have

$$\begin{aligned}
&E\|(\widehat{\Phi}_2 z)(\tau_2) - (\widehat{\Phi}_2 z)(\tau_1)\|^2 \\
&\quad \le 6(t_{k+1} - s_k) \int_{s_k}^{\tau_1-\epsilon} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 E\|f(s, z_s + \widehat{\phi}_s)\|^2 ds \\
&\qquad +6(t_{k+1} - s_k) \int_{\tau_1-\epsilon}^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 E\|f(s, z_s + \widehat{\phi}_s)\|^2 ds \\
&\qquad +6(t_{k+1} - s_k) \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|^2 E\|f(s, z_s + \widehat{\phi}_s)\|^2 ds \\
&\qquad +12H(t_{k+1} - s_k)^{2H-1} \times \\
&\qquad\quad \times \int_{s_k}^{\tau_1-\epsilon} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 E\|g(s, z_s + \widehat{\phi}_s)\|^2_{L^0_Q(\mathcal{K},\mathcal{H})} ds \\
&\qquad +12H(t_{k+1} - s_k)^{2H-1} \times \\
&\qquad\quad \times \int_{\tau_1-\epsilon}^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 E\|g(s, z_s + \widehat{\phi}_s)\|^2_{L^0_Q(\mathcal{K},\mathcal{H})} ds \\
&\qquad +12H(t_{k+1} - s_k)^{2H-1} \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|^2 E\|g(s, z_s + \widehat{\phi}_s)\|^2_{L^0_Q(\mathcal{K},\mathcal{H})} ds \\
&\quad \le 6(t_{k+1} - s_k) \int_{s_k}^{\tau_1-\epsilon} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 \alpha_{q'}(s) ds \\
&\qquad +6(t_{k+1} - s_k) \int_{\tau_1-\epsilon}^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 \alpha_{q'}(s) ds \\
&\qquad +6M(t_{k+1} - s_k) \int_{\tau_1}^{\tau_2} \alpha_{q'}(s) ds \\
&\qquad +12H(t_{k+1} - s_k)^{2H-1} \int_{s_k}^{\tau_1-\epsilon} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 \alpha_{1q'}(s) ds \\
&\qquad +12H(t_{k+1} - s_k)^{2H-1} \int_{\tau_1-\epsilon}^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 \alpha_{1q'}(s) ds \\
&\qquad +12MH(t_{k+1} - s_k)^{2H-1} \int_{\tau_1}^{\tau_2} \alpha_{1q'}(s) ds.
\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and ϵ sufficiently small, since the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology [25]. This proves the equicontinuity. Here, we consider the case $0 < \tau_1 < \tau_2 \le T$, since the cases $\tau_1 < \tau_2 \le 0$ or $\tau_1 \le 0 \le \tau_2 \le T$ are easier to handle.

Step 5. $(\widehat{\Phi}_2 \mathcal{B}_q)(t)$ is precompact in $\mathcal{H}$

As a consequence of Steps 3 to 4, together with the Arzelá-Ascoli theorem, it suffices to show that $\widehat{\Phi}_2$ maps $\mathcal{B}_q$ into a precompact set in $\mathcal{H}$.

Let $s_k < t < t_{k+1}$ be fixed and let ϵ be a real number satisfying $s_k < \epsilon < t$. For $z \in \mathcal{B}_q$ we define

$$(\widehat{\Phi}_{2\epsilon} z)(t) = S(\epsilon) \int_{s_k}^{t-\epsilon} S(t-s-\epsilon) f(s, z_s + \widehat{\phi}_s) ds + S(\epsilon) \int_{s_k}^{t-\epsilon} S(t-s-\epsilon) g(s, z_s + \widehat{\phi}_s) dB^H_Q(s).$$

Since $S(t)$ is a compact operator, the set

$$Y_\epsilon(t) = \{\widehat{\Phi}_{2\epsilon}(z)(t) : \ z \in \mathcal{B}_q\}$$

is precompact in $\mathcal{H}$ for every ϵ, $s_k < \epsilon < t$. Moreover, for every $z \in \mathcal{B}_q$ we have

$$
\begin{aligned}
&E\|(\widehat{\Phi}_2 z)(t)-(\widehat{\Phi}_{2\epsilon} z)(t)\|^2\\
&\quad\leq 4(t_{k+1}-s_k)\int_{t-\epsilon}^{t}\|S(t-s)\|^2E\|f(s,z_s+\widehat{\phi}_s)\|^2ds\\
&\qquad+8H(t_{k+1}-s_k)^{2H-1}\int_{t-\epsilon}^{t}\|S(t-s)\|^2E\|g(s,z_s+\widehat{\phi}_s)\|^2_{L_Q^0(\mathcal{K},\mathcal{H})}ds\\
&\quad\leq 4M(t_{k+1}-s_k)\int_{t-\epsilon}^{t}\alpha_{q'}(s)ds\\
&\qquad+8MH(t_{k+1}-s_k)^{2H-1}\int_{t-\epsilon}^{t}\alpha_{1q'}(s)ds.
\end{aligned}
$$

Thus, there are precompact sets arbitrarily close to the set $Y_\epsilon(t)=\{\widehat{\Phi}_{2\epsilon}(z)(t):\ z\in\mathcal{B}_q\}$. Hence the set $Y(t)=\{\widehat{\Phi}_2(z)(t):\ z\in\mathcal{B}_q\}$ is precompact in $\mathcal{H}$, and therefore, the operator $\widehat{\Phi}_2$ is completely continuous .

Step 6 : A priori bounds.
Now it remains to show that the set

$$\Xi=\{z\in\mathcal{D}'_{\mathcal{F}_T}: z=\lambda\widehat{\Phi}_2(z)+\lambda\widehat{\Phi}_1\left(\frac{z}{\lambda}\right), \text{for some } 0<\lambda<1\}$$

is bounded.

Case 1: For each $t\in[0,t_1]$

$$z(t)=\int_0^t S(t-s)f(s,z_s+\widehat{\phi}_s)ds+\int_0^t S(t-s)g(s,z_s+\widehat{\phi}_s)dB_Q^H(s).$$

This implies, for each $t\in[0,t_1]$,

$$
\begin{aligned}
E\|z(t)\|^2 \leq& 2Mt_1\int_0^t p(s)\psi(\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}})ds\\
&+4MHt_1^{2H-1}\int_0^t p_1(s)\psi_1(\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}})ds.
\end{aligned}
\tag{7}
$$

But

$$
\begin{aligned}
\|z_t+\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} &\leq 2(\|z_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}})\\
&\leq 4\widetilde{K}^2\sup_{s\in[0,t_1]}E\|z(s)\|^2+4\widetilde{K}^2ME\|\widehat{\phi}(0)\|^2+4\widetilde{N}^2\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}.
\end{aligned}
$$

If we set $w(t)$ the right hand side of the above inequality we have that

$$\|z_t+\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}\leq w(t),$$

and therefore (7) becomes

$$E\|z(t)\|^2 \leq 2Mt_1\int_0^t p(s)\psi(w(s))ds+4MHt_1^{2H-1}\int_0^t p_1(s)\psi_1(w)ds, \tag{8}$$

Using (8) in the definition of w, we have that

$$w(t) \leq \begin{array}{l} 4\widetilde{K}^2\left(2Mt_1\int_0^t p(s)\psi(w(s))ds + 4MHt_1^{2H-1}\int_0^t p_1(s)\psi_1(w)ds\right) \\ +4\widetilde{K}^2ME\|\widehat{\phi}(0)\|^2 + 4\widetilde{N}^2\|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}. \end{array} \tag{9}$$

Thus, we obtain

$$w(t) \leq K_0 + K_1\int_0^t p(s)\psi(w(s))ds + K_2\int_0^t p_1(s)\psi_1(w(s))ds. \tag{10}$$

where $K_0 = 4\widetilde{K}^2ME\|\widehat{\phi}(0)\|^2 + 4\widetilde{N}^2\|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}$, $K_1 = 8\widetilde{K}^2Mt_1$, and $K_2 = 16\widetilde{K}^2MHt_1^{2H-1}$.

Let us denote the right-hand side of the inequality (10) by $v(t)$. Then we have

$$v(0) = K_0,\ w(t) \leq v(t),\ t \in [0, t_1],$$

and

$$v'(t) = K_1p(t)\psi(w(t)) + K_2p_1(t)\psi_1(w(t)),\ t \in [0, t_1].$$

Using the increasing character of ψ and ψ_1 we obtain

$$v'(t) \leq K_1p(t)\psi(v(t)) + K_2p_1(t)\psi_1(v(t)),\ \text{for } a.e.\ t \in [0, t_1].$$

This implies, for each $t \in J$, we have

$$\int_{v(0)}^{v(t)} \frac{ds}{\psi(s)+\psi_1(s)} \leq K_1\int_0^{t_1} p(s)ds + K_2\int_0^{t_1} p_1(s)ds < \int_{K_0}^{\infty} \frac{ds}{\psi(s)+\psi_1(s)}.$$

By Lemma 2.7,

$$v(t) \leq \Gamma_0^{-1}\left(\int_0^{t_1}(K_1p(t) + K_2p_1(t))dt\right), \quad t \in [0, t_1]$$

where

$$\Gamma_0(x) = \int_{K_0}^{x} \frac{du}{\psi(u)+\psi_1(u)}.$$

Hence, there exists $M_{t_0} > 0$ such that

$$\|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} < M_{t_0}.$$

From equation (8)

$$E\|z(t)\|^2 \leq 2Mt_1\int_0^{t_1} p(s)\psi(M_{t_0})ds + 4MHt_1^{2H-1}\int_0^{t_1} p_1(s)\psi_1(M_{t_0})ds. \tag{11}$$

Thus

$$\|z\|^2_{\mathcal{D}_{\mathcal{F}'_T}} \leq L_{t_0}.$$

Cas 2: For each $t \in (t_k, s_k], k = 1, \cdots, m$

$$z(t) = h_k(t, z_t + \widehat{\phi}_t)$$

This implies, for each $t \in (t_k, s_k]$,

$$E\|z(t)\|^2 \leq L_{h_k}\|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \tag{12}$$

If

$$\|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq w(t),$$

and therefore (12) becomes

$$E|z(t)|^2 \leq L_{h_k} w(t). \tag{13}$$

Using (13) in the definition of $w(t)$, we have that

$$w(t) \leq \quad 4\widetilde{K}^2 L_{h_k} w(t) + 4\widetilde{K}^2 ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2 \|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}. \tag{14}$$

Thus, we obtain

$$w(t) \leq \frac{4\widetilde{K}^2 ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2 \|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}}{1 - 4\widetilde{K}^2 L_{h_k}} = M_{t_k}. \tag{15}$$

This implies there is a constant $M_{t_k} > 0$ such that

$$w(t) \leq M_{t_k}, \quad t \in (s_k, t_k].$$

So

$$\|z_t + \widehat{\phi}_t\|_{\mathcal{D}_{\mathcal{F}_0}} \leq M_{t_k}, \quad t \in (s_k, t_k].$$

From equation (13) we obtain that

$$E|z(t)|^2 \leq L_{h_k} M_{t_k} \tag{16}$$

Thus

$$\|z\|^2_{\mathcal{D}_{\mathcal{F}'_T}} \leq L_{t_k}.$$

Cas 3: For each $t \in [s_k, t_{k+1}]$, $k = 1 \cdots, m$

$$z(t) = S(t - s_k) h_k(s_k, z_{s_k} + \widehat{\phi}_{s_k}) + \int_0^t S(t-s) f(s, z_s + \widehat{\phi}_s) ds$$

$$+ \int_0^t S(t-s) g(s, z_s + \widehat{\phi}_s) dB^H_Q(s).$$

This implies, for each $t \in J$,

$$\begin{aligned} E\|z(t)\|^2 \leq & 3ML_{h_k} \|z_{s_k} \\ & + \widehat{\phi}_{s_k}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 3M(t_{k+1} - s_k) \int_0^t p(s) \psi(\|z_s + \widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}}) ds \\ & + 6MH(t_{k+1} - s_k)^{2H-1} \int_0^t p_1(s) \psi_1(\|z_s + \widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}}) ds. \end{aligned} \tag{17}$$

But

$$\begin{aligned} \|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} & \leq 2(\|z_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\ & \leq 4\widetilde{K}^2 \sup_{s \in [0,T]} E|z(s)|^2 + 4\widetilde{K}^2 ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2 \|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}. \end{aligned}$$

If we set $w(t)$ the right hand side of the above inequality we have that

$$\|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq w(t),$$

and therefore (17) becomes

$$E\|z(t)\|^2 \leq 3ML_{h_k} w(t) + 3M(t_{k+1} - s_k) \int_{s_k}^t p(s) \psi(w(s)) ds$$

$$+ 6MH(t_{k+1}-s_k)^{2H-1}\int_{s_k}^{t} p_1(s)\psi_1(w)ds. \tag{18}$$

Using (18) in the definition of w, we have that

$$\begin{aligned} w(t) \leq \;& 4\widetilde{K}^2\Big(3ML_{h_k}w(t) + 3M(t_{k+1}-s_k)\int_{s_k}^{t} p(s)\psi(w(s))ds \\ & + 4MH(t_{k+1}-s_k)^{2H-1}\textstyle\int_{s_k}^{t} p_1(s)\psi_1(w)ds\Big) + 4\widetilde{K}^2ME|\widehat{\phi}(0)|^2 \\ & +4\widetilde{N}^2\|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}. \end{aligned} \tag{19}$$

Thus, we obtain

$$w(t) \leq \bar{K}_0 + \bar{K}_1\int_{s_k}^{t} p(s)\psi(w(s))ds + \bar{K}_2\int_{s_k}^{t} p_1(s)\psi_1(w(s))ds. \tag{20}$$

where

$$\bar{K}_0 = \frac{4\widetilde{K}^2ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2\|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}}{1-12\widetilde{K}^2ML_{h_k}}, \quad \bar{K}_1 = \frac{12\widetilde{K}^2M(t_{k+1}-s_k)}{1-12\widetilde{K}^2ML_{h_k}},$$

$$\bar{K}_2 = \frac{24\widetilde{K}^2MHt_1^{2H-1}}{1-12\widetilde{K}^2ML_{h_k}}.$$

Let us denote by $v(t)$ the right-hand side of inequality (20). Then we have

$$v(0) = \bar{K}_0,\;\; w(t) \leq v(t),\;\; t \in [s_k, t_{k+1}], k = 1, \cdots, m$$

and

$$v'(t) = \bar{K}_1 p(t)\psi(w(t)) + \bar{K}_2 p_1(t)\psi_1(w(t)),\;\; t \in [s_k, t_{k+1}].$$

Using the increasing character of ψ and ψ_1 we obtain

$$v'(t) \leq \bar{K}_1 p(t)\psi(v(t)) + \bar{K}_2 p_1(t)\psi_1(v(t)),\;\; \text{for } a.e.\ t \in [s_k, t_{k+1}].$$

This implies, for each $t \in J$, we have

$$\int_{v(0)}^{v(t)} \frac{ds}{\psi(s)+\psi_1(s)} \leq \bar{K}_1\int_{s_k}^{t_{k+1}} p(s)ds + \bar{K}_2\int_{s_k}^{t_{k+1}} p_1(s)ds < \int_{\bar{K}_0}^{\infty}\frac{ds}{\psi(s)+\psi_1(s)}.$$

By Lemma 2.7, we have

$$v(t) \leq \Gamma_1^{-1}\left(\int_{s_k}^{t_{k+1}} (K_1p(t) + K_2p_1(t))dt\right), \quad t \in [s_k, t_{k+1}]$$

where

$$\Gamma_1(x) = \int_{\bar{K}_0}^{x}\frac{d(u)}{\psi(u)+\psi_1(u)}.$$

Hence, there exists $M_{t_{k+1}} > 0$ such that

$$\|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} < M_{t_{k+1}}.$$

From equation (18) we obtain that

$$E\|z(t)\|^2 \leq 3ML_{h_k}M_{t_{k+1}} + 3M(t_{k+1}-s_k)\int_{s_k}^{t_{k+1}} p(s)\psi(M_{t_{k+1}})ds$$

$$+ 6MH(t_{k+1} - s_k)^{2H-1} \int_{s_k}^{t_{k+1}} p_1(s)\psi_1(M_{t_{k+1}})ds. \quad (21)$$

Thus, there exists $L_{t_{k+1}} > 0$ such that

$$\|z\|^2_{\mathcal{D}_{\mathcal{F}'_T}} \leq L_{t_{k+1}}.$$

This implies that the set Ξ is bounded.

As a consequence of Theorem 2.8 we deduce that $\widehat{\Phi}$ has a fixed point, since $y(t) = z(t) + \widehat{\phi}(t)$, $t \in (-\infty, T]$. Then y is a fixed point of the operator Ψ which is a mild solution of the problem (1)-(3). □

4. **Continuous dependence of mild solutions.** In this section we will prove that continuous dependence of mild solutions with respect to the initial data.

Theorem 4.1. *Suppose that assumptions* $(H1)-(H3)$ *are fulfilled and the following inequalities hold as well:*

$$L_2 = \max\{8M\widetilde{K}^2(t_1^2 L_f + Ht_1^{2H} L_g), 4L_{h_k}\widetilde{K}^2, 12M\widetilde{K}^2(L_{h_k} + t_1^2 L_f + Ht_1^{2H} L_g)\} < 1.$$

Then for each $\phi, \phi^* \in \mathcal{D}_{\mathcal{F}_0}$ *and* $y(t) = z(t) + \widehat{\phi}(t)$, $y^*(t) = z^*(t) + \widehat{\phi}^*(t)$ *the corresponding mild solutions of the system (1)-(3), the following inequalities hold:*

$$\|z - z^*\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq \frac{8M\alpha^2(M+1)(t_1^2 L_f + Ht_1^{2H} L_g)}{1 - 8M\widetilde{K}^2(t_1^2 L_f + Ht_1^{2H} L_g)} \|\widehat{\phi} - \widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}, t \in [0, t_1],$$

$$\|z - z^*\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq \frac{4L_{h_k}\alpha^2(M+1)}{1 - 4L_{h_k}\widetilde{K}^2} \|\widehat{\phi} - \widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}, t \in (t_k, s_k], k = 1, 2, \cdots, m$$

and

$$\|z - z^*\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq \frac{12M\alpha^2(M+1)(l_{h_k} + t_1^2 L_f + Ht_1^{2H} L_g)}{1 - 12M\widetilde{K}^2(L_{h_k} + t_1^2 L_f + Ht_1^{2H} L_g)} \|\widehat{\phi} - \widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}},$$

for $t \in (s_k, t_{k+1}], k = 1, 2, \cdots, m.$

Proof. Estimating for $t \in [0, t_1]$, we have

$$\begin{aligned} E\|z(t) - z^*(t)\|^2 &\leq 2E\left\|\int_0^t S(t-s)[f(s, z_s + \widehat{\phi}_s) - f(s, z^*_s + \widehat{\phi}_s)]ds\right\|^2 \\ &\quad + 2E\left\|\int_0^t S(t-s)[g(s, z_s + \widehat{\phi}_s) - g(s, z^*_s + \widehat{\phi}^*_s)]dB^H_Q(s)\right\|^2. \end{aligned}$$

But

$$\begin{aligned} \|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} &\leq 2(\|z_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\ &\leq 4\widetilde{K}^2 \sup_{s\in[0,T]} E\|z(s)\|^2 + 4\widetilde{K}^2 ME\|\widehat{\phi}(0)\|^2 + 4\widetilde{N}^2\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ &\leq 4\widetilde{K}^2 \sup_{s\in[0,T]} E\|z(s)\|^2 + 4\alpha^2(M+1)\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}, \end{aligned}$$

where $\alpha^2 = \max\{\widetilde{K}^2, \widetilde{N}^2\}$.

$$\begin{aligned} E\|z(t) - z^*(t)\|^2 &\leq 2Mt_1^2 L_f(4\widetilde{K}^2\|z - z^*\|^2_{\mathcal{D}'_{\mathcal{F}_T}} + 4\alpha^2(M+1)\|\widehat{\phi} - \widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\ &\quad + 2MHt_1^{2H} L_g(4\widetilde{K}^2\|z - z^*\|^2_{\mathcal{D}'_{\mathcal{F}_T}} + 4\alpha^2(M+1)\|\widehat{\phi} - \widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \end{aligned}$$

which implies that

$$\|z-z^*\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq \frac{8M\alpha^2(M+1)(t_1^2L_f+Ht_1^{2H}L_g)}{1-8M\widetilde{K}^2(t_1^2L_f+Ht_1^{2H}L_g)}\|\widehat{\phi}-\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}.$$

For $t\in(t_k,s_k]$, $k=1,2,\cdots,m$, we have

$$\begin{aligned} E\|z(t)-z^*(t)\|^2 &\leq E\left\|h_k(t,z_t+\widehat{\phi}_t)-h_k(t,z_t^*+\widehat{\phi}_t^*)\right\|^2 \\ &\leq L_{h_k}(4\widetilde{K}^2\|z-z^*\|^2_{\mathcal{D}'_{\mathcal{F}_T}}+4\alpha^2(M+1)\|\widehat{\phi}-\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \end{aligned}$$

which implies that

$$\|z-z^*\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq \frac{4L_{h_k}\alpha^2(M+1)}{1-4L_{h_k}\widetilde{K}^2}\|\widehat{\phi}-\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}.$$

In a similar way, when $t\in[s_k,t_{k+1}]$, we have

$$\begin{aligned} E\|z(t)-z^*(t)\|^2 &\leq 3E\|S(t-s_k)[h(s_k,z_{s_k}+\widehat{\phi}_{s_k})-h(s_k,z^*_{s_k}+\widehat{\phi}^*_{s_k})]\|^2 \\ &\quad +3E\left\|\int_{s_k}^t S(t-s)[f(s,z_s+\widehat{\phi}_s)-f(s,z_s^*+\widehat{\phi}_s^*)]ds\right\|^2 \\ &\quad +3E\left\|\int_{s_k}^t S(t-s)[g(s,z_s+\widehat{\phi}_s)-g(s,z_s^*+\widehat{\phi}_s^*)]dB_Q^H(s)\right\|^2 \\ &\leq 3ML_{h_k}(4\widetilde{K}^2\|z-z^*\|^2_{\mathcal{D}'_{\mathcal{F}_T}}+4\alpha^2(M+1)\|\widehat{\phi}-\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\ &\quad +3Mt_1^2L_f(4\widetilde{K}^2\|z-z^*\|^2_{\mathcal{D}'_{\mathcal{F}_T}}+4\alpha^2(M+1)\|\widehat{\phi}-\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\ &\quad +3MHt_1^{2H}L_g(4\widetilde{K}^2\|z-z^*\|^2_{\mathcal{D}'_{\mathcal{F}_T}}+4\alpha^2(M+1)\|\widehat{\phi}-\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}), \end{aligned}$$

which implies that

$$\|z-z^*\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq \frac{12M\alpha^2(M+1)(l_{h_k}+t_1^2L_f+Ht_1^{2H}L_g)}{1-12M\widetilde{K}^2(L_{h_k}+t_1^2L_f+Ht_1^{2H}L_g)}\|\widehat{\phi}-\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}.$$

□

5. **An example.** Consider the following stochastic partial differential equation with delays and impulsive effects

$$\begin{cases} \frac{\partial}{\partial t}u(t,\xi) &= \frac{\partial^2}{\partial\xi^2}u(t,\xi)+F(t,u_t(.,\xi)) \\ &\quad +\sigma(t,u_t(.,\xi))\frac{dB_Q^H}{dt}, \quad (t,\xi)\in\bigcup_{i=0}^m[s_i,t_{i+1}]\times[0,\pi], \\ u(t,\xi) &= H_k(t,u_t(.,\xi)), \quad k=1,\cdots,m, \\ u_t(.,0) &= u_t(.,\pi)=0, t\in[0, \\ u(t,\xi) &= \phi(t,\xi), -\infty\leq t\leq 0, 0\leq\xi\leq\pi, \end{cases} \tag{22}$$

where B_Q^H denotes a fractional Brownian motion, the impulse times t_k satisfy $0=t_0=s_0<t_1\leq s_1<t_2<\cdots<t_m\leq s_m<t_{m+1}=T$. As for u_t we mean the segment solution which is defined in the usual way, that is, if $u(\cdot,\cdot,\cdot):(-\infty,T]\times\times[0,\pi]\times\Omega\to\mathcal{H}$, then for any $t\geq 0$, $u_t(\cdot,\cdot,\cdot):(-\infty,0]\times[0,\pi]\times\Omega\to\mathcal{H}$ is given by:

$$u_t(\theta,\xi,\omega)=u(t+\theta,\xi,\omega),\ \text{for}\ \theta\in(-\infty,0],\ \omega\in\Omega,$$

$F,G:[0,T]\times\mathcal{D}_{\mathcal{F}_0}\to\mathbb{R}$ are continuous functions.

Let

$$y(t)(\xi) = u(t,\xi,.)\ t \in J,\ \xi \in [0,\pi],$$

$$h_k(t,\phi)(\xi) = H_k(t,\phi(\theta,\xi)),\ \theta \in (-\infty,0],\ \xi \in [0,\pi],\ k = 1,\cdots,m,$$

$$f(t,\phi)(\xi) = F(t,\phi(\theta,\xi)),\ \theta \in (-\infty,0],\ \xi \in [0,\pi],$$

$$\phi(\theta)(\xi) = \phi(\theta,\xi),\ \theta \in (-\infty,0],\ \xi \in [0,\pi],$$

Take $\mathcal{K} = \mathcal{H} = L^2([0,\pi])$. We define the operator A by $Au = u^{''}$, with domain $D(A) = \{u \in \mathcal{H}, u^{''} \in \mathcal{H} \quad \text{and} \quad u(0) = u(\pi) = 0\}$.

Take $\mathcal{K} = \mathcal{H} = L^2([0,\pi])$. We define the operator A by $Au = \frac{\partial^2}{\partial\xi^2}u$. with domain $D(A) = \{u \in \mathcal{H}, \frac{\partial u}{\partial\xi}, \frac{\partial^2 u}{\partial\xi^2} \in \mathcal{H} \quad \text{and} \quad u(0) = u(\pi) = 0\}$.

Then, it is well known that

$$Az = -\sum_{n=1}^{\infty} n^2 \langle z, e_n\rangle e_n,\ z \in \mathcal{H},$$

and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}$, which is given by

$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}\langle u, e_n\rangle e_n$, $u \in \mathcal{H}$, and $e_n(u) = (2/\pi)^{1/2}\sin(nu), n = 1,2,\cdots$, is the orthogonal set of eigenvectors of A. It is well known that $\{S(t)\}$, $t \in J$, is compact, and there exists a constant $M \geq 1$ such that $\|S(t)\|^2 \leq M$.

In order to define the operator $Q : \mathcal{K} \longrightarrow \mathcal{K}$, we choose a sequence $\{\sigma_n\}_{n\geq 1} \subset \mathbb{R}^+$, set $Qe_n = \sigma_n e_n$, and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty.$$

Define the process $B_Q^H(s)$ by

$$B_Q^H = \sum_{n=1}^{\infty} \sqrt{\sigma_n}\gamma_n^H(t)e_n,$$

where $H \in (1/2,1)$, and $\{\gamma_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional fractional Brownian motions mutually independent.

In the case of $t \in (-\infty,T]$. Assume now that

- **(i):** For all $k = 0,1\cdots,m$, the function $f : [s_k,t_{k+1}] \times \mathcal{D}_{\mathcal{F}_0} \longrightarrow \mathcal{H}$ defined by $f(t,u)(\cdot) = F(t,u(\cdot))$ is continuous and we impose suitable conditions on F to verify assumption $(H1)$. For example we take

$$F(t,\phi) = t + \frac{\phi}{1 + \|\phi\|_{\mathcal{D}_{\mathcal{F}_0}}},\ t \in [0,T],\ \phi \in \mathcal{D}_{\mathcal{F}_0}.$$

- **(ii):** For all $k = 0,1\cdots,m$, the function $g : [s_k,t_{k+1}] \times \mathcal{D}_{\mathcal{F}_0} \longrightarrow L_Q^0(\mathcal{K},\mathcal{H})$ defined by $g(t,u)(.) = G(t,u(.))$ is continuous and it is easy to impose suitable conditions on G to make assumption $(H2)$ hold. We take

$$G(t,\phi) = t^2 + \sin\phi, \quad t \in [0,T],\ \phi \in \mathcal{D}_{\mathcal{F}_0}.$$

(iii): For all $k = 1, 2\cdots, m$, the function $h_k : (t_k, s_k] \times \mathcal{D}_{\mathcal{F}_0} \longrightarrow \mathcal{H}$ defined by $h_k(t, u)(.) = H_k(t, u(.))$ is continuous and it is easy to impose suitable conditions on H_k to make assumption $(H3)$ hold.

$$H_k(t, \phi)) = K_k\phi, \ \xi \in \Omega, \ k = 1, \cdots, m, \quad t \in [0, T], \ \phi \in \mathcal{D}_{\mathcal{F}_0},$$

where $K_k \in \mathbb{R}, \ k = 1, \dots, m.$

Thus the problem (22) can be written in the abstract form

$$\begin{cases} dy(t) = A[y(t) + f(t, y_t)]dt + g(t, y_t)B_Q^H(t), \quad t \in J_k = (s_k, t_{k+1}], k = 0, \cdots, m \quad , \\ y(t) = h_k(t, y_t) \ \ t \in (t_k, s_k], k = 1, \cdots, m, \\ y(t) = \phi(t), \quad \text{for a.e. } t \in (-\infty, 0]. \end{cases}$$

Thanks to these assumptions, it is straightforward to check that (H1)-(H3) hold and thus assumptions in Theorem 3.2 are fulfilled, ensuring that system (22) possesses a mild solution on $(-\infty, T]$.

REFERENCES

[1] H. M. Ahmed, Semilinear neutral fractional stochastic integro-differential equations with non-local conditions, *J. Theoret. Probab.* **28** (2015), no. 2, 667–680.

[2] E. Alos, O. Mazet, D.Nualart, Stochastic calculus with respect to Gaussian processes, *Ann. Probab.* **29** (2001), 766–801.

[3] C. Avramescu, Some remarks on a fixed point theorem of Krasnoselskii, *Electron. J. Qual. Theory Differ. Equ.* **5** (2003), 1–15.

[4] J. Bao, Z. Hou, Existence of mild solutions to stochastic neutral partial functional differential equations with non-Lipschitz coefficients, *Comput. Math. Appl.* 59 (2010), no. 1, 207–214.

[5] I. Bihari, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, *Acta Math. Acad. Sci. Hungar.* **7** (1956), 81–94.

[6] A. Boudaoui, T. Caraballo, A. Ouahab, Existence of mild solutions to stochastic delay evolution equations with a fractional Brownian motion and impulses, *Stoch. Anal. Appl.* **33** (2015), no. 2, 244–258.

[7] A. Boudaoui, T. Caraballo, A. Ouahab, Impulsive stochastic functional differential inclusions driven by a fractional Brownian motion with infinite delay. *Math. Meth. Appl. Sci.* **39** (2016), no. 6, 1435–1451. doi: 10.1002/mma.3580.

[8] A. Boudaoui, T. Caraballo, A. Ouahab, Impulsive neutral functional differential equations driven by a fractional Brownian motion with unbounded delay, *Appl. Anal.* **95** (2016), no. 9, 2039–2062.

[9] B. Boufoussi, S. Hajji, Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space, *Statist. Probab. Lett.* **82** (2012), no. 8, 1549–1558.

[10] G. Cao, K. He, X. Zhang, Successive approximations of infinite dimensional SDES with jump, *Stoch. Dyn.* **5** (2005), no. 4, 609–619.

[11] T. Caraballo, M. J. Garrido-Atienza, T. Taniguchi, The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion, *Nonlinear Anal.* **74** (2011), 3671–3684.

[12] T. Caraballo, Mamadou A. Diop, Neutral stochastic delay partial functional integro-differential equations driven by a fractional Brownian motion, *Front. Math. China* **8** (2013), no. 4, 745–760.

[13] M.M. El-Borai, K. EI-Said EI-Nadi, H.A. Fouad, On some fractional stochastic delay differential equations, *Comput. Math. Appl.*, 59 (2010), no. 3, 1165–1170.

[14] G. R. Gautam, J. Dabas, Existence result of fractional functional integrodifferential equation with not instantaneous impulse, *Int. J. Adv. Appl. Math. Mech* **1.3** (2014), 11–21.

[15] T.E. Govindan, Almost sure exponential stability for stochastic neutral partial functional differential equations, *Stochastics* **77** (2005), no. 2, 139–154.

[16] J. R. Graef, J. Henderson, A. Ouahab, *Impulsive differential inclusions. A fixed point approach.* De Gruyter Series in Nonlinear Analysis and Applications, 20. De Gruyter, Berlin, 2013.

[17] J. K. Hale, J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.*, **21** (1978), no. 1, 11–41.

[18] E. Hernández, D. O'Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.* **141** (2013), no. 5, 1641–1649.

[19] F. Jiang, Y. Shen, A note on the existence and uniqueness of mild solutions to neutral stochastic partial functional differential equations with non-Lipschitz coefficients, *Comput. Math. Appl.* **61** (2011), no. 6, 1590–1594.

[20] V. Lakshmikantham, D. Bainov, P. Simeonov, *Theory of Impulsive Differential Equations*. Series in Modern Applied Mathematics, 6. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[21] X. Li, M. Bohner, An Impulsive Delay Differential Inequality and Applications, *Comput. Math. Appl.* **64** (2012), no. 6, 1875–1881.

[22] X. Li, X. Fu, On the global exponential stability of impulsive functional differential equations with infinite delays or finite delays, *Commun. Nonlinear Sci. Numer. Simul.* **19** (2014), no. 3, 442–447.

[23] Y. Mishura *Stochastic Calculus for Fractional Brownian Motion and Related Topics*. Lecture Notes in Mathematics, **1929**. Springer-Verlag, Berlin, 2008.

[24] D. Nualart *The Malliavin Calculus and Related Topics.* 2nd ed. Probability and its Applications (New York). Springer-Verlag, Berlin, 2006.

[25] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences, **44**. Springer-Verlag, New York, 1983.

[26] M. Pierri, D. O'Regan, V. Rolnik, Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses, *Appl. Math. Comp.*, **219** (2013), no. 12, 6743–6749.

[27] R. Sakthivel, J. Luo, Asymptotic stability of impulsive stochastic partial differential equations with infinite delays, *J. Math. Anal. Appl.* **356** (2009), no. 1, 1–6.

[28] A.M. Samoilenko, N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore 1995.

[29] G. Shen, Y. Ren, Neutral stochastic partial differential equations with delay driven by Rosenblatt process in a Hilbert space, *J. Korean Statist. Soc.* **44** (2015), no. 1, 123–133. doi: 10.1016/j.jkss.2014.06.002.

[30] T. Taniguchi, Successive approximations to solutions of stochastic differential equations, *J. Differential Equations* **96** (1992), no. 1, 152–169.

[31] S. Tindel, C. Tudor, F. Viens, Stochastic evolution equations with fractional Brownian motion, *Probab. Theory Related Fields*, **127** (2003), no. 2, 186–204.

[32] J. R. Wang, Y. Zhou, Z. Lin, On a new class of impulsive fractional differential equations, *Appl. Math. Comput.* **242** (2014), 649–657.

[33] Z. Yan, X. Yan, Existence of solutions for impulsive partial stochastic neutral integro-differential equations with state-dependent delay, *Collect. Math.* **64** (2013), no. 2, 235–250.

[34] Q. Zhu, Asymptotic stability in the pth moment for stochastic differential equations with Levy noise, *J. Math. Anal. Appl.* **416** (2014), no. 1, 126–142.

E-mail address: ahmedboudaoui@gmail.com
E-mail address: caraball@us.es
E-mail address: agh_ouahab@yahoo.fr