Programa de Doctorado "Matemáticas"

PHD DISSERTATION

ON COCYCLE AND UNIFORM ATTRACTORS FOR MULTI-VALUED AND RANDOM NON-AUTONOMOUS DYNAMICAL SYSTEMS

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Abstract

In this thesis we study the long time behavior of multi-valued and random dynamical systems in terms of global attractors.

We begin with cocycle, pullback and uniform attractors for multi-valued non-autonomous dynamical systems. We first consider the relationship between the three attractors to find that they imply each other under suitable conditions. Then, for generalized dynamical systems, we find that these attractors can be characterized by complete trajectories, which implies that the uniform attractor is lifted invariant, though it has no standard invariance by definition. Finally, we study both upper and lower semi-continuity of these attractors. A weak equi-attraction method is introduced to study the lower semi-continuity, and we show with an example the advantages of this method. A reaction-diffusion system and a scalar ordinary differential inclusion are studied as applications.

Then we go to the random (but single-valued) case.

Firstly, we study cocycle attractors for autonomous random dynamical systems (RDS) and non-autonomous random dynamical systems (NRDS) with only a so-called quasi strong-to-weak (abbrev. quasi-S2W) continuity. It is shown that such continuity is equivalent to the closed-graph property for mappings taking values in weakly compact spaces. Moreover, it is inheritable: if a mapping is quasi-S2W continuous in some space, then so it is automatically in more regular subspaces. Moreover, by establishing some existence criteria for cocycle attractors we see that the quasi-S2W continuity is adequate to derive the measurability of the cocycle attractor. These observations generalize known existence theorems for cocycle attractors on one hand, and enable us to study cocycle attractors in more regular spaces without further proving the system's continuity on the other hand. Applying to bi-spatial cocycle attractor theory, we establish an existence theorem indicating that the measurability of bi-spatial attractors is valid in regularity space, not only in the basic phase space as in the previous literature.

Secondly, for NRDS we compare cocycle attractors with autonomous and non-autonomous attraction universes, and then for autonomous ones we establish some existence criteria and characterization. We also study the upper semi-continuity of cocycle attractors with respect to non-autonomous symbols to find that a cocycle attractor is upper semi-continuous in symbols if and only if it is uniformly compact.

Thirdly, we establish a (random) uniform attractor theory for NRDS. We define a uniform attractor as the minimal compact uniformly pullback attracting random set. About the definition we observe that the uniform pullback attraction of a uniform attractor in fact implies a uniform forward attraction in probability, and implies also an almost uniform pullback attraction for discrete time-sequences. Though no invariance is required by definition, uniform attractor can have a negative semi-invariance.

We further study the existence of uniform attractors and the relationship between uniform and cocycle attractors. To overcome the measurability difficulty, the symbol space is required to be Polish which is shown fulfilled by locally integrable forcing if the symbol space is defined as the hull of the forcing. For the relationship between uniform and cocycle attractors we find that the uniform attractor of a continuous NRDS is composed of states involved in the cocycle attractor on one hand, and can be regarded as the cocycle attractor of a corresponding multi-valued (but autonomous) RDS on the other hand. Moreover, uniform attractors for continuous NRDS are shown to be determined by uniformly attracting nonrandom compact sets.

Cocycle and uniform attractors for reaction-diffusion equation, Ginzburg-Landau equation and 2D Navier-Stokes equation with scalar white noise are studied as applications.

Resumen

En esta tesis estudiamos el comportamiento a largo plazo de sistemas dinámicos multivaluados y aleatorios en términos de sus atractores globales.

Comenzamos con el estudio de los atractores cociclo, pullback y uniforme para sistemas dinámicos no autónomos multivaluados. En primer lugar consideramos la relación entre estos tres tipos de atractores para encontrar que, bajo condiciones adecuadas, se implican entre sí. Encontramos además que estos atractores pueden caracterizarse por trayectorias (soluciones globales), lo que implica que el atractor uniforme tiene una propiedad de invarianza (*lifted invariance*), aunque, por definición, no posee la invarianza estándar. Finalmente, estudiamos tanto la semicontinuidad superior como inferior de estos atractores. Se introduce un equi-atracción débil para estudiar la semicontinuidad inferior, y se muestra con un ejemplo las ventajas de este método. Un sistema de reacción-difusión y una inclusión diferencial ordinaria escalar se estudian como aplicaciones.

A continuación estudiamos el caso aleatorio (pero univaluado), en el marco de los sistemas dinámicos aleatorios (RDS, por sus siglas en inglés).

En primer lugar, se estudian los atractores cociclo para RDS y sistemas dinámicos aleatorios no autónomos (NRDS) con sólo una continuidad llamada *cuasi fuerte a débil* (abreviadamente cuasi-S2W). Esta continuidad se muestra heredable: si una aplicación es cuasi-S2W continua en algún espacio, entonces lo es automáticamente en espacios más regulares. Además, al establecer algunos criterios de existencia para los atractores cociclo, vemos que la continuidad cuasi-S2W es suficiente para derivar la medibilidad del atractor cociclo. Estas observaciones generalizan los teoremas de existencia conocidos para los atractores cociclo, por un lado, y, por otro, nos permiten estudiar estos atractores en espacios regulares sin demostrar la continuidad del sistema. Aplicando estos resultados a la teoría bi-espacial de atractores cociclos, establecemos un teorema de existencia que indica que la medibilidad de los atractores bi-espaciales es válida en espacio más regulares, no sólo en el espacio de fases básico como previamente en la literatura.

En segundo lugar, para NRDS se comparan los atractores cociclos con universos de atracción autónomos y no autónomos, y luego para universos autónomos se establecen algunos criterios de existencia y caracterización. También estudiamos la semicontinuidad superior de estos atractores con respecto a los símbolos no autónomos, para hallar que un atractor cociclo es semicontinuo superiormente respecto a los símbolos si y sólo si es uniformemente compacto.

En tercer lugar, establecemos una teoría de atractores uniformes (aleatorios) para NRDS. Definimos un atractor uniforme como el menor conjunto aleatorio compacto uniformemente atrayente. En cuanto a la definición, observamos que la propiedad de atracción uniforme de un atractor uniforme, de hecho, implica una atracción uniforme hacia adelante en probabilidad, e implica también una at-

racción pullback casi uniforme para sucesiones de tiempo discretas. Aunque no se requiere invarianza por definición, el atractor uniforme posee una semi-invarianza negativa.

Estudiamos la existencia de atractores uniformes, y la relación entre los atractores uniformes y los atractores para los productos cruzados aleatorios (*random skew-products*). Para superar la dificultad de la medibilidad de los conjuntos aleatorios, se requiere que el espacio de símbolos sea Polish, que se tiene para funciones localmente integrables cuando el espacio de símbolos se define como la clausura de las mismas. Para la relación entre los atractores uniformes y cociclos encontramos, por un lado, que el atractor uniforme de un NRDS continuo se compone de estados involucrados en el atractor cociclo, y que, por el otro, puede ser descrito como el atractor cociclo de un RDS multivaluado (pero autónomo). Además, los atractores uniformes para NRDS continuos aparecen determinados (como en el caso de RDS autónomos) por la atracción uniforme de conjuntos compactos no aleatorios.

Como aplicaciones se estudian la existencia y caracterización de atractores cociclo y uniformes para la ecuación de reacción-difusión, la ecuación de Ginzburg-Landau y la ecuación bidimensional de Navier-Stokes con ruido blanco escalar.

Contents

Introduction

Non-autonomous dynamical systems describe some evolution phenomena in the real world with changing forcing field [19, 55]. Multi-valued non-autonomous dynamical systems [3, 18, 26, 72, 91] are introduced mainly to deal with situations where for some initial data more than one solution can be generated, while random non-autonomous dynamical systems are for problems with stochastic perturbation involved [1, 33, 34].

In this work we study global attractors for multi-valued and random dynamical systems, including cocycle attractors, pullback attractors and uniform attractors. The main tasks will be three-folds. The first is to study the three kinds of attractors on their existence criteria, relationship and robustness for multi-valued (but nonrandom) dynamical systems. The second is to study cocycle attractors for random dynamical systems (RDS) to establish some new existence theorems standing on a so-called quasi strong-to-weak (abbrev. quasi-S2W) continuity of the system. The third is to establish a uniform attractor theory for non-autonomous random dynamical systems (abbrev. NRDS). Correspondingly, we split the introduction into three parts. The first part covers Chapter 1, the second covers Chapters 2-4 and the third covers Chapter 5.

Part I: Attractors for multi-valued nonautonomous dynamical systems

In Chapter 1 we study the attractors for multi-valued non-autonomous dynamical systems. The first aim is to establish the relationship between different types of "attractors", including pullback attractors for multi-valued processes, cocycle and uniform attractors for multi-valued cocycles and global attractors for multi-valued skew-product semiflows. This topic is interesting due to the fact that, given a non-autonomous model such as a differential inclusion or a differential equation without uniqueness of solutions, it is possible to define a multi-valued process, a cocycle and a skew-product semiflow with the above mentioned attractors. Though each of these attractors has advantages over others in describing the long time behavior of solutions, there should be some relationship between them when they are describing the same model. Due to this fact, the relationship of attractors for *single-valued* dynamical systems has attracted much attention, see [5, 21, 24, 55]. In this part, we extend such results to multi-valued dynamical systems.

The second purpose is to characterize the attractors by complete trajectories, which is already well known in the single-valued case, see [19] and the references therein. Here we work with multi-valued cocycles instead of processes mainly because, as will be shown latter, cocycle attractors have a very close relationship with uniform attractors. We find that both cocycle and uniform attractors can be characterized by complete trajectories. This leads to a *lifted-invariance* (see Definition 1.2.10, also [5]) of uniform attractors, though uniform attractors are not invariant by definition (only satisfies a minimality property instead).

The third purpose is to study the robustness of attractors under perturbations. As mentioned above, we study both cocycle and uniform attractors, noticing that related results for uniform attractors are seldom seen in the literature until the year 2014 [5]. We split this part into two: upper and lower semi-continuity. As in the single-valued case [17, 20, 40], the upper semi-continuity is much easier to check than the lower semi-continuity, since the latter often has a close relationship with the structure of attractors (see [5, 19, 20, 69]). To study the lower semi-continuity, we introduce a property of weak equi-attraction to generalize the standard equi-attraction method introduced by Li and Kloeden [58, 59]. The weak equi-attracting property is shown to be more appropriate to treat the lower-semi-continuity of attractors, while the standard one is hard to check because of the multi-valued feature of the systems.

Finally, we study a reaction-diffusion system and a scalar differential inclusion as applications. Remarkably, for the scalar differential inclusion we study the lower semi-continuity of attractors by using the method of weak equi-attraction, which is also taken as an example to highlight an advantage of the method in dealing with lower semi-continuity of attractors for multi-valued dynamical systems.

Part II: Cocycle attractors for random dynamical systems with quasi-S2W continuity

It is known that for *non-random* dynamical systems with closed graph, the existence of a global attractor does not require the system to be continuous (here and hereafter w.r.t. initial data), cf. [44, 30, 71, 40]. But it is not the case for RDS. As mentioned in [28, p.18], due to the attracting property of a random cocycle attractor, the distance between trajectories and the attractor should be able to be treated at least as a random variable. This basic demand leads to the measurability problem of random attractors, which, however, at least so far, needs the continuity of the system, see [40, Remark 2.26].

Cocycle attractors[1] for RDS have attracted much attention ever since the concept was introduced [34, 42, 33]. Particularly for the measurability problem, [34, 42, 33] gave sufficient conditions for the measurability w.r.t. $\bar{\mathcal{F}}$, where $\bar{\mathcal{F}}$ denotes the $\mathcal{P}$-completion of the sigma-algebra $\mathcal{F}$ of probability space $(\Omega, \mathcal{F}, \mathcal{P})$, while recently in [83, 84], in a non-autonomous framework the authors developed a method to ensure the attractor to be measurable w.r.t. $\mathcal{F}$, so that the attractor can be conveniently studied in a non-complete subspace $\tilde{\Omega}$ with full measure. However, all the publications mentioned above were for continuous RDS or NRDS only. The non-continuous problem seems still open.

On the other hand, the continuity of many important systems is unclear or requires restrictive conditions. Take the reaction-diffusion (RD) equation as an example. Under some general conditions, the solution of the RD equation is known continuous in L^2, but the continuity in L^p with $p > 2$ is unclear and that in H_0^1 needs more restrictive conditions, see Section 2.3. As a consequence, cocycle

[1] Note that cocycle attractors for RDS and NRDS are popularly called *random attractors* in the literature. As we shall introduce a concept of random uniform attractor for NRDS, we call them (random) cocycle attractors to avoid confusion.

attractors in L^p or H_0^1 cannot be studied by present theory unless the continuity problem in such spaces is proved. For instance, in [7] the authors studied the cocycle attractor in H_0^1, with technical efforts paid to derive the continuity of the system in H_0^1; nevertheless, such efforts are unnecessary at all for non-random cases according to bi-spatial attractor theory, see [38] for a discussion.

In Chapter 2, we try to find out a continuity condition easy to satisfy, but meeting the measurability demand. The *quasi strong-to-weak (abbrev. quasi-S2W) continuity* is introduced.

A mapping G from a metric space $\mathcal{M}$ to a Banach space X is called quasi-S2W continuous if $G(m_j) \rightharpoonup G(m_0)$ whenever $\{G(m_j)\}$ is bounded in X and $m_j \to m_0$. It is shown that such quasi-S2W continuity "is very close to" the closed condition and they are equivalent in weakly compact Banach spaces. Moreover, it ensures the $(\mathcal{B}(\mathcal{M}), \mathcal{B}(X))$-measurability of the mapping, where $\mathcal{B}(\cdot)$ denotes the Borel sigma-algebra. This enables us to prove the measurability of a cocycle attractor with only the quasi-S2W continuity of the system, leading to our existence theorem, Theorem 2.2.9, for cocycle attractors. Roughly, this theorem shows that, in order to obtain the cocycle attractor by establishing a pullback absorbing set (which belongs to the attraction universe) and a compact pullback attracting set as usual, an NRDS on some separable Banach space only needs to have the quasi-S2W continuity.

The real novelty of the quasi-S2W continuity lies in its inheritableness: if a mapping is quasi-S2W continuous in some space, say L^2, then so it is automatically in regular spaces, say H_0^1, see Propositions 2.1.3-2.1.5. This property makes Theorem 2.2.9 powerful especially in the study of cocycle attractors in more regular spaces. For example, the RD equation talked above is continuous and of course quasi-S2W continuous in L^2, and then by the inheritableness it is quasi-S2W continuous in H_0^1 so that, in order to study the cocycle attractor in H_0^1, by Theorem 2.2.9 we have to do nothing on the continuity in H_0^1, just like the non-random case.

Applying the main idea of Theorem 2.2.9 to bi-spatial cocycle attractor theory, we obtain the measurability of bi-spatial cocycle attractors w.r.t. the Borel sigma-algebra of regular spaces. Noticing that, in the literature, the measurability of bi-spatial cocycle attractors is assumed measurable only in the basic phase space, while the (pullback) attraction is expected to take place in more regular spaces, see, e.g. [62, 63, 93], the result is new and of certain significance.

Notice that results in Chapter 2 are applicable only for autonomous RDS. In Chapter 3 and Chapter 4, we study cocycle attractors for non-autonomous random dynamical systems (NRDS) which are usually generated by stochastic evolution equations with time-dependent terms, called the *(non-autonomous) symbol* of the equation. As indicated by name, non-autonomous symbols lead to all the non-autonomous features of dynamical behaviors of a non-autonomous dynamical system [26, 19, 55]. Since the long time behavior of a non-autonomous dynamical system is determined not only by lapsing time, but also by the initial time when the system is started, it is useful to consider time-translations of non-autonomous symbols. Hence, the concept of *symbol space*, containing translations of symbols, was introduced [26].

Attractors for NRDS generated by evolution equations with both deterministic time-dependent terms and stochastic pertubations were first studied in [35, 83] where a general framework was established. Typically, an NRDS is a measurable map $\phi : \mathbb{R}^+ \times \Omega \times \Sigma \times X \mapsto X$ with two base flows $\{\vartheta_t\}_{t\in\mathbb{R}}$ and $\{\theta_t\}_{t\in\mathbb{R}}$ acting on Ω and Σ, respectively, where X denotes the phase space, $\mathbb{R}^+ = [0, \infty)$ the space of lapsing time, $(\Omega, \mathcal{F}, \mathcal{P})$ a probability space and Σ the symbol space [37, 83]. When some continuity is mentioned to an NRDS, it usually means the continuous dependence on $x \in X$. If an NRDS ϕ is continuous in both $x \in X$ and $\sigma \in \Sigma$, then it is said to be *jointly continuous*.

In Chapter 3 we study cocycle attractors for NRDS, mainly to generalize the quasi-S2W continuous RDS theory introduced in Chapter 2 to non-autonomous cases. We follow the definitions introduced by [83] and establish some existence criteria for cocycle attractors for NRDS with only quasi-S2W continuity, which develops existence theorems in [83, 84]. Results are then generalized to bi-spatial cocycle attractor theory to obtain the measurability of bi-spatial attractors in regular spaces which seems also new in the literature.

In Chapter 4 we study cocycle attractors whose attraction universe is *autonomous*, i.e., containing only *autonomous* random sets. Here, by autonomous random sets we mean random sets independent of non-autonomous symbols. Given an NRDS ϕ and an autonomous universe $\mathcal{D}_X$ composed of some class of autonomous random sets, the cocycle attractor for ϕ with attraction universe $\mathcal{D}_X$ is a non-autonomous random set $A = \{A_\sigma(\omega)\}_{\sigma\in\Sigma,\omega\in\Omega}$ (not belonging to $\mathcal{D}_X$) such that

- A pullback attracts each random set D belonging to $\mathcal{D}_X$ under ϕ, i.e.,

$$\lim_{t\to\infty} \operatorname{dist}_X(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)), A_\sigma(\omega)) = 0, \quad \forall\omega \in \Omega, \sigma \in \Sigma,$$

  where and hereafter "dist_X" (or simply "dist") denotes the Hausdorff semi-distance between sets in X, namely,

$$\operatorname{dist}_X(A, B) := \sup_{a\in A} \inf_{b\in B} \|a - b\|_X, \quad \forall A, B \in 2^X \setminus \emptyset;$$

- A is the minimal compact non-autonomous random set satisfying the above condition;

- A is invariant under ϕ, i.e.,

$$\phi(t, \omega, \sigma, A_\sigma(\omega)) = A_{\theta_t\sigma}(\vartheta_t\omega), \quad \forall t \geqslant 0, \omega \in \Omega, \sigma \in \Sigma.$$

Clearly, since the cocycle attractor studied here no longer belongs to its attraction universe, it is generally different from that studied in Chapter 3, or in [83, 84] and many others considering non-autonomous systems. In this part we establish existence theorem, Theorem 4.3.2, for cocycle attractors with autonomous attraction universes. The relationship, see Proposition 4.2.1, and differences between autonomous and non-autonomous attraction universes are highlighted.

The continuous dependence of cocycle attractors in non-autonomous symbols is studied in Section 4.4. It is shown that the upper semi-continuity of the mapping $\sigma \mapsto A_\sigma(\omega)$, from symbol space Σ to sections of cocycle attractors, has a close relationship with the compactness of $\cup_{\sigma\in\Sigma}A(\omega)$ for each ω, see Theorem 4.4.2.

As applications, the $(L^2, L^p \cap H_0^1)$-cocycle attractor for a stochastic reaction-diffusion equation is studied in Section 2.3, the (L^2, H_0^1)-cocycle attractor for a stochastic Ginzburg-Landau equation with translation bounded external forcing is studied in Section 3.3, and the tempered cocycle attractor for a stochastic 2D Navier-Stokes equation with translation bounded external forcing is studied in Section 4.5, respectively.

Part III: Uniform attractors for non-autonomous random dynamical systems

For nonrandom non-autonomous dynamical systems, there are typically three kinds of global attractors which have drawn much attention in recent years: pullback attractors, cocycle attractors and uniform attractors [19, 55, 75, 26]. Each of these three attractors has its own interesting features on one hand, and has close relationship with the others on the other hand. More precisely, the pullback attractor and cocycle attractor for a non-autonomous dynamical system are directly related (so that we will not talk about pullback attractors though our results are valid for pullback attractors as well), while the uniform attractor is exactly the union of elements involved in the pullback/cocycle attractor [6, 5, 21, 39, 36].

Cocycle attractors for NRDS are studied in Chapter 3 and Chapter 4, where the study does in fact cover pullback attractors since the symbol space could be taken as the real line. In Chapter 5, we establish a theory of (random) uniform attractor for NRDS with Σ compact (see Section 5.1). Considering the random features of NRDS, we give the definition as follows.

Definition. An (autonomous) random set $\mathscr{A} \in \mathcal{D}_X$ is said to be the (random) $\mathcal{D}_X$-uniform attractor for an NRDS ϕ if

(I) $\mathscr{A}$ uniformly (pullback) attracts each $D \in \mathcal{D}_X$ under ϕ, namely,

$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} \operatorname{dist}_X\big(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)), \mathscr{A}(\omega)\big) = 0, \quad \forall \omega \in \Omega;$$

(II) $\mathscr{A}$ is the minimal compact (autonomous) random set satisfying (I).

From the definition it is clear that the random uniform attractor could be regarded as a random generalization of nonrandom uniform attractor concept [26, 25]. Indeed, when Ω is a singleton, ϕ reduces to a nonrandom non-autonomous dynamical system, where the uniformly *pullback* attracting property (I) is equivalent to the uniformly *forward* attracting property that

$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} \operatorname{dist}_X(\phi(t, \sigma, D), \mathscr{A}) = 0.$$

Because of the random features involved, the equivalence between pullback and forward uniform attractions fails for random uniform attractors, just like cocycle attractors for autonomous RDS, see [1, 34, 33, 42]. However, it is proved that the uniform pullback attraction implies a uniform forward attraction in probability, which makes a random uniform attractor $\mathscr{A}$ still possible to describe the forward dynamical behaviors of an NRDS (see Proposition 5.1.7):

(III) $\mathscr{A}$ is uniformly forward $\mathcal{D}_X$-attracting in probability in the sense that

$$\lim_{t\to\infty} \mathcal{P}\left\{\omega \in \Omega : \sup_{\sigma\in\Sigma} \operatorname{dist}_X(\phi(t, \omega, \sigma, D(\omega)), \mathscr{A}(\vartheta_t\omega)) > \varepsilon\right\} = 0, \quad \forall \varepsilon > 0,\ D \in \mathcal{D}_X.$$

Such an interesting connection between pullback attraction and forward attracting in probability property was first introduced in [34, 33, 73] for cocycle attractors of *autonomous* RDS. Here we show that such a connection holds for uniform attractors for NRDS. But note that it fails for cocycle attractors for NRDS, because even for nonrandom non-autonomous dynamical systems, pullback and forward attractions are not equivalent in general, see, e.g., [56, 54, 23].

Compared with $\mathcal{D}_X$-cocycle attractor A, in addition to the forward attracting in probability property (III), $\mathcal{D}_X$-uniform attractor $\mathscr{A}$ has the following more properties.

Firstly, the random uniform attractor is determined by uniformly attracting nonrandom compact sets (see Proposition 5.4.11), that is,

(IV) if $\mathfrak{D}$ is the collection of nonempty compact subsets of X and $\mathfrak{D} \subset \mathcal{D}_X$, and $\mathfrak{A}$ is the $\mathfrak{D}$-uniform attractor, then $\mathcal{P}(\mathscr{A} = \mathfrak{A}) = 1$, provided that ϕ is jointly continuous.

This result is a generalization of the analogous statement for cocycle attractors of (autonomous) RDS established in [31] via Poincaré recurrence theorem. To prove (IV), we make use of multi-valued RDS theory which is usually used to deal with dynamical systems without uniqueness, see e.g. [91, 10, 80, 72, 51]. It is shown that, given any jointly continuous NRDS ϕ, the mapping Φ defined by

$$\Phi(t, \omega, x) := \bigcup_{\sigma \in \Sigma} \phi(t, \omega, \sigma, x)$$

is a continuous multi-valued RDS (see Proposition 5.4.4), called the multi-valued RDS generated by NRDS ϕ. Moreover, the $\mathcal{D}_X$-uniform attractor $\mathscr{A}$ of ϕ is exactly the $\mathcal{D}_X$-cocycle attractor of the multi-valued RDS Φ (see Theorem 5.4.5). Then we prove (IV) by showing that the cocycle attractor for the multi-valued RDS Φ generated by the NRDS ϕ is determined by attracting compact sets.

Secondly, even though by definition we have no invariance property for random uniform attractors (also for nonrandom uniform attractors), inspired by [13, 39] we have the following negative semi-invariance property (see Proposition 5.2.5):

(V) $\mathscr{A}$ is negatively semi-invariant in the sense that

$$\mathscr{A}(\vartheta_t \omega) \subseteq \Phi(t, \omega, \mathscr{A}(\omega)) \quad \text{for each } t \geqslant 0,\ \omega \in \Omega,$$

provided that ϕ is jointly continuous, where Φ is the multi-valued RDS generated by ϕ.

Thirdly, while cocycle attractors are pullback attracting for each single $\omega \in \Omega$, the random uniform attractor $\mathscr{A}$ can attract almost uniformly (w.r.t. $\omega \in \Omega$) for discrete time sequences (see Proposition 5.1.9):

(VI) For each $t_n \to \infty$ and any $\varepsilon > 0$, there exists an $F \in \mathcal{F}$ (depending on $\{t_n\}_{n\in\mathbb{N}}$ and ε) with $\mathcal{P}(F) < \varepsilon$ such that, for any $D \in \mathcal{D}_X$,

$$\sup_{\sigma \in \Sigma} \operatorname{dist}_X(\phi(t_n, \vartheta_{-t_n}\omega, \sigma, D(\vartheta_{-t_n}\omega)), \mathscr{A}(\omega)) \xrightarrow{n\to\infty} 0, \quad \text{uniformly for all } \omega \in \Omega \setminus F.$$

This result is an application of Egoroff's theorem, and clearly holds for cocycle attractors of autonomous RDS as a particular case.

Fourthly, the random uniform attractor $\mathscr{A}$ can be composed of states involved in the cocycle attractor A which makes it possible to learn uniform attractors via cocycle attractors (see Theorem 5.3.13):

(VII) uniform attractor $\mathscr{A}$ and cocycle attractor A for a jointly continuous NRDS ϕ have the relation

$$\mathscr{A}(\omega) = \bigcup_{\sigma \in \Sigma} A_\sigma(\omega), \quad \forall \omega \in \Omega.$$

Note that analogous results for nonrandom dynamical systems were established in most recent works [5, 21, 39]. However, we remark that the theory in this paper is established in a rather different way due to the difficulty arising from the measurability problem. We first construct a proper attraction universe $\mathcal{D}_\mathbb{X}$ composed of *proper random sets* (see Definition 5.3.1), and then study the $\mathcal{D}_\mathbb{X}$-cocycle attractor $\mathbb{A}$ for the skew-product cocycle generated by the NRDS ϕ and the base flow $\{\theta_t\}_{t\in\mathbb{R}}$ as a bridge. By developing the relationship between uniform attractor $\mathscr{A}$ and the cocycle attractor $\mathbb{A}$ and that between $\mathbb{A}$ and the cocycle attractor A, we conclude the relationship (VII) between uniform and cocycle attractors, $\mathscr{A}$ and A.

Existence criteria and characterization of random uniform attractors are established as well. Similar to nonrandom cases, random uniform attractor is shown to have a close relationship to compact uniformly attracting random sets (see Theorems 5.2.5 & 5.3.14), and can be characterized by omega-limit sets and complete trajectories (see Proposition 5.3.17). However, in order to prove the measurability, we require the symbol space to be Polish, i.e., a complete metric space with a countable dense subset, which cannot be seen in nonrandom attractor theory. The Polish condition is so related to the stochastic features (see [1, 32]) that it is also crucial for further analysis of random uniform attractors, but it is shown in Section 5.5.1 to be general enough to cover usual applications.

In the final section the tempered uniform and cocycle attractors for a reaction-diffusion equation with both translation-bounded forcing and additive white noise are studied.

Part I

Attractors for multi-valued systems

Chapter 1

On attractors for multi-valued non-autonomous dynamical systems

In this chapter we study cocycle, pullback and uniform attractors for multi-valued dynamical systems. The structure of this chapter is as follows. In Section 1.1 we first introduce the concepts of cocycles and processes, and then go deeper into the relationship between different attractors. In Section 1.2, we characterize the attractors for generalized cocycles by complete trajectories, and introduce lifted-invariance for uniform attractors. In Section 1.3 we study both the upper and lower semi-continuity of attractors. Applications are given in the last section.

1.1 Relationship between different attractors

Let (X, d) be a complete metric space and $P(X)$ be the set of all nonempty subsets of X, $\mathcal{B}(X)$ the collection of all bounded non-empty subsets of X, and $C(X) \subset P(X)$ the collection of all non-empty closed subsets of X. Let $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}_d = \{(t, \tau) \in \mathbb{R}^2 : t \geqslant \tau\}$. In the following, for any mapping $f : D \to V$ we denote $f(C) := \cup_{c \in C} f(c)$, $\forall C \subset D$.

1.1.1 Equivalence between multi-valued cocycles and processes

In this part, we show the equivalence between cocycles and processes. This equivalence gives a connection between a cocycle and a process, and usually is true when the cocycle and process are referred to the same model. We begin with the definition of multi-valued cocycles and processes.

Denote by (Σ, ρ) a metric space, endowed with a family $\{\theta_t\}_{t \in \mathbb{R}}$ of translation operators on Σ satisfying

- θ_0 = identity operator on Σ;
- $\theta_t \Sigma = \Sigma, \quad \forall t \in \mathbb{R}$;
- $\theta_s \circ \theta_t = \theta_{t+s}, \quad \forall t, s \in \mathbb{R}$;

- $(t, \sigma) \mapsto \theta_t\sigma$ is continuous.

Such a family $\{\theta_t\}_{t\in\mathbb{R}}$ is called a driving system on Σ. A typical example is given by the time shift $\theta_t\sigma(\cdot) := \sigma(\cdot + t)$ in $C(\mathbb{R}, \mathbb{R})$.

Remark 1.1.1. (i) In applications, the space Σ contains all the terms in an evolution equation leading to all the non-autonomous features and called the *non-autonomous symbol* or simply called the symbol of the equation. The space Σ itself is called the *symbol space*.

(ii) That whether or not the symbol space Σ is compact sometimes makes a great difference. Here, we shall not require Σ to be compact or even bounded most generally, so that Σ can be taken as $\mathbb{R}$ in Chapter 3. But in the uniform attractor part, Chapter 5, we shall require Σ to be compact for simplicity.

Definition 1.1.2. A multi-valued mapping $\Phi : \mathbb{R}^+ \times \Sigma \times X \to C(X)$, $(t, \sigma, x) \mapsto \Phi(t, \sigma, x)$, is called a multi-valued cocycle on (X, Σ) if:

1. $\Phi(0, \sigma, \cdot)$ is the identity operator on X, $\quad \forall\sigma \in \Sigma$;
2. $\Phi(t + s, \sigma, x) \subseteq \Phi(t, \theta_s\sigma, \Phi(s, \sigma, x))$, $\quad \forall t, s \geqslant 0, \sigma \in \Sigma, x \in X$.

If, moreover, we have an equality in the second property, then Φ is called strict.

Definition 1.1.3. The family of multi-valued mappings $\{U_\sigma : \mathbb{R}_d \times X \mapsto C(X)\}_{\sigma\in\Sigma}$ is said to be a family of multi-valued processes (family of MP), if for all $\sigma \in \Sigma$, $\tau \in \mathbb{R}$:

1. $U_\sigma(\tau, \tau, x) = x, \quad \forall x \in X;$
2. $U_\sigma(t, \tau, x) \subseteq U_\sigma(t, s, U_\sigma(s, \tau, x)), \quad \forall t \geqslant s \geqslant \tau,\ x \in X;$
3. it satisfies the translation identity property:
$$U_\sigma(t + h, \tau + h, x) = U_{\theta_h\sigma}(t, \tau, x), \quad \forall t \geqslant \tau,\ h \in \mathbb{R},\ \sigma \in \Sigma,\ x \in X.$$

For each $\sigma \in \Sigma$, U_σ is called a process (driven by σ). The family of MP is called strict if we have an equality in the second property.

Given a non-autonomous differential inclusion, to investigate its dynamical behavior one can associate it with either a multi-valued cocycle or a family of MP. The two approaches are closely related as implied by the following proposition.

Proposition 1.1.4. *The following statements hold:*

1. *Suppose that Φ is a multi-valued cocycle on (X, Σ). Then the family $\{U_\sigma\}_{\sigma\in\Sigma}$ is a family of MP, where*
$$U_\sigma(t, \tau, x) := \Phi(t - \tau, \theta_\tau\sigma, x), \quad \forall (t, \tau) \in \mathbb{R}_d,\ \sigma \in \Sigma,\ x \in X. \tag{1.1}$$

2. *Suppose that* $\{U_\sigma\}_{\sigma\in\Sigma}$ *is a family of MP, then* Φ *is a multi-valued cocycle, where*

$$\Phi(t,\sigma,x) := U_\sigma(t,0,x), \quad \forall t \geqslant 0,\ \sigma\in\Sigma,\ x\in X. \tag{1.2}$$

Proof. The proof is straightforward by definition and thus omitted here. □

Definition 1.1.5. Suppose that Φ is a multi-valued cocycle, and $\{U_\sigma\}_{\sigma\in\Sigma}$ is a family of MP. Then Φ and $\{U_\sigma\}_{\sigma\in\Sigma}$ are called equivalent non-autonomous dynamical systems if (1.1) and (1.2) are satisfied.

We recall that a multivalued map $F : D(F) \subset X \to 2^Y$, where X, Y are metric spaces, is said to be upper semicontinuous if for any $x \in D(F)$ and any neighborhood $\mathcal{O}$ of $F(x)$ there exists $\delta > 0$ such that $F(y) \subset \mathcal{O}$ whenever $d(y,x) < \delta$. This upper semicontinuity implies that

$$\text{dist}(F(x), F(x_0)) \to 0 \text{ as } x \to x_0,$$

and they are equivalent if F has compact values.

Throughout this chapter, we will often assume that either $x \mapsto \Phi(t,\sigma,x)$ or $(\sigma,x) \mapsto \Phi(t,\sigma,x)$ is upper semicontinuous. Nevertheless, such conditions are not optimal and might be weakened by using a closed graph condition instead, see [30, 40] for a discussion.

1.1.2 On the concepts of an "attractor"

Now we define different type of attractors and prove their existence and properties under suitable assumptions.

a). Cocycle attractors for multi-valued cocycles

Definition 1.1.6. A non-autonomous set $D = \{D(\sigma)\}_{\sigma\in\Sigma}$ in X is a mapping D: $\Sigma \to P(X), \sigma \mapsto D(\sigma)$.

Definition 1.1.7. A non-autonomous set D is called compact/bounded/closed (in X) if for every $\sigma \in \Sigma$, $D(\sigma)$ is compact/bounded/closed (in X). D is called backwards bounded if

$$\bigcup_{t\geqslant T} D(\theta_{-t}\sigma) \text{ is bounded for each } T \in \mathbb{R}.$$

Definition 1.1.8. Let Φ be a multi-valued cocycle on (X,Σ). A non-autonomous set $A = \{A(\sigma)\}_{\sigma\in\Sigma}$ is called the cocycle attractor for Φ if:

1. A is compact in X;

2. A pullback attracts every bounded subsets B of X, that is,

$$\lim_{t\to\infty} \text{dist}(\Phi(t,\theta_{-t}\sigma,B), A(\sigma)) = 0, \quad \forall \sigma\in\Sigma; \tag{1.3}$$

3. A is negatively invariant under Φ, that is,

$$A(\theta_t\sigma) \subset \Phi(t, \sigma, A(\sigma)), \quad \forall \sigma \in \Sigma,\ t \geqslant 0; \tag{1.4}$$

4. A is minimal among all closed non-autonomous sets satisfying (1.3), that is, if $A' = \{A'(\sigma)\}_{\sigma\in\Sigma}$ is closed and satisfies (1.3), then

$$A(\sigma) \subset A'(\sigma), \quad \forall \sigma \in \Sigma.$$

If, moreover, we have an identity in (1.4), then the cocycle attractor is called strictly invariant.

Note that if a non-autonomous set $A = \{A(\sigma)\}_{\sigma\in\Sigma}$ satisfies the first three properties in Definition 1.1.8 and is backwards bounded, then A must satisfy the minimality property and thereby A is the cocycle attractor. In this case, A is called a backwards bounded cocycle attractor. We refer to [38] for a study of backwards bounded and backwards compact pullback attractors.

Definition 1.1.9. Suppose that Φ is a multi-valued cocycle on (X, Σ). A non-autonomous set D is called pullback attracting under Φ if

$$\lim_{t\to\infty} \Big(\mathrm{dist}(\Phi(t, \theta_{-t}\sigma, B), D(\sigma))\Big) = 0, \quad \forall \sigma \in \Sigma,\ B \in \mathcal{B}(X).$$

In particular, we say that $D(\sigma)$ pullback attracts B driven by σ. The multi-valued cocycle Φ is called pullback asymptotically compact if there exists a compact non-autonomous set D in X which is pullback attracting under Φ.

Definition 1.1.10. Suppose that Φ is a multi-valued cocycle on (X, Σ). Then a set $K \subset X$ is called uniformly attracting under Φ if

$$\lim_{t\to\infty} \Big(\sup_{\sigma\in\Sigma} \mathrm{dist}(\Phi(t, \theta_{-t}\sigma, B), K)\Big) = 0, \quad \forall B \in \mathcal{B}(X), \tag{1.5}$$

which is equivalent to

$$\lim_{t\to\infty} \Big(\sup_{\sigma\in\Sigma} \mathrm{dist}(\Phi(t, \sigma, B), K)\Big) = 0, \quad \forall B \in \mathcal{B}(X), \tag{1.6}$$

due to the invariance of Σ under θ. The multi-valued cocycle Φ is called uniformly asymptotically compact if there exists a compact set K which is uniformly attracting under Φ.

It is straightforward to obtain the following proposition.

Proposition 1.1.11. *If a multi-valued cocycle Φ is uniformly asymptotically compact, then it is pullback asymptotically compact.*

Suppose that Φ is a multi-valued cocycle on (X, Σ). Let us define

$$\mathcal{W}(B, \sigma) = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} \Phi(t, \theta_{-t}\sigma, B)}, \quad \forall B \in \mathcal{B}(X),\ \sigma \in \Sigma.$$

Proposition 1.1.12. *$y \in \mathcal{W}(B, \sigma)$ if and only if there exist sequences $\{x_n\}_{n\in\mathbb{N}}$ and $t_n \to \infty$ such that $x_n \in \Phi(t_n, \theta_{-t_n}\sigma, B)$ and $x_n \to y$.*

Proposition 1.1.13. *Suppose that Φ is a multi-valued cocycle on (X, Σ) with a compact pullback attracting non-autonomous set D and that the map $x \mapsto \Phi(t, \sigma, x)$ is upper semicontinuous for fixed t, σ. Then, for each $B \in \mathcal{B}(X)$ and $\sigma \in \Sigma$, $\mathcal{W}(B, \sigma)$ is non-empty, compact and negatively invariant. Moreover, it is the minimal closed set pullback attracting B driven by σ.*

Proof. To see that $\mathcal{W}(B, \sigma)$ is non-empty, let us take sequences $t_n \to \infty$ and $x_n \in \Phi(t_n, \theta_{-t_n}\sigma, B)$. Then by the pullback attracting property of D we have

$$\lim_{n\to\infty} \operatorname{dist}(x_n, D(\sigma)) \leqslant \lim_{n\to\infty} \operatorname{dist}(\Phi(t_n, \theta_{-t_n}\sigma, B), D(\sigma)) = 0.$$

Since $D(\sigma)$ is compact, there exists a $y \in D(\sigma)$ such that, up to a subsequence,

$$x_n \to y.$$

Therefore, by Proposition 1.1.12 we have $y \in \mathcal{W}(B, \sigma)$ and thereby $\mathcal{W}(B, \sigma)$ is non-empty.

We prove the pullback attracting property by contradiction. If this is not true, then there exist an $\epsilon > 0$ and $x_n \in \Phi(t_n, \theta_{-t_n}\sigma, B)$ with $t_n \to \infty$ such that

$$\operatorname{dist}(x_n, \mathcal{W}(B, \sigma)) > \epsilon, \quad \forall n \in \mathbb{N}. \tag{1.7}$$

However, by the pullback attracting property and the compactness of $D(\sigma)$, arguing as above there exists $y \in \mathcal{W}(B, \sigma)$ such that $x_n \to y$, which contradicts (1.7).

We then prove the negative invariance. By Proposition 1.1.12, for any $y \in \mathcal{W}(B, \theta_t\sigma)$, $t \geqslant 0$, there exists a sequence $x_n \in \Phi(t_n, \theta_{-t_n}\theta_t\sigma, B)$ with $t_n \to \infty$ such that $x_n \to y$. Since

$$\Phi(t_n, \theta_{-t_n}\theta_t\sigma, B) \subset \Phi(t, \sigma, \Phi(t_n - t, \theta_{t-t_n}\sigma, B)), \quad \forall t_n > t,$$

$x_n \in \Phi(t, \sigma, z_n)$ for some $z_n \in \Phi(t_n - t, \theta_{t-t_n}\sigma, B)$. By the pullback attracting and the compactness of $D(\sigma)$ again there exists a converging subsequence $\{z_{n_k}\}$ such that $z_{n_k} \to z$ for some $z \in \mathcal{W}(B, \sigma)$. Therefore, since $x \mapsto \Phi(t, \sigma, x)$ is upper semi-continuous and Φ has closed values, we have

$$y \in \Phi(t, \sigma, z) \subset \Phi(t, \sigma, \mathcal{W}(B, \sigma)),$$

and thereby the negative invariance of $\mathcal{W}(B, \sigma)$ is proved.

Since D is pullback attracting and compact, $\mathcal{W}(B, \sigma) \subset D(\sigma)$ is compact.

To prove the minimality property, we suppose that there is another closed pullback attracting set $A'(B, \sigma)$. Take $y \in \mathcal{W}(B, \sigma)$, so there exists $x_n \in \Phi(t_n, \theta_{-t_n}\sigma, B)$ with some $t_n \to \infty$ such that $x_n \to y$. However, since $A'(B, \sigma)$ pullback attracts B and $A'(B, \sigma)$ is closed, x_n converges to some point in $A'(B, \sigma)$ and thereby $y \in A'(B, \sigma)$. The proof is complete. □

Remark 1.1.14. *The above results were previously obtained by Caraballo et al. [13, Lemma 5 & Theorem 6] for multi-valued processes, where more general two-parameterized systems were considered.*

Theorem 1.1.15. *Suppose that Φ is a multi-valued cocycle on (X, Σ) and that the map $x \mapsto \Phi(t, \sigma, x)$ is upper semicontinuous for fixed t, σ. If Φ has a compact pullback attracting non-autonomous set D, then it has the unique cocycle attractor $A = \{A(\sigma)\}_{\sigma \in \Sigma}$ defined by*

$$A(\sigma) = \overline{\bigcup_{B \in \mathcal{B}(X)} \mathcal{W}(B, \sigma)}, \quad \forall \sigma \in \Sigma. \tag{1.8}$$

Proof. The fact that A is non-empty, minimal and pullback attracting follows from the properties of the omega-limit sets $\mathcal{W}(B, \sigma)$ immediately, see Proposition 1.1.13. Also, since D is pullback attracting and compact, for each $B \in \mathcal{B}(X)$ we have $\mathcal{W}(B, \sigma) \subset D(\sigma)$. Therefore, $A(\sigma)$ is compact since it is a closed subset of a compact set.

Let us prove that A is negatively invariant. Take an arbitray $y \in A(\theta_t \sigma)$, $t \geqslant 0$. There exists a sequence $y_n \in \mathcal{W}(B_n, \theta_t \sigma)$, $B_n \in \mathcal{B}(X)$, such that $y_n \to y$ in X. Since the $\mathcal{W}$-limit sets $\mathcal{W}(B_n, \sigma)$ are negatively invariant, we can find $x_n \in \mathcal{W}(B_n, \sigma)$ such that $y_n \in \Phi(t, \sigma, x_n)$. The compacity of $A(\sigma)$ implies, up to a subsequence, that $x_n \to x \in A(\sigma)$. Hence, since $x \mapsto \Phi(t, \sigma, x)$ is upper semicontinuous and has closed values, we get $y \in \Phi(t, \sigma, x) \subset \Phi(t, \sigma, A(\sigma))$. □

Proposition 1.1.16. *Backwards bounded cocycle attractors for strict multi-valued cocycles are invariant.*

Proof. Let $A = \{A(\sigma)\}_{\sigma \in \Sigma}$ be the backwards bounded cocycle attractor for cocycle Φ, which satisfies

$$\bigcup_{t \geqslant T} A(\theta_{-t} \sigma) \subset B_T \in \mathcal{B}(X), \ \forall T \in \mathbb{R}.$$

Let us fix some T. Then, by the negative invariance of A and the strictness of Φ we have

$$\begin{aligned}
\Phi(t, \sigma, A(\sigma)) &\subset \Phi(t, \sigma, \Phi(s, \theta_{-s}\sigma, A(\theta_{-s}\sigma))) \\
&= \Phi(t + s, \theta_{-s}\sigma, A(\theta_{-s}\sigma)) \\
&\subset \Phi(t + s, \theta_{-s}\sigma, B_T) \\
&= \Phi(t + s, \theta_{-s-t}\theta_t\sigma, B_T), \ \forall s \geqslant T.
\end{aligned}$$

Letting $s \to \infty$ and taking the limit we have $\Phi(t, \sigma, A(\sigma)) \subset A(\theta_t \sigma)$ by the pullback attracting property and the compactness of A. □

b). Uniform attractors for multi-valued cocycles

Definition 1.1.17. Suppose that Φ is a multi-valued cocycle on (X, Σ). A subset $\mathscr{A} \subset X$ is called the uniform attractor for Φ if:

1. $\mathscr{A}$ is compact in X;
2. $\mathscr{A}$ is uniformly attracting in the sense of Definition 1.1.10;
3. $\mathscr{A}$ is the minimal closed set satisfying (1.5).

Given a multi-valued cocycle Φ, we denote by

$$\mathcal{W}(B,\Sigma) = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\Phi(t,\theta_{-t}\Sigma,B)} = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\bigcup_{\sigma\in\Sigma}\Phi(t,\theta_{-t}\sigma,B)}.$$

Note that, in general, $\mathcal{W}(B,\Sigma) \neq \overline{\cup_{\sigma\in\Sigma}\mathcal{W}(B,\sigma)}$. More precisely, we have

$$\begin{aligned}\overline{\bigcup_{\sigma\in\Sigma}\mathcal{W}(B,\sigma)} &= \overline{\bigcup_{\sigma\in\Sigma}\bigcap_{s\in\mathbb{N}}\overline{\bigcup_{t\geqslant s}\Phi(t,\theta_{-t}\sigma,B)}} \subseteq \bigcap_{s\in\mathbb{N}}\overline{\bigcup_{\sigma\in\Sigma}\overline{\bigcup_{t\geqslant s}\Phi(t,\theta_{-t}\sigma,B)}}\\ &\subseteq \bigcap_{s\in\mathbb{N}}\overline{\overline{\bigcup_{\sigma\in\Sigma}\bigcup_{t\geqslant s}\Phi(t,\theta_{-t}\sigma,B)}} = \bigcap_{s\in\mathbb{N}}\overline{\overline{\bigcup_{t\geqslant s}\Phi(t,\theta_{-t}\Sigma,B)}}\\ &= \bigcap_{s\in\mathbb{N}}\overline{\bigcup_{t\geqslant s}\Phi(t,\theta_{-t}\Sigma,B)} = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\Phi(t,\theta_{-t}\Sigma,B)} = \mathcal{W}(B,\Sigma).\end{aligned}$$

In a similar way to Proposition 1.1.12 we have the following property of $\mathcal{W}(B,\Sigma)$.

Proposition 1.1.18. *A point $y \in X$ belongs to $\mathcal{W}(B,\Sigma)$ if and only if there exist sequences $t_n \to \infty$ and $x_n \in \Phi(t_n,\theta_{-t_n}\Sigma,B)$ such that $x_n \to y$.*

Proposition 1.1.19. *Suppose that Φ is a multi-valued cocycle on (X,Σ). If Φ is uniformly asymptotically compact, then for each $B \in \mathcal{B}(X)$, $\mathcal{W}(B,\Sigma)$ is non-empty, compact and uniformly attracting B. Moreover, $\mathcal{W}(B,\Sigma)$ is minimal among all closed sets uniformly attracting B.*

Proof. The non-empty, compact and uniformly attracting properties are proved in a similar way to Proposition 1.1.13. To see the minimality property, we suppose K to be another closed set which is uniformly attracting. Let $y \in \mathcal{W}(B,\Sigma)$. Then by Proposition 1.1.18 we can find a sequence $x_n \in \Phi(t_n,\theta_{-t_n}\Sigma,B)$, where $t_n \to \infty$, such that $x_n \to y$. Therefore, by the uniform attraction of K,

$$\text{dist}(x_n,K) \leqslant \text{dist}\big(\Phi(t_n,\theta_{-t_n}\Sigma,B),K\big) \to 0, \quad \text{as } n\to\infty,$$

which implies $y \in \bar{K}$ and then $\mathcal{W}(B,\Sigma) \subset \bar{K} = K$ since K is closed. Thus the minimality property is proved. □

Now we are ready to state our result on the existence of uniform attractors.

Theorem 1.1.20. *Suppose that Φ is a multi-valued cocycle on (X,Σ). If Φ is uniformly asymptotically compact, then it has the uniform attractor*

$$\mathscr{A} = \overline{\bigcup_{B\in\mathcal{B}(X)}\mathcal{W}(B,\Sigma)}.$$

Proof. The uniform attracting and the minimal properties follow from the analogous properties of $\mathcal{W}(B,\Sigma)$. In order to check that $\mathscr{A}$ is compact, it suffices to notice that in view of Proposition 1.1.19 if K is a closed uniformly attracting set, for any $B \in \mathcal{B}(X)$ we have $\mathcal{W}(B,\Sigma) \subset K$. Hence, the theorem is proved. □

c). Global attractors for skew product semiflows

For the two metric spaces (X, d) and (Σ, ρ), denote by $\mathcal{X} = X \times \Sigma$ the skew product space with the metric $\rho_{\mathcal{X}}$ defined by

$$\rho_{\mathcal{X}}\big(\{x_1\} \times \{\sigma_1\}, \{x_2\} \times \{\sigma_2\}\big) = d(x_1, x_2) + \rho(\sigma_1, \sigma_2).$$

Clearly, each subset $\mathbb{B}$ of $\mathcal{X}$ has the form $\mathbb{B} = \cup_{\sigma\in\Sigma} B(\sigma) \times \{\sigma\}$, where each $B(\sigma)$ is a subset, possibly empty, of X. This leads to the projectors $P_\sigma : \mathcal{X} \to X$ given by $P_\sigma(\mathbb{B}) = B(\sigma)$, $\forall \sigma \in \Sigma$. Denote also $P_X\mathbb{B} = \cup_{\sigma\in\Sigma} P_\sigma \mathbb{B}$ $(\subset X)$.

The following proposition indicates that any multi-valued cocycle generates a multi-valued (skew product) semiflow. The proof is straightforward and thereby omitted here.

Proposition 1.1.21. *Let θ be a driving system on the metric space (Σ, ρ) and Φ be a multi-valued cocycle on (X, Σ). Then the family $\Pi = \{\Pi(t,\cdot)\}_{t\geqslant 0}$ of mappings $\Pi(t,\cdot) : \mathcal{X} \to C(\mathcal{X})$, $\{x\} \times \{\sigma\} \mapsto \Phi(t, \sigma, x) \times \{\theta_t\sigma\}$ is a multi-valued semiflow (generated by (Φ, θ)), namely, satisfying*

1. *$\Pi(0,\cdot)$ is the identity operator on $\mathcal{X}$;*
2. *$\Pi(t + s, \nu) \subset \Pi(t, \Pi(s, \nu))$, $\quad \forall t, s \in \mathbb{R}^+$, $\nu \in \mathcal{X}$.*

If, moreover, Φ is strict, then we have an identity in the second condition and then Π is called strict. If for each $t \in \mathbb{R}^+$, the mapping $\nu \mapsto \Pi(t, \nu)$ is upper semi-continuous, then Π is called upper semi-continuous.

Definition 1.1.22. A subset $\mathbb{A} \subset \mathcal{X}$ is called the global attractor for a multi-valued skew product semiflow Π if:

1. $\mathbb{A}$ is compact in $\mathcal{X}$;
2. $\mathbb{A}$ (forwards) attracts every bounded subsets of $\mathcal{X}$, that is,
$$\lim_{t\to\infty} \operatorname{dist}_{\mathcal{X}}(\Pi(t, \mathbb{B}), \mathbb{A}) = 0, \quad \forall \mathbb{B} \in \mathcal{B}(\mathcal{X});$$
3. $\mathbb{A}$ is negatively invariant under Π, that is,
$$\mathbb{A} \subset \Pi(t, \mathbb{A}), \quad \forall t \geqslant 0.$$

If we have an equality in the last condition, then $\mathbb{A}$ is called (strictly) invariant.

Remark 1.1.23. If the global attractor $\mathbb{A}$ of a multi-valued skew product semiflow Π exists, it is unique and has the representation

$$\mathbb{A} = \bigcup_{\mathbb{B}\in\mathcal{B}(\mathcal{X})} \bigg(\bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} \Pi(t, \mathbb{B})}\bigg). \tag{1.9}$$

This is verified by checking the inclusion relations in both directions. Therefore, $\upsilon \in \mathbb{A}$ if and only if there exists a $\mathbb{B} \in \mathcal{B}(\mathcal{X})$ such that for some sequences $t_n \to \infty$ and $\upsilon_n \in \Pi(t_n, \mathbb{B})$ it holds $\upsilon_n \to \upsilon$.

Remark 1.1.24. The negative invariance implies the minimality property, that is, $\mathbb{A}$ is the minimal among closed attracting sets. Indeed, for any closed and attracting set $\mathbb{K}$, we have

$$\operatorname{dist}(\mathbb{A}, \mathbb{K}) \leqslant \operatorname{dist}(\Pi(t, \mathbb{A}), \mathbb{K}) \to 0, \quad \text{as } t \to \infty,$$

which implies $\mathbb{A} \subset \mathbb{K}$.

The following result was proved in [72, Theorem 4].

Theorem 1.1.25. *Suppose the multi-valued skew product semiflow Π is upper semi-continuous. If there is a compact subset $\mathbb{K} \subset \mathcal{X}$ such that*

$$\lim_{t\to\infty} \operatorname{dist}_{\mathcal{X}}(\Pi(t, \mathbb{B}), \mathbb{K}) = 0, \quad \forall \mathbb{B} \in \mathcal{B}(\mathcal{X}),$$

then Π has a unique global attractor $\mathbb{A}$ defined by

$$\mathbb{A} = \mathcal{W}(\mathbb{K}) = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} \Pi(t, \mathbb{K})}.$$

Proposition 1.1.26. *The global attractor $\mathbb{A}$ for a strict multi-valued skew product semiflow Π must be invariant.*

Proof. We only need to prove $\Pi(t, \mathbb{A}) \subset \mathbb{A}$, $\forall t \geqslant 0$. It suffices to observe that, by the negative invariance of $\mathbb{A}$ and the strictness of Π, it holds

$$\begin{aligned}\operatorname{dist}(\Pi(t, \mathbb{A}), \mathbb{A}) &\leqslant \operatorname{dist}(\Pi(t, \Pi(s, \mathbb{A})), \mathbb{A}) \\ &= \operatorname{dist}(\Pi(t + s, \mathbb{A}), \mathbb{A}) \to 0, \quad \text{as } s \to \infty.\end{aligned}$$

Since $\mathbb{A}$ is compact, we have $\Pi(t, \mathbb{A}) \subset \mathbb{A}$. □

d). Pullback attractors for multi-valued processes

Given a family of upper semi-continuous multi-valued processes $\{U_\sigma\}_{\sigma\in\Sigma}$ on $\mathbb{R}_d \times X$ (see Definition 1.1.3), we study the pullback attractor A_σ for each process U_σ.

Definition 1.1.27. For each $\sigma \in \Sigma$, the family $A_\sigma = \{A_\sigma(t)\}_{t\in\mathbb{R}}$ is called a pullback attractor for the process U_σ if:

1. $A_\sigma(t)$ is a compact subset of X, $\forall t \in \mathbb{R}$;
2. for each $t \in \mathbb{R}$, $A_\sigma(t)$ pullback attracts every bounded subset B of X at t under U_σ, i.e.,
$$\lim_{\tau\to-\infty} \operatorname{dist}(U_\sigma(t, \tau, B), A_\sigma(t)) = 0; \tag{1.10}$$
3. $A_\sigma(t)$ is negatively invariant under U_σ, i.e.,
$$A_\sigma(t) \subseteq U_\sigma(t, \tau, A_\sigma(\tau)), \quad \forall t \geqslant \tau. \tag{1.11}$$

4. A_σ satisfies the minimality property, that is, if A'_σ is a closed family satisfying the pullback attracting property (1.10), then

$$A_\sigma(t) \subset A_\sigma(t)', \quad \forall t \in \mathbb{R}.$$

If an equality in (1.11) holds, then the pullback attractor A_σ is called (strictly) invariant. The family $\{A_\sigma\}_{\sigma\in\Sigma}$ is called a family of pullback attractors for the family $\{U_\sigma\}_{\sigma\in\Sigma}$ of MP if each A_σ is a pullback attractor for U_σ.

Definition 1.1.28. A pullback attractor A_σ is called backwards bounded if $\cup_{t\leqslant T} A_\sigma(t)$ is bounded for every $T \in \mathbb{R}$.

Definition 1.1.29. The multi-valued process U_σ is called pullback asymptotically compact if there exists a family $D_\sigma = \{D_\sigma(t)\}_{t\in\mathbb{R}}$ which is pullback asymptotically attracting under U_σ, that is,

$$\lim_{\tau\to-\infty} \operatorname{dist}(U_\sigma(t,\tau,B), D_\sigma(t)) = 0, \quad \forall t \in \mathbb{R},\ B \in \mathcal{B}(X),$$

and each $D_\sigma(t)$ is a compact subset of X.

Theorem 1.1.30. *Suppose that U_σ is a multi-valued process such that the map $x \mapsto U_\sigma(t,s,x)$ is upper semicontinuous. If U_σ has a compact pullback attracting set D_σ, then it has a unique pullback attractor $A_\sigma = \{A_\sigma(t)\}_{t\in\mathbb{R}}$ defined by*

$$A_\sigma(t) = \overline{\bigcup_{B\in\mathcal{B}(X)} \mathcal{W}_\sigma(B,t)}, \quad \forall t \in \mathbb{R}, \tag{1.12}$$

where

$$\mathcal{W}_\sigma(B,t) = \bigcap_{s\leqslant t} \overline{\bigcup_{\tau\leqslant s} U_\sigma(t,\tau,B)}, \quad \forall B \in \mathcal{B}(X),\ t \in \mathbb{R}.$$

Proof. The proof is established in a similar way to Theorem 1.1.15. See also [13, Theorems 18, 43], where a two-parameter process was considered. □

Proposition 1.1.31. *The backwards bounded pullback attractor A_σ for a strict multi-valued process U_σ must be invariant.*

Proof. We choose $T \in \mathbb{R}$ and $B_T \in \mathcal{B}(X)$ such that

$$\bigcup_{t\leqslant T} A_\sigma(t) \subset B_T.$$

Then by the negative invariance of A_σ and the strictness of U_σ we have, for all $t \geqslant \tau \geqslant s$ and $s \leqslant T$,

$$\begin{aligned}\operatorname{dist}(U_\sigma(t,\tau,A_\sigma(\tau)), A_\sigma(t)) &\leqslant \operatorname{dist}(U_\sigma(t,\tau,U_\sigma(\tau,s,A_\sigma(s)), A_\sigma(t)) \\ &\leqslant \operatorname{dist}(U_\sigma(t,s,B_T), A_\sigma(t)) \to 0 \quad \text{as } s \to -\infty.\end{aligned}$$

Hence $U_\sigma(t,\tau,A_\sigma(\tau)) \subset A(t)$ and thereby the invariance follows. □

The following proposition allows us to keep focusing only on one parameter, either σ or t. This is due to the translation identity of multi-valued processes, see Definition 1.1.3.

Proposition 1.1.32. *Suppose that a family $\{U_\sigma\}_{\sigma\in\Sigma}$ of MP has the family $\{A_\sigma\}_{\sigma\in\Sigma}$ of pullback attractors. Then it satisfies the translation identity property, that is,*

$$A_{\theta_s\sigma}(t) = A_\sigma(t+s), \quad \forall t,s \in \mathbb{R},\ \sigma \in \Sigma.$$

Proof. It suffices to prove that $A_\sigma(t+s) \subset A_{\theta_s\sigma}(t)$ holds for every $t,s \in \mathbb{R}$ and $\sigma \in \Sigma$, since the inverse will be clear if we take $\sigma = \theta_{-s}\tilde{\sigma}$.

Clearly, since $\{A_\sigma\}_{\sigma\in\Sigma}$ is a family of pullback attractors, $A_\sigma(t+s)$ and $A_{\theta_s\sigma}(t)$ are both compact. Let

$$A'_\sigma(\cdot) := A_{\theta_s\sigma}(\cdot - s).$$

Then by the translation identity of $\{U_\sigma\}_{\sigma\in\Sigma}$ we have

$$\begin{aligned}
&\lim_{\tau\to-\infty} \operatorname{dist}\big(U_\sigma(t+s,\tau+s,B), A'_\sigma(t+s)\big) \\
&\quad= \lim_{\tau\to-\infty} \operatorname{dist}\big(U_\sigma(t+s,\tau+s,B), A_{\theta_s\sigma}(t)\big) \\
&\quad= \lim_{\tau\to-\infty} \operatorname{dist}\big(U_{\theta_s\sigma}(t,\tau,B), A_{\theta_s\sigma}(t)\big) \\
&\quad= 0, \quad \forall B \in \mathcal{B}(X),
\end{aligned}$$

and

$$\begin{aligned}
A'_\sigma(r) = A_{\theta_s\sigma}(r-s) &\subset U_{\theta_s\sigma}(r-s,\tau-s,A_{\theta_s\sigma}(\tau-s)) \\
&= U_\sigma(r,\tau,A'_\sigma(\tau)), \quad \forall (r,\tau) \in \mathbb{R}_d.
\end{aligned}$$

Therefore, A'_σ satisfies the first three properties in Definition 1.1.27. Hence, $A_\sigma(t+s) \subset A'_\sigma(t+s) = A_{\theta_s\sigma}(t)$ since the pullback attractor A_σ is minimal. The proof is complete. □

1.1.3 Relationship between different attractors

In this section we study the relationship between cocycle and uniform attractors for multi-valued cocycles and global attractors for skew product flows.

a). Global and uniform attractors

Theorem 1.1.33. *Suppose that Φ is a multi-valued cocycle on (X,Σ) and Π is the multi-valued skew product semiflow generated by (Φ,θ). Then:*

1. *if Π has a global attractor $\mathbb{A}$, then $\mathscr{A} := P_X\mathbb{A}$ is the uniform attractor of Φ;*
2. *if Φ has a uniform attractor $\mathscr{A}$ and the map $(\sigma,x) \mapsto \Phi(t,\sigma,x)$ is upper semi-continuous for any fixed $t \geqslant 0$, then Π has a global attractor.*

Proof. 1. The compactness of $\mathscr{A}$ follows from the compactness of $\mathbb{A}$ directly. Now we prove the uniformly attracting property. For each $\sigma \in \Sigma$, $x \in X$ and $t \geqslant 0$, since $\mathscr{A} = P_X\mathbb{A} = \cup_{\sigma' \in \Sigma} P_{\sigma'}\mathbb{A}$, it follows that

$$\begin{aligned} \operatorname{dist}_X(x, \mathscr{A}) &= \inf_{\sigma' \in \Sigma} \operatorname{dist}_X\big(x, P_{\sigma'}\mathbb{A}\big) \\ &\leqslant \inf_{\sigma' \in \Sigma} \big(\operatorname{dist}_X\big(x, P_{\sigma'}\mathbb{A}\big) + \rho(\theta_t\sigma, \sigma')\big) \\ &= \operatorname{dist}_{\mathcal{X}}\Big(\{x\} \times \{\theta_t\sigma\}, \cup_{\sigma' \in \Sigma} P_{\sigma'}\mathbb{A} \times \{\sigma'\}\Big). \end{aligned}$$

Hence, for any bounded set D in X we have

$$\begin{aligned} \sup_{\sigma \in \Sigma} \operatorname{dist}_X\big(\Phi(t, \sigma, D), \mathscr{A}\big) &= \sup_{\sigma \in \Sigma} \sup_{x \in \Phi(t,\sigma,D)} \operatorname{dist}_X(x, \mathscr{A}) \\ &\leqslant \sup_{\sigma \in \Sigma} \operatorname{dist}_{\mathcal{X}}\Big(\Phi(t, \sigma, D) \times \{\theta_t\sigma\}, \cup_{\sigma' \in \Sigma} P_{\sigma'}\mathbb{A} \times \{\sigma'\}\Big) \\ &= \sup_{\sigma \in \Sigma} \operatorname{dist}_{\mathcal{X}}\Big(\Pi\big(t, D \times \{\sigma\}\big), \mathbb{A}\Big) \\ &= \operatorname{dist}_{\mathcal{X}}\Big(\Pi\big(t, D \times \Sigma\big), \mathbb{A}\Big) \to 0, \quad \text{as } t \to \infty, \end{aligned} \tag{1.13}$$

since $D \times \Sigma$ is a bounded set in $\mathcal{X}$, which is attracted by $\mathbb{A}$. Thus, the uniform attraction of $\mathscr{A}$ is proved.

To see the minimality property, let $\mathscr{A}'$ be another closed uniformly attracting set for Φ. Then $\mathbb{A}' := \mathscr{A}' \times \Sigma$ is a closed attracting set for the skew product semi-flow Π. Indeed, for any bounded set $\mathbb{B}$ in $\mathcal{X}$, by the uniform attraction of $\mathscr{A}'$, we have

$$\begin{aligned} \operatorname{dist}_{\mathcal{X}}\big(\Pi(t, \mathbb{B}), \mathbb{A}'\big) &= \operatorname{dist}_{\mathcal{X}}\big(\Pi(t, \cup_{\sigma \in \Sigma} P_\sigma\mathbb{B} \times \{\sigma\}), \mathbb{A}'\big) \\ &= \sup_{\sigma \in \Sigma} \operatorname{dist}_{\mathcal{X}}\big(\Pi(t, P_\sigma\mathbb{B} \times \{\sigma\}), \mathscr{A}' \times \Sigma\big) \\ &= \sup_{\sigma \in \Sigma} \operatorname{dist}_{\mathcal{X}}\big(\Phi(t, \sigma, P_\sigma\mathbb{B}) \times \{\theta_t\sigma\}, \mathscr{A}' \times \Sigma\big) \\ &\leqslant \sup_{\sigma \in \Sigma} \operatorname{dist}_X\big(\Phi(t, \sigma, \cup_{\sigma' \in \Sigma} P_{\sigma'}\mathbb{B}), \mathscr{A}'\big) \xrightarrow{t \to \infty} 0. \end{aligned} \tag{1.14}$$

Therefore, by the minimality property of the global attractor $\mathbb{A}$ as indicated by Remark 1.1.24, we have $\mathscr{A} = P_X\mathbb{A} \subset P_X\mathbb{A}' = \mathscr{A}'$. Hence, $\mathscr{A}$ is minimal.

2. Clearly, Π is upper semi-continuous. By Theorem 1.1.25, it suffices to notice that, in view of (1.14), the set $\mathscr{A} \times \Sigma$ is a compact attracting set for Π. The proof is complete. □

b). Global and cocycle attractors

In this part, we study the relationship between global attractors of the multi-valued semiflow Π generated by (Φ, θ) and cocycle attractors of the multi-valued cocycle Φ.

Theorem 1.1.34. *Suppose that Φ is a multi-valued cocycle on (X, Σ) and Π is the multi-valued skew product semiflow generated by (Φ, θ). If Π has a global attractor $\mathbb{A}$, then the non-autonomous set $A = \{A(\sigma)\}_{\sigma \in \Sigma}$ given by $A(\sigma) = P_\sigma\mathbb{A}$, $\forall \sigma \in \Sigma$, is the cocycle attractor of Φ.*

Proof. The compactness of A follows from the compactness of $\mathbb{A}$. Now we prove the negative invariance of A. For any $\sigma \in \Sigma$ and $t \geqslant 0$, let $y \in A(\theta_t \sigma)$. Then by the negative invariance of $\mathbb{A}$ we have

$$\{y\} \times \{\theta_t \sigma\} \in \mathbb{A} \subset \Pi(t, \mathbb{A}).$$

Therefore, there is an $\{x\} \times \{\bar{\sigma}\} \in \mathbb{A}$, which implies $x \in A(\bar{\sigma})$, such that

$$\{y\} \times \{\theta_t \sigma\} \in \Pi(t, \{x\} \times \{\bar{\sigma}\}) = \Phi(t, \bar{\sigma}, x) \times \{\theta_t \bar{\sigma}\}.$$

Therefore, $\bar{\sigma} = \sigma$ and $y \in \Phi(t, \sigma, x) \subset \Phi(t, \sigma, A(\sigma))$. Thus $A(\theta_t \sigma) \subset \Phi(t, \sigma, A(\sigma))$.

Let us prove the pullback attracting property of A by contradiction. Suppose that for some $B \in \mathcal{B}(X)$, $\sigma \in \Sigma$ and $\epsilon > 0$, there exists a sequence $t_n \to \infty$ such that

$$\operatorname{dist}(\Phi(t_n, \theta_{-t_n}\sigma, B), A(\sigma)) \geqslant \epsilon, \quad \forall n \in \mathbb{N}.$$

Then there is a sequence $x_n \in \Phi(t_n, \theta_{-t_n}\sigma, B)$ such that

$$\operatorname{dist}(x_n, A(\sigma)) \geqslant \epsilon, \quad \forall n \in \mathbb{N}. \tag{1.15}$$

In view of (1.13) we have

$$\begin{aligned} \operatorname{dist}(x_n, P_X \mathbb{A}) &\leqslant \operatorname{dist}(\Phi(t_n, \theta_{-t_n}\sigma, B), P_X \mathbb{A}) \\ &\leqslant \sup_{\sigma \in \Sigma} \operatorname{dist}(\Phi(t_n, \sigma, B), P_X \mathbb{A}) \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, there is a $y \in P_X \mathbb{A}$ such that, up to a subsequence,

$$x_n \to y.$$

Denote $\sigma_n := \theta_{-t_n}\sigma$. Then the sequence $\big\{\{x_n\} \times \{\sigma\}\big\}_{n \in \mathbb{N}}$ is such that

$$\begin{gathered} \{x_n\} \times \{\sigma\} \in \Phi(t_n, \sigma_n, B) \times \{\theta_{t_n}\sigma_n\} \subset \Pi(t_n, (B \times \Sigma)), \ \forall n \in \mathbb{N}, \\ \{x_n\} \times \{\sigma\} \to \{y\} \times \{\sigma\} \quad \text{as } n \to \infty. \end{gathered}$$

Therefore, by Remark 1.1.23 we have $\{y\} \times \{\sigma\} \in \mathbb{A}$ and thereby $y \in A(\sigma)$, which contradicts (1.15).

Finally, let us show the minimality property of A. Note that since any element of A comes from the global attractor $\mathbb{A}$, $\cup_{\sigma \in \Sigma} A(\sigma)$ is bounded in X. Then for each closed and pullback attracting non-autonomous set $A' = \{A'(\sigma)\}_{\sigma \in \Sigma}$ it must hold for every $\sigma \in \Sigma$ that

$$\begin{aligned} \operatorname{dist}(A(\sigma), A'(\sigma)) &\leqslant \operatorname{dist}(\Phi(t, \theta_{-t}\sigma, A(\theta_{-t}\sigma)), A'(\sigma)) \\ &\leqslant \operatorname{dist}(\Phi(t, \theta_{-t}\sigma, \cup_{\sigma \in \Sigma} A(\sigma)), A'(\sigma)) \to 0, \quad \text{as } t \to \infty, \end{aligned}$$

which implies $A(\sigma) \subset A'(\sigma)$ as $A'(\sigma)$ is closed. The minimality property is proved. □

Theorem 1.1.34 can be interpreted as follows.

Corollary 1.1.35. *Suppose the multi-valued cocycle Φ has a cocycle attractor $A = \{A(\sigma)\}_{\sigma\in\Sigma}$. If the skew product semi-flow Π generated by (Φ,θ) has the global attractor $\mathbb{A}$, then*

$$\mathbb{A} = \bigcup_{\sigma\in\Sigma} A(\sigma)\times\{\sigma\}. \tag{1.16}$$

Let us prove now the converse statement of Theorem 1.1.34.

Theorem 1.1.36. *Assume that Φ is a multi-valued cocycle on (X,Σ) with cocycle attractor $A = \{A(\sigma)\}_{\sigma\in\Sigma}$, and that the map $(\sigma,x)\mapsto\Phi(t,\sigma,x)$ is upper semicontinuous for any fixed $t\geqslant 0$. Let Π be the multi-valued skew product semiflow generated by (Φ,θ). If Φ is uniformly asymptotically compact, then Π possesses a global attractor $\mathbb{A}$ satisfying (1.16).*

Proof. The existence of the global attractor $\mathbb{A}$ follows from Theorems 1.1.20 and 1.1.33. Equality (1.16) is a consequence of Corollary 1.1.35. □

c). Cocycle and uniform attractors

In this part, let us study the relationship between cocycle and uniform attractors of a multi-valued cocycle Φ.

Theorem 1.1.37. *Suppose that Φ is a multi-valued cocycle on (X,Σ) and that the map $(\sigma,x)\mapsto\Phi(t,\sigma,x)$ is upper semicontinuous for any fixed $t\geqslant 0$. If Φ has a uniform attractor $\mathscr{A}$, then it has a cocycle attractor $A = \{A(\sigma)\}_{\sigma\in\Sigma}$ satisfying*

$$\bigcup_{\sigma\in\Sigma} A(\sigma) = \mathscr{A}.$$

Proof. By Theorem 1.1.33, the existence of the uniform attractor $\mathscr{A}$ implies the existence of the global attractor $\mathbb{A}$ of the skew product semiflow generated by (Φ,θ), which satisfies the relation $\mathscr{A} = P_X\mathbb{A}$. On the other hand, Theorem 1.1.34 implies that Φ has a cocycle attractor $A = \{A(\sigma)\}_{\sigma\in\Sigma}$ such that $A(\sigma) = P_\sigma\mathbb{A}$. Hence, $\cup_{\sigma\in\Sigma}A(\sigma) = P_X\mathbb{A} = \mathscr{A}$. The proof is complete. □

Remark 1.1.38. The relationship between uniform, cocycle and global attractors for single-valued dynamical systems was studied by Kloeden and Rasmussen [55, Section 3.4] and Bortolan et al. [5, Section 3].

It is interesting to draw the following conclusions.

Corollary 1.1.39. *Suppose Φ is a multi-valued cocycle on (X,Σ) and the map $(\sigma,x)\mapsto\Phi(t,\sigma,x)$ is upper semicontinuous for any fixed $t\geqslant 0$. If Φ has a uniform attractor $\mathscr{A}$, then it has a cocycle attractor $\{A(\sigma)\}_{\sigma\in\Sigma}$ and the multi-valued semiflow Π generated by (Φ,θ) has the global attractor $\mathbb{A}$. Moreover, they satisfy the relations*

$$\mathscr{A} = \text{Image } A,$$

$$\mathbb{A} = \text{Graph } A,$$

where $A:\sigma\mapsto A(\sigma)$ is the set-valued mapping identified by the cocycle attractor $\{A(\sigma)\}_{\sigma\in\Sigma}$.

By the relationship between uniform and cocycle attractors, it is straightforward to check the negative invariance of the uniform attractor, which was studied in [13].

Corollary 1.1.40. *Suppose Φ is a multi-valued cocycle on (X, Σ) and the map $(\sigma, x) \mapsto \Phi(t, \sigma, x)$ is upper semicontinuous for any fixed $t \geqslant 0$. If Φ has the uniform attractor $\mathscr{A}$, then $\mathscr{A}$ is negatively invariant, namely,*

$$\mathscr{A} \subset \Phi_\Sigma(t, \mathscr{A}), \quad \forall t \geqslant 0,$$

where $\Phi_\Sigma(t, x) := \cup_{\sigma\in\Sigma}\Phi(t, \sigma, x)$ for all $t \geqslant 0$ and $x \in X$.

Proof. From Theorem 1.1.37 and the negative invariance of cocycle attractors it follows that

$$\mathscr{A} = \bigcup_{\sigma\in\Sigma} A(\theta_t\sigma) \subset \bigcup_{\sigma\in\Sigma} \Phi(t, \sigma, A(\sigma)) \subset \bigcup_{\sigma\in\Sigma} \Phi(t, \sigma, \mathscr{A}) = \Phi_\Sigma(t, \mathscr{A}), \quad \forall t \geqslant 0,$$

whence we have the result. □

We obseve that in [55, Section 3.4] by uniform attractors the authors meant cocycle attractors $\{A(\sigma)\}_{\sigma\in\Sigma}$ which are both forwards uniformly attracting, i.e.

$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} \operatorname{dist}(\Phi(t, \sigma, B), A(\theta_t\sigma)) = 0, \quad \forall B \in \mathcal{B}(X), \tag{1.17}$$

and backwards uniformly attracting, i.e.

$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} \operatorname{dist}(\Phi(t, \theta_{-t}\sigma, B), A(\sigma)) = 0, \quad \forall B \in \mathcal{B}(X). \tag{1.18}$$

Now let us show the relationship between the uniform attractor in the sense of Definition 1.1.17 and the uniformly attracting cocycle attractor, under the condition that the map $(\sigma, x) \mapsto \Phi(t, \sigma, x)$ is upper semicontinuous for any fixed $t \geqslant 0$.

First note that since Σ is invariant, i.e., $\theta_t\Sigma = \Sigma$, $\forall t \geqslant 0$, (1.17) is equivalent to (1.18). Indeed, if (1.17) holds, then for each $\epsilon > 0$ there exists a time T such that

$$\sup_{\sigma\in\Sigma} \operatorname{dist}(\Phi(t, \sigma, B), A(\theta_t\sigma)) < \epsilon, \quad \forall t \geqslant T.$$

Hence, for each $t \geqslant T$, let $\sigma' = \theta_t\sigma$ and we obtain

$$\sup_{\sigma'\in\Sigma} \operatorname{dist}(\Phi(t, \theta_{-t}\sigma', B), A(\sigma')) = \sup_{\sigma\in\Sigma} \operatorname{dist}(\Phi(t, \sigma, B), A(\theta_t\sigma)) < \epsilon,$$

which implies (1.18). In a similar way one can conclude that (1.18) implies (1.17). Therefore, it is sensible to say that the cocycle attractor $\{A(\sigma)\}_{\sigma\in\Sigma}$ is uniformly attracting if either (1.17) or (1.18) is satisfied.

Clearly, if a cocycle attractor $\{A(\sigma)\}_{\sigma\in\Sigma}$ is uniformly attracting, then the set $\mathscr{A} \subset X$ defined by

$$\mathscr{A} := \bigcup_{\sigma\in\Sigma} A(\sigma) \tag{1.19}$$

is the uniform attractor of Φ, provided that $\cup_{\sigma\in\Sigma}A(\sigma)$ is pre-compact. Indeed, it follows easily using Definition 1.1.10 that $\mathscr{A}$ is uniformly attracting, and since $\mathscr{A}$ is pre-compact, the uniform attractor exists by Theorem 1.1.20. Moreover, $\mathscr{A}$ is exactly the uniform attractor by Theorem 1.1.37. On the other hand, if the cocycle Φ has a uniform attractor $\mathscr{A}$, then (1.19) holds by Theorem 1.1.37, and hence $\cup_{\sigma\in\Sigma}A(\sigma)$ is compact. However, the cocycle attractor is not necessarily uniformly attracting in the sense of (1.17) or (1.18) if no additional conditions are assumed.

Therefore, we conclude that the existence of a uniform attractor $\mathscr{A}$ is a necessary but not a sufficient condition for the existence of a uniformly attracting cocycle attractor $\{A(\sigma)\}_{\sigma\in\Sigma}$ with $\cup_{\sigma\in\Sigma}A(\sigma)$ being compact.

d). Cocycle and pullback attractors

In this section, in order to to study the relationship between cocycle attractors and pullback attractors we assume that Φ and $U = \{U_\sigma\}_{\sigma\in\Sigma}$ are a multi-valued cocycle and a family of MP which are equivalent in the sense of Definition 1.1.5. That is, Φ and $U = \{U_\sigma\}_{\sigma\in\Sigma}$ satisfy

$$U_\sigma(t,\tau,x) = \Phi(t-\tau,\theta_\tau\sigma,x), \quad \forall (t,\tau)\in\mathbb{R}_d,\ \sigma\in\Sigma,\ x\in X, \tag{1.20}$$

$$\Phi(t,\sigma,x) = U_\sigma(t,0,x), \quad \forall t\geqslant 0,\ \sigma\in\Sigma,\ x\in X. \tag{1.21}$$

We note that all the results in this section are independent of the continuity of Φ or U.

First we will prove that the existence of the cocycle attractor for Φ implies the existence of the pullback attractor for U.

Theorem 1.1.41. *If $A = \{A(\sigma)\}_{\sigma\in\Sigma}$ is the cocycle attractor of Φ, then $\{A_\sigma\}_{\sigma\in\Sigma}$ is a family of pullback attractors for the family of MP equivalent to Φ, where each $A_\sigma = \{A_\sigma(t)\}_{t\in\mathbb{R}}$ is given by*

$$A_\sigma(t) = A(\theta_t\sigma), \quad \forall t\in\mathbb{R}, \sigma\in\Sigma.$$

Proof. We will prove that for each $\sigma\in\Sigma$, $A_\sigma = \{A_\sigma(t)\}_{t\in\mathbb{R}}$ is a pullback attractor for the process U_σ. First the compactness of $A_\sigma(t)$ is clear since $A(\sigma)$ is compact for every $\sigma\in\Sigma$. For every $t\in\mathbb{R}$, the pullback attraction follows from

$$\begin{aligned}\lim_{\tau\to-\infty}\operatorname{dist}(U_\sigma(t,\tau,B),A_\sigma(t)) &= \lim_{\tau\to-\infty}\operatorname{dist}(\Phi(t-\tau,\theta_\tau\sigma,B),A(\theta_t\sigma))\\ &= \lim_{s\to\infty}\operatorname{dist}(\Phi(s,\theta_{-s}\theta_t\sigma,B),A(\theta_t\sigma))\\ &= 0, \quad \forall B\in\mathcal{B}(X),\end{aligned}$$

where we have used (1.20) and the attraction property of the cocycle attractor.

The negative invariance is obtained from

$$A_\sigma(t) = A(\theta_t\sigma) \subseteq \Phi(t-\tau,\theta_\tau\sigma,A(\theta_\tau\sigma)) = U_\sigma(t,\tau,A_\sigma(\tau)), \quad \forall t\geqslant\tau.$$

If there is another family of closed sets $\{A'_\sigma\}_{\sigma\in\Sigma}$ which is pullback attracting, then $A' = \{A'(\sigma)\}_{\sigma\in\Sigma}$ with $A'(\sigma) := A'_\sigma(0)$ forms a pullback attracting family for Φ (see the proof of Theorem 1.1.42

below), so by the minimality of the family $\{A(\sigma)\}_{\sigma\in\Sigma}$ we have $A(\sigma)\subset A'(\sigma)$ for any $\sigma\in\Sigma$, and then

$$A_\sigma(t)=A(\theta_t\sigma)\subset A'(\theta_t\sigma)=A'_\sigma(t),$$

which proves the minimality of $\{A_\sigma(t)\}_{t\in\mathbb{R}}$. □

Theorem 1.1.42. *Suppose that the family of MP $U=\{U_\sigma\}_{\sigma\in\Sigma}$ has a family $\{A_\sigma\}_{\sigma\in\Sigma}$ of pullback attractors. Then $A=\{A(\sigma)\}_{\sigma\in\Sigma}$ is the cocycle attractor for the cocycle Φ equivalent to U, where*

$$A(\sigma):=A_\sigma(0),\quad \forall\sigma\in\Sigma.$$

Proof. The compactness of $A(\sigma)$ is clear since A_σ is a pullback attractor for U_σ. The attraction property follows from

$$\begin{aligned}\lim_{s\to\infty}\operatorname{dist}(\Phi(s,\theta_{-s}\sigma,B),A(\sigma))&=\lim_{s\to\infty}\operatorname{dist}(U_{\theta_{-s}\sigma}(s,0,B),A(\sigma))\\&=\lim_{s\to\infty}\operatorname{dist}(U_\sigma(0,-s,B),A_\sigma(0))\\&=0,\quad\forall B\in\mathcal{B}(X),\end{aligned}$$

where we have used (1.21), the translation identity and the attraction property of the pullback attractor. To prove the negative invariance, it suffices to see that, by Proposition 1.1.32,

$$\begin{aligned}A(\theta_t\sigma)&=A_{\theta_t\sigma}(0)=A_{\theta_\tau\sigma}(t-\tau)\\&\subseteq U_{\theta_\tau\sigma}(t-\tau,0,A_{\theta_\tau\sigma}(0))\\&=\Phi(t-\tau,\theta_\tau\sigma,A(\theta_\tau\sigma)),\quad\forall t\geqslant\tau.\end{aligned}$$

The minimality property is proved in a similar way to the proof of the previous theorem. The proof is complete. □

Corollary 1.1.43. *The cocycle attractor $A=\{A(\sigma)\}_{\sigma\in\Sigma}$ and the family $\{A_\sigma\}_{\sigma\in\Sigma}$ of pullback attractors imply each other. Moreover,*

$$A(\theta_t\sigma)=A_\sigma(t),\quad\forall t\in\mathbb{R},\sigma\in\Sigma.$$

In particular,

$$A(\sigma)=A_\sigma(0),\quad\forall\sigma\in\Sigma.$$

1.2 Characterization and lifted-invariance

In this section we study the so-called generalized cocycles, for which the cocycle attractors will be shown to have a structure in terms of complete trajectories. On the other hand, uniform attractors are proved to have a lifted-invariance property though in Definition 1.1.17 there is no invariance assumed.

Generalized non-autonomous dynamical systems were first introduced by Ball [2], and then studied by Ball [3], Simsen and Gentile [76] and Kapustyan et al. [50] for generalized multi-valued semiflows and Kapustyan et al. [51] and Capulato and Simsen [8] for generalized multi-valued processes. Most recently, Caraballo et al. [16] characterized pullback attractors for multi-valued processes by (backwards) bounded complete trajectories.

A generalized cocycle is generated by a collection $\mathcal{R}=\{\mathcal{R}_{\sigma,x}\}_{\sigma\in\Sigma,x\in X}$ of functions satisfying:

(H1) For each $\sigma \in \Sigma$ and $x \in X$, $\mathcal{R}_{\sigma,x}$ is a non-empty family of continuous functions $\gamma : [0,\infty) \to X$ such that $\gamma(0) = x$;

(H2) For every $\gamma \in \mathcal{R}_{\sigma,x}$ with $\sigma \in \Sigma$ and $x \in X$, $\tilde{\gamma}(\cdot) := \gamma(\cdot + s) \in \mathcal{R}_{\theta_s\sigma,\gamma(s)}$, $\forall s \geqslant 0$.

The following properties of $\{\mathcal{R}_{\sigma,x}\}_{\sigma\in\Sigma,x\in X}$ are often needed for further studies.

(H3) (Concatenation property) if $\varphi \in \mathcal{R}_{\sigma,x}$, $\psi \in \mathcal{R}_{\theta_s\sigma,\varphi(s)}$ for some $s > 0$, then the function γ, defined by

$$\gamma(t) = \begin{cases} \varphi(t), & t \in [0,s), \\ \psi(t-s), & t \in [s,+\infty), \end{cases}$$

belongs to $\mathcal{R}_{\sigma,x}$.

(H4) For any $\gamma_n \in \mathcal{R}_{\sigma,x_n}$ with $x_n \to x_0$, $\sigma \in \Sigma$, there exists a $\gamma_0 \in \mathcal{R}_{\sigma,x_0}$ such that, up to a subsequence, $\gamma_n(t) \to \gamma_0(t)$, $\forall t \geqslant 0$.

Condition (H4) can be strengthened as follows.

(H5) For any $\gamma_n \in \mathcal{R}_{\sigma_n,x_n}$ with $x_n \to x_0$, $\sigma_n \to \sigma$, there exists a $\gamma_0 \in \mathcal{R}_{\sigma,x_0}$ such that, up to a subsequence, $\gamma_n(t) \to \gamma_0(t)$, $\forall t \geqslant 0$.

Proposition 1.2.1. *If (H1) and (H2) hold, then the mapping $\Phi : \mathbb{R}^+ \times \Sigma \times X \to X$ defined by*

$$\Phi(t,\sigma,x) = \bigcup_{\gamma\in\mathcal{R}_{\sigma,x}} \gamma(t), \quad \forall t \in \mathbb{R}^+, \sigma \in \Sigma, x \in X, \tag{1.22}$$

is a multi-valued cocycle over (X,Σ), called the ***generalized cocycle*** *generated by $\{\mathcal{R}_{\sigma,x}\}_{\sigma\in\Sigma,x\in X}$. If (H3) holds, then Φ is strict. Moreover, if (H4) holds, then the map $x \mapsto \Phi(t,\sigma,x)$ is upper semi-continuous, while (H5) implies that the map $(\sigma,x) \mapsto \Phi(t,\sigma,x)$ is upper semicontinuous.*

Proof. It is clear that $\Phi(0,\sigma,x) = x$ for all $\sigma \in \Sigma$ and $x \in X$ by (H1). Let us prove

$$\Phi(t+s,\sigma,x) \subseteq \Phi(t,\theta_s\sigma,\Phi(s,\sigma,x)), \quad \forall t,s \geqslant 0, \sigma \in \Sigma, x \in X.$$

Take $y \in \Phi(t+s,\sigma,x)$. Then by (1.22) there is a $\gamma \in \mathcal{R}_{\sigma,x}$ such that $y = \gamma(t+s) = \tilde{\gamma}(t)$, where $\tilde{\gamma}(\cdot) = \gamma(\cdot + s) \in \mathcal{R}_{\theta_s\sigma,\gamma(s)}$ by (H2). Therefore,

$$y \in \bigcup_{\tilde{\gamma}\in\mathcal{R}_{\theta_s\sigma,\gamma(s)}} \tilde{\gamma}(t) = \Phi(t,\theta_s\sigma,\gamma(s)). \tag{1.23}$$

It is clear that

$$\gamma(s) \in \Phi(s,\sigma,x),$$

which along with (1.23) implies $y \in \Phi(t,\theta_s\sigma,\Phi(s,\sigma,x))$.

Let (H3) be satisfied. We take $y \in \Phi(t, \theta_s\sigma, \Phi(s, \sigma, x))$. Then there exist $\gamma_1 \in \mathcal{R}_{\sigma,x}$, $\gamma_2 \in \mathcal{R}_{\theta_s\sigma,\gamma_1(s)}$ such that $\gamma_2(t) = y$. By (H3) the map

$$\gamma(r) = \begin{cases} \gamma_1(r), & r \in [0, s), \\ \gamma_2(r - s), & r \in [s, +\infty), \end{cases}$$

belongs to $\mathcal{R}_{\sigma,x}$, and thus $y = \gamma(t + s) \in \Phi(t + s, \sigma, x)$.

The upper semi-continuity of $x \mapsto \Phi(t, \sigma, x)$ (respectively, $(\sigma, x) \mapsto \Phi(t, \sigma, x)$) follows easily from (H4) (respectively, (H5)). □

1.2.1 Characterization of cocycle attractors for generalized cocycles

In this part we characterize cocycle attractors of generalized cocycles by complete trajectories. Note that a similar characterization for pullback attractors was studied by [16].

Definition 1.2.2. For any $x \in X$ and $\sigma \in \Sigma$, the continuous map $\xi : \mathbb{R} \to X$ is called a complete trajectory of $\mathcal{R}$ through x driven by σ if $\xi(0) = x$, and

$$\xi(\cdot + s)|_{[0,+\infty)} \in \mathcal{R}_{\theta_s\sigma,\xi(s)}, \ \forall s \in \mathbb{R}.$$

Correspondingly, for a cocycle Φ, the continuous map $\xi : \mathbb{R} \to X$ is called a complete trajectory of Φ through x driven by σ if $\xi(0) = x$, and

$$\xi(t) \in \Phi(t - s, \theta_s\sigma, \xi(s)), \quad \forall t \geqslant s.$$

Complete trajectories have the following translation property.

Proposition 1.2.3. *Suppose that ξ is a complete trajectory of $\mathcal{R}$ (resp. Φ) driven by σ, then $\tilde{\xi} : \mathbb{R} \to X$ defined by $\tilde{\xi}(\cdot) = \xi(\cdot + r)$, $\forall r \in \mathbb{R}$, is a complete trajectory of $\mathcal{R}$ (resp. Φ) driven by $\theta_r\sigma$.*

Proof. For trajectories of Φ, it suffices to observe that

$$\tilde{\xi}(t) = \xi(t + r) \in \Phi(t - s, \theta_{s+r}\sigma, \xi(s + r)) = \Phi(t - s, \theta_s\theta_r\sigma, \tilde{\xi}(s)), \quad t \geqslant s.$$

For trajectories of $\mathcal{R}$ the proof is rather similar. □

In general, complete trajectories of $\mathcal{R}$ and those of the generalized cocycle Φ generated by $\mathcal{R}$ are not identical. In particular, we have the following result.

Proposition 1.2.4. *Suppose that $\mathcal{R}$ is a family satisfying (H1) and (H2) and Φ is a generalized cocycle generated by $\mathcal{R}$. Then any complete trajectory of $\mathcal{R}$ is a complete trajectory of Φ, the inverse being true if (H3) and (H4) hold.*

Proof. The proof is quite similar to the autonomous case [50, Lemma 8]. □

Before characterizing cocycle attractors by complete trajectories, let us define bounded complete trajectories.

Definition 1.2.5. If a complete trajectory ξ is such that

$$\bigcup_{t \leqslant T} \xi(t) \in \mathcal{B}(X), \quad \forall T \in \mathbb{R},$$

then ξ is called backwards bounded. ξ is called (completely) bounded if

$$\bigcup_{t \in \mathbb{R}} \xi(t) \in \mathcal{B}(X).$$

Proposition 1.2.6. *Suppose that the cocycle Φ has a cocyle attractor $A = \{A(\sigma)\}_{\sigma \in \Sigma}$ and ξ is a backwards bounded complete trajectory of Φ driven by σ. Then*

$$\xi(t) \in A(\theta_t \sigma), \quad \forall t \in \mathbb{R}.$$

Proof. For each $t \in \mathbb{R}$, it suffices to observe that

$$\begin{aligned} \operatorname{dist}(\xi(t), A(\theta_t\sigma)) &\leqslant \operatorname{dist}\big(\Phi(t-s, \theta_s\sigma, \xi(s)), A(\theta_t\sigma)\big) \\ &\leqslant \operatorname{dist}\Big(\Phi\big(t-s, \theta_s\sigma, \cup_{r \leqslant t}\xi(r)\big), A(\theta_t\sigma)\Big) \to 0, \text{ as } s \to -\infty. \end{aligned}$$

□

Theorem 1.2.7. *Suppose that (H1), (H2) and (H4) hold and the cocycle Φ defined by (1.22) has a backwards bounded cocycle attractor $A = \{A(\sigma)\}_{\sigma \in \Sigma}$. Then*

$$A(\theta_t\sigma) = \big\{\xi(t) : \xi \textit{ is a backwards bounded complete trajectory of } \mathcal{R} \textit{ driven by } \sigma\big\}$$

for each $t \in \mathbb{R}, \sigma \in \Sigma$.

Proof. Since every complete trajectory of $\mathcal{R}$ is a complete trajectory of Φ, by Proposition 1.2.6 we only need to prove the inclusion

$$A(\sigma) \subset \big\{\xi(0) : \xi \text{ is a backwards bounded complete trajectory of } \mathcal{R} \text{ driven by } \sigma\big\}$$

for each $\sigma \in \Sigma$. Indeed, if this is true, then for each $x \in A(\theta_t\sigma)$ we have a backwards bounded complete trajectory $\tilde{\xi}$ driven by $\tilde{\sigma} := \theta_t\sigma$ such that $\tilde{\xi}(0) = x$. Denote $\xi(\cdot) = \tilde{\xi}(\cdot - t)$. Then by Proposition 1.2.3 we see that ξ is a backwards bounded trajectory driven by $\theta_{-t}\tilde{\sigma} = \sigma$ with $\xi(t) = x$. Then we have the result.

Let $x \in A(\sigma)$ and $0 < s_n \to \infty$. Then by the negative invariance of A we have

$$x \in A(\sigma) \subset \Phi(s_n, \theta_{-s_n}\sigma, A(\theta_{-s_n}\sigma)).$$

Therefore, by (1.22), there exists a family $\gamma_n \in \mathcal{R}_{\theta_{-s_n}\sigma, y_n}$ with $y_n \in A(\theta_{-s_n}\sigma)$ such that

$$\gamma_n(s_n) = x, \quad \gamma_n(0) = y_n,$$

and

$$\gamma_n(t+s_n) \in \Phi(t+s_n, \theta_{-s_n}\sigma, y_n) \subset \Phi(t+s_n, \theta_{-s_n}\sigma, A(\theta_{-s_n}\sigma)). \tag{1.24}$$

For every $n \in \mathbb{N}$ we define the mapping $\tilde{\gamma}_n : [0,\infty) \to X$ by

$$\tilde{\gamma}_n(\cdot) = \gamma_n(\cdot + s_n).$$

Then by (H2) we have

$$\tilde{\gamma}_n \in \mathcal{R}_{\sigma,x}, \ \tilde{\gamma}_n(0) = x, \quad \forall n \in \mathbb{N}.$$

Therefore, by (H4) there is a $\xi_0 \in \mathcal{R}_{\sigma,x}$ such that, up to a subsequence,

$$\tilde{\gamma}_n(t) = \gamma_n(t+s_n) \to \xi_0(t), \quad \forall t \geqslant 0. \tag{1.25}$$

Next let us define the mapping $\tilde{\gamma}_n^1$ by

$$\tilde{\gamma}_n^1(\cdot) = \gamma_n(\cdot + s_n - 1).$$

Then by (H2) again we have

$$\tilde{\gamma}_n^1 \in \mathcal{R}_{\theta_{-1}\sigma, \gamma_n(s_n-1)}, \ \tilde{\gamma}_n^1(0) = \gamma_n(s_n-1), \quad \forall n \in \mathbb{N}.$$

Recall that $\gamma_n \in \mathcal{R}_{\theta_{-s_n}\sigma, y_n}$ with $y_n \in A(\theta_{-s_n}\sigma)$. Therefore

$$\tilde{\gamma}_n^1(0) = \gamma_n(s_n-1) \in \Phi(s_n-1, \theta_{-s_n}\sigma, y_n) \subset \Phi(s_n-1, \theta_{-s_n}\sigma, A(\theta_{-s_n}\sigma))$$

has a convergent subsequence since A is backwards bounded. Therefore, by (H4) again we obtain the existence of a $\tilde{\gamma}_0^1 \in \mathcal{R}_{\theta_{-1}\sigma, \tilde{\gamma}_0^1(0)}$ such that

$$\tilde{\gamma}_n^1(t) = \gamma_n(t+s_n-1) \to \tilde{\gamma}_0^1(t), \quad \forall t \geqslant 0. \tag{1.26}$$

Define $\xi_1 : [-1,\infty) \to X$ by

$$\xi_1(s) = \tilde{\gamma}_0^1(s+1), \quad s \geqslant -1. \tag{1.27}$$

Then (1.25), (1.26) and (1.27) imply that

$$\xi_1(s) = \xi_0(s), \quad \forall s \geqslant 0.$$

Besides, by $\tilde{\gamma}_0^1 \in \mathcal{R}_{\theta_{-1}\sigma, \tilde{\gamma}_0^1(0)}$, (1.27) and (H2) it follows that

$$\xi_1(\cdot + s)|_{[0,\infty)} = \tilde{\gamma}_0^1(\cdot + s + 1)|_{[0,\infty)} \in \mathcal{R}_{\theta_s\sigma, \xi_1(s)}, \quad \forall s \geqslant -1.$$

Continuing in such a way we obtain a sequence of functions $\xi_n : [-n,\infty) \to X$ such that

$$\xi_m(s) = \xi_n(s), \quad \forall s \geqslant \max\{-m,-n\}, m,n \in \mathbb{N},$$
$$\xi_n(\cdot + s)|_{[0,\infty)} \in \mathcal{R}_{\theta_s\sigma, \xi_n(s)}, \quad \forall s \geqslant -n.$$

Denote by $\xi(t)$ the common value of $\xi_n(t)$ for every $t \in \mathbb{R}$. Then $\xi(t)$ is such that

$$\xi(0) = x,$$
$$\xi(\cdot + s)|_{[0,\infty)} \in \mathcal{R}_{\theta_s\sigma, \xi(s)}, \quad \forall s \in \mathbb{R},$$

which shows that ξ is a complete trajectory of $\mathcal{R}$ through x and driven by σ. □

Theorem 1.2.8. *Suppose (H1), (H2) and (H4) hold and the cocycle Φ defined by (1.22) has a completely bounded cocycle attractor $A = \{A(\sigma)\}_{\sigma\in\Sigma}$. Then*

$$A(\theta_t\sigma) = \big\{\xi(t) : \xi \text{ is a completely bounded complete trajectory of } \mathcal{R} \text{ driven by } \sigma\big\}$$

for all $t \in \mathbb{R}$, $\sigma \in \Sigma$.

Proof. The result is a direct consequence of Theorem 1.2.7 and the boundedness of A. □

By Proposition 1.2.6 and the relationship given in Proposition 1.2.4 between trajectories of $\mathcal{R}$ and Φ, it is straightforward to obtain the following result.

Corollary 1.2.9. *Suppose (H1), (H2) and (H4) hold and the cocycle Φ defined by (1.22) has a backwards (or completely) bounded cocycle attractor $A = \{A(\sigma)\}_{\sigma\in\Sigma}$. Then, for all $t \in \mathbb{R}, \sigma \in \Sigma$,*

$$A(\theta_t\sigma) = \big\{\xi(t) : \xi \text{ is a backwards (or completely) bounded complete trajectory of } \Phi \text{ driven by } \sigma\big\}.$$

1.2.2 Lifted-invariant sets and uniform attractors

From Definition 1.1.17 of a uniform attractor it is interesting to observe that no invariance with respect to the maps Φ is assumed on uniform attractors (although we proved its negative invariance with respect to the bigger map $\Phi_\Sigma(\cdot,) := \cup_{\sigma\in\Sigma}\Phi(\cdot,\sigma,\cdot)$ in Corollary 1.1.40). Thus, in this sense there is a big difference between uniform attractors and other attractors, such as pullback/cocycle attractors and global attractors for skew product semiflows. However, as will be shown in this section, the uniform attractor of the cocycle Φ generated by (1.22) satisfies a lifted-invariance property.

Definition 1.2.10. A subset $E \subset X$ is called lifted-invariant (under Φ) if for every $x \in E$, there exists a bounded complete trajectory ξ of $\mathcal{R}$ (driven by some $\sigma \in \Sigma$) such that $\xi(0) = x$ and $\cup_{t\in\mathbb{R}}\xi(t) \subset E$.

Remark 1.2.11. Without the boundedness property, the lifted-invariance is also known as weakly invariance or quasi-invariance in the literature, see, e.g., [3, 41, 57]. Here we call it lifted-invariance following [5].

Proposition 1.2.12. *Suppose that Φ is a multi-valued cocycle over (X, Σ) with a uniform attractor $\mathscr{A}$. Then for any backwards bounded complete trajectory ξ of $\mathcal{R}$ it holds*

$$\bigcup_{t\in\mathbb{R}} \xi(t) \subset \mathscr{A}.$$

Proof. Let ξ be a backwards bounded complete trajectory driven by some $\sigma \in \Sigma$. Now we prove for each $t \in \mathbb{R}$ that $\xi(t) \in \mathscr{A}$. Notice that since ξ is a complete trajectory of Φ, it is a complete trajectory of Φ by Proposition 1.2.4, and by the backwards boundedness there is a bounded set B such that

$\cup_{s\leqslant t}\xi(s) \subset B$. Then, by the uniformly attracting property of the uniform attractor, see Definition 1.1.10, we have

$$\begin{aligned}\operatorname{dist}(\xi(t),\mathscr{A}) &\leqslant \operatorname{dist}\Big(\Phi\big(t-s,\theta_s\sigma,\xi(s)\big),\mathscr{A}\Big)\\ &= \operatorname{dist}\Big(\Phi\big(t-s,\theta_{-(t-s)}\circ\theta_t\sigma,\xi(s)\big),\mathscr{A}\Big)\\ &\leqslant \operatorname{dist}\Big(\Phi\big(t-s,\theta_{-(t-s)}\circ\theta_t\sigma,B\big),\mathscr{A}\Big)\to 0, \quad \text{as } s\to-\infty.\end{aligned}$$

Therefore, $\xi(t)\in\mathscr{A}$ as $\mathscr{A}$ is compact and the proposition is proved. □

The following proposition is a direct corollary of Proposition 1.2.12.

Proposition 1.2.13. *Suppose that Φ is a multi-valued cocycle over (X,Σ) with a uniform attractor $\mathscr{A}$. Then for any lifted-invariant set $E\subset X$ it holds $E\subseteq\mathscr{A}$.*

Proposition 1.2.13 implies that if the uniform attractor of a multi-valued cocycle is lifted-invariant, then it is the largest lifted-invariant set. The following result shows that, under conditions (H1), (H2) and (H5), the uniform attractor of the generalized cocycle Φ must be lifted-invariant.

Theorem 1.2.14. *Suppose that (H1), (H2) and (H5) hold, and Φ is the generalized cocycle defined by (1.22). Then if Φ has a uniform attractor $\mathscr{A}$, $\mathscr{A}$ is lifted-invariant.*

Proof. By Theorem 1.1.37, we have $\mathscr{A}=\cup_{\sigma\in\Sigma}A(\sigma)$, where $\{A(\sigma)\}_{\sigma\in\Sigma}$ is the completely bounded cocycle attractor of Φ such that, in view of Theorems 1.2.8-1.2.9,

$$\begin{aligned}A(\sigma) &= \big\{\xi(0):\xi \text{ is a completely bounded complete trajectory of } \mathcal{R} \text{ driven by } \sigma\big\}\\ &= \big\{\xi(0):\xi \text{ is a completely bounded complete trajectory of } \Phi \text{ driven by } \sigma\big\}.\end{aligned}$$

So, for every $x\in\mathscr{A}$ there exists a bounded complete trajectory ξ of $\mathcal{R}$ (driven by some $\sigma\in\Sigma$) through x, and, moreover, ξ is included in $\mathscr{A}$ by Proposition 1.2.12. Hence, $\mathscr{A}$ is lifted-invariant. □

As a consequence of Proposition 1.2.12 and Theorem 1.2.14, we obtain the following corollary, which was proved at first in [51] but using complete trajectories of Φ.

Corollary 1.2.15. *Suppose that (H1), (H2) and (H5) hold, and that Φ is the generalized cocycle defined by (1.22). If Φ has a uniform attractor $\mathscr{A}$, then*

$$\mathscr{A}=\{\xi(0):\xi(\cdot) \textit{ is a completely bounded trajectory of } \mathcal{R}\}.$$

1.3 Robustness of cocycle and uniform attractors

Let $\Phi_\infty(t)$ be a multivalued semiflow defined on X, namely, satisfying:

- $\Phi_\infty(0,\cdot)$ is the identity on X;

- $\Phi_\infty(t+s,x) \subseteq \Phi_\infty(t,\Phi_\infty(s,x))$ for all $t,s \geqslant 0$ and $x \in X$.

Suppose that $\Phi_\infty(t)$ that has the global attractor A_∞ which is defined as a compact subset of X such that:

- $A_\infty \subset \Phi(t,A_\infty), \quad \forall t \geqslant 0$;
- $\lim_{t\to\infty} \operatorname{dist}(\Phi(t,B),A_\infty) = 0$ for all bounded subsets B of X.

Consider the sequence $\{\Phi_n\}_{n\in\mathbb{N}}$ of non-autonomously perturbed cocycles of Φ_∞. Assume that each cocycle Φ_n has a cocycle attractor $A_n = \{A_n(\sigma)\}_{\sigma\in\Sigma}$. We now study the upper and lower semi-continuity of these cocycle attractors towards the global attractor A_∞ as $n \to \infty$ (i.e., as the perturbation vanishes).

Definition 1.3.1. Given a complete metric space Λ and a collection of cocycle attractors $\{A_\lambda\}_{\lambda\in\Lambda}$, where each A_λ has the form $A_\lambda = \{A_\lambda(\sigma)\}_{\sigma\in\Sigma}$, we say that

1. $\{A_\lambda\}_{\lambda\in\Lambda}$ is upper semi-continuous at $\lambda_0 \in \Lambda$ if $\operatorname{dist}(A_\lambda(\sigma),A_{\lambda_0}(\sigma)) \xrightarrow{\lambda\to\lambda_0} 0$ for each $\sigma \in \Sigma$;
2. $\{A_\lambda\}_{\lambda\in\Lambda}$ is lower semi-continuous at $\lambda_0 \in \Lambda$ if $\operatorname{dist}(A_{\lambda_0}(\sigma),A_\lambda(\sigma)) \xrightarrow{\lambda\to\lambda_0} 0$ for each $\sigma \in \Sigma$;
3. $\{A_\lambda\}_{\lambda\in\Lambda}$ is continuous at $\lambda_0 \in \Lambda$ if it is both upper and lower semi-continuous at λ_0.

The next proposition allows us to consider only the case with discrete parameters.

Proposition 1.3.2. *Suppose that $A_\lambda \subset X$ forms a family of bounded sets indexed by $\lambda \in I \subset \mathbb{R}$. Then*

$$\operatorname{dist}(A_\lambda,A_{\lambda_0}) \xrightarrow{\lambda\to\lambda_0} 0 \quad (\textit{resp.}\ \operatorname{dist}(A_{\lambda_0},A_\lambda) \xrightarrow{\lambda\to\lambda_0} 0)$$

if and only if for any $\{\lambda_n\}_{n\in\mathbb{N}} \subset I$ with $\lambda_n \to \lambda_0$ it holds

$$\operatorname{dist}(A_{\lambda_n},A_{\lambda_0}) \xrightarrow{n\to\infty} 0 \quad (\textit{resp.}\ \operatorname{dist}(A_{\lambda_0},A_{\lambda_n}) \xrightarrow{n\to\infty} 0).$$

1.3.1 Upper semi-continuity

First, we have the following theorem.

Theorem 1.3.3. *Suppose that $\{\Phi_n\}_{n\in\mathbb{N}}$ is a family of cocycles with its corresponding family $\{A_n\}_{n\in\mathbb{N}}$ of backwards bounded cocycle attractors and Φ_∞ is a multi-valued semiflow with the global attractor A_∞. Assume the following conditions:*

(A1) for each $\sigma \in \Sigma$, $t \in \mathbb{R}$ and compact set $K \subset X$,

$$\lim_{n\to\infty} \sup_{x\in K} \operatorname{dist}\big(\Phi_n(t,\sigma,x),\Phi_\infty(t,x)\big) = 0;$$

(A2) $\{A_n\}_{n\in\mathbb{N}}$ *is uniformly backwards bounded, that is, for each* $\sigma \in \Sigma$ *and* $T \in \mathbb{R}$ *there is a bounded* $B \subset X$ *such that*

$$\bigcup_{n\in\mathbb{N}} \bigcup_{t\geqslant T} A_n(\theta_{-t}\sigma) \subset B;$$

(A3) for each $\sigma \in \Sigma$,

$$\bigcup_{n\in\mathbb{N}} A_n(\sigma) \quad \textit{is precompact.}$$

Then the family $\{A_n\}_{n\in\mathbb{N}}$ *is upper semi-continuous, that is,*

$$\lim_{n\to\infty} \operatorname{dist}(A_n(\sigma), A_\infty) = 0, \quad \forall \sigma \in \Sigma.$$

Proof. For each $\sigma \in \Sigma$ we denote

$$D_\sigma(t) := \overline{\bigcup_{n\in\mathbb{N}} A_n(\theta_{-t}\sigma)}, \quad \forall t \geqslant 0.$$

Then each $D_\sigma(t)$ is compact by (A3) and, by (A2), there is a bounded set B such that

$$\bigcup_{t>0} D_\sigma(t) \subset B.$$

Hence, by the negative invariance of cocycle attractors we have

$$\begin{aligned}
\operatorname{dist}(A_n(\sigma), A_\infty) &\leqslant \operatorname{dist}\big(\Phi_n(t, \theta_{-t}\sigma, A_n(\theta_{-t}\sigma)), A_\infty\big) \\
&\leqslant \operatorname{dist}\big(\Phi_n(t, \theta_{-t}\sigma, D_\sigma(t)), A_\infty\big) \\
&\leqslant \operatorname{dist}\big(\Phi_n(t, \theta_{-t}\sigma, D_\sigma(t)), \Phi_\infty(t, D_\sigma(t))\big) + \operatorname{dist}\big(\Phi_\infty(t, B), A_\infty\big), \ \forall t > 0.
\end{aligned}$$

For any $\epsilon > 0$, there exists a $t = t(\epsilon) > 0$ such that $\operatorname{dist}(\Phi_\infty(t, B), A_\infty) < \epsilon/2$. Also, by (A1) there is an $N = N(\epsilon, t, \sigma)$ such that

$$\operatorname{dist}\big(\Phi_n(t, \theta_{-t}\sigma, D_\sigma(t)), \Phi_\infty(t, D_\sigma(t))\big) < \epsilon/2, \quad \forall n \geqslant N.$$

Thus, $\operatorname{dist}(A_n(\sigma), A_\infty) < \epsilon$ for all $n \geqslant N = N(\epsilon, t(\epsilon), \sigma)$. The proof is complete. □

As an application of the relationship between cocycle and uniform attractors given by Theorem 1.1.37, we have the following result.

Theorem 1.3.4. *Suppose that* $\{\Phi_n\}_{n\in\mathbb{N}}$ *is a family cocycles with its corresponding family* $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ *of uniform attractors and a family* $\{A_n\}_{n\in\mathbb{N}}$ *of cocycle attractors.* Φ_∞ *is a multi-valued semiflow with the global attractor* A_∞. *Assume that the map* $(\sigma, x) \mapsto \Phi(t, \sigma, x)$ *is upper semicontinuous for any fixed* $t \geqslant 0$ *and the following conditions, which are stronger than (A1)-(A2):*

(A4) for each compact set $K \subset X$ *and* $t \in \mathbb{R}$,

$$\sup_{\sigma\in\Sigma} \sup_{x\in K} \operatorname{dist}\big(\Phi_n(t, \sigma, x), \Phi_\infty(t, x)\big) \to 0 \quad \textit{as } n \to \infty;$$

(A5) there is a compact set $K \subset X$ such that

$$\bigcup_{n\in\mathbb{N}}\bigcup_{\sigma\in\Sigma} A_n(\sigma) \subset K.$$

Then the family $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ is upper semi-continuous, that is,

$$\mathrm{dist}(\mathscr{A}_n, A_\infty) \to 0, \quad \textit{as } n \to \infty.$$

Proof. By Theorem 1.1.37 we have $\cup_{\sigma\in\Sigma} A_n(\sigma) = \mathscr{A}_n$, $\forall n \in \mathbb{N}$, and (A5) implies that the union $\cup_{\sigma\in\Sigma, n\in\mathbb{N}} A_n(\sigma)$ is precompact. Then with an analogous proof to that in Theorem 1.3.3 we obtain that $\sup_{\sigma\in\Sigma} \mathrm{dist}(A_n(\sigma), A_\infty) \to 0$ as $n \to \infty$, from which we have

$$\mathrm{dist}(\mathscr{A}_n, A_\infty) \leqslant \sup_{\sigma\in\Sigma} \mathrm{dist}(A_n(\sigma), A_\infty) \to 0, \quad \text{as } n \to \infty.$$

□

1.3.2 Lower semi-continuity

In this section we will prove some results concerning the lower semicontinuity of attractors, which together with the upper semicontinuity gives us the continuity of attractors.

a). Unstable manifolds

The lower semi-continuity of attractors often has an inner relation with the structure of attractors. To see this, we begin with the definition of a complete trajectory of a multivalued semiflow. For the corresponding definition for cocycles see Definition 1.2.2.

Definition 1.3.5. A complete trajectory ξ_∞ through x of an autonomous semigroup Φ_∞ on X is a single-valued continuous mapping $\xi_\infty(\cdot) : \mathbb{R} \to X$ satisfying $\xi_\infty(0) = x$ and

$$\xi_\infty(t) \in \Phi_\infty(t-s, \xi_\infty(s)), \quad \forall t \geqslant s.$$

A complete trajectory ξ_∞ of Φ_∞ is called backwards bounded if

$$\bigcup_{t\leqslant T} \xi_\infty(t) \in \mathcal{B}(X), \quad \forall T \in \mathbb{R}.$$

Definition 1.3.6. Let ξ^* be a backwards bounded complete trajectory of a cocycle Φ driven by $\sigma \in \Sigma$. The unstable manifold of ξ^* for Φ is the set defined by

$$W^u(\xi^*) = \Big\{(\tau, \zeta) \in \mathbb{R} \times X : \text{ there is a backwards bounded complete trajectory } \xi \\ \text{of } \Phi \text{ driven by } \sigma \text{ with } \xi(\tau) = \zeta \text{ and } \lim_{t\to-\infty} \mathrm{dist}(\xi(t), \xi^*(t)) = 0\Big\}.$$

The section of an unstable manifold at time τ is denoted by

$$W^u(\xi^*)(\tau) = \{\zeta : (\tau, \zeta) \in W^u(\xi^*)\}.$$

The local unstable manifold at time τ of a global solution ξ^* is defined by

$$\begin{aligned} W^u_\delta(\xi^*)(\tau) = \big\{\zeta \in X :\ &\text{there is a backwards bounded complete trajectory } \xi \\ &\text{of } \Phi \text{ driven by } \sigma \text{ with } \xi(\tau) = \zeta,\ \lim_{t\to-\infty} \operatorname{dist}(\xi(t), \xi^*(t)) = 0 \\ &\text{and } \operatorname{dist}(\xi(s), \xi^*(s)) < \delta \text{ for all } s \leqslant \tau\big\} \end{aligned}$$

for some δ sufficiently small.

The corresponding definitions for a semigroup Φ_∞ are as follows.

Definition 1.3.7. Let ξ^*_∞ be a backwards bounded complete trajectory of a semigroup Φ_∞. The unstable manifold of ξ^*_∞ for Φ_∞ is the set defined by

$$\begin{aligned} W^u(\xi^*_\infty) = \big\{(\tau, \zeta) \in \mathbb{R} \times X :\ &\text{there is a backwards bounded complete trajectory} \\ &\xi_\infty \text{ of } \Phi_\infty \text{ with } \xi_\infty(\tau) = \zeta \text{ and } \lim_{t\to-\infty} \operatorname{dist}(\xi_\infty(t), \xi^*_\infty(t)) = 0\big\}. \end{aligned}$$

The local unstable manifold at time τ of a global solution ξ^*_∞ is defined by

$$\begin{aligned} W^u_\delta(\xi^*_\infty)(\tau) = \big\{\zeta \in X :\ &\text{there is a backwards bounded complete trajectory } \xi_\infty \\ &\text{of } \Phi_\infty \text{ with } \xi_\infty(\tau) = \zeta,\ \lim_{t\to-\infty} \operatorname{dist}(\xi_\infty(t), \xi^*_\infty(t)) = 0 \\ &\text{and } \operatorname{dist}(\xi_\infty(s), \xi^*_\infty(s)) < \delta \text{ for all } s \leqslant \tau\big\} \end{aligned}$$

for some δ sufficiently small.

Theorem 1.3.8. *Suppose that $\{\Phi_n\}_{n\in\mathbb{N}}$ is a family of cocycles with a family $\{A_n\}_{n\in\mathbb{N}}$ of strictly invariant cocycle attractors, and that Φ_∞ is a multi-valued semi-group with the global attractor A_∞ satisfying:*

(A1') for each $\sigma \in \Sigma$, $t \in \mathbb{R}^+$ and $x_n \to x$ in X,

$$\operatorname{dist}\big(\Phi_\infty(t, x), \Phi_n(t, \sigma, x_n)\big) \to 0 \quad \text{as } n \to \infty.$$

If

- *there is a sequence $\{\xi^*_{\infty,j}\}_{j\in\mathbb{N}}$ of backwards bounded complete trajectories of Φ_∞ such that*

$$A_\infty = \overline{\bigcup_{j\in\mathbb{N}} W^u(\xi^*_{\infty,j})(0)}; \tag{1.28}$$

- *for each $\sigma \in \Sigma$ and every complete trajectory $\xi^*_{\infty,j}$ in (1.28) there is a sequence $\{\xi^{*,n}_{\sigma,j}\}_{n\in\mathbb{N}}$, with $\xi^{*,n}_{\sigma,j}$ a backwards bounded trajectory of Φ_n driven by σ, such that their local unstable manifolds behave lower semi-continuously, that is, there are $\delta_j > 0$ and $T_j > 0$ satisfying*

$$\operatorname{dist}\Big(W^u_{\delta_j}(\xi^*_{\infty,j})(s), W^u_{\delta_j}(\xi^{*,n}_{\sigma,j})(s)\Big) \xrightarrow{n\to\infty} 0, \quad \forall s < -T_j, \tag{1.29}$$

then the sequence of cocycle attractors is lower semi-continuous, that is,

$$\lim_{n\to\infty} \operatorname{dist}(A_\infty, A_n(\sigma)) = 0, \quad \forall \sigma \in \Sigma.$$

Proof. To prove the lower semi-continuity it suffices to prove that for each $\sigma \in \Sigma$ and $x \in A_\infty$ there is a sequence $x_n \in A_n(\sigma)$ such that $x_n \to x$.

By (1.28), for any $x \in A_\infty$ and $\epsilon > 0$ there exists an $x_\epsilon \in W^u(\xi^*_{\infty,j})(0)$ for some $j \in \mathbb{N}$ such that

$$\operatorname{dist}(x, x_\epsilon) < \epsilon/2.$$

By the definition of $W^u(\xi^*_{\infty,j})(0)$ we have a backwards bounded complete trajectory $\xi_{\infty,j}$ of Φ_∞ such that

$$\xi_{\infty,j}(0) = x_\epsilon, \quad \text{and} \quad \lim_{s\to-\infty} \operatorname{dist}(\xi_{\infty,j}(s), \xi^*_{\infty,j}(s)) = 0.$$

Let $T > T_j$ be so large that

$$z_j := \xi_{\infty,j}(-T) \in W^u_{\delta_j}(\xi^*_{\infty,j})(-T).$$

Then by the lower semi-continuity of the local unstable manifolds, there is a sequence $\{z^n_j\}_{n\in\mathbb{N}}$ with $z^n_j \in W^u_{\delta_j}(\xi^{*,n}_{\sigma,j})(-T)$ such that

$$z^n_j \to z_j \quad \text{as } n \to \infty.$$

Therefore, by (A1'), there is an $n_0 \in \mathbb{N}$ such that

$$\operatorname{dist}(\Phi_\infty(T, z_j), \Phi_n(T, \theta_{-T}\sigma, z^n_j)) < \epsilon/2 \quad \text{for all } n \geqslant n_0.$$

By the definition of complete trajectory of Φ_∞ we have $x_\epsilon = \xi_{\infty,j}(0) \in \Phi_\infty(T, z_j)$. Also, since $z^n_j \in A(\theta_{-T}\sigma)$ by Proposition 1.2.6, the strict invariance of A_n implies that

$$\Phi_n(T, \theta_{-T}\sigma, z^n_j) \subset A_n(\sigma).$$

Thus, there is $x_n \in A_n(\sigma)$ such that

$$\operatorname{dist}(x_n, x) \leqslant \operatorname{dist}(x_n, x_\epsilon) + \operatorname{dist}(x_\epsilon, x) < \epsilon.$$

Since ϵ is arbitrary, we have completed the proof. □

Remark 1.3.9. It is important to observe the following. In the single-valued case, if Φ_∞ is a continuous cocycle, then the condition (A1') is equivalent to

$$\sup_{x\in E} \operatorname{dist}(\Phi_\infty(t, x), \Phi_n(t, \sigma, x)) \xrightarrow{n\to\infty} 0 \quad \text{for all compact set } E \text{ and } t \geqslant 0,\ \sigma \in \Sigma, \tag{1.30}$$

while, in the multi-valued case, (A1') implies (1.30) if Φ_∞ is upper semi-continuous, and (1.30) implies (A1') if Φ_∞ is lower semi-continuous. In applications, (A1'), as well as the lower semi-continuity of Φ_∞, is often more difficult to verify than (1.30). This makes it difficult to prove the lower semi-continuity of attractors for multi-valued dynamical systems.

When the semiflow Φ_∞ is generated by a collection of functions $\mathcal{K}$, the results in this section can be reformulated in terms of complete trajectories of $\mathcal{K}$.

Namely, we are going to consider multivalued semiflows generated by a collection of continuous functions $\mathcal{K} = \{\mathcal{K}_x\}_{x\in X}$ satisfying:

(K1) for each $x \in X$, $\mathcal{K}_x$ is a non-empty set of continuous functions $\gamma : [0, \infty) \to X$ such that $\gamma(0) = x$;

(K2) $\widetilde{\gamma}(\cdot) = \gamma(\cdot+s) \in \mathcal{K}_{\gamma(s)}$, for every $x \in X$, $s \geqslant 0$ and $\gamma \in \mathcal{K}_x$.

The map Φ_∞ is defined by

$$\Phi_\infty(t, x) = \{y : y = \gamma(t), \gamma \in \mathcal{K}_x\}.$$

It is well-known that (K1)-(K2) imply that Φ_∞ is a multivalued semiflow [50].

Definition 1.3.10. A complete trajectory ξ_∞ through x of $\mathcal{K}$ is a single-valued continuous mapping $\xi_\infty(\cdot) : \mathbb{R} \to X$ satisfying $\xi_\infty(0) = x$ and

$$\xi_\infty(\cdot + s) \mid_{[0,\infty)} \in \mathcal{K}_{\xi_\infty(s)}, \forall s \in \mathbb{R}.$$

It is blatantly obvious that a complete trajectory of $\mathcal{K}$ is a complete trajectory of Φ_∞, the converse being true only under additional assumptions [51, Lemma 12.2].

We can easily rewrite Definitions 1.3.6 and 1.3.7 in terms of complete trajectories of $\mathcal{K}$. Theorem 1.3.8 remains valid if we make use of complete trajectories of $\mathcal{K}$ instead of Φ_∞.

Theorem 1.3.11. *Suppose that $\{\Phi_n\}_{n\in\mathbb{N}}$ is a family of cocycles with a family $\{A_n\}_{n\in\mathbb{N}}$ of strictly invariant cocycle attractors, and that Φ_∞ is a multi-valued semi-group with the global attractor A_∞ satisfying (A1'). Assume that:*

- *there is a sequence $\{\xi^*_{\infty,j}\}_{j\in\mathbb{N}}$ of backwards bounded complete trajectories of $\mathcal{K}$ such that (1.28) holds;*

- *for each $\sigma \in \Sigma$ and every complete trajectory $\xi^*_{\infty,j}$ in (1.28) there is a sequence $\{\xi^{*,n}_{\sigma,j}\}_{n\in\mathbb{N}}$, with $\xi^{*,n}_{\sigma,j}$ a backwards bounded trajectories of $\mathcal{R}$ driven by σ, such that their local unstable manifolds behave lower semi-continuously, that is, there are $\delta_j > 0$ and $T_j > 0$ satisfying (1.29).*

Then, the sequence of cocycle attractors is lower semi-continuous, that is,

$$\lim_{n\to\infty} \operatorname{dist}(A_\infty, A_n(\sigma)) = 0, \quad \forall \sigma \in \Sigma.$$

b). Weak equi-attraction

Now we characterize the continuity of attractors by the property of equi-attraction, instead of using the inner structure of attractors. As will be shown later, this method is competent to deal with some cases where Theorem 1.3.8 is powerless.

To begin with, we introduce the concept of weakly equi-attraction as a generalization of equi-attraction first introduced by Li and Kloeden [58, 59].

Definition 1.3.12. Suppose that $\{\Phi_n\}_{n\in\mathbb{N}}$ is a collection of cocycles with the corresponding family $\{A_n\}_{n\in\mathbb{N}}$ of cocycle attractors. If

$$\lim_{(t,n)\to(\infty,\infty)} \operatorname{dist}(\Phi_n(t,\theta_{-t}\sigma,K),A_n(\sigma))=0 \quad \text{for each } \sigma\in\Sigma \text{ and } K\in\mathcal{K}(X),$$

then we say that $\{A_n\}_{n\in\mathbb{N}}$ is weakly equi-pullback-attracting (under $\{\Phi_n\}_{n\in\mathbb{N}}$). If

$$\lim_{t\to\infty}\sup_{n\in\mathbb{N}} \operatorname{dist}(\Phi_n(t,\theta_{-t}\sigma,B),A_n(\sigma))=0 \quad \text{for each } \sigma\in\Sigma \text{ and } B\in\mathcal{B}(X),$$

then we say that $\{A_n\}_{n\in\mathbb{N}}$ is equi-pullback-attracting (under $\{\Phi_n\}_{n\in\mathbb{N}}$).

Clearly, if $\{A_n\}_{n\in\mathbb{N}}$ is equi-pullback-attracting, then it is weakly equi-pullback-attracting. The following result shows that, under some conditions, the weak equi-attraction implies the lower semi-continuity.

Theorem 1.3.13. *Suppose that $\{\Phi_n\}_{n\in\mathbb{N}}$ is a collection of cocycles with the corresponding family $\{A_n\}_{n\in\mathbb{N}}$ of cocycle attractors, and that Φ_∞ is a multi-valued semiflow with a global attractor A_∞. If*

$$\operatorname{dist}\big(\Phi_\infty(t,x),\Phi_n(t,\sigma,x)\big)\xrightarrow{n\to\infty}0,\quad \forall t\geqslant 0,\ \sigma\in\Sigma,\ x\in X, \tag{1.31}$$

then the fact that $\{A_n\}_{n\in\mathbb{N}}$ is weak equi-pullback-attracting implies the lower semi-continuity

$$\operatorname{dist}(A_\infty,A_n(\sigma))\xrightarrow{n\to\infty}0,\quad \forall\sigma\in\Sigma.$$

Proof. We prove the result by contradiction. If this is not the case for some $\sigma\in\Sigma$, then there must be a sequence $\{x_n\}_{n\in\mathbb{N}}\subset A_\infty$ such that

$$\operatorname{dist}(x_n,A_n(\sigma))>\delta,\quad \forall n\in\mathbb{N},$$

for some $\delta>0$. Since A_∞ is compact, up to a subsequence, there is an $x_\infty\in A_\infty$ such that

$$\operatorname{dist}(x_n,x_\infty)<\frac{\delta}{2},\quad \forall n\geqslant n_0,$$

for some $n_0\in\mathbb{N}$ large enough. Therefore, it follows that

$$\operatorname{dist}(x_\infty,A_n(\sigma))>\frac{\delta}{2},\quad \forall n\geqslant n_0. \tag{1.32}$$

However, from the weak equi-attraction property it follows the existence of a $T > 0$ and an $N_0 \geqslant n_0$ satisfying

$$\operatorname{dist}(\Phi_n(T, \theta_{-T}\sigma, A_\infty), A_n(\sigma)) < \frac{\delta}{4}, \quad \forall n \geqslant N_0.$$

Also, from the negative invariance of A_∞ we have an $x_T \in A_\infty$ such that $x_\infty \in \Phi_\infty(T, x_T)$ and

$$\operatorname{dist}(\Phi_n(T, \theta_{-T}\sigma, x_T), A_n(\sigma)) \leqslant \operatorname{dist}(\Phi_n(T, \theta_{-T}\sigma, A_\infty), A_n(\sigma)) < \frac{\delta}{4}, \quad \forall n \geqslant N_0.$$

Moreover, by (1.31), there exists an $N \geqslant N_0$ such that

$$\operatorname{dist}\Big(\Phi_\infty(T, x_T), \Phi_N(T, \theta_{-T}\sigma, x_T)\Big) < \frac{\delta}{4}.$$

Therefore,

$$\begin{aligned}
\operatorname{dist}(x_\infty, A_N(\sigma)) &\leqslant \operatorname{dist}(\Phi_\infty(T, x_T), A_N(\sigma)) \\
&\leqslant \operatorname{dist}\Big(\Phi_\infty(T, x_T), \Phi_N(T, \theta_{-T}\sigma, x_T)\Big) + \operatorname{dist}(\Phi_N(T, \theta_{-T}\sigma, x_T), A_N(\sigma)) \\
&< \frac{\delta}{2},
\end{aligned}$$

which contradicts (1.32). Thus the lower semi-continuity holds. □

If the cocycles have uniform attractors, we also have the following result.

Theorem 1.3.14. *Assume that either the conditions of Theorem 1.3.8 or those of Theorem 1.3.13 hold. If the maps $(\sigma, x) \mapsto \Phi_n(t, \sigma, x)$ are upper semi-continuous and each cocycle Φ_n has a uniform attractor $\mathscr{A}_n$, $n = 1, 2, \cdots$, then*

$$\lim_{n\to\infty} \operatorname{dist}(A_\infty, \mathscr{A}_n) = 0.$$

Proof. By Theorem 1.1.37 we have $\cup_{\sigma\in\Sigma} A_n(\sigma) = \mathscr{A}_n$, and thereby

$$\operatorname{dist}(A_\infty, \mathscr{A}_n) \leqslant \operatorname{dist}(A_\infty, A_n(\sigma)), \quad \forall \sigma \in \Sigma.$$

From Theorem 1.3.8 and Theorem 1.3.13 the result follows. □

Remark 1.3.15. It is actually natural that weak equi-attraction can replace the role of equi-attraction. This is because the continuity of attractors takes care only of those large n that are close to infinity, and thereby only the equi-attraction of those A_n with large n contributes to the continuity property of attractors.

Remark 1.3.16. Equi-attraction was first introduced to study the continuity of global attractors by Li and Kloeden [58, 59], and further studied in [20, 52, 60, 47]. In particular, Li and Kloeden [60] studied the relationship between equi-attraction and continuity of attractors for *strict* multi-valued autonomous dynamical systems which are continuous in the sense of Hausdorff distance. Such strong assumptions are not required here and the multi-valued cocycles considered here involve some multi-valued features, the proof is less straightforward and different from the single-valued case (see also [20]).

1.4 Applications

1.4.1 Attractors for a multi-valued reaction-diffusion equation

In this part, we study the non-autonomous reaction-diffusion equation

$$\begin{cases} \frac{\partial u}{\partial t} - a\triangle u + f(u) = g(x,t), \\ u|_{\partial\mathcal{O}} = 0, \\ u|_{t=\tau} = u_\tau, \end{cases} \tag{1.33}$$

where $(t,x) \in (\tau,\infty)\times\mathcal{O}$ with $\mathcal{O} \subset \mathbb{R}^N$ a bounded open domain with smooth boundary, the unknown $u = (u^1(t,x),\cdots,u^d(t,x)) : (\tau,\infty)\times\mathcal{O} \to \mathbb{R}^d$ is d-dimensional, a is a real $d\times d$ matrix with a positive symmetric part $\frac{a+a^t}{2} \geqslant \beta I$ for some $\beta > 0$ and the nonlinear term $f = (f^1(u),\cdots,f^d(u)) \in C(\mathbb{R}^d;\mathbb{R}^d)$ satisfies

$$\sum_{j=1}^{d} |f^j(u)|^{\frac{p_j}{p_j-1}} \leqslant c_1\left(1 + \sum_{j=1}^{d}|u^j|^{p_j}\right), \tag{1.34}$$

$$\sum_{j=1}^{d} f^j(u)u^j \geqslant \alpha_1 \sum_{j=1}^{d}|u^j|^{p_j} - c_2, \tag{1.35}$$

where $p_j \geqslant 2$ and α_1, c_1, c_2 are all positive constants.

The equation (1.33) models many important real phenomena, as it clearly covers at least complex Ginzburg-Landau equation, the Fitz-Hugh-Nagumo equation and the Lotka-Volterra system if the parameters are properly fixed, see [80, 49]. Mathematically, as will be shown later, this is a multi-valued dynamical system as for each initial data there are (possibly) more that one solution. Because of these interesting features, this model drew much attention recently. Valero and Kapustyan [80, 49] studied the Kneser property of weak solutions and the existence and connectedness of uniform attractors with the condition that the external force $g(x,t)$ is translation compact, i.e.

$$\text{the symbol space } \Sigma = \overline{\{\theta_s g(\cdot) : s\in\mathbb{R}\}}^{L^{2,w}_{loc}(\mathbb{R},H)} \text{ is compact in } L^{2,w}_{loc}(\mathbb{R},H), \tag{1.36}$$

where $\theta_s g(\cdot) = g(\cdot + s)$ is the translation operator, whereas Kapustyan et al. [51] analyzed the structure of uniform attractors, and Cui et al. [38] studied the regularity of the associated pullback attractors.

Throughout this section we assume that (1.34)-(1.36) are satisfied. It is easy to see that $\theta_t\Sigma = \Sigma$ for all $t \in \mathbb{R}$. In the following, we shall study the relationship between different attractors of system (1.33). Denote $H = (L^2(\mathcal{O}))^d$ and $V = (H^1_0(\mathcal{O}))^d$ with norms given by

$$\|u\|_H = \left(\sum_{j=1}^{d}\|u^j\|^2_{L^2(\mathcal{O})}\right)^{1/2}, \quad \|u\|_V = \left(\sum_{j=1}^{d}\|u^j\|^2_{H^1_0(\mathcal{O})}\right)^{1/2}.$$

Let $P = (p_1, p_2, \cdots, p_d)$ and $L^P(\mathcal{O}) = L^{p_1}(\mathcal{O}) \times \cdots \times L^{p_d}(\mathcal{O})$ with the norm

$$\|u\|^{\mathbf{P}}_{L^{\mathbf{P}}(\mathcal{O})} = \sum_{j=1}^{d} \|u^j\|^{p_j}_{L^{p_j}(\mathcal{O})}.$$

In order to define a multi-valued cocycle, we take an arbitrary element σ in the symbol space Σ and solve problem (1.33) but replacing g by σ.

Definition 1.4.1. The function $u = u(t,x) \in L^2_{loc}(\tau, \infty; V) \cap L^P_{loc}(\tau, \infty; L^P(\mathcal{O}))$ is called a (weak) solution of problem (1.33) on (τ, ∞) driven by $\sigma \in \Sigma$ if, for all $T > \tau$ and $v \in V \cap L^P(\mathcal{O})$,

$$\int_{\mathcal{O}} u(t,x)v(x)\mathrm{d}x + \int_{\mathcal{O}} \Big(a\nabla u(t,x)\nabla v(x) + f(t,u(t,x))v(x) - \sigma(t,x)v(x)\Big)\mathrm{d}x = 0$$

holds in the sense of scalar distributions on (τ, T).

The next result shows that problem (1.33) generates a strict multi-valued process.

Lemma 1.4.2. *[80] Under conditions (1.34)-(1.36), for each $\tau \in \mathbb{R}$ and $u_\tau \in H$, there exists at least one weak solution of (1.33) on (τ, ∞) and any weak solution belongs to $C([\tau, \infty); H)$. Moreover, the family $\mathcal{R} = \{\mathcal{R}_{\sigma,x}\}_{\sigma\in\Sigma, x\in H}$ given by*

$$\mathcal{R}_{\sigma,x} = \{u(\cdot) \in C([0,\infty); H) : u \textit{ is a solution of } (1.33) \textit{ driven by } \sigma \textit{ and } u(0) = x\}$$

satisfies (H1)-(H3), (H5) and thereby Proposition 1.2.1 implies that the mapping Φ defined by

$$\Phi(t,\sigma,x) = \{u(t) : u(\cdot) \in \mathcal{R}_{\sigma,x}\}, \quad \forall t \in \mathbb{R}^+,\ \sigma \in \Sigma,\ x \in H$$

is a strict multivalued cocycle and, moreover, the map $(\sigma, x) \mapsto \Phi(t,\sigma,x)$ is upper semicontinuous for any fixed $t \geqslant 0$.

To study different attractors of the cocycle, we first rewrite a well-known result on uniform attractors in terms of cocycles.

Lemma 1.4.3. *[51, Theorem 12.4] Suppose that conditions (1.34)-(1.36) hold. Then the cocycle Φ has the uniform attractor $\mathscr{A}$, which consists of bounded complete trajectories of Φ, that is,*

$$\mathscr{A} = \{\xi(0) :\ \xi(\cdot) \textit{ is a bounded complete trajectory of } \Phi\}.$$

Now, thanks to the previous analysis, we are able to prove the following result.

Theorem 1.4.4. *Suppose that conditions (1.34)-(1.36) hold. Then the cocycle Φ has a lifted invariant uniform attractor $\mathscr{A}$, a cocycle attractor $A = \{A(\sigma)\}_{\sigma\in\Sigma}$ and a family $\{A_\sigma\}_{\sigma\in\Sigma}$ of pullback attractors of the equivalent family of multivalued processes $\{U_\sigma\}_{\sigma\in\Sigma}$, which consist of bounded complete trajectories of $\mathcal{R}$, that is,*

$$\begin{aligned}
\mathscr{A} &= \{\xi(0) :\ \xi(\cdot) \textit{ is a bounded complete trajectory of } \mathcal{R}\},\\
A(\sigma) &= \{\xi(0) :\ \xi(\cdot) \textit{ is a bounded complete trajectory of } \mathcal{R} \textit{ driven by } \sigma\},\\
A_\sigma(t) &= \{\xi(t) :\ \xi(\cdot) \textit{ is a bounded complete trajectory of } \mathcal{R} \textit{ driven by } \sigma\}.
\end{aligned}$$

Besides, the cocycle and the pullback attractors are uniformly compact, that is,

$$\bigcup_{\sigma\in\Sigma} A(\sigma) \text{ and } \bigcup_{\sigma\in\Sigma}\bigcup_{t\in\mathbb{R}} A_\sigma(t) \text{ are compact.}$$

Moreover, the three attractors are related as follows:

$$\mathscr{A} = \bigcup_{\sigma\in\Sigma} A(\sigma), \tag{1.37}$$

$$A_\sigma(t) = A(\theta_t\sigma), \quad \forall t\in\mathbb{R},\ \sigma\in\Sigma. \tag{1.38}$$

Proof. From Lemma 1.4.3 and Theorem 1.1.37 we obtain the existence of the uniform and cocycle attractors $\mathscr{A}$, A and the equality (1.37) as well. Therefore, by the compactness of $\mathscr{A}$ and Theorem 1.1.41 we have the uniform compactness of A and $\{A_\sigma\}_{\sigma\in\Sigma}$. The lifted invariance of the uniform attractor follows from Theorem 1.2.14 and the characterizations by complete trajectories of $\mathcal{R}$ and (1.38) follow from Theorems 1.2.7, 1.1.41 and Corollary 1.2.15. The proof is complete. □

Remark 1.4.5. An advantage of characterizing attractors by complete trajectories of $\mathcal{R}$ instead of complete trajectories of Φ is that any complete trajectory of $\mathcal{R}$ is a solution of the evolution equation, while a complete trajectory of Φ might not be so. Therefore, estimates of solutions are always applicable to complete trajectories of $\mathcal{R}$. Particularly in this model, complete trajectories of Φ and $\mathcal{R}$ coincide by Proposition 1.2.4 and Lemma 1.4.2.

Let us also consider the associated skew-product semiflow $\Pi : \mathbb{R}^+\times H\times\Sigma \to H\times\Sigma$ generated by (Φ,θ).

Theorem 1.4.6. *Suppose that conditions (1.34)-(1.36) hold. Then the semiflow Π possesses a global attractor $\mathbb{A}$, which has the following relationship with the uniform attractor $\mathscr{A}$ and the cocyle attractor $\{A(\sigma)\}_{\sigma\in\Sigma}$:*

$$\mathscr{A} = P_H\mathbb{A}, \tag{1.39}$$

$$A(\sigma) = P_\sigma\mathbb{A},\ \forall\sigma\in\Sigma. \tag{1.40}$$

Proof. The existence of $\mathbb{A}$ and (1.39) follow from Theorem 1.1.33, while equality (1.40) is by Theorem 1.1.34. □

1.4.2 Lower semi-continuity of cocycle attractors for a scalar differential inclusion

In this section, we study the robustness of cocycle attractors for the following scalar differential inclusion

$$\begin{cases} \dfrac{du}{dt} + \lambda u \in \sigma(t)H_0(u), \quad t\geqslant s, \\ u(s) = u_s, \end{cases} \tag{1.41}$$

where

$$H_0(u) = \begin{cases} -1, & \text{if } u < 0, \\ [-1, 1], & \text{if } u = 0, \\ 1, & \text{if } u > 0, \end{cases}$$

is the Heaviside function, which is the subdifferential of the absolute value $|u|$, and $\sigma \in \Sigma$, where

$$\Sigma = cl_{C(\mathbb{R},\mathbb{R})}\{b(\cdot + s) : s \in \mathbb{R}\}$$

with $b : \mathbb{R} \to \mathbb{R}^+$ being a continuous function satisfying:

1. $0 < b_0 \leqslant \sup_{t\in\mathbb{R}} b(t) \leqslant b_1$;
2. the set $\{b(\cdot + s) : s \in \mathbb{R}\}$ is equicontinuous in any interval $[t_1, t_2]$.

The translation operator θ is defined by $\theta_t\sigma = \sigma(\cdot + t)$.

Ascoli-Arzelà's theorem implies that the set Σ is compact in the metrizable space $C(\mathbb{R}, \mathbb{R})$ and it is easy to see that $\theta_t\Sigma = \Sigma$ for any $t \in \mathbb{R}$.

Definition 1.4.7. For any $\sigma \in \Sigma$, the function $u \in C([s, +\infty), \mathbb{R})$ is called a solution of (1.41) (driven by σ) if $\frac{du}{dt} \in L^\infty_{loc}([s, +\infty), \mathbb{R})$ and there exists $h \in L^\infty_{loc}([s, +\infty), \mathbb{R})$ such that $h(t) \in H_0(u(t))$, for a.e. $t > s$, and

$$\frac{du}{dt} + \lambda u = \sigma(t)h(t) \quad \text{for } a.e.\ t > s.$$

The following lemma shows that the uniqueness of solutions fails when the initial data is zero.

Lemma 1.4.8. *[16]. For any $u_s \neq 0$ the problem* (1.41) *has the unique solution given by*

$$u(t) = e^{-\lambda(t-s)}u_s + \int_s^t e^{-\lambda(t-\tau)}\sigma(\tau)\,\mathrm{d}\tau.$$

For $u_s = 0$, there are infinitely many solutions given by

$$u_\infty \equiv 0,$$

$$u_r^+(t) = \begin{cases} 0, & s \leqslant t \leqslant r, \\ \int_r^t e^{-\lambda(t-\tau)}\sigma(\tau)\,\mathrm{d}\tau, & t > r, \end{cases}$$

$$u_r^-(t) = \begin{cases} 0, & s \leqslant t \leqslant r, \\ -\int_r^t e^{-\lambda(t-\tau)}\sigma(\tau)\,\mathrm{d}\tau, & t > r, \end{cases}$$

where $r \geqslant s$ is arbitrary, and these are the only possible solutions.

Let $\mathcal{R} = \{\mathcal{R}_s^\sigma\}$ be the collection of all solutions of (1.41) driven by σ and starting at s.

The recent work by Caraballo et al. [16] shows that (1.41) generates a strict multi-valued process, which possesses a compact pullback attractor composed by complete trajectories. Specifically, we have the following result.

Lemma 1.4.9. *[16] For any $\sigma \in \Sigma$, the mapping $(t, s, u_s) \mapsto U_\sigma(t, s, u_s)$ given by*

$$U_\sigma(t,s,u_s) = \big\{y \in \mathbb{R} : \text{there exists a continuous function } \eta \in \mathcal{R}_s^\sigma \text{ such that } y = \eta(t) \text{ and } u_s = \eta(s)\big\}$$

is a strict multi-valued process which has a closed graph. Moreover, each process U_σ has a unique pullback attractor $A_\sigma = \{A_\sigma(t)\}_{t\in\mathbb{R}}$, which is invariant and satisfies

$$A_\sigma(t) = \{\xi(t) : \xi \text{ is a bounded complete trajectory of } \mathcal{R} \text{ driven by } \sigma\},$$

where a complete trajectory $\gamma : \mathbb{R} \to \mathbb{R}$ of $\mathcal{R}$ driven by σ is defined as a continuous function such that $\gamma\,|_{[s,\infty)} \in \mathcal{R}_s^\sigma$ for all $s \in \mathbb{R}$.

Moreover, in [16] all possible bounded complete trajectories are given. Therefore, the structure of the pullback attractor is clear.

Lemma 1.4.10. *[16] Suppose that ξ is a bounded complete trajectory of $\mathcal{R}$ with $\sigma \in \Sigma$, then either $\xi(t) \equiv 0$, $\xi(t) = \pm\xi_M(t)$, or $\xi(t) = \xi_s^\pm(t)$ for some $s \in \mathbb{R}$, where*

$$\xi_M(t) = \int_{-\infty}^{t} e^{-\lambda(t-\tau)}\sigma(\tau)\,\tau, \quad \forall t \in \mathbb{R},$$

and ξ_s^+ and ξ_s^- are bounded complete trajectories such that $\xi_s^+(t) = \xi_s^-(t) = 0$ for all $t \leqslant s$, and

$$|\xi_s^+(t) - \xi_M(t)| \to 0 \quad \text{as } t \to \infty,$$
$$|\xi_s^-(t) + \xi_M(t)| \to 0 \quad \text{as } t \to \infty.$$

Remark 1.4.11. The complete trajectories $\xi(t) \equiv 0$, $\xi(t) = \pm\xi_M(t)$ are called non-autonomous equilibria. Lemma 1.4.10 shows that each pullback attractor is composed of 0, $\pm\xi_M(t)$ and all the heteroclinic connections going from 0 to $\pm\xi_M(t)$.

The collection $\mathcal{R}$ satisfies the following properties:

(K1) For any $x \in X,\ \sigma \in \Sigma,\ \tau \in \mathbb{R}$ there exists $\varphi(\cdot) \in \mathcal{R}_\tau^\sigma$ such that $\varphi(\tau) = x$;

(K2) $\varphi(\cdot)|_{[s,+\infty)} \in \mathcal{R}_s^\sigma$, for any $\varphi(\cdot) \in \mathcal{R}_\tau^\sigma,\ s \geqslant \tau$;

(K3) If $\psi \in \mathcal{R}_\tau^\sigma$, $\varphi \in \mathcal{R}_s^\sigma$, $s > \tau$, are such that $\psi(s) = \varphi(s)$, then the mapping

$$\xi(t) = \begin{cases} \psi(t), & t \in [\tau, s], \\ \varphi(t), & t > s, \end{cases} \tag{1.42}$$

belongs to $\mathcal{R}_\tau^\sigma$;

(K4) For any $\varphi_n \in \mathcal{R}_x^\sigma$, $\sigma \in \Sigma$, there exists $\varphi_0 \in \mathcal{R}_x^\sigma$ such that, up to a subsequence, $\varphi_n(t) \to \varphi_0(t)$, $\forall t \geqslant 0$;

(K5) $\varphi(\cdot + h) \in \mathcal{R}_{\tau}^{\theta(h)\sigma}$, for any $h \geqslant 0,\ \varphi(\cdot) \in \mathcal{R}_{\tau+h}^{\sigma}$.

Properties (K1)-(K4) were proved in [16], while (K5) is straightforward to check. Hence, it follows from [51, Lemma 12.2] that $\gamma : \mathbb{R} \to \mathbb{R}$ is a complete trajectory of $\mathcal{R}$ driven by σ if and only if γ is a complete trajectory of U_σ, which means that γ is continuous and

$$\gamma(t) \in U_\sigma(t, s, \gamma(s)) \quad \text{for all } t \geqslant s.$$

Consider the cocycle generated by (1.41). By the equivalence argument in Proposition 1.1.4 and the properties of the processes, we know that (1.41) generates a strict and closed multi-valued cocycle Φ given by

$$\Phi(t, \sigma, r) := U_\sigma(t, 0, r), \quad \forall t \geqslant 0,\ \sigma \in \Sigma,\ r \in \mathbb{R}.$$

Now let us establish the existence of the cocycle attractor for Φ by Theorem 1.1.42.

Lemma 1.4.12. *The cocycle Φ generated by* (1.41) *has a cocycle attractor $A = \{A(\sigma)\}_{\sigma\in\Sigma}$ given by*

$$\begin{aligned} A(\sigma) &= \{\xi(0) : \xi \textit{ is a bounded complete trajectory (driven by } \sigma\textit{) of } \Phi\} \\ &= \{\pm\xi_M(0)\} \cup W^u(0)(t)|_{t=0}. \end{aligned}$$

Proof. First we note that a complete trajectory $\xi(t)$ of U_σ is a complete trajectory driven by σ of Φ. This is because

$$\xi(t) \in U_\sigma(t, s, x(s)) = U_{\theta_s\sigma}(t-s, 0, \xi(s)) = \Phi(t-s, \theta_s\sigma, \xi(s)), \quad \forall t \geqslant s,$$

where we have used the translation identity.

From Theorem 1.1.42 it follows the relationship between the cocyle and the pullback attractor, that is,

$$A(\sigma) = A_\sigma(0).$$

Therefore, by Lemma 1.4.9 the first identity follows. To see the second, it suffices to observe from Lemma 1.4.10 that all bounded complete trajectories except $\pm\xi_M(t)$ build up $W^u(0)(t)$. □

To study the lower semi-continuity of cocycle attractors, we consider

$$\sigma_n(t) = \frac{1}{n}\sigma(t) + b, \quad \forall t \in \mathbb{R},\ n \in \mathbb{N},$$

where $b > 0$ is a constant and $\sigma \in \Sigma$. Clearly, $\sigma_n(t) \to b$ as $n \to \infty$ uniformly for $t \in \mathbb{R}$.

In the following, we denote by Φ_n the cocycle generated by (1.41) driven by $\sigma_n(t)$, and Φ_∞ the autonomous semiflow driven by $\sigma_\infty(t) \equiv b$. The cocycle attractor of Φ_n and the global attractor of Φ_∞ are denoted by $A_n = \{A_n(\sigma)\}_{\sigma\in\Sigma}$ and A_∞, respectively.

Recall that, generally, there are two possible methods to study the lower semi-continuity of cocycle attractors: one relies on studying the structure of attractors applying Theorem 1.3.8, whereas the other one is the weak equi-attraction method analyzed in Theorem 1.3.13. However, the following lemma implies that Theorem 1.3.8 is not applicable to this example.

Lemma 1.4.13. *The condition (A1') in Theorem 1.3.8 does not hold for (1.41). In fact, for each $\sigma \in \Sigma$, $t > 0$ and any sequence $x_n > 0$ with $x_n \to 0$ we have*

$$\liminf_{n\to\infty} \operatorname{dist}\big(\Phi_\infty(t,0), \Phi_n(t,\sigma,x_n)\big) \geqslant b - be^{-\lambda t}.$$

Proof. By Lemma 1.4.8 we know that $0 \in \Phi_\infty(t,0)$, and

$$\Phi_n(t,\sigma,x_n) = U_{\sigma_n}(t,0,x_n) = e^{-\lambda t}x_n + \int_0^t e^{-\lambda(t-\tau)}\sigma_n(\tau)d\tau.$$

Therefore, for any $t > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned}\operatorname{dist}\big(\Phi_\infty(t,0), \Phi_n(t,\sigma,x_n)\big) &\geqslant \operatorname{dist}\big(0, \Phi_n(t,\sigma,x_n)\big)\\ &= \left|e^{-\lambda t}x_n + \int_0^t e^{-\lambda(t-\tau)}\sigma_n(\tau)\,\mathrm{d}\tau\right|\\ &> \left|\int_0^t e^{-\lambda(t-\tau)} b\,\mathrm{d}\tau\right| = b - be^{-\lambda t},\end{aligned}$$

which completes the proof. □

Next we shall prove the lower semi-continuity of attractors by the weak equi-attraction method following Theorem 1.3.13.

Lemma 1.4.14. *The family $\{\Phi_n\}$ of cocycles is weakly equi-pullback-attracting.*

Proof. Given a $K \in \mathcal{K}(\mathbb{R})$, without loss of generality we assume that $K = K_1 \cup \{0\} \cup K_2$, where K_1 contains all positive elements of K and K_2 contains all negative ones. We will conclude the lemma if we prove that

$$\lim_{(t,n)\to(\infty,\infty)} \operatorname{dist}(\Phi_n(t,\theta_{-t}\sigma,K), A_n(\sigma)) = 0 \quad \text{for each } \sigma \in \Sigma \tag{1.43}$$

holds for $K = K_1,\ \{0\},\ K_2$, respectively. Note that by Lemma 1.4.10 and Lemma 1.4.12 we know that $\xi_M^n(0) \in A_n(\sigma)$, where

$$\xi_M^n(0) = \int_{-\infty}^0 e^{\lambda\tau}\sigma_n(\tau)\,d\tau.$$

Since Φ_n are single-valued on K_1 and, by Lemma 1.4.8,

$$\Phi_n(t,\theta_{-t}\sigma,x) = U_{\sigma_n}(0,-t,x) = e^{-\lambda t}x + \int_{-t}^0 e^{\lambda\tau}\sigma_n(\tau)\,\mathrm{d}\tau, \quad \forall x \in K_1,$$

we have

$$\begin{aligned}\operatorname{dist}(\Phi_n(t,\theta_{-t}\sigma,K_1), A_n(\sigma)) &\leqslant \sup_{x\in K_1} \operatorname{dist}(\Phi_n(t,\theta_{-t}\sigma,x), \xi_M(0))\\ &= \sup_{x\in K_1}\left|e^{-\lambda t}x - \int_{-\infty}^{-t} e^{\lambda\tau}\sigma_n(\tau)\,\mathrm{d}\tau\right|\\ &\to 0, \quad \text{as } (t,n)\to(\infty,\infty).\end{aligned}$$

Therefore, (1.43) holds for K_1.

The case of K_2 is rather similar to K_1 by considering $-\xi_M(0)$ instead of $\xi_M(0)$.

Now we consider the case $K = \{0\}$. Since each cocycle Φ_n is strict, the cocycle attractor $\mathscr{A}_n$ is invariant, that is, $\Phi_n(t, \theta_{-t}\sigma, A_n(\theta_{-t}\sigma)) = A_n(\sigma)$ for every $t \geqslant 0$ and $\sigma \in \Sigma$. Therefore, since $\xi(t) \equiv 0$ is a bounded complete trajectory of Φ_n and thereby belongs to $A_n(\theta_t\sigma)$ for each $t \in \mathbb{R}$, we have

$$\begin{aligned}\operatorname{dist}(\Phi_n(t, \theta_{-t}\sigma, 0), A_n(\sigma)) &\leqslant \operatorname{dist}(\Phi_n(t, \theta_{-t}\sigma, A_n(\theta_{-t}\sigma)), A_n(\sigma)) \\ &= 0, \quad \forall t \geqslant 0,\ n \in \mathbb{N}.\end{aligned}$$

This implies that (1.43) holds for $\{0\}$. The proof is complete. □

Theorem 1.4.15. *The family $\{A_n\}_{n\in\mathbb{N}}$ of cocycle attractors is lower semi-continuous with respect to the global attractor A_∞, that is,*

$$\operatorname{dist}(A_\infty, A_n(\sigma)) \xrightarrow{n\to\infty} 0, \quad \forall \sigma \in \Sigma.$$

Proof. By Theorem 1.3.13 and Lemma 1.4.14, it suffices to prove (1.31), that is, for each $t \geqslant 0$ and $\sigma \in \Sigma$,

$$\operatorname{dist}\big(\Phi_\infty(t, x), \Phi_n(t, \sigma, x)\big) \xrightarrow{n\to\infty} 0, \quad \forall x \in \mathbb{R}. \tag{1.44}$$

Since for all $x \neq 0$ the solutions of (1.41) is single-valued, (1.44) is clear by Lemma 1.4.8 as $\sigma_n(t) \to b$ for each $t \geqslant 0$. Also, for the case $x = 0$, it is clear that

$$\operatorname{dist}\big(\Phi_\infty(t, 0), \Phi_n(t, \sigma, 0)\big) \leqslant \sup_{0\leqslant r<t} \left| \int_r^t e^{-\lambda(t-\tau)}(\sigma_n(\tau) - b)\, \mathrm{d}\tau \right| \xrightarrow{n\to\infty} 0.$$

Therefore, we have (1.44) and then the theorem follows. □

Remark 1.4.16. The map Φ_n in this example is multi-valued only at the point $x = 0$. In this case, condition (A1') in Theorem 1.3.8 fails, while condition (1.31) in Theorem 1.3.13 is quite easy to check. Hence, the weak equi-attraction method has been shown more appropriate in this application.

Part II

Random cocycle attractors

Chapter 2

Cocycle attractors for quasi strong-to-weak continuous random dynamical systems

In this chapter, we mainly obtain existence criteria for cocycle attractors for RDS with only quasi strong-to-weak continuity. Results in this chapter not only generalize known existence theorems for cocycle attractors, but also solve the measurability problem for bi-spatial cocycle attractors which seems still open in the literature.

For converging sequences " $\to$ " denotes the usual convergence while " $\rightharpoonup$ " the weak convergence.

2.1 Quasi strong-to-weak continuity

In this section, we introduce the quasi strong-to-weak continuity, and prove two important properties for quasi strong-to-weak continuous mappings: the inheritability and the measurability.

Recall that a mapping $G : \eta \mapsto G(\eta)$ from a complete metric space $\mathcal{M}$ to a Banach space X is said to be

- *strongly continuous*, or simply called *continuous*, if $G(\eta_n) \to G(\eta)$ whenever $\eta_n \to \eta$;
- *strong-to-weak continuous* (from $\mathcal{M}$ to X), if $G(\eta_n) \rightharpoonup G(\eta)$ whenever $\eta_n \to \eta$.

Definition 2.1.1. A mapping $G : \eta \mapsto G(\eta)$ from a complete metric space $\mathcal{M}$ to a Banach space X is said to be *quasi strong-to-weak (abbrev. quasi-S2W) continuous* from $\mathcal{M}$ to X at η, if $G(\eta_n) \rightharpoonup G(\eta)$ whenever $\{G(\eta_n)\}_{n\in\mathbb{N}}$ is bounded in X and $\eta_n \to \eta$. G is called quasi-S2W continuous from $\mathcal{M}$ to X if it is quasi-S2W continuous at every point in $\mathcal{M}$. Particularly when $\mathcal{M} = X$, the mapping G is called quasi-S2W continuous in X if G is quasi-S2W continuous from X to X.

Clearly, quasi-S2W continuity is weaker than strong and strong-to-weak continuities:

$$\text{strong continuity} \;\Rightarrow\; \text{strong-to-weak continuity} \;\Rightarrow\; \text{quasi-S2W continuity}.$$

Note that when $\mathcal{M} = X$, the strong-to-weak continuity reduces to the norm-to-weak continuity introduced in [94, 66], which is also termed as weak continuity in [85], while the quasi-S2W continuity reduces to the quasi-continuity introduced in [64], see also [45]. We shall see later that the quasi-S2W continuity defined here is not only a generalization of the quasi-continuity introduced in [64] but also more suitable and powerful especially for proving the generation of RDS and the measurability of cocycle attractors.

Recall that a mapping G from $\mathcal{M}$ to X is said to be *with closed graph* or simply *closed* if $\eta_n \to \eta$ and $G(\eta_n) \to z$ imply $z = G(\eta)$. The following proposition tells that the quasi-S2W continuity "is very close to" the closedness property, and if X is weakly compact, then they are equivalent.

Proposition 2.1.2. *A mapping from $\mathcal{M}$ to X is closed if and only if*

(i) $G(\eta_n) \rightharpoonup G(\eta)$ whenever $\{G(\eta_n)\}_{n\in\mathbb{N}}$ is precompact in X and $\eta_n \to \eta$;

or, equivalently,

(ii) $G(\eta_n) \to G(\eta)$ whenever $\{G(\eta_n)\}_{n\in\mathbb{N}}$ is precompact in X and $\eta_n \to \eta$.

Proof. We first prove by contradiction that the closedness implies (ii). Suppose (ii) does not hold, then there exists an $\varepsilon > 0$ and a subsequence $\{\eta_{n_k}\}$ such that

$$\|G(\eta_{n_k}) - G(\eta)\|_X \geqslant \varepsilon, \quad \forall k \in \mathbb{N}. \tag{2.1}$$

But $\{G(\eta_{n_k})\}$ is precompact, hence there exists a subsequence, denoted by itself after relabling, such that $G(\eta_{n_k}) \to z$ for some $z \in X$. By the closedness of G we have $z = G(\eta)$, contradicting (2.1). Hence, the closedness implies (ii).

In the same way we conclude that (i) implies (ii). Indeed, by (i) we have $G(\eta_{n_k}) \rightharpoonup G(\eta)$, which along with $G(\eta_{n_k}) \to z$ implies $G(\eta_{n_k}) \to z = G(\eta)$ since the weak and the strong limits are identical; this contradicts (2.1) as well. As (ii) is clearly stronger than both (i) and the closedness, we have the proposition. □

2.1.1 Inheritability of quasi-S2W continuity

Now we prove that quasi-S2W continuity is inheritable: roughly speaking, if a mapping is quasi-S2W continuous in some basic space then so it is in more regular subspaces.

Proposition 2.1.3. *Let $\mathcal{M}$ be a complete metric space, and let $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be two Banach spaces with dual spaces Y^*, Z^*, respectively, such that*

$$Z \hookrightarrow Y \quad \text{and} \quad Y^* \hookrightarrow Z^*,$$

where the injection $i : Z \to Y$ is continuous and its adjoint $i^* : Y^* \to Z^*$ is a dense injective. Suppose $G : \mathcal{M} \to Y$ is a mapping. Then if G is quasi-S2W continuous from $\mathcal{M}$ to Y, so it is from $\mathcal{M}$ to Z at points where G takes values in Z.

Proof. Let $\{\eta_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ converging to some η such that $G(\eta) \in Z$, and $\{G(\eta_n)\}_{n\in\mathbb{N}}$ be bounded in Z. We need to prove that $G(\eta_n) \rightharpoonup G(\eta)$ in Z, i.e. for any $z^* \in Z^*$,

$$\big\langle z^*, G(\eta_n) - G(\eta)\big\rangle_{Z^*} \to 0. \tag{2.2}$$

By the boundedness of $\{G(\eta_n)\}_{n\in\mathbb{N}}$ in Z, let M be a constant such that

$$\|G(\eta_n) - G(\eta)\|_Z \leqslant M \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

Since $i^* : Y^* \to Z^*$ is dense, for any $\varepsilon > 0$ and $z^* \in Z^*$ there exists a $y_\varepsilon^* \in Y^*$ such that

$$\|i^*(y_\varepsilon^*) - z^*\|_{Z^*} < \frac{\varepsilon}{2M}. \tag{2.4}$$

On the other hand, since the injection $i : Z \to Y$ is continuous, $\{G(\eta_n)\}_{n\in\mathbb{N}}$ is bounded in Y. Hence, thanks to the quasi-S2W continuity of G from $\mathcal{M}$ to Y, there exists an $N_0 \in \mathbb{N}$ such that

$$\big|\big\langle y_\varepsilon^*, i(G(\eta_n) - G(\eta))\big\rangle_{Y^*}\big| < \frac{\varepsilon}{2}, \quad \forall n \geqslant N_0. \tag{2.5}$$

Therefore, from (2.3)-(2.5) it follows that

$$\begin{aligned}
\big|\big\langle z^*, G(\eta_n) - G(\eta)\big\rangle_{Z^*}\big| &\leqslant \big|\big\langle i^*(y_\varepsilon^*) - z^*, G(\eta_n) - G(\eta)\big\rangle_{Z^*}\big| \\
&\quad + \big|\big\langle i^*(y_\varepsilon^*), G(\eta_n) - G(\eta)\big\rangle_{Z^*}\big| \\
&\leqslant \|i^*(y_\varepsilon^*) - z^*\|_{Z^*}\|G(\eta_n) - G(\eta)\|_Z \\
&\quad + \big|\big\langle y_\varepsilon^*, i(G(\eta_n) - G(\eta))\big\rangle_{Y^*}\big| \\
&< \varepsilon, \quad \forall n \geqslant N_0,
\end{aligned}$$

which implies (2.2) and the proof is complete. □

Taking $\mathcal{M} = Y$ in Proposition 2.1.3 we have the following corollaries which themselves are interesting and useful.

Proposition 2.1.4. *Let Y, Z be two Banach spaces as in Proposition 2.1.3. Suppose $G : Y \to Y$ is a mapping. Then if G is quasi-S2W continuous in Y, so it is from Y to Z at points where G takes values in Z.*

Proposition 2.1.5. *Let Y, Z be two Banach spaces as in Proposition 2.1.3. Suppose $G : Y \to Y$ is a mapping and maps Z into Z. Then if G is quasi-S2W continuous from Y to Y, so it is from Z to Z.*

In applications, the spaces Y and Z satisfying conditions in Propositions 2.1.3-2.1.5 refer to basic and regularity spaces, respectively. Examples are $Y = L^2(\mathcal{O})$ while $Z = L^p(\mathcal{O})$ or $H_0^1(\mathcal{O})$ or $H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$, etc, where $p \geqslant 2$ and $\mathcal{O}$ denotes some bounded domain in $\mathbb{R}^N$. Many evolution equations modeling important physical phenomena are known continuous in $L^2(\mathcal{O})$ but not clearly known whether or not they are continuous in more regular spaces, unless at least more restrictive conditions are assumed, see [75]. Propositions 2.1.3-2.1.5 indicate that the quasi S2W-continuity is expectable for such cases, and, what is more interesting, it will be shown sufficient to study cocycle attractors.

It is interesting to note the case $Y = L^2(\mathcal{O})$ with $Z = L^p(\mathcal{O})$, $p \geqslant 2$, for unbounded domains. When the domain $\mathcal{O}$ is unbounded, Propositions 2.1.3-2.1.5 are still applicable though the relation $L^p(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$ fails, only to notice that for each $r, s \geqslant 1$ the space $L^r(\mathcal{O}) \cap L^s(\mathcal{O})$ is a Banach space continuously and densely embedded into $L^r(\mathcal{O})$, see Lemma A.1.6. So by taking $Z = L^2(\mathcal{O}) \cap L^p(\mathcal{O})$ one is able to apply Propositions 2.1.3-2.1.5 to study the cocycle attractor in Z. This observation is useful especially in the study of bi-spatial cocycle attractors, which will be presented in Section 2.2.3.

2.1.2 Measurability of quasi-S2W continuous mappings

It is well known that the strong continuity of a mapping implies the measurability of the mapping with respect to Borel sigma-algebra. The following proposition, inspired by [85, Theorem 2.3], indicates that the quasi-S2W continuity is already sufficient to ensure such measurability.

We always denote by $\mathcal{B}(\cdot)$ the Borel sigma-algebra of a metric space.

Theorem 2.1.6. *Suppose that $\mathcal{M}$ is a complete metric space and $(X, \|\cdot\|_X)$ is a separable Banach space. If a mapping $G : \mathcal{M} \mapsto X$ is quasi-S2W continuous, then G is $(\mathcal{B}(M), \mathcal{B}(X))$-measurable.*

Proof. We first prove that the inverse image of every closed ball in X under G is a closed set in $\mathcal{M}$. Consider the closed r-neighborhood $\bar{B}$ of any an $x \in X$. Suppose $\eta_n \in G^{-1}(\bar{B})$ is a sequence converging to some $\eta \in \mathcal{M}$. Since $\eta_n \to \eta$ and $G(\eta_n) \in \bar{B}$ is bounded, by the quasi-S2W continuity of G we know that $G(\eta_n)$ converge weakly to $G(\eta)$ in X, which follows that

$$\|G(\eta) - x\|_X \leqslant \liminf_{n\to\infty} \|G(\eta_n) - x\|_X \leqslant r.$$

Hence, $G(\eta)$ is in the r-neighborhood $\bar{B}$ as well, and therefore $\eta \in G^{-1}(\bar{B})$ as desired.

Now let B be any open ball in X with center x_0 and radius r_0. Then $B = \cup_{m\in\mathbb{N}} \bar{B}_m$ with $\bar{B}_m$ the closed $(r_0 - 1/m)$-neighborhood of x_0, and $G^{-1}(B) = \cup_{m=1} G^{-1}(\bar{B}_m)$. Since in the first paragraph we have proved that each $G^{-1}(\bar{B}_m)$ is closed in $\mathcal{M}$ and thereby $G^{-1}(\bar{B}_m) \in \mathcal{B}(\mathcal{M})$, we have $G^{-1}(B) \in \mathcal{B}(\mathcal{M})$.

For any open set O in X, since X is separable and hence has a countable dense subset $\{x_m\}_{m\in\mathbb{N}}$, there exist a subset $\{x_{m_k}\}_{k\in\mathbb{N}}$ and a sequence of rational numbers $\{r_{m_k}\}_{k\in\mathbb{N}}$ such that $O = \cup_{k\in\mathbb{N}} B_k$ where B_k is the open ball with center y_{m_k} and radius r_{m_k}. As we have proved that each $G^{-1}(B_k)$ belongs to $\mathcal{B}(\mathcal{M})$, $G^{-1}(O) = \cup_{k\in\mathbb{N}} G^{-1}(B_k) \in \mathcal{B}(\mathcal{M})$. The proof is complete. □

2.2 Existence criteria

In this section, we study the measurability and the existence of cocycle attractors for quasi-S2W continuous RDS, and then generalize the analysis to bi-spatial cocycle attractors for which the measurability is considered in more regular spaces.

2.2.1 Preliminaries: random dynamical systems and cocycle attractors

Throughout this thesis, we always denote by $(\Omega, \mathcal{F}, \mathcal{P})$ a probability space, which is unnecessarily $\mathcal{P}$-complete, endowed with a flow $\{\vartheta_t\}_{t\in\mathbb{R}}$ satisfying

- ϑ_0 = identity operator on Ω;
- $\vartheta_t\Omega = \Omega, \quad \forall t \in \mathbb{R}$;
- $\vartheta_s \circ \vartheta_t = \vartheta_{t+s}, \quad \forall t, s \in \mathbb{R}$;
- $(t, \omega) \mapsto \vartheta_t\omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable;
- $\mathcal{P}$-preserving: $\mathcal{P}(\vartheta_t F) = \mathcal{P}(F), \quad \forall t \leqslant 0, F \in \mathcal{F}$.

For the ease of notations, we often use ϑ, instead of ϑ_t, when describing universal properties valid for every $t \in \mathbb{R}$.

It is convenient to work on a full measure subspace $\tilde{\Omega}$, instead of Ω. In the following, we shall not distinguish $\tilde{\Omega}$ from Ω, that is, by saying that a statement holds for all $\omega \in \Omega$ we mean that it holds on $\tilde{\Omega}$.

Definition 2.2.1. Let X be a complete metric space. The mapping $\phi : \mathbb{R}^+ \times \Omega \times X \mapsto X$ is called a random dynamical system (RDS for short) on X with base flow $\{\vartheta_t\}_{t\in\mathbb{R}}$ on Ω, if

(1) ϕ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(2) $\phi(0, \omega, \cdot)$ is the identity on X;

(3) it holds the cocycle property

$$\phi(t + s, \omega, x) = \phi\big(t, \theta_s\omega, \phi(s, \omega, x)\big).$$

When some continuity of NRDS (and RDS) is mentioned, the continuity is referred to x in the phase space X, i.e., of the mapping $x \mapsto \phi(t, \omega, \sigma, x)$, except otherwise clearly stated.

Next we define the so-called random set, which plays a central role in the study of RDS.

Definition 2.2.2. For any complete metric space X, a set-valued mapping D: $\Omega \mapsto 2^X \setminus \emptyset$, $\omega \mapsto D(\omega)$ is said to be an *(autonomous) random set (in X)* if it is *measurable*, namely, the mapping $\omega \to \mathrm{dist}_X(x, D(\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for each $x \in X$. If each image $D(\omega)$ is closed (resp. bounded or compact) in X, then D is called a closed (resp. bounded or compact) random set in X.

Note that the measurability in Definition 2.2.2 is stronger than to require the mapping $\omega \mapsto D(\omega)$ to be measurable in the sense that

$$\mathrm{graph}(D) = \{(\omega, x) \in \Omega \times X : x \in D(\omega)\}$$

being $\mathcal{F} \times \mathcal{B}(X)$-measurable. In fact, the two arguments are equivalent if $\mathcal{F}$ is $\mathcal{P}$-complete, see [28, Chapter 1.3]. But in order to characterize attracting properties of attractors, generally a set-valued mapping D is said to be measurable if and only if it satisfies Definition 2.2.2. However, it is worth pointing out that if $D(\cdot)$ is a single-valued mapping, then the two measurabilities of D are equivalent, since the distance mapping $\text{dist}(x, \cdot)$ is $(X, \mathbb{R})$-continuous.

Given two random sets D_1, D_2, write $D_1 \subset D_2$ if $D_1(\omega) \subset D_2(\omega)$ for all $\omega \in \Omega$, and we then say D_1 is smaller than D_2.

Throughout this thesis, for any random set D in some metric space X, we denote by $\mathcal{N}_\varepsilon(D(\cdot))$ the random set in X identified by the closed ε-neighborhood of D, i.e.,

$$\mathcal{N}_\varepsilon(D(\omega)) = \big\{x \in X : \text{dist}_X(x, D(\omega)) \leqslant \varepsilon\big\}, \quad \forall \omega \in \Omega.$$

Recall that dist_X (or simply dist) denotes the Hausdorff semi-distance between sets in X.

Lemma 2.2.3. *The closed ε-neighborhood $\mathcal{N}_\varepsilon(D)$ of a random set D is a closed random set in X.*

Proof. It suffices to observe that for any $a > 0$ and $x \in X$ we have

$$\{\omega \in \Omega : \text{dist}_X(x, \mathcal{N}_\varepsilon(D(\omega))) \leqslant a\} = \{\omega \in \Omega : \text{dist}_X(x, D(\omega)) \leqslant a + \varepsilon\}$$

belonging to $\mathcal{F}$ as D is measurable. □

We need the next lemma on measurability, see [22, Chapter III], [48, Chapter 2.2] and [10, Lemma 2.2].

Lemma 2.2.4. *(I) Let $\{D_n\}_{n\in\mathbb{N}}$ be a family of closed random sets in a Polish space X. Then*

$$\omega \to \overline{\bigcup_{n\in\mathbb{N}} D_n(\omega)}$$

is a closed random set in X. If in addition $\{D_n\}_{n\in\mathbb{N}}$ is decreasing and every sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in D_n(\omega)$ is precompact, then

$$\bigcap_{n\in\mathbb{N}} D_n(\omega)$$

is non-empty and measurable.

(II) For any closed random set D in X there exist a set of countable random variables f_n, $n \in \mathbb{N}$, such that $f_n(\omega) \in D(\omega)$ for all $\omega \in \Omega$ and

$$D(\omega) = \overline{\bigcup_{n\in\mathbb{N}} f_n(\omega)}.$$

Denote by $\mathcal{D}_X$ a collection of some random sets in Polish space X which is

- neighborhood-closed, i.e. for each $D \in \mathcal{D}_X$ there exits an $\varepsilon > 0$ such that the closed ε-neighborhood $\mathcal{N}_\varepsilon(D)$ belongs to $\mathcal{D}_X$, and

- inclusion-closed, i.e., if $D \in \mathcal{D}_X$ then each random set smaller than D belongs to $\mathcal{D}_X$.

An example of the universe $\mathcal{D}_X$ is the collection of all the bounded random sets in X.

Definition 2.2.5. Given two random sets D and B, it is said that D pullback attracts B (in X under RDS ϕ) if for each $\omega \in \Omega$ it holds that

$$\begin{gathered} \text{for any } \varepsilon > 0 \text{ there exists a } T > 0 \text{ such that} \\ \operatorname{dist}_X(\phi(s, \theta_{-s}\omega, B(\theta_{-s}\omega)), D(\omega)) < \varepsilon \quad \text{for all } s \geqslant T. \end{gathered} \tag{2.6}$$

If D pullback attracts every element in $\mathcal{D}_X$ under the topology of X, i.e., (2.6) holds for each $B \in \mathcal{D}_X$, then D is said $\mathcal{D}_X$-pullback attracting (in X).

Definition 2.2.6. [83] A random set $\mathscr{A} = \{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ is called a *(random) cocycle attractor with attraction universe* $\mathcal{D}_X$ for an RDS ϕ if

(i) $\mathscr{A}$ belongs to $\mathcal{D}_X$ and is compact;

(ii) $\mathscr{A}$ is $\mathcal{D}_X$-pullback attracting under ϕ;

(iii) $\mathscr{A}$ is invariant, i.e.,

$$\phi(s, \omega, \mathscr{A}(\omega)) = \mathscr{A}(\vartheta_s\omega), \quad \forall s \in \mathbb{R}^+, \omega \in \Omega.$$

A cocycle attractor with attraction universe $\mathcal{D}_X$ is often called a $\mathcal{D}_X$-cocycle attractor or simply a cocycle attractor without mentioning the universe $\mathcal{D}_X$ when no confusion occurs. Note that the property $\mathscr{A} \in \mathcal{D}_X$ ensures the minimality and uniqueness of the attractor among compact random sets satisfying (ii) and (iii).

2.2.2 Measurability and existence criteria

Let X be a Polish Banach space and ϕ an RDS on X. Now standing on the quasi-S2W continuity of an RDS we prove the measurability and some existence criteria for the cocycle attractor. Let us review for strongly continuous RDS, that [34, 33] established some existence criteria for cocycle attractors, while [83, 84] investigated the non-autonomous cases. In [64], the authors studied cocycle attractors for quasi-continuous RDS but with the measurability problem untouched.

Omega-limit sets play a central role in the study of attractors. For any random set B in X, the omega-limit set $\mathcal{W}(\cdot, B)$ of B in X is defined as

$$\mathcal{W}(\omega, B) = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} \phi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega))}, \quad \forall \omega \in \Omega,$$

where the closure is taken under the norm-topology of the phase space X. Note that it is generally unclear whether or not the omega-limit set of a random set is a random set. But Lemma 2.2.8 will show that $\mathcal{W}(\cdot, B)$ would be a random set if B pullback attracts itself.

The following statement is well known.

Lemma 2.2.7. *For each $\omega \in \Omega$ and $D \in \mathcal{D}_X$, $y \in \mathcal{W}(\omega, D)$ if and only if there exit sequences $t_n \to \infty$ and $x_n \in \phi(t_n, \vartheta_{-t_n}\omega, D(\vartheta_{-t_n}\omega))$ such that $x_n \to y$ in X.*

The following result for omega-limit sets of random sets is crucial.

Lemma 2.2.8. *Suppose that ϕ is a quasi-S2W continuous RDS and B a random set in X. If there exists a compact random set K in X pullback attracting B under ϕ, then*

(I) for each $\omega \in \Omega$, $\mathcal{W}(\omega, B)$ is nonempty, compact, and has the invariant property

$$\phi(t, \omega, \mathcal{W}(\omega, B)) = \mathcal{W}(\vartheta_t\omega, B), \quad t \in \mathbb{R}^+; \tag{2.7}$$

(II) $\mathcal{W}(\cdot, B)$ pullback attracts B, i.e. satisfying (2.6), and is included in any closed non-autonomous random set D which pullback attracts B

$$\mathcal{W}(\omega, B)) \subset D(\omega), \quad \forall \omega \in \Omega; \tag{2.8}$$

(III) if moreover B pullback attracts itself in X, then $\mathcal{W}(\cdot, B)$ is a random set.

Proof. (I) Let $t_n \to \infty$ and $x_n \in B(\vartheta_{-t_n}\omega)$. Since K is compact and pullback attracts B, there exists a $y \in K(\omega)$ such that, up to a subsequence,

$$\phi(t_n, \vartheta_{-t_n}\omega, x_n) \to y$$

which indicates that $y \in \mathcal{W}(\omega, B)$ by Lemma 2.2.7. Hence, $\mathcal{W}(\omega, B)$ is nonempty.

To see the compactness, take arbitrarily a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{W}(\omega, B)$. Then by Lemma 2.2.7 we have sequences $x_n \in B(\vartheta_{-t_n}\omega)$ and $t_n \to \infty$ such that

$$\mathrm{dist}_X(\phi(t_n, \vartheta_{-t_n}\omega, x_n), y_n) \leqslant 1/n, \quad \forall n \in \mathbb{N}.$$

On the other hand, as K pullback attracts B and by Lemma 2.2.7 again, there exists a $y \in \mathcal{W}(\omega, B)$ such that

$$\phi(t_n, \vartheta_{-t_n}\omega, x_n) \to y$$

in a subsequence sense. Hence, $y_n \to y$ and $\mathcal{W}(\omega, B)$ is compact.

To prove the invariance, we claim that

$$\begin{aligned}\mathcal{W}(\omega, B) &= \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega))}^{S} \\ &= \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega))}^{W}, \quad \forall \omega \in \Omega,\end{aligned} \tag{2.9}$$

where and hereafter the indicator "S" (resp. "W") nearby the over-line indicates the strong (resp. weak) topology under which the closure is taken. Indeed, for any y lying in the right-hand side term, there exist sequences $t_n \to \infty$ and $x_n \in B(\vartheta_{-t_n}\omega))$ such that $\phi(t_n, \vartheta_{-t_n}\omega, x_n) \rightharpoonup y$. On the other hand, since B is pullback attracted by K and K is compact, $\phi(t_n, \vartheta_{-t_n}\omega, x_n)$ converges to some $z \in X$ strongly in a subsequence sense, which implies that

$$y = z \in \cap_{s\geqslant 0}\overline{\cup_{t\geqslant s}\phi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega))}^S.$$

Hence,

$$\bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega))}^W \subset \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega))}^S,$$

and then (2.9) follows since the inverse inclusion is trivial.

Now we prove the invariance. As K pullback attracts B, for any $t \geqslant 0$ and $\omega \in \Omega$, there exists a time $T > 0$ such that

$$\begin{aligned}\phi(t+\eta, \vartheta_{-\eta}\omega, B(\vartheta_{-\eta}\omega)) &= \phi(t+\eta, \vartheta_{-t-\eta}\vartheta_t\omega, B(\vartheta_{-t-\eta}\vartheta_t\omega)) \\ &\subset \mathcal{N}_1(K(\vartheta_t\omega)), \quad \forall \eta \geqslant T,\end{aligned} \tag{2.10}$$

where $\mathcal{N}_1(\cdot)$ denotes the closed 1-neighborhood. As it holds $f(\cap_\alpha A_\alpha) \subset \cap_\alpha f(A_\alpha)$ for an arbitrary function f, we observe that

$$\begin{aligned}\phi(t, \omega, \mathcal{W}(\omega, B)) &= \phi\big(t, \omega, \cap_{s\geqslant T}\overline{\cup_{\eta\geqslant s}\phi(\eta, \vartheta_{-\eta}\omega, B(\vartheta_{-\eta}\omega))}^S\big) \\ &\subset \bigcap_{s\geqslant T}\phi\big(t, \omega, \overline{\cup_{\eta\geqslant s}\phi(\eta, \vartheta_{-\eta}\omega, B(\vartheta_{-\eta}\omega))}^S\big),\end{aligned} \tag{2.11}$$

where $T > 0$ is given in (2.10). Take arbitrarily $y \in \phi\big(t, \omega, \overline{\cup_{\eta\geqslant s}\phi(\eta, \vartheta_{-\eta}\omega, B(\vartheta_{-\eta}\omega))}^S\big)$ for any an $s \geqslant T$. Then there exist an $x \in \overline{\cup_{\eta\geqslant s}\phi(\eta, \vartheta_{-\eta}\omega, B(\vartheta_{-\eta}\omega))}^S$ and a sequence

$$x_n \in \phi(\eta_n, \vartheta_{-\eta_n}\omega, B(\vartheta_{-\eta_n}\omega)) \quad \text{with} \ \ \eta_n \geqslant s,$$

such that $y = \phi(t, \omega, x)$ and $x_n \to x$. Note that, by (2.10),

$$\begin{aligned}\phi(t, \omega, x_n) &\in \phi\big(t, \omega, \phi(\eta_n, \vartheta_{-\eta_n}\omega, B(\vartheta_{-\eta_n}\omega))\big) \\ &= \phi\big(t+\eta_n, \vartheta_{-\eta_n}\omega, B(\vartheta_{-\eta_n}\omega)\big) \\ &\subset \mathcal{N}_1(K(\vartheta_t\omega)).\end{aligned} \tag{2.12}$$

Hence, the sequence $\phi(t, \omega, x_n)$ is bounded and converges weakly to $\phi(t, \omega, x) = y$ by the quasi-S2W continuity of ϕ, which follows that

$$y \in \overline{\bigcup_{\eta\geqslant s}\phi\big(t, \omega, \phi(\eta, \vartheta_{-\eta}\omega, B(\vartheta_{-\eta}\omega))\big)}^W.$$

Since y was taken arbitrarily, we have

$$\phi\big(t, \omega, \overline{\cup_{\eta\geqslant s}\phi(\eta, \vartheta_{-\eta}\omega, B(\vartheta_{-\eta}\omega))}^S\big) \subset \overline{\bigcup_{\eta\geqslant s}\phi\big(t, \omega, \phi(\eta, \vartheta_{-\eta}\omega, B(\vartheta_{-\eta}\omega))\big)}^W,$$

which along with (2.11) and (2.9) implies that

$$\begin{aligned}\phi(t,\omega,\mathcal{W}(\omega,B)) &\subset \bigcap_{s\geqslant T}\phi\big(t,\omega,\overline{\cup_{\eta\geqslant s}\phi(\eta,\vartheta_{-\eta}\omega,B(\vartheta_{-\eta}\omega))}^{S}\big)\\ &\subset \bigcap_{s\geqslant T}\overline{\bigcup_{\eta\geqslant s}\phi\big(t,\omega,\phi(\eta,\vartheta_{-\eta}\omega,B(\vartheta_{-\eta}\omega))\big)}^{W}\\ &= \bigcap_{s\geqslant T}\overline{\bigcup_{\eta\geqslant s}\phi(t+\eta,\vartheta_{-\eta}\omega,B(\vartheta_{-\eta}\omega))}^{W}\\ &= \bigcap_{s\geqslant t+T}\overline{\bigcup_{\eta\geqslant s}\phi(\eta,\vartheta_{-\eta}\circ\vartheta_t\omega,B(\vartheta_{-\eta}\circ\vartheta_t\omega))}^{W} = \mathcal{W}(\vartheta_t\omega,B).\end{aligned}$$

To see $\mathcal{W}(\vartheta_t\omega,B)\subset\phi(t,\omega,\mathcal{W}(\omega,B))$, let $y\in\mathcal{W}(\vartheta_t\omega,B)$. Then by Lemma 2.2.7 there exists a sequence

$$\begin{aligned}x_n &\in \phi(t_n,\vartheta_{-t_n}\vartheta_t\omega,B(\vartheta_{t-t_n}\omega))\\ &= \phi(t,\omega,\phi(t_n-t,\vartheta_{t-t_n}\omega,B(\vartheta_{t-t_n}\omega))\end{aligned}$$

with $t<t_n\to\infty$ such that $x_n\to y$. Suppose $z_n\in\phi(t_n-t,\vartheta_{t-t_n}\omega,B(\vartheta_{t-t_n}\omega))$ is such that $x_n=\phi(t,\omega,z_n)$. Since B is attracted by K, for any $\omega\in\Omega$, there exists a time $T_0>0$ such that

$$\phi(\eta,\vartheta_{-\eta}\omega,B(\vartheta_{-\eta}\omega))\subset\mathcal{N}_1(K(\omega)),\quad \forall\eta\geqslant T_0. \tag{2.13}$$

Hence, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Moreover, by the pullback attracting and compact properties of K, there exists a $z\in X$ such that $z_n\to z$ up to a subsequence, which implies that $z\in\mathcal{W}(\omega,B))$ by Lemma 2.2.7. Hence, by the quasi-S2W continuity of ϕ we have

$$x_n=\phi(t,\omega,z_n)\rightharpoonup\phi(t,\omega,z).$$

Thus, by the uniqueness of a limit, we have $y=\phi(t,\omega,z)\in\phi(t,\omega,\mathcal{W}(\omega,B))$. The invariance is clear.

(II) We prove that $\mathcal{W}(\cdot,B)$ pullback attracts B by contradiction. Suppose there exist an $\omega\in\Omega$, $\varepsilon_0>0$, sequences $t_n\to\infty$ and $x_n\in B(\vartheta_{-t_n}\omega)$ such that for all $n\in\mathbb{N}$,

$$\mathrm{dist}_X(\phi(t_n,\vartheta_{-t_n}\omega,x_n),\mathcal{W}(\omega,B))\geqslant\varepsilon_0. \tag{2.14}$$

Then as the compact random set K pullback attracts B, there is a $y\in K(\omega)$ such that $\phi(t_n,\vartheta_{-t_n}\omega,x_n)$ converges to y up to a subsequence. By Lemma 2.2.7 we have $y\in\mathcal{W}(\omega,B)$, contradicting (2.14). Hence, $\mathcal{W}(\cdot,B)$ indeed pullback attracts B.

Now we prove (2.8). Take arbitrarily a $y\in\mathcal{W}(\omega,B)$, then by Lemma 2.2.7 we have a sequence $x_n\in\phi(t_n,\vartheta_{-t_n}\omega,B(\vartheta_{-t_n}\omega))$ with $t_n\to\infty$ such that $x_n\to y$. By the pullback attracting property and the closedeness of D we know $y\in D(\omega)$. Hence, $\mathcal{W}(\omega,B)\subset D(\omega)$.

(III) Let us prove the measurability of the mapping $\omega \mapsto \mathcal{W}(\omega, B)$. First, let us prove that

$$\begin{aligned}\mathcal{W}(\omega, B) &= \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m=n}^{\infty} \phi(m, \vartheta_{-m}\omega, B(\vartheta_{-m}\omega))}^{S} \\ &= \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m=n}^{\infty} \phi(m, \vartheta_{-m}\omega, B(\vartheta_{-m}\omega))}^{W}, \quad \forall \omega \in \Omega.\end{aligned} \tag{2.15}$$

Since B pullback attracts itself, by (2.8) we have $\mathcal{W}(\omega, B) \subset B(\omega)$ for each $\omega \in \Omega$. Hence, by the invariance (2.7) of $\mathcal{W}(\omega, B)$, we have

$$\begin{aligned}\mathcal{W}(\omega, B) &= \phi(m, \vartheta_{-m}\omega, \mathcal{W}(\vartheta_{-m}\omega, B)) \\ &\subset \phi(m, \vartheta_{-m}\omega, B(\vartheta_{-m}\omega)), \quad \forall m \in \mathbb{N}.\end{aligned}$$

Therefore,

$$\mathcal{W}(\omega, B) \subset \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m=n}^{\infty} \phi(m, \vartheta_{-m}\omega, B(\vartheta_{-m}\omega))}^{S},$$

and thereby, since the inverse inclusion is straightforward, the first identity of (2.15) holds. Similarly to (2.9) we have the second identity of (2.15).

By (2.15) it is elementary to check that

$$\mathcal{W}(\omega, B) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m=n}^{\infty} \overline{\phi(m, \vartheta_{-m}\omega, B(\vartheta_{-m}\omega))}^{W}}^{W}. \tag{2.16}$$

Recall from (2.13) that for any $\omega \in \Omega$, there exists a time $T_0 > 0$ such that

$$\phi(\eta, \vartheta_{-\eta}\omega, B(\vartheta_{-\eta}\omega)) \subset \mathcal{N}_1(K(\omega)), \quad \forall \eta \geqslant T_0.$$

Then, since B is a non-empty closed random set, by Lemma 2.2.4 (II) there exists a sequence $\{f_j\}_{j\in\mathbb{N}}$ of measurable functions such that $B(\vartheta_{-m}\omega) = \overline{\cup_{j\in\mathbb{N}} f_j(\vartheta_{-m}\omega)}^{S}$, which makes

$$\overline{\phi(m, \vartheta_{-m}\omega, B(\vartheta_{-m}\omega))}^{W} = \overline{\bigcup_{j\in\mathbb{N}} \phi(m, \vartheta_{-m}\omega, f_j(\omega))}^{W}, \quad \forall m \geqslant T_0, \tag{2.17}$$

where we have used the quasi-S2W continuity of $x \mapsto \phi(m, \vartheta_{-m}\omega, x)$. Hence, by (2.16) and (2.17) we have

$$\begin{aligned}\mathcal{W}(\omega, B) &= \bigcap_{n\geqslant T_0} \overline{\bigcup_{m=n}^{\infty} \overline{\bigcup_{j\in\mathbb{N}} \phi(m, \vartheta_{-m}\omega, f_j(\omega))}^{W}}^{W} \\ &= \bigcap_{n\geqslant T_0} \overline{\bigcup_{m=n}^{\infty} \bigcup_{j\in\mathbb{N}} \phi(m, \vartheta_{-m}\omega, f_j(\omega))}^{W} \\ &= \bigcap_{n\geqslant T_0} \overline{\bigcup_{m=n}^{\infty} \bigcup_{j\in\mathbb{N}} \phi(m, \vartheta_{-m}\omega, f_j(\omega))}^{S},\end{aligned}$$

where the last identity is established similar to (2.15).

Since each $\phi(m, \vartheta_{-m}\omega, x)$ is $(\mathcal{F}, \mathcal{B}(X))$-measurable in ω and $(\mathcal{B}(X), \mathcal{B}(X))$ measurable in x, the mapping $\omega \mapsto \phi(m, \vartheta_{-m}\omega, f_j(\omega))$ is $(\mathcal{F}, \mathcal{B}(X))$-measurable, and thereby, as a single-valued mapping, it is measurable in the sense of Definition 2.2.2 as well. Denote by

$$D_n(\omega) = \overline{\bigcup_{m=n}^{\infty} \bigcup_{j\in\mathbb{N}} \phi(m, \vartheta_{-m}\omega, f_j(\omega))}^{S}, \quad \forall n \in \mathbb{N},\ \omega \in \Omega.$$

Then by Lemma 2.2.4 (I) we know each $D_n(\cdot)$ is measurable.

On the other hand, D_n is clearly decreasing and every sequence $\{x_n\}$ inside $\mathcal{W}(\tau, \omega, B)$ is precompact since $\mathcal{W}(\omega, B)$ is compact itself. By Lemma 2.2.4 (I) we conclude that $\mathcal{W}(\omega, B) = \overline{\bigcap_{n\in\mathbb{N}} D_n(\omega)}$ is measurable. The proof is complete. □

Now, thanks to Lemma 2.2.8, we are ready to establish an existence theorem for cocycle attractors of quasi-S2W continuous RDS via omega-limit sets.

Theorem 2.2.9. *Suppose ϕ is a quasi-S2W continuous RDS on X. If ϕ has a compact $\mathcal{D}_X$-pullback attracting set K and a $\mathcal{D}_X$-pullback absorbing set $B \in \mathcal{D}_X$, then ϕ has a $\mathcal{D}_X$-cocycle attractor $\mathscr{A} \in \mathcal{D}_X$ given by*

$$\mathscr{A}(\omega) = \mathcal{W}(\omega, B), \quad \forall \omega \in \Omega.$$

Proof. By Lemma 2.2.8 it is clear that $\mathscr{A}$ is a compact non-autonomous random set which is invariant and pullback attracts B. Moreover, $\mathscr{A}$ is smaller than B and hence belongs to $\mathcal{D}_X$ since $\mathcal{D}_X$ is inclusion-closed. We now prove the $\mathcal{D}_X$-attracting property. Since $\mathscr{A}$ pullback attracts B, for each $\varepsilon > 0$ and $\omega \in \Omega$ fixed, there is a time $T > 0$ such that

$$\operatorname{dist}\Big(\phi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega)), \mathscr{A}(\omega)\Big) < \varepsilon, \quad \forall t \geqslant T.$$

On the other hand, for each $D \in \mathcal{D}_X$ and $\omega \in \Omega$, there is a time $T_D(\omega) > 0$ such that

$$\bigcup_{t\geqslant T_D(\omega)} \phi(t, \vartheta_{-t}\omega, D(\vartheta_{-t}\omega)) \subset B(\omega)$$

as B is a $\mathcal{D}_X$-pullback absorbing set. Hence,

$$\begin{aligned}
&\operatorname{dist}\Big(\phi(t+T, \vartheta_{-t-T}\omega, D(\vartheta_{-t-T}\omega)), \mathscr{A}(\omega)\Big)\\
&\quad = \operatorname{dist}\Big(\phi(T, \vartheta_{-T}\omega, \phi(t, \vartheta_{-t}\vartheta_{-T}\omega, D(\vartheta_{-t}\vartheta_{-T}\omega)), \mathscr{A}(\omega)\Big)\\
&\quad \leqslant \operatorname{dist}\Big(\phi(T, \vartheta_{-T}\omega, B(\vartheta_{-T}\omega)), \mathscr{A}(\omega)\Big)\\
&\quad < \varepsilon, \quad \forall t \geqslant T_D(\vartheta_{-T}\omega),
\end{aligned}$$

which indicates that $\mathscr{A}$ pullback attracts D. The proof is complete. □

Remark 2.2.10. Theorem 2.2.9 requires a compact $\mathcal{D}_X$-pullback attracting set K (which unnecessarily belongs to $\mathcal{D}_X$). This condition can be replaced by some dynamical compactnesses, such as asymptotic compactness, flattening and squeezing properties, etc., [68, 53, 38, 30], each of which ensures the omega-limit set of a $\mathcal{D}_X$-absorbing set $B \in \mathcal{D}_X$ to be a compact $\mathcal{D}_X$-pullback attracting set. This observation will be presented in a more general non-autonomous framework, see Theorem 3.2.7.

2.2.3 The bi-spatial case

In this section we go into bi-spatial cocycle attractor theory. Bi-spatial attractors are known not only belonging to a more regular space, but also pullback attracting in the topology of that space, and hence recently have drawn much attention, see, e.g. [62–64, 87, 93] and references therein. Now, we establish some theorems ensuring the measurability of a bi-spatial cocycle attractor without proving the continuity of the system in that more regular space, which seems new in the literature.

In order to study bi-spatial cocycle attractors for RDS, let Y be another separable Banach space such that $Y \hookrightarrow X$ continuously and $X^* \hookrightarrow Y^*$ densely, where X^* and Y^* are dual spaces of X and Y, respectively. Denote by $\mathcal{D}_Y$ some inclusion- and neighborhood-closed universe of random sets in Y such that $\mathcal{D}_Y \subset \mathcal{D}_X$.

Definition 2.2.11. An RDS ϕ on X is said to be (X, Y)-dissipative (on the universe $\mathcal{D}_X$) if

- ϕ is an RDS when restricted on Y, i.e. satisfying Definition 2.2.1 with X replaced by Y;
- there exists a random set $\mathbf{B}$ in Y which belongs to $\mathcal{D}_X$ and is $\mathcal{D}_X$-pullback absorbing.

Note that $\mathbf{B}$ could belong to $\mathcal{D}_Y$ $(\subset \mathcal{D}_X)$, in which case some analysis would be more intuitive and straightforward.

Definition 2.2.12. Given an (X, Y)-dissipative RDS ϕ, a random set $\mathscr{A}$ in Y is called the (X, Y)-*cocycle attractor with attraction universe* $\mathcal{D}_X$ for ϕ if

(I) $\mathscr{A} \in \mathcal{D}_X$ and $\mathscr{A}$ is a compact random set in Y;

(II) $\mathscr{A}$ pullback attracts elements in $\mathcal{D}_X$ in the topology of Y, i.e., for each $D \in \mathcal{D}_X$ and any $\varepsilon > 0$ there exists a $T > 0$ such that $\operatorname{dist}_Y(\phi(s, \theta_{-s}\omega, D(\theta_{-s}\omega)), \mathscr{A}(\omega)) < \varepsilon$ holds for all $s \geqslant T$;

(III) $\mathscr{A}$ is invariant under ϕ.

Remark 2.2.13. (1) Notice that the (X, Y)-cocycle attractor here is required to be *measurable* in Y, not only measurable in X as described in [62, 63, 40, 93], etc. As the pullback attraction of an (X, Y)-attractor is expected under the distance of Y, the mapping $\omega \mapsto \operatorname{dist}_Y(x, \mathscr{A}(\omega))$ should be a random variable so that the attraction in Y makes sense.

(2) Clearly, an (X, Y)-cocycle attractor, if exists, must be the cocycle attractor of ϕ in X (cf. Definition 2.2.6), showing higher regularity and stronger pullback attracting ability (in the topology of Y).

(3) If an (X, Y)-cocycle attractor belongs to $\mathcal{D}_Y$, then it must be the $\mathcal{D}_Y$-cocycle attractor in Y (cf. Definition 2.2.6), showing a broader attraction universe ($\mathcal{D}_X$, not only $\mathcal{D}_Y$).

The following existence theorem for cocycle attractors requires only the continuity of the RDS in less regular spaces. We write the omega-limit set in X as $\mathcal{W}(\cdot, B)^X$ to indicate the X-topology.

Theorem 2.2.14. *Suppose that ϕ is a quasi-S2W continuous RDS on X. If ϕ is an RDS on Y, and has a $\mathcal{D}_Y$-pullback absorbing set $B \in \mathcal{D}_Y$ and a compact $\mathcal{D}_Y$-pullback attracting set K in Y, then ϕ has a $\mathcal{D}_Y$-cocycle attractor $\mathscr{A} \in \mathcal{D}_Y$ in Y given by*

$$\mathscr{A}(\omega) = \mathcal{W}(\omega, B)^Y.$$

Proof. Since the RDS ϕ is quasi-S2W continuous in X, so it is in Y by Proposition 2.1.5. Therefore, the theorem follows directly from Theorem 2.2.9. □

Now we establish an existence theorem for bi-spatial cocycle attractors.

Theorem 2.2.15. *Suppose that ϕ is a quasi-S2W continuous RDS on X, and is (X, Y)-dissipative on $\mathcal{D}_X$ (with a $\mathcal{D}_X$-pullback absorbing set $\mathbf{B}$ which is a random set in Y belonging to $\mathcal{D}_X$ but unnecessarily belonging to $\mathcal{D}_Y$). Then if there exists a compact random set K in Y which is $\mathcal{D}_X$-pullback attracting under ϕ under the topology of Y, then ϕ has a (X, Y)-cocycle attractor $\mathscr{A} \in \mathcal{D}_X$, with attraction universe $\mathcal{D}_X$, given by*

$$\mathscr{A}(\omega) = \mathcal{W}(\omega, \mathbf{B})^X = \mathcal{W}(\omega, \mathbf{B})^Y.$$

If, moreover, $\mathbf{B} \in \mathcal{D}_Y$, then $\mathscr{A} \in \mathcal{D}_Y$.

Proof. By Theorem 2.2.9 it is clear that ϕ has a $\mathcal{D}_X$-cocycle attractor $\mathscr{A}$ in X given by $\mathscr{A}(\cdot) = \mathcal{W}(\cdot, \mathbf{B})^X$. Now we claim that $\mathcal{W}(\cdot, \mathbf{B})^X = \mathcal{W}(\cdot, \mathbf{B})^Y$. Indeed, for any $y \in \mathcal{W}(\omega, \mathbf{B})^X$, there exist sequences $t_n \to \infty$ and $x_n \in \mathbf{B}(\vartheta_{-t_n}\omega)$ such that $\phi(t_n, \vartheta_{-t_n}\omega, x_n) \to y$ in X. On the other hand, since $\mathbf{B}$ is attracted by a compact random set K in Y, there exits a $y' \in Y$ such that, up to a subsequence, $\phi(t_n, \vartheta_{-t_n}\omega, x_n) \to y'$ in Y. By the uniqueness of a limit we have $y = y' \in \mathcal{W}(\omega, \mathbf{B})^Y$, and thereby $\mathcal{W}(\cdot, \mathbf{B})^X = \mathcal{W}(\cdot, \mathbf{B})^Y$. Hence $\mathscr{A}(\cdot) = \mathcal{W}(\cdot, \mathbf{B})^Y$, which along with Lemma 2.2.8 implies that $\mathscr{A}(\cdot)$ is a compact random set in Y and pullback attracts $\mathbf{B}$ in the topology of Y. Since $\mathbf{B}$ is $\mathcal{D}_X$-pullback absorbing, we know that $\mathscr{A}$ is $\mathcal{D}_X$-pullback attracting in Y. Since $\mathscr{A}$ is clearly invariant, it is indeed the (X, Y)-cocycle attractor. If $\mathbf{B}$ is in $\mathcal{D}_X$ or $\mathcal{D}_Y$, then so is $\mathscr{A}$ due to the inclusion-closedness of $\mathcal{D}_X$ and $\mathcal{D}_Y$ as $\mathscr{A} \subset \mathbf{B}$. □

Remark 2.2.16. Similar to Theorem 2.2.9, the compact $\mathcal{D}_X$-pullback attracting set K in Theorem 2.2.15 can be replaced by other (bi-spatial) dynamical compactnesses, such as the (X, Y)-pullback asymptotic compactness on $\mathcal{D}_X$: for any $D \in \mathcal{D}_X$, $\omega \in \Omega$ and sequence $t_n \to \infty$, the sequence $\{\phi(t_n, \vartheta_{-t_n}\omega, x_n)\}$ with $x_n \in D(\vartheta_{-t_n}\omega)$ has a convergent subsequence in Y.

2.3 Applications to a stochastic reaction-diffusion equation

In this section we study a stochastic reaction-diffusion equation on $\mathbb{R}^N$ as an example to illustrate how our theoretical results contribute to prove the measurability and existence of cocycle attractors, especially in more regular spaces. Under general conditions, insufficient to derive the continuity of the system in $L^p \cap H^1$ (cf. [75, p. 227]), we are now able to study the cocycle attractor in $L^p \cap H^1$ which seems new in the literature; nevertheless, we are more interested to derive a stronger result, the $(L^2, L^p \cap H^1)$-cocycle attractor.

To make our theoretical results more convenient to apply, we begin with some analysis for the widely used Wiener probability space and a solution of an Ornstein-Uhlenbeck equation.

2.3.1 Wiener probability space and the continuity of a random variable

Now we define a typical probability space, the Wiener probability space $(\Omega, \mathcal{F}, \mathcal{P})$, given by

$$\Omega = \{\omega \in C(\mathbb{R}; \mathbb{R}) : \omega(0) = 0\},$$

endowed with the compact open topology given by the complete metric

$$\rho(\omega, \omega') := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\omega - \omega'\|_n}{1 + \|\omega - \omega'\|_n}, \quad \|\omega - \omega'\|_n := \sup_{-n \leqslant t \leqslant n} |\omega(t) - \omega'(t)|,$$

and $\mathcal{F}$ the Borel sigma-algebra induced by the compact-open topology of Ω, $\mathcal{P}$ the two-sided Wiener measure on $(\Omega, \mathcal{F})$ which is given by the distribution of a two-sided Wiener process with covariance $q > 0 : \mathbb{E}\omega(t)^2 = q|t|$. The Wiener probability space is widely used to depict white noises describing by Wiener processes. Define the translation-operator group $\{\vartheta_t\}_{t \in \mathbb{R}}$ on Ω by

$$\vartheta_t \omega = \omega(\cdot + t) - \omega(t), \quad \forall t \in \mathbb{R}, \ \omega \in \Omega.$$

Then $\mathcal{P}$ is ergodic and invariant under ϑ, see for instance [42].

On the other hand, in view of [11, Lemma 11], there exists a full measure subspace $\tilde{\Omega} \subset \Omega$ whose each member has subexponential growth as $|t| \to \infty$. This implies that for each $\varepsilon > 0$ and $\omega \in \tilde{\Omega}$ there exists a positive constant $C(\varepsilon, \omega)$ such that

$$|\omega(t)| \leqslant C(\varepsilon, \omega) e^{\varepsilon |t|}, \quad \forall t \in \mathbb{R}.$$

It is clear that such $\tilde{\Omega}$ is invariant under ϑ, and in the sequel we restrict ourselves on $\tilde{\Omega}$ and write $\tilde{\Omega}$ as Ω.

For $\lambda > 0$, set

$$z(\omega) := -\lambda \int_{-\infty}^{0} e^{\lambda \tau} \omega(\tau) \, \mathrm{d}\tau, \quad \forall \omega \in \Omega. \tag{2.18}$$

Then z is well-defined on Ω and $z(\vartheta_t \omega)$ is continuous in t for every $\omega \in \Omega$ satisfying the tempered property, that is, for any $\varepsilon > 0$,

$$\lim_{t \to \infty} e^{-\varepsilon t} |z(\vartheta_{-t} \omega)| = 0, \quad \forall \omega \in \Omega. \tag{2.19}$$

Moreover, $(t, \omega) \mapsto z(\vartheta_t \omega)$ is a stationary solution of the one-dimensional Ornstein-Uhlenbeck equation

$$\mathrm{d}z(\vartheta_t \omega) + \lambda z(\vartheta_t \omega) \mathrm{d}t = \mathrm{d}\omega. \tag{2.20}$$

In fact, we have the following lemma.

Lemma 2.3.1. *(See [9, 12, 86]) There exists a ϑ-invariant subset $\tilde{\Omega} \in \mathcal{F}$ of full measure such that*

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{t} = 0 \quad \textit{for all } \omega \in \tilde{\Omega},$$

and, for such ω, the random variable given by (2.18) is well defined. Moreover, for $\omega \in \tilde{\Omega}$, the mapping

$$(t, \omega) \mapsto z(\vartheta_t \omega)$$

is a stationary solution of (2.20) with continuous trajectories. In addition, for $\omega \in \tilde{\Omega}$,

$$\lim_{t\to\pm\infty} \frac{|z(\vartheta_t\omega)|}{|t|} = 0, \quad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\vartheta_s\omega)\,\mathrm{d}s = 0, \tag{2.21}$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t |z(\vartheta_s\omega)|\,\mathrm{d}s = \mathbb{E}|z| < \infty. \tag{2.22}$$

For later purpose, set

$$\Omega_N = \left\{\omega \in \Omega : |\omega(t)| \leqslant N e^{\frac{\lambda}{2}|t|},\ \forall t \in \mathbb{R}\right\}, \quad \forall N \in \mathbb{N}. \tag{2.23}$$

Clearly, we have

$$\Omega = \bigcup_{N\in\mathbb{N}} \Omega_N.$$

Proposition 2.3.2. *(i) Each Ω_N is a closed set in Ω;*

(ii) For each Ω_N, the mapping $\omega \mapsto z(\omega)$ is continuous from (Ω_N, ρ) to $\mathbb{R}$.

Proof. (i) Suppose that $\{\omega_n\}_{n\in\mathbb{N}} \subset \Omega_N$ is a sequence approaching to some ω in Ω. Let us prove $\omega \in \Omega_N$ by contradiction. If it is not the case, then there exits a $T \in \mathbb{R}$, without loss of generality, assumed lying in $[k, k+1)$ for some $k \in \mathbb{N}$, such that

$$|\omega(T)| > N e^{\frac{\lambda}{2}|T|}. \tag{2.24}$$

On the other hand, since $\omega_n \to \omega$ in Ω, we have

$$\|\omega_n - \omega\|_{k+1} = \sup_{-(k+1)\leqslant t\leqslant k+1} |\omega_n(t) - \omega(t)| \to 0, \quad \text{as } n \to \infty,$$

and in particular $|\omega_n(T) - \omega(T)| \to 0$. Hence, $\omega(T)$ is no greater than $Ne^{\frac{\lambda}{2}|T|}$ as so is not $\omega_n(T)$, which contradicts (2.24).

(ii) Suppose that $\{\omega_n\}_{n\in\mathbb{N}} \subset \Omega_N$ is a sequence approaching to some ω in Ω_N. Given any $\varepsilon > 0$, let T be a negative integer such that

$$4Ne^{\frac{\lambda T}{2}} < \frac{\varepsilon}{2}. \tag{2.25}$$

Since $\omega_n \to \omega$ in Ω_N, there exists an $N_* \in \mathbb{N}$ such that $\|\omega_n - \omega\|_{|T|} < \frac{\varepsilon}{2(1-e^{\lambda T})}$ for all $n \geqslant N_*$. Hence,

$$\begin{aligned} |z(\omega_n) - z(\omega)| &\leqslant \left|\lambda \int_T^0 e^{\lambda\tau}(\omega_n(\tau) - \omega(\tau))\,\mathrm{d}\tau\right| + \left|\lambda \int_{-\infty}^T e^{\lambda\tau}(\omega_n(\tau) - \omega(\tau))\,\mathrm{d}\tau\right| \\ &\leqslant \|\omega_n - \omega\|_{|T|}\left|\lambda \int_T^0 e^{\lambda\tau}\,\mathrm{d}\tau\right| + 2N\left|\lambda \int_{-\infty}^T e^{\lambda\tau}e^{\frac{\lambda}{2}|\tau|}\,\mathrm{d}\tau\right| \\ &= \|\omega_n - \omega\|_{|T|}(1 - e^{\lambda T}) + 4Ne^{\frac{\lambda T}{2}} < \varepsilon, \quad \forall n \geqslant N_*. \end{aligned} \tag{2.26}$$

Hence, $z(\omega_n) \to z(\omega)$ as $\omega_n \to \omega$ in Ω_N. The proof is complete. □

Remark 2.3.3. Actually we have proved in Proposition 2.3.2 a Lusin's continuity for $z(\cdot)$, that is, for each $\delta > 0$ there is a closed set $F \subset \Omega$ such that $\mathcal{P}(\Omega \setminus F) < \delta$ and $z(\cdot)$ is continuous on F. Indeed, as $\{\Omega_N\}_{N\in\mathbb{N}}$ is increasing and $\cup_{N\in\mathbb{N}}\Omega_N = \Omega$, by the countable additivity of the measure $\mathcal{P}$ we have $\mathcal{P}(\Omega_N) \to 1$. Hence, Proposition 2.3.2 implies that for each $\delta > 0$ there exists an $\Omega_N \in \mathcal{F}$ (in the form of (2.23)) with $\mathcal{P}(\Omega \setminus \Omega_N) < \delta$ such that $z(\cdot)$ is continuous on Ω_N.

Corollary 2.3.4. *For each Ω_N, the mapping $\omega \mapsto z(\vartheta_t\omega)$ is continuous on Ω_N uniformly in t on bounded intervals, that is, for any $(T_1, T_2) \subset \mathbb{R}$ being bounded,*

$$\sup_{s\in(T_1,T_2)} |z(\vartheta_s\omega_k) - z(\vartheta_s\omega_0)| \to 0 \quad \text{as } \omega_k \to \omega_0 \text{ in } \Omega_N.$$

Proof. Suppose $I = (T_1, T_2) \subset \mathbb{R}$ is a bounded interval. Then for any $s \in I$ it holds

$$\begin{aligned} |z(\vartheta_s\omega_k) - z(\vartheta_s\omega_0)| &= \lambda\left|\int_{-\infty}^0 e^{\lambda\tau}\Big(\big(\omega_k(\tau+s) - \omega_k(s)\big) - \big(\omega_0(\tau+s) - \omega_0(s)\big)\Big)\mathrm{d}\tau\right| \\ &\leqslant \lambda\left|\int_{-\infty}^0 e^{\lambda\tau}\Big(\big(\omega_k(\tau+s) - \omega_0(\tau+s)\Big)\mathrm{d}\tau\right| + |\omega_k(s) - \omega_0(s)| \\ &\leqslant \lambda e^{-\lambda s}\left|\int_{-\infty}^s e^{\lambda\tau}\big(\big(\omega_k(\tau) - \omega_0(\tau)\big)\mathrm{d}\tau\right| + \|\omega_k - \omega_0\|_{m_I}, \end{aligned}$$

where m_I is a positive number no less than $|T_1|$ and $|T_2|$. Since

$$\begin{aligned} \lambda e^{-\lambda s}\left|\int_{-\infty}^s e^{\lambda\tau}\big(\big(\omega_k(\tau) - \omega_0(\tau)\big)\mathrm{d}\tau\right| &= \lambda e^{-\lambda s}\left|\int_{-\infty}^0 + \int_0^s\right| \\ &\leqslant e^{-\lambda s}|z(\omega_k) - z(\omega_0)| + |1 - e^{-\lambda s}|\|\omega_k - \omega_0\|_{m_I}, \end{aligned}$$

by Proposition 2.3.2 we conclude that

$$\begin{aligned} \sup_{s\in I}|z(\vartheta_s\omega_k) - z(\vartheta_s\omega_0)| &\leqslant e^{-\lambda T_1}|z(\omega_k) - z(\omega_0)| + (2 + e^{-\lambda T_1})\|\omega_k - \omega_0\|_{m_I} \\ &\to 0, \quad \text{as } \omega_k \to \omega_0 \text{ in } \Omega_N. \end{aligned}$$

The proof is complete. □

The next lemma, along with Prop. 2.3.2 (ii), gives a clear view of the measurability of $z(\cdot)$.

Lemma 2.3.5. *A mapping f from Ω to a metric space $\mathcal{M}$ is $(\mathcal{F}, \mathcal{B}(\mathcal{M}))$-measurable if and only if for all $N \in \mathbb{N}$ the restriction $f|_{\Omega_N}$ of f on Ω_N is $(\mathcal{B}(\Omega_N), \mathcal{B}(\mathcal{M}))$-measurable.*

Proof. The fact $\Omega = \cup_{N\in\mathbb{N}} \Omega_N$ gives for any $M \in \mathcal{B}(\mathcal{M})$ that

$$\begin{aligned}\{\omega \in \Omega : f(\omega) \in M\} &= \bigcup_{N\in\mathbb{N}} \{\omega \in \Omega_N : f(\omega) \in M\} \\ &= \bigcup_{N\in\mathbb{N}} \{\omega \in \Omega_N : f|_{\Omega_N}(\omega) \in M\}\end{aligned}$$

and that

$$\begin{aligned}\{\omega \in \Omega_N : f|_{\Omega_N}(\omega) \in M\} &= \{\omega \in \Omega_N : f(\omega) \in M\} \\ &= \Omega_N \bigcap \{\omega \in \Omega : f(\omega) \in M\}, \quad \forall N \in \mathbb{N}.\end{aligned}$$

Moreover, as any closed subset U of the subspace Ω_N has the form $\Omega_N \cap V$ for some closed subset V of Ω, Proposition 2.3.2 (i) implies that $\mathcal{B}(\Omega_N) \subset \mathcal{B}(\Omega) = \mathcal{F}$ and $\mathcal{B}(\Omega_N) = \{\Omega_N \cap F : F \in \mathcal{F}\}$. Hence, if each $f|_{\Omega_N}$ is $(\mathcal{B}(\Omega_N), \mathcal{B}(\mathcal{M}))$-measurable, then $F_N := \{\omega \in \Omega_N : f|_{\Omega_N}(\omega) \in M\} \in \mathcal{B}(\Omega_N) \subset \mathcal{F}$. So $\{\omega \in \Omega : f(\omega) \in M\} = \cup_{N\in\mathbb{N}} F_N \in \mathcal{F}$, that is, f is $(\mathcal{F}, \mathcal{B}(\mathcal{M}))$-measurable. Conversely, if f is $(\mathcal{F}, \mathcal{B}(\mathcal{M}))$-measurable, then $F := \{\omega \in \Omega : f(\omega) \in M\} \in \mathcal{F}$ so that $\{\omega \in \Omega_N : f|_{\Omega_N}(\omega) \in M\} = \Omega_N \cap F \in \mathcal{B}(\Omega_N)$, that is, $f|_{\Omega_N}$ is $(\mathcal{B}(\Omega_N), \mathcal{B}(\mathcal{M}))$-measurable as desired. □

We shall see later (in Proposition 2.3.9 in the next application subsection) that Lemma 2.3.5 along with our quasi-S2W continuous results could contribute to prove the measurability of an RDS generated by a stochastic differential equation with white noises.

2.3.2 $(L^2, L^p \cap H^1)$-cocycle attractor for a reaction-diffusion equation

We study the following stochastic reaction-diffusion equation defined on $\mathbb{R}^N$, $N \in \mathbb{N}$,

$$\mathrm{d}u + (\lambda u - \triangle u)\mathrm{d}t = f(x, u)\mathrm{d}t + g(x)\mathrm{d}t + h(x)\mathrm{d}\omega \tag{2.27}$$

with initial value condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{2.28}$$

where $\lambda > 0$ is a constant, $g(x) \in L^2(\mathbb{R}^n)$, $h(x) \in W^{2,p}(\mathcal{O})$ for some $p \geqslant 2$ and ω comes from Ω studied in Section 2.3.1. The nonlinear term $f(x, u)$ is assumed to satisfy the following standard conditions

$$f(x, s)s \leqslant -\alpha_1 |s|^p + \psi_1(x), \tag{2.29}$$

$$|f(x, s)| \leqslant \alpha_2 |s|^{p-1} + \psi_2(x), \tag{2.30}$$

$$\frac{\partial f}{\partial s}(x, s) \leqslant \alpha_3, \tag{2.31}$$

$$\left|\frac{\partial f}{\partial x}(x, s)\right| \leqslant \psi_3(x), \tag{2.32}$$

where α_j are positive constants, $\psi_1 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\psi_2 \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\psi_3 \in L^2(\mathbb{R}^N) \cap L^{\frac{p}{p-2}}(\mathbb{R}^N)$.

Consider the following reaction-diffusion equation with random coefficients

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} + \lambda v - \triangle v = f(x, v + hz(\vartheta_t\omega)) + g(x) + z(\vartheta_t\omega)\triangle h(x), \\ v(x,t)|_{t=0} = v_0(x). \end{cases} \tag{2.33}$$

Similar to [4] we have the following existence result.

Lemma 2.3.6. *Under conditions (2.29)-(2.32), for each $v_0 \in L^2(\mathbb{R}^N)$ and $\omega \in \Omega$ there exists a unique solution*

$$v(\cdot, \omega, v_0) \in C([0,\infty), L^2(\mathbb{R}^N)) \cap L^p_{loc}(0,\infty; L^p(\mathbb{R}^N)) \cap L^2_{loc}(0,\infty; H^1(\mathbb{R}^N))$$

satisfying (2.33) such that v is continuous in L^2 with respect to initial data.

As a classical parabolic system, the stochastic reaction-diffusion equation has been considerably studied. In terms of cocycle attractors, [4] studied the existence of cocycle attractor in L^2, while [62, 64, 93] studied the (L^2, L^p)-cocycle attractor and [65] the (L^2, H^1)-cocycle attractor, etc. But notice that, since we are working on arbitrary $p \geqslant 2$ and $N \in \mathbb{N}$, without additional conditions the solution v is not continuous (w.r.t. initial data) neither in H^1 nor in L^p, and also the mapping $t \mapsto v(t)$ is not $(\mathbb{R}, L^p)$-continuous. As a consequence, though the regularity of the attractor was studied in [93, 62, 92, 78, 65], etc, the measurability of the cocycle attractor obtained there was only in L^2. Now, thanks to our theoretical analysis for quasi-S2W continuous RDS, we are able to prove that the cocycle attractor is in fact an $(L^2, H^1 \cap L^p)$-cocycle attractor, showing that the cocycle attractor is a random set in $H^1 \cap L^p$.

For simplicity we write

$$X = \left(L^2(\mathbb{R}^N), \|\cdot\|_X\right), \quad Y = \left(H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N), \|\cdot\|_Y\right),$$

with $\|v\|_Y = \|\nabla v\|_X + \|v\|_X + \|v\|_{L^p}$. Then Y is a Banach space continuously and densely embedded into X with $X^* \subset Y^*$, see Lemma A.1.6 and the discussion after Proposition 2.1.5.

Define

$$\phi(t,\omega,u_0) = u(t,\omega,u_0) := v(t,\omega,u_0 - hz(\omega)) - hz(\vartheta_t\omega).$$

Then clearly u is the solution of (2.27) and ϕ defines a continuous RDS in X.

In the following, **denote by $\mathcal{D}_X$ (resp. $\mathcal{D}_Y$) the collection of tempered random sets in X (resp. in Y)**, i.e. $D \in \mathcal{D}_X$ if and only if for any $\varepsilon > 0$

$$\lim_{t\to\infty} e^{-\varepsilon t}\|D(\vartheta_{-t}\omega)\|_X = 0, \quad \forall \omega \in \Omega.$$

a). (X,Y)-dissipation of the reaction-diffusion equation

Now we prove that the RDS ϕ generated by the stochastic reaction-diffusion equation is (X, Y)-dissipative on $\mathcal{D}_X$, which is necessary to study the (X, Y)-cocycle attractor for ϕ. First recall the following uniform estimates of solutions.

Lemma 2.3.7. *(See [4, Lemmas 4.1 & 4.5] and [62, Lemma 4.3]) Under conditions* (2.29)-(2.32)*, for each $D \in \mathcal{D}_X$ there exists a random variable $T_D(\omega) > 0$ and a tempered random variable $r(\omega) > 0$ such that the solution u of the equation* (2.27) *satisfies*

$$\|u(t, \vartheta_{-t}\omega, D(\vartheta_{-t}\omega))\|_Y \leqslant r(\omega), \quad \forall t \geqslant T_D(\omega).$$

Lemma 2.3.8. *For the solution v of* (2.33)*, the mapping $\omega \mapsto v(t, \omega, v_0)$ is continuous from each (Ω_N, ρ) to X.*

Proof. For any $t \in \mathbb{R}^+$ and $v_0 \in X$ fixed, denote by v_1 and v_2 the solutions $v(t, \omega_1, v_0)$ and $v(t, \omega_2, v_0)$, respectively, for any $\omega_1, \omega_2 \in \Omega$. Then by (2.33) we know the difference $\mathsf{v} := v_1 - v_2$ satisfies

$$\begin{aligned} \frac{\mathrm{d}\mathsf{v}}{\mathrm{d}t} + \lambda \mathsf{v} - \triangle \mathsf{v} &= f(x, v_1 + hz(\vartheta_t\omega_1) - f(x, v_2 + hz(\vartheta_t\omega_2) \\ &\quad + \triangle h(z(\vartheta_t\omega_1) - z(\vartheta_t\omega_2)) \\ &\leqslant \alpha_3|\mathsf{v} + h(z(\vartheta_t\omega_1) - z(\vartheta_t\omega_2))| + \triangle h(z(\vartheta_t\omega_1) - z(\vartheta_t\omega_2)). \end{aligned}$$

Taking the inner product with v in X, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathsf{v}\|_X^2 \leqslant c\|\mathsf{v}\|_X^2 + c|z(\vartheta_t\omega_1) - z(\vartheta_t\omega_2)|^2,$$

where c is a positive constant depending only on $\|h\|^2$ and $\|\triangle h\|^2$, and Gronwall's lemma gives

$$\begin{aligned} \|\mathsf{v}(T)\|_X^2 &\leqslant ce^{cT} \int_0^T e^{-cs}|z(\vartheta_s\omega_1) - z(\vartheta_s\omega_2)|^2 \,\mathrm{d}s \\ &\leqslant ce^{cT} \int_0^T |z(\vartheta_s\omega_1) - z(\vartheta_s\omega_2)|^2 \,\mathrm{d}s, \quad \forall T \in \mathbb{R}^+. \end{aligned}$$

By Corollary 2.3.4 we know that $\|\mathsf{v}(T)\|_X^2 \to 0$ as $\omega_1 \to \omega_2$ in Ω_N. □

Proposition 2.3.9. *The RDS ϕ generated by reaction-diffusion equation* (2.27) *is (X, Y)-dissipative on $\mathcal{D}_X$, i.e. satisfying Definition 2.2.11.*

Proof. Firstly, define

$$\mathbf{B}(\omega) := \big\{u \in Y : \|u\|_Y \leqslant r(\omega)\big\}, \tag{2.34}$$

where $r(\omega)$ is the tempered random variable in Lemma 2.3.7. Then $\mathbf{B} \in \mathcal{D}_Y \subset \mathcal{D}_X$, and is a $\mathcal{D}_X$-pullback absorbing set by Lemma 2.3.7. Hence, we only need to prove that ϕ is an RDS in Y.

When restricted on Y, i.e. when $u_0 \in Y$, it is known (by Gronwall's technique) that $\phi(\cdot, \cdot, u_0)$ maps $\mathbb{R}^+ \times \Omega$ into Y. Moreover, we claim that when restricted on Y, ϕ defines an RDS in Y. Indeed, since the map $t \mapsto \phi(t, \omega, u_0)$ is continuous from $\mathbb{R}^+$ to X, by Proposition 2.1.3 we know it is quasi-S2W continuous from $\mathbb{R}^+$ to Y and thereby $(\mathcal{B}(\mathbb{R}^+), \mathcal{B}(Y))$-measurable by Theorem 2.1.6. Similarly, since, in view of Lemma 2.3.8, $\omega \mapsto \phi(t, \omega, u_0)$ is continuous from each Ω_N to X w.r.t. the compact-open topology, it is quasi-S2W continuous from each Ω_N to Y by Proposition 2.1.3. So it is $(\mathcal{B}(\Omega_N), \mathcal{B}(Y))$-measurable for each $N \in \mathbb{N}$ by Theorem 2.1.6 and then $(\mathcal{F}, \mathcal{B}(Y))$-measurable by Lemma 2.3.5. Finally, since $u_0 \mapsto \phi(t, \omega, u_0)$ is continuous in X, it is quasi-S2W continuous in Y according to Proposition 2.1.5 and thereby $(\mathcal{B}(Y), \mathcal{B}(Y))$-measurable. In other words, we have seen that ϕ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(Y), \mathcal{B}(Y))$-measurable and is indeed an RDS in Y. □

b). The (X,Y)-cocycle attractor

The $\mathcal{D}_X$-cocycle attractor $\mathscr{A}$ in X was established in [4]. Moreover, in view of [62, 65, 78] the cocycle attractor $\mathscr{A}$ is in fact compact in Y (i.e. each image $\mathscr{A}(\omega)$ is a compact set in Y) and pullback attracts $\mathcal{D}_X$ in the topology of Y. These results are summarized in the following lemma.

Lemma 2.3.10. *Suppose conditions (2.29)-(2.32) hold. Then the RDS ϕ generated by the stochastic reaction-diffusion equation (2.27) has a $\mathcal{D}_X$-cocycle attractor $\mathscr{A} \in \mathcal{D}_Y$ such that*

(i) for each $\omega \in \Omega$, $\mathscr{A}(\omega)$ is a compact set in Y;

(ii) $\mathscr{A}$ pullback attracts $\mathcal{D}_X$ in the topology of Y.

Proof. The existence of $\mathcal{D}_X$-cocycle attractor $\mathscr{A}$ in X was established in [4]. In [62] the authors proved that $\mathscr{A}$ pullback attracts $\mathcal{D}_X$ in the topology of $L^p(\mathbb{R}^N)$ and each $\mathscr{A}(\omega)$ is compact in $L^p(\mathbb{R}^N)$. First in [78] with improvements in [65] it was proved that $\mathscr{A}$ pullback attracts $\mathcal{D}_X$ in the topology of $H^1(\mathbb{R}^N)$ and each $\mathscr{A}(\omega)$ is compact in $H^1(\mathbb{R}^N)$. Finally, since $\mathscr{A}$ is smaller than $\mathbf{B}$ defined by (2.34), $\mathscr{A} \in \mathcal{D}_Y$ as $\mathcal{D}_Y$ is inclusion-bounded. Hence, this lemma is clear. □

Now we show the measurability of the attractor $\mathscr{A}$ in Y, i.e. the mapping $\omega \mapsto \mathrm{dist}_Y(y, \mathscr{A}(\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}^+))$-measurable for each $y \in Y$.

Theorem 2.3.11. *Under conditions (2.29)-(2.32), the $\mathcal{D}_X$-cocycle attractor $\mathscr{A}$ for the RDS ϕ generated by the stochastic reaction-diffusion equation (2.27) is a compact random set in Y. In fact, $\mathscr{A}$ is the (X, Y)-cocycle attractor for ϕ with attraction universe $\mathcal{D}_X$ in the sense of Definition 2.2.12.*

Proof. Thanks to the analysis in Section 2.3.2 and Lemma 2.3.10, $\mathscr{A}$ is indeed the (X, Y)-cocycle attractor with attraction universe $\mathcal{D}_X$ by Theorem 2.2.15. □

Remark 2.3.12. The existence of (X, Y)-cocycle attractor for the stochastic reaction-diffusion equation implies that the equation has a $\mathcal{D}_Y$-cocycle attractor in Y and a $\mathcal{D}_{H^1}$-cocycle attractor in H^1. Even the last result alone is new in the literature, since our conditions are too general to ensure the continuity of the system in H^1 so that usual existence theorems for cocycle attractors are not applicable. Notice that the recent publication [7] obtained the cocycle attractor in H^1, with techniques paid to obtain the continuity in H^1 (of course, this continuity property is interesting itself). Here, thanks to our theoretical results for quasi-S2W continuous RDS, we obtained the attractor in H^1 and even in $Y = L^p \cap H^1$ without proving further continuities than the continuity in L^2.

Chapter 3

Cocycle attractors for non-autonomous random dynamical systems I: non-autonomous attraction universe case

In this chapter we generalize the results in Chapter 2 to non-autonomous cocycle attractors. Notice that cocycle attractors studied in Chapter 2 belong to their attraction universes so that are always attracted by themselves. In this chapter we restrict ourselves to a similar situation, studying cocycle attractors with non-autonomous attraction universes for non-autonomous random dynamical systems (NRDS). The aim is to generalize the autonomous theory established in Chapter 2 to a non-autonomous one. Autonomous attraction universe case will be studied in the next chapter.

3.1 Preliminaries

In this section we give some basic definitions related to NRDS. We often regard autonomous RDS as a particular NRDS, and all the definitions for NRDS will adapt to RDS.

Definition 3.1.1. Suppose X is a complete metric space. The mapping $\phi(t, \omega, \sigma, x) : \mathbb{R}^+ \times \Omega \times \Sigma \times X \mapsto X$ is called a non-autonomous random dynamical system (NRDS for short) on X with base flows $\{\vartheta_t\}_{t\in\mathbb{R}}$ and $\{\theta_t\}_{t\in\mathbb{R}}$, if

(1) ϕ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(\Sigma) \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(2) $\phi(0, \omega, \sigma, \cdot)$ is the identity on X for each ω and σ fixed;

(3) it holds the cocycle property

$$\phi(t + s, \omega, \sigma, x) = \phi(t, \vartheta_s\omega, \theta_s\sigma) \circ \phi(s, \omega, \sigma, x), \quad \forall t, s \in \mathbb{R}^+, \omega \in \Omega, \sigma \in \Sigma.$$

When some continuity of NRDS (and RDS) is mentioned, the continuity is referred to x in the phase space X, i.e., of the mapping $x \mapsto \phi(t, \omega, \sigma, x)$, except otherwise clearly stated.

For simplicity, we often speak of NRDS without mentioning its base flows.

In order to emphazise the dependence on symbols, random sets satisfying Definition 2.2.2 are often called autonomous. Now we define non-autonomous random sets.

Definition 3.1.2. Suppose X is a complete metric space. A two parameterized mapping $\hat{D}$: $\Sigma \times \Omega \to 2^X \setminus \emptyset$, $(\sigma, \omega) \mapsto \hat{D}_\sigma(\omega)$ is called a *non-autonomous random set* in X if, for each $\sigma \in \Sigma$, $\hat{D}_\sigma(\cdot)$ is a random set in the sense of Definition 2.2.2. A non-autonomous random set is said to be closed (or bounded, compact, etc) if each $\hat{D}_\sigma$ is closed (or bounded, compact, etc).

In the following, let X be a Polish Banach space and ϕ an NRDS on X. **Denote by $\hat{\mathcal{D}}_X$ some class of non-autonomous random sets in** X which is

- neighborhood-closed, i.e. for each $\hat{D} \in \hat{\mathcal{D}}_X$ there exits an $\varepsilon > 0$ such that the closed ε-neighborhood $\mathcal{N}_\varepsilon(\hat{D})$ belongs to $\hat{\mathcal{D}}_X$, and
- inclusion-closed, i.e., if $\hat{D} \in \hat{\mathcal{D}}_X$ then each non-autonomous random set smaller than $\hat{D}$ belongs to $\hat{\mathcal{D}}_X$.

An example of the universe $\hat{\mathcal{D}}_X$ is the collection of all the bounded non-autonomous random sets in X.

Omega-limit sets play an important role in the study of attractors. For each non-empty non-autonomous random set $\hat{D}$ and $\sigma \in \Sigma$, the random omega-limit set $\mathcal{W}(\cdot, \sigma, \hat{D})$ of $\hat{D}$ driven by σ is defined by

$$\mathcal{W}(\omega, \sigma, \hat{D}) = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \hat{D}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega))}, \quad \forall \omega \in \Omega.$$

It is straightforward to have the following lemma.

Lemma 3.1.3. *For each $\omega \in \Omega$, $\sigma \in \Sigma$ and non-autonomous random set $\hat{D}$, $y \in \mathcal{W}(\omega, \sigma, \hat{D})$ if and only if there exit sequences $t_n \to \infty$ and $x_n \in \phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, \hat{D}_{\theta_{-t}\sigma}(\vartheta_{-t_n}\omega))$ such that $x_n \to y$.*

Definition 3.1.4. Given an NRDS ϕ, a non-autonomous random set $\hat{K} = \{\hat{K}_\sigma(\cdot)\}_{\sigma \in \Sigma}$ is called $\hat{\mathcal{D}}_X$-pullback attracting under ϕ if for each $\hat{D} \in \hat{\mathcal{D}}_X$,

$$\lim_{t \to \infty} \operatorname{dist}\Big(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \hat{D}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega)), \hat{K}_\sigma(\omega)\Big) = 0, \quad \forall \sigma \in \Sigma, \omega \in \Omega,$$

while it is called $\hat{\mathcal{D}}_X$-pullback absorbing if, for each $\hat{D} \in \hat{\mathcal{D}}_X$, $\sigma \in \Sigma$ and $\omega \in \Omega$, there exists a time $T_{\hat{D}}(\omega, \sigma) > 0$ such that

$$\bigcup_{t \geqslant T_{\hat{D}}(\omega, \sigma)} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \hat{D}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega)) \subset \hat{K}_\sigma(\omega).$$

Note that a $\hat{\mathcal{D}}_X$-pullback attracting/absorbing set does not necessarily belong to $\hat{\mathcal{D}}_X$.

Definition 3.1.5. A non-autonomous random set $\hat{A} = \{\hat{A}_\sigma(\cdot)\}_{\sigma \in \Sigma}$ is called a $\hat{\mathcal{D}}_X$*-random cocycle attractor* of the NRDS ϕ if

(1) $\hat{A} \in \hat{\mathcal{D}}_X$;

(2) every $\hat{A}_\sigma(\cdot)$ is a compact random set,

(3) $\hat{A}$ is $\hat{\mathcal{D}}_X$-pullback attracting;

(4) $\hat{A}$ is invariant under ϕ, that is,

$$\phi(t, \omega, \sigma, \hat{A}_\sigma(\omega)) = \hat{A}_{\theta_t\sigma}(\vartheta_t\omega), \quad \forall t \in \mathbb{R}^+.$$

Since $\hat{A}$ pullback attracts itself as it belongs to $\hat{\mathcal{D}}_X$, the minimal property follows directly from the invariance of $\hat{A}$, and thereby a $\hat{\mathcal{D}}_X$-attractor must be unique.

For $\hat{\mathcal{D}}_X$-random cocycle attractors, [83, 84] studied the existence and characterization by complete trajectories. The following existence result is well known.

Lemma 3.1.6. *[83, 84] Suppose ϕ is a continuous NRDS with a compact $\hat{\mathcal{D}}_X$-pullback attracting set $\hat{K}$ and a closed $\hat{\mathcal{D}}$-pullback absorbing set $\hat{B} \in \hat{\mathcal{D}}_X$. Then ϕ has a unique $\hat{\mathcal{D}}_X$-random cocycle attractor $\hat{A} \in \hat{\mathcal{D}}_X$ given by*

$$\hat{A}_\sigma(\omega) = \mathcal{W}(\omega, \sigma, \hat{B}).$$

3.2 Existence results under quasi-S2W continuity of NRDS

3.2.1 A first result

In this section, we establish some existence criteria for cocycle attractors for quasi-S2W continuous NRDS, generalizing Lemma 3.1.6 and also Theorem 2.2.9.

Lemma 3.2.1. *Suppose that ϕ is a quasi S2W-continuous NRDS and $\hat{B}$ is a non-autonomous random set in X. If there exists a compact non-autonomous random set $\hat{K}$ pullback attracting $\hat{B}$, then*

(i) $\mathcal{W}(\cdot, \cdot, \hat{B})$ is nonempty compact, and has the invariant property

$$\phi(t, \omega, \sigma, \mathcal{W}(\omega, \sigma, \hat{B})) = \mathcal{W}(\vartheta_t\omega, \theta_t\sigma, \hat{B}), \quad t \in \mathbb{R}^+, \sigma \in \Sigma, \omega \in \Omega, \tag{3.1}$$

and is included in any closed non-autonomous random set $\hat{D}$ which pullback attracts B

$$\mathcal{W}(\omega, \sigma, \hat{B}) \subset \hat{D}_\sigma(\omega), \quad \forall \omega \in \Omega, \sigma \in \Sigma; \tag{3.2}$$

(ii) if, moreover, $\hat{B}$ pullback attracts itself, then $\mathcal{W}(\cdot, \cdot, \hat{B})$ is a non-autonomous random set.

Proof. We prove in a similar way to Lemma 2.2.8.

(i) Let $t_n \to \infty$ and $x_n \in \hat{B}_{\theta_{-t_n}\sigma}(\vartheta_{-t_n}\omega)$. Then since $\hat{B}$ is pullback attracted by $\hat{K}$, by the compactness of $\hat{K}$ there exists a $y \in \hat{K}_\sigma(\omega)$ such that, up to a subsequence,

$$\phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, x_n) \to y$$

which indicates that $y \in \mathcal{W}(\omega,\sigma,\hat{B})$ by Lemma 3.1.3. Hence, $\mathcal{W}(\omega,\sigma,\hat{B})$ is not empty.

To see the compactness, take arbitrarily a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{W}(\omega,\sigma,\hat{B})$. Then by Lemma 3.1.3 we have sequences $x_n \in \hat{B}_{\theta_{-t_n}\sigma}(\vartheta_{-t_n}\omega)$ and $t_n \to \infty$ such that

$$\operatorname{dist}(\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma,x_n),y_n) \leqslant 1/n, \quad \forall n \in \mathbb{N}.$$

On the other hand, by the pullback attraction of $\hat{K}$ and Lemma 3.1.3 again, there exists $y \in \mathcal{W}(\omega,\sigma,\hat{B})$ such that

$$\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma,x_n) \to y$$

in a subsequence sense. Hence, $y_n \to y$ and $\mathcal{W}(\omega,\sigma,\hat{B})$ is compact.

To prove the invariance property, we notice that

$$\begin{aligned}\mathcal{W}(\omega,\sigma,\hat{B}) &= \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\hat{B}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega))}^{S} \\ &= \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\hat{B}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega))}^{W}, \quad \sigma\in\Sigma,\omega\in\Omega,\end{aligned} \tag{3.3}$$

where and hereafter the indicator "S" (resp. "W") nearby the over-line indicates the strong (resp. weak) topology under which the closure is taken. Indeed, for any y lying in the right-hand side term, there exist sequences $t_n \to \infty$ and $x_n \in \hat{B}_{\theta_{-t_n}\sigma}(\vartheta_{-t_n}\omega))$ such that $\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma,x_n) \rightharpoonup y$. On the other hand, since $\hat{B}$ is pullback attracted by $\hat{K}$ and $\hat{K}$ is compact, $\phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma,x_n)$ converge to some $z \in X$ strongly in a subsequence sense, which implies that

$$y = z \in \cap_{s\geqslant 0}\overline{\cup_{t\geqslant s}\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,B_{\theta_{-t}\sigma}(\vartheta_{-t}\omega))}^{S}.$$

So,

$$\bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\hat{B}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega))}^{S} \supset \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\hat{B}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega))}^{W},$$

and then (3.3) follows since the $\subset$ inclusion is trivial.

Now we prove the invariance property. By the pullback attraction property of $\hat{K}$, for any $t \geqslant 0$, $\sigma \in \Sigma$ and $\omega \in \Omega$, there exists a time $T > 0$ such that

$$\begin{aligned}\phi(t+\eta,\vartheta_{-\eta}\omega,\theta_{-\eta}\sigma,\hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega)) &= \phi(t+\eta,\vartheta_{-t-\eta}\vartheta_t\omega,\theta_{-t-\eta}\theta_t\sigma,\hat{B}_{\theta_{-t-\eta}\theta_t\sigma}(\vartheta_{-t-\eta}\vartheta_t\omega)) \\ &\subset \mathcal{N}_1(\hat{K}_{\theta_t\sigma}(\vartheta_t\omega)), \quad \forall \eta \geqslant T.\end{aligned} \tag{3.4}$$

Since for arbitrary function f it holds $f(\cap_\alpha A_\alpha) \subset \cap_\alpha f(A_\alpha)$, we observe that

$$\begin{aligned}\phi(t,\omega,\sigma,\mathcal{W}(\sigma,\omega,\hat{B})) &= \phi\big(t,\omega,\sigma,\cap_{s\geqslant T}\overline{\cup_{\eta\geqslant s}\phi(\eta,\vartheta_{-\eta}\omega,\theta_{-\eta}\sigma,\hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega))}^{S}\big) \\ &\subset \bigcap_{s\geqslant T}\phi\big(t,\omega,\sigma,\overline{\cup_{\eta\geqslant s}\phi(\eta,\vartheta_{-\eta}\omega,\theta_{-\eta}\sigma,\hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega))}^{S}\big),\end{aligned} \tag{3.5}$$

where $T > 0$ is given satisfying (3.4). For any $s \geqslant T$ and

$$y \in \phi\big(t, \omega, \sigma, \overline{\cup_{\eta \geqslant s} \phi(\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, \hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega))}^{S}\big),$$

there exist an $x \in \overline{\cup_{\eta \geqslant s} \phi(\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, \hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega))}^{S}$ and a sequence

$$x_n \in \phi(\eta_n, \vartheta_{-\eta_n}\omega, \theta_{-\eta_n}\sigma, \hat{B}_{\theta_{-\eta_n}\sigma}(\vartheta_{-\eta_n}\omega)) \quad \text{with } \eta_n \geqslant s,$$

such that $y = \phi(t, \omega, \sigma, x)$ and $x_n \to x$. Note that, by (3.4),

$$\begin{aligned} \phi(t, \omega, \sigma, x_n) &\in \phi\big(t, \omega, \sigma, \phi(\eta_n, \vartheta_{-\eta_n}\omega, \theta_{-\eta_n}\sigma, \hat{B}_{\theta_{-\eta_n}\sigma}(\vartheta_{-\eta_n}\omega))\big) \\ &= \phi\big(t + \eta_n, \vartheta_{-\eta_n}\omega, \theta_{-\eta_n}\sigma, \hat{B}_{\theta_{-\eta_n}\sigma}(\vartheta_{-\eta_n}\omega)\big) \\ &\subset \mathcal{N}_1(\hat{K}_{\theta_t\sigma}(\vartheta_t\omega)). \end{aligned} \tag{3.6}$$

Hence, as the 1-neighborhood $\mathcal{N}_1(\hat{K}_{\theta_t\sigma}(\vartheta_t\omega))$ is bounded, the sequence $\phi(t, \omega, \sigma, x_n)$ is bounded and converges weakly to $\phi(t, \omega, \sigma, x) = y$ by the quasi S2W-continuity of ϕ, from which it follows that

$$y \in \overline{\bigcup_{\eta \geqslant s} \phi\big(t, \omega, \sigma, \phi(\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, \hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega))\big)}^{W}.$$

Since y was taken arbitrarily, we have

$$\begin{aligned} &\phi\big(t, \omega, \sigma, \overline{\cup_{\eta \geqslant s} \phi(\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, \hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega))}^{S}\big) \\ &\qquad \subset \overline{\bigcup_{\eta \geqslant s} \phi\big(t, \omega, \sigma, \phi(\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, \hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega))\big)}^{W}, \end{aligned}$$

which along with (3.5) and (3.3) implies that

$$\begin{aligned} \phi(t, \omega, \sigma, \mathcal{W}(\omega, \sigma, \hat{B})) &\subset \bigcap_{s \geqslant T} \phi\big(t, \omega, \sigma, \overline{\cup_{\eta \geqslant s} \phi(\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, \hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega))}^{S}\big) \\ &\subset \bigcap_{s \geqslant T} \overline{\bigcup_{\eta \geqslant s} \phi\big(t, \omega, \sigma, \phi(\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, \hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega))\big)}^{W} \\ &= \bigcap_{s \geqslant T} \overline{\bigcup_{\eta \geqslant s} \phi(t + \eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, \hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega))}^{W} \\ &= \bigcap_{s \geqslant t+T} \overline{\bigcup_{\eta \geqslant s} \phi(\eta, \vartheta_{-\eta} \circ \vartheta_t\omega, \theta_{-\eta} \circ \theta_t\sigma, \hat{B}_{\theta_{-(t+\eta)} \circ \theta_t\sigma}(\vartheta_{-\eta} \circ \vartheta_t\omega))}^{W} \\ &= \mathcal{W}(\vartheta_t\omega, \theta_t\sigma, \hat{B}), \quad \forall t \in \mathbb{R}^+, \omega \in \Omega, \sigma \in \Sigma. \end{aligned}$$

To see $\mathcal{W}(\vartheta_t\omega, \theta_t\sigma, \hat{B}) \subset \phi(t, \omega, \sigma, \mathcal{W}(\omega, \sigma, \hat{B}))$, let $y \in \mathcal{W}(\vartheta_t\omega, \theta_t\sigma, \hat{B})$. Then by Lemma 3.1.3 there exists a sequence

$$\begin{aligned} x_n &\in \phi(t_n, \vartheta_{-t_n}\vartheta_t\omega, \theta_{-t_n}\theta_t\sigma, \hat{B}_{\theta_{t-t_n}\sigma}(\vartheta_{t-t_n}\omega)) \\ &= \phi(t, \omega, \sigma, \phi(t_n - t, \vartheta_{t-t_n}\omega, \theta_{t-t_n}\sigma, \hat{B}_{\theta_{t-t_n}\sigma}(\vartheta_{t-t_n}\omega)) \end{aligned}$$

with $t < t_n \to \infty$ such that $x_n \to y$. Suppose $z_n \in \phi(t_n - t, \vartheta_{t-t_n}\omega, \theta_{t-t_n}\sigma, \hat{B}_{\theta_{t-t_n}\sigma}(\vartheta_{t-t_n}\omega))$ is such that $x_n = \phi(t, \omega, \sigma, z_n)$. By the pullback attraction of K, for any $\sigma \in \Sigma$ and $\omega \in \Omega$, there exists a time $T_0 > 0$ such that

$$\phi(\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, \hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega)) \subset \mathcal{N}_1(\hat{K}_\sigma(\omega)), \quad \forall \eta \geqslant T_0. \tag{3.7}$$

Hence, the sequence x_n is bounded. Moreover, by the pullback attracting and compact properties of $\hat{K}$, there exists a $z \in X$ such that $z_n \to z$ up to a subsequence, which implies that $z \in \mathcal{W}(\omega, \sigma, \hat{B}))$ by Lemma 3.1.3. Hence, by the quasi S2W-continuity of ϕ we have

$$x_n = \phi(t, \omega, \sigma, z_n) \rightharpoonup \phi(t, \omega, \sigma, z).$$

Thus, by the uniqueness of a limit, we have $y = \phi(t, \omega, \sigma, z) \in \phi(t, \omega, \sigma, \mathcal{W}(\omega, \sigma, \hat{B}))$. The invariance is clear.

Finally, we prove (3.2). Take arbitrarily $y \in \mathcal{W}(\omega, \sigma, \hat{B})$, then by Lemma 3.1.3 we have a sequence

$$x_n \in \phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, \hat{B}_{\theta_{-t_n}\sigma}(\vartheta_{-t_n}\omega))$$

with $t_n \to \infty$ such that $x_n \to y$. By the pullback attracting property and the closedeness of $\hat{D}$ we know $y \in \hat{D}_\sigma(\omega)$. Therefore, $\mathcal{W}(\omega, \sigma, \hat{B}) \subset \hat{D}_\sigma(\omega)$ and (3.2) follows.

(ii) Let us prove the measurability of the mapping $\omega \mapsto \mathcal{W}(\omega, \sigma, \hat{B})$. First, let us prove that

$$\begin{aligned}\mathcal{W}(\omega, \sigma, \hat{B}) &= \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m=n}^{\infty} \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, \hat{B}_{\theta_{-t}\sigma}(\vartheta_{-m}\omega))}^{S} \\ &= \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m=n}^{\infty} \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, \hat{B}_{\theta_{-t}\sigma}(\vartheta_{-m}\omega))}^{W}, \quad \forall \omega \in \Omega, \sigma \in \Sigma.\end{aligned} \tag{3.8}$$

Since $\hat{B}$ pullback attracts itself, by (3.2) we have $\mathcal{W}(\omega, \sigma, \hat{B}) \subset \hat{B}_\sigma(\omega)$ for each $\sigma \in \Sigma$ and $\omega \in \Omega$. Hence, by the invariance (3.1) of $\mathcal{W}(\omega, \sigma, \hat{B})$, we have

$$\begin{aligned}\mathcal{W}(\omega, \sigma, \hat{B}) &= \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, \mathcal{W}(\vartheta_{-m}\omega, \theta_{-m}\sigma, \hat{B})) \\ &\subset \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, \hat{B}_{\theta_{-m}\sigma}(\vartheta_{-m}\omega)), \quad \forall m \in \mathbb{N}.\end{aligned}$$

Therefore,

$$\mathcal{W}(\omega, \sigma, \hat{B}) \subset \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m=n}^{\infty} \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, \hat{B}_{\theta_{-m}\sigma}(\vartheta_{-m}\omega))}^{S},$$

and thereby, since the inverse inclusion is straightforward, the first identity of (3.8) holds. Similarly to (3.3) we have the second identity. Hence, (3.8) holds true.

By (3.8) it is elementary to check that

$$\mathcal{W}(\omega, \sigma, \hat{B}) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m=n}^{\infty} \overline{\phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, \hat{B}_{\theta_{-t}\sigma}(\vartheta_{-m}\omega))}^{W}}^{W}. \tag{3.9}$$

Recall from (3.7) that for any $\sigma \in \Sigma$ and $\omega \in \Omega$, there exists a time $T_0 > 0$ such that

$$\phi(\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, \hat{B}_{\theta_{-\eta}\sigma}(\vartheta_{-\eta}\omega)) \subset \mathcal{N}_1(\hat{K}_\sigma(\omega)), \quad \forall \eta \geqslant T_0.$$

Then, since $\hat{B}$ is a non-empty closed random set, by Lemma 2.2.4 (II) there exists a sequence $\{f_j\}_{j\in\mathbb{N}}$ of measurable functions such that $\hat{B}_{\theta_{-t}\sigma}(\vartheta_{-m}\omega) = \overline{\cup_{j\in\mathbb{N}} f_j(\vartheta_{-m}\omega)}^S$, which makes

$$\overline{\phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, \hat{B}_{\theta_{-t}\sigma}(\vartheta_{-m}\omega))}^W = \overline{\bigcup_{j\in\mathbb{N}} \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, f_j(\omega))}^W, \ \forall m \geqslant T_0, \tag{3.10}$$

as $x \to \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, x)$ is quasi S2W-continuous. Hence, by (3.9) and (3.10) we have

$$\begin{aligned}
\mathcal{W}(\omega, \sigma, \hat{B}) &= \bigcap_{n\geqslant T_0} \overline{\bigcup_{m=n}^{\infty} \overline{\bigcup_{j\in\mathbb{N}} \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, f_j(\omega))}^W}^W \\
&= \bigcap_{n\geqslant T_0} \overline{\bigcup_{m=n}^{\infty} \bigcup_{j\in\mathbb{N}} \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, f_j(\omega))}^W \\
&= \bigcap_{n\geqslant T_0} \overline{\bigcup_{m=n}^{\infty} \bigcup_{j\in\mathbb{N}} \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, f_j(\omega))}^S,
\end{aligned}$$

where the last identity is established similar to (3.8).

As each $\phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, x)$ is $(\mathcal{F}, \mathcal{B}(X))$-measurable in ω and $(\mathcal{B}(X), \mathcal{B}(X))$ measurable in x, the mapping $\omega \mapsto \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, f_j(\omega))$ is $(\mathcal{F}, \mathcal{B}(X))$-measurable. Hence, as a single-valued mapping, $\omega \mapsto \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, f_j(\omega))$ is measurable in the sense of Definition 2.2.2 as well. Denote by

$$D_n(\omega) = \overline{\bigcup_{m=n}^{\infty} \bigcup_{j\in\mathbb{N}} \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma, f_j(\omega))}^S, \quad \forall n \in \mathbb{N},\ \omega \in \Omega.$$

Then by Lemma 2.2.4 (I) we know each $D_n(\cdot)$ is measurable.

On the other hand, clearly, D_n is decreasing and each sequence $\{x_n\}$ inside $\mathcal{W}(\omega, \sigma, B)$ is pre-compact since $\mathcal{W}(\omega, \sigma, B)$ is compact itself. By Lemma 2.2.4 (I) we conclude that $\mathcal{W}(\omega, \sigma, B) = \cap_{n\in\mathbb{N}} \overline{D_n(\omega)}$ is measurable. The proof is complete. □

Theorem 3.2.2. *Suppose that ϕ is a quasi S2W-continuous NRDS on X. If ϕ has a compact $\hat{\mathcal{D}}_X$-pullback attracting set $\hat{K}$ and a $\hat{\mathcal{D}}_X$-pullback absorbing set $\hat{B} \in \hat{\mathcal{D}}_X$, then ϕ has a $\hat{\mathcal{D}}_X$-cocycle attractor $\hat{A} \in \hat{\mathcal{D}}_X$ given by*

$$\hat{A}_\sigma(\omega) = \mathcal{W}(\omega, \sigma, \hat{B}), \quad \forall \sigma \in \Sigma,\ \omega \in \Omega.$$

Proof. By Lemma 3.2.1 it is clear that $\hat{A}$ is a compact non-autonomous random set which is invariant and pullback attracts $\hat{B}$. Moreover, $\hat{A}$ is smaller than $\hat{B}$ and hence belongs to $\hat{\mathcal{D}}_X$ since $\hat{\mathcal{D}}_X$ is inclusion-closed. We now prove the $\hat{\mathcal{D}}$-attracting property. Since $\hat{A}$ pullback attracts $\hat{B}$, for each $\varepsilon > 0$ and $\sigma \in \Sigma$, $\omega \in \Omega$ fixed, there is a time $T > 0$ such that

$$\operatorname{dist}\Big(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \hat{B}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega)), \hat{A}_\sigma(\omega)\Big) < \varepsilon, \quad \forall t \geqslant T.$$

On the other hand, for each $\hat{D} \in \hat{\mathcal{D}}_X$ and $\omega \in \Omega$, $\sigma \in \Sigma$, there is a time $T_{\hat{D}}(\omega, \sigma) > 0$ such that

$$\bigcup_{t \geqslant T_{\hat{D}}(\omega,\sigma)} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \hat{D}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega)) \subset \hat{B}_\sigma(\omega)$$

as $\hat{B}$ is a $\hat{\mathcal{D}}_X$-pullback absorbing set. Hence,

$$\begin{aligned}
&\operatorname{dist}\Big(\phi(t+T, \vartheta_{-t-T}\omega, \theta_{-t-T}\sigma, \hat{D}_{\theta_{-t-T}\sigma,}(\vartheta_{-t-T}\omega)), \hat{A}_\sigma(\omega)\Big)\\
&\quad = \operatorname{dist}\Big(\phi(T, \vartheta_{-T}\omega, \theta_{-T}\sigma, \phi(t, \vartheta_{-t}\vartheta_{-T}\omega, \theta_{-t}\theta_{-T}\sigma, \hat{D}_{\theta_{-t-T}\sigma}(\vartheta_{-t}\vartheta_{-T}\omega)), \hat{A}_\sigma(\omega)\Big)\\
&\quad \leqslant \operatorname{dist}\Big(\phi(T, \vartheta_{-T}\omega, \theta_{-T}\sigma, \hat{B}_{\theta_{-T}\sigma}\vartheta_{-T}\omega)), \hat{A}_\sigma(\omega)\Big)\\
&\quad < \varepsilon, \quad \forall t \geqslant T_{\hat{D}}(\vartheta_{-T}\omega, \theta_{-T}\sigma),
\end{aligned}$$

which indicates that $\hat{A}$ pullback attracts $\hat{D}$. The proof is complete. □

The following result indicates a close relationship between compact attracting sets and the cocycle attractor.

Theorem 3.2.3. *Suppose ϕ is a quasi-S2W continuous NRDS with a compact $\hat{\mathcal{D}}_X$-pullback attracting set $\hat{K} \in \hat{\mathcal{D}}_X$. Then ϕ has a unique $\hat{\mathcal{D}}_X$-random cocycle attractor $\hat{A} \in \hat{\mathcal{D}}_X$ given by*

$$\hat{A}_\sigma(\omega) = \mathcal{W}(\omega, \sigma, \hat{K}). \tag{3.11}$$

Proof. Since $\hat{K}$ is a compact $\hat{\mathcal{D}}_X$-pullback attracting set, by the neighborhood-closedness of $\hat{\mathcal{D}}_X$, the closed ε-neighborhood $\hat{K}^\varepsilon$ of $\hat{K}$, that is,

$$\hat{K}^\varepsilon_\sigma(\omega) := \{x \in X : \operatorname{dist}(x, \hat{K}_\sigma(\omega)) \leqslant \varepsilon\}, \quad \forall \sigma \in \Sigma, \omega \in \Omega$$

for some $\varepsilon > 0$, is a measurable closed $\hat{\mathcal{D}}_X$-pullback absorbing set of ϕ belonging to $\hat{\mathcal{D}}_X$. Hence, by Theorem 3.2.2 we know ϕ has a unique $\hat{\mathcal{D}}_X$-random cocycle attractor $\hat{A}$ with the form

$$\hat{A}_\sigma(\omega) = \mathcal{W}(\omega, \sigma, \hat{K}^\varepsilon).$$

Clearly, $\hat{A}_\sigma(\omega) \supseteq \mathcal{W}(\omega, \sigma, \hat{K})$ since $\hat{K}^\varepsilon \supseteq \hat{K}$. Thus to prove (3.11) we shall prove $\hat{A}_\sigma(\omega) \subseteq \mathcal{W}(\omega, \sigma, \hat{K})$. Note that $\mathcal{W}(\omega, \sigma, \hat{A}) \subseteq \mathcal{W}(\omega, \sigma, \hat{K})$ since $\hat{A} \subset \hat{K}$ by the minimal property of $\hat{A}$. Therefore, by the invariance of $\hat{A}$, we have

$$\begin{aligned}
\hat{A}_\sigma(\omega) &= \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \hat{A}_\sigma(\omega)} = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \hat{A}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega))}\\
&= \mathcal{W}(\omega, \sigma, \hat{A}) \subseteq \mathcal{W}(\omega, \sigma, \hat{K}),
\end{aligned}$$

which completes the proof. □

3.2.2 Alternative dynamical compactnesses

Theorem 3.2.3 implies a direct relationship between compact attracting sets and cocycle attractors. However, the existence of a compact attracting set is often nontrivial to establish. Therefore, several dynamical compactnesses have been introduced in attractor theory, such as asymptotic compactness, pullback omega-limit compactness, asymptotic contraction, flattening and squeezing properties [68, 53, 38, 30, 77, 88], etc. These dynamical compactnesses are sometimes more convenient to use especially in cases where Sobolev compactness embeddings are not available.

Definition 3.2.4. An NRDS ϕ on X is called $\hat{\mathcal{D}}_X$-(pullback) flattening if for each $\hat{D} \in \hat{\mathcal{D}}_X$, $\varepsilon > 0$ $\sigma \in \Sigma$ and $\omega \in \Omega$ there exist a $T_0 = T_0(\hat{D}, \varepsilon, \sigma, \omega) > 0$ and a finite-dimensional subspace X_ε of X such that

(i) $\cup_{t \geqslant T_0} P_\varepsilon \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \hat{D}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega))$ is bounded, and

(ii) $\|(I - P_\varepsilon) \cup_{t \geqslant T_0} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \hat{D}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega))\| < \varepsilon$,

where $P_\varepsilon : X \mapsto X_\varepsilon$ is a bounded projection.

Definition 3.2.5. An NRDS ϕ on X is called $\hat{\mathcal{D}}_X$-(pullback) omega-limit compact if for each $\hat{D} \in \hat{\mathcal{D}}_X$, $\varepsilon > 0$, $\sigma \in \Sigma$ and $\omega \in \Omega$ there exists a $T_1 = T_1(\hat{D}, \varepsilon, \sigma, \omega) > 0$ such that

$$\kappa\left(\bigcup_{t \geqslant T_1} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \hat{D}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega)) \right) < \varepsilon,$$

where κ denotes the Kuratowski measure [68] of noncompactness of sets defined as

$$\kappa(B) = \inf \left\{ \delta : B \text{ has a finite cover by balls of } X \text{ of diameter less than } \delta \right\}, \quad \forall B \subset X.$$

Definition 3.2.6. An NRDS ϕ on X is called $\hat{\mathcal{D}}_X$-(pullback) asymptotically compact if for each $\hat{D} \in \hat{\mathcal{D}}_X$, $\sigma \in \Sigma$, $\omega \in \Omega$ and any sequences $0 < t_n \to \infty$ and $x_n \in \hat{D}_{\theta_{-t_n}\sigma}(\vartheta_{-t_n}\omega)$, the set $\{\phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, x_n)\}_{n \in \mathbb{N}}$ is precompact in X.

The following theorem implies that these dynamical compactnesses could replace the requirement of a compact $\hat{\mathcal{D}}_X$-attracting set $\hat{K}$ in Theorems 3.2.2 and 3.2.3.

Theorem 3.2.7. *Suppose that X is a uniformly convex Banach space (particularly, a Hilbert space). The following dynamical compactness properties of an NRDS ϕ on X are equivalent:*

(i) $\hat{\mathcal{D}}_X$-(pullback) flattening;

(ii) $\hat{\mathcal{D}}_X$-(pullback) omega-limit compactness;

(iii) $\hat{\mathcal{D}}_X$-(pullback) asymptotically compactness,

where the uniformly convex property of X is only for the relation (iii)⇒(i). Moreover, each of these dynamical compactnesses implies that the omega-limit set $\mathcal{W}(\cdot, \cdot, \hat{B})$ of a $\hat{\mathcal{D}}_X$-pullback absorbing set $\hat{B} \in \hat{\mathcal{D}}_X$ is a compact $\hat{\mathcal{D}}_X$-pullback attracting random set.

Proof. Similar to, e.g., [53, Theorems 4.5 & 4.6], or [30, Section 2]. □

3.2.3 The bi-spatial case

In this part, we study bi-spatial cocycle attractors for NRDS, generalizing analogous arguments in Section 2.2.3.

Let $(Y, \|\cdot\|_Y)$ be another separable Banach space such that $Y \hookrightarrow X$ continuously and $X^* \hookrightarrow Y^*$ densely, where X^* and Y^* are dual spaces of X and Y, respectively. Denote by $\hat{\mathcal{D}}_Y$ some inclusion- and neighborhood-closed universe of non-autonomous random sets in Y such that $\hat{\mathcal{D}}_Y \subset \hat{\mathcal{D}}_X$.

Definition 3.2.8. An NRDS ϕ on X is said to be (X, Y)-dissipative (on the universe $\hat{\mathcal{D}}_X$) if

- ϕ is an NRDS when restricted on Y, i.e. satisfying Definition 3.1.1 with X replaced by Y;
- there exists a non-autonomous random set $\hat{\mathrm{B}}$ in Y which belongs to $\hat{\mathcal{D}}_X$ and is $\hat{\mathcal{D}}_X$-pullback absorbing.

Note that we did not require $\hat{\mathrm{B}}$ to belong to $\hat{\mathcal{D}}_Y (\subset \hat{\mathcal{D}}_X)$.

Definition 3.2.9. Given an (X, Y)-dissipative NRDS ϕ, a non-autonomous random set $\hat{A}$ in Y is called the (X, Y)*-cocycle attractor with attraction universe* $\hat{\mathcal{D}}_X$ for ϕ if

(I) $\hat{A} \in \hat{\mathcal{D}}_X$ and $\hat{A}$ is a compact non-autonomous random set in Y;

(II) $\hat{A}$ pullback attracts elements in $\hat{\mathcal{D}}_X$ in the topology of Y, i.e., for each $\hat{D} \in \hat{\mathcal{D}}_X$ and any $\varepsilon > 0$ there exists a $T > 0$ such that $\mathrm{dist}_Y(\phi(s, \vartheta_{-s}\omega, \theta_{-t}\sigma, \hat{D}_{\theta_{-t}\sigma}(\theta_{-s}\omega)), \hat{A}_\sigma(\omega)) < \varepsilon$ holds for all $s \geqslant T$;

(III) $\hat{A}$ is invariant under ϕ.

Remark 3.2.10. Notice that the (X, Y)-random cocycle attractor here is required to be *measurable* in Y, instead of only to be measurable in X as described in [63, 40, 61], etc. As the pullback attraction of an (X, Y)-attractor is expected under the distance of Y, the mapping $\omega \mapsto \mathrm{dist}_Y(x, \hat{A}_\sigma(\omega))$ should be a random variable so that the attraction in Y makes sense.

The following existence theorem for cocycle attractors requires only the continuity of the NRDS in less regular spaces. We write the omega-limit set in X as $\mathcal{W}(\cdot, \cdot, B)^X$ to indicate the X-topology.

Theorem 3.2.11. *Suppose that ϕ is a quasi-S2W continuous RDS on X. If ϕ is an NRDS on Y, and has a $\hat{\mathcal{D}}_Y$-pullback absorbing set $\hat{B} \in \hat{\mathcal{D}}_Y$ and a compact $\hat{\mathcal{D}}_Y$-pullback attracting set $\hat{K}$ in Y, then ϕ has a $\hat{\mathcal{D}}_Y$-cocycle attractor $\hat{A} \in \hat{\mathcal{D}}_Y$ in Y given by*

$$\hat{A}_\sigma(\omega) = \mathcal{W}(\omega, \sigma, B)^Y.$$

Proof. Since the RDS ϕ is quasi-S2W continuous in X, so it is in Y by Proposition 2.1.5. Therefore, the theorem follows directly from Theorem 3.2.2. □

Now we establish an existence criterion for bi-spatial cocycle attractors.

Theorem 3.2.12. *Suppose that ϕ is a quasi-S2W continuous NRDS on X, and is (X, Y)-dissipative on $\hat{\mathcal{D}}_X$ (with a $\hat{\mathcal{D}}_X$-pullback absorbing set $\hat{\mathbf{B}}$ which is a non-autonomous random set in Y belonging to $\hat{\mathcal{D}}_X$ but unnecessarily belonging to $\hat{\mathcal{D}}_Y$). Then if there exists a compact non-autonomous random set $\hat{K}$ in Y which is $\hat{\mathcal{D}}_X$-pullback attracting under ϕ under the topology of Y, then ϕ has an (X, Y)-cocycle attractor $\hat{A} \in \hat{\mathcal{D}}_X$ with attraction universe $\hat{\mathcal{D}}_X$, given by*

$$\hat{A}_\sigma(\omega) = \mathcal{W}(\omega, \sigma, \hat{\mathbf{B}})^X = \mathcal{W}(\omega, \sigma, \hat{\mathbf{B}})^Y.$$

If, moreover, $\hat{\mathbf{B}} \in \hat{\mathcal{D}}_Y$, then $\hat{A} \in \hat{\mathcal{D}}_Y$.

Proof. By Theorem 3.2.2 it is clear that ϕ has a $\hat{\mathcal{D}}_X$-cocycle attractor $\hat{A}$ in X given by $\hat{A}_\sigma(\cdot) = \mathcal{W}(\cdot, \sigma, \hat{\mathbf{B}})^X$. Now we claim that $\mathcal{W}(\cdot, \sigma, \hat{\mathbf{B}})^X = \mathcal{W}(\cdot, \sigma, \hat{\mathbf{B}})^Y$. Indeed, for a $y \in \mathcal{W}(\omega, \sigma, \hat{\mathbf{B}})^X$, there exist sequences $t_n \to \infty$ and $x_n \in \hat{\mathbf{B}}(\vartheta_{-t_n}\omega)$ such that $\phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, x_n) \to y$ in X. On the other hand, since $\hat{\mathbf{B}}$ is attracted by a compact random set $\hat{K}$ in Y, there exits a $y' \in Y$ such that, up to a subsequence, $\phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, x_n) \to y'$ in Y. By the uniqueness of a limit we have $y = y' \in \mathcal{W}(\omega, \sigma, \hat{\mathbf{B}})^Y$, and thereby $\mathcal{W}(\cdot, \sigma, \hat{\mathbf{B}})^X = \mathcal{W}(\cdot, \sigma, \hat{\mathbf{B}})^Y$. Hence $\hat{A}_\sigma(\cdot) = \mathcal{W}(\cdot, \sigma, \hat{\mathbf{B}})^Y$, which along with Lemma 3.2.1 and Theorem 3.2.2 implies that $\hat{A}$ is a compact random set in Y and pullback attracts $\hat{\mathcal{D}}_X$ in the topology of Y. Since $\hat{A}$ is clearly invariant, it is indeed the (X, Y)-cocycle attractor. If $\hat{B}$ is in $\hat{\mathcal{D}}_X$ or $\hat{\mathcal{D}}_Y$, then so is $\hat{A}$ due to the inclusion-closedness of $\hat{D}_X$ and $\hat{D}_Y$ as $\hat{A} \subset \hat{\mathbf{B}}$. □

3.3 Applications to a stochastic Ginzburg-Landau equation

Given $\tau \in \mathbb{R}$ and $t \geqslant \tau$, consider the following Ginzburg-Landau equation

$$\mathrm{d}u = \left[\left(\lambda + \mathrm{i}\alpha(t)\right)\triangle u - \left(\kappa + \mathrm{i}\beta(t)\right)|u|^2 u + \delta u + g(x,t)\right]\mathrm{d}t + h(x)\mathrm{d}\omega(t), \tag{3.12}$$

defined on $\mathcal{I} := (0, 1) \subseteq \mathbb{R}$, with initial-boundary value conditions

$$u(x, \tau) = u_0(x), \quad u(x, \cdot)|_{\partial\mathcal{I}} \equiv 0, \tag{3.13}$$

where the unknown $u(x,t)$ is a complex-valued function, dispersion coefficients $\alpha(t)$, $\beta(t)$ and external force $g(x,t)$ are all time-dependent and real-valued functions. λ, κ and δ are positive constants and $h(x) \in H_0^1(\mathcal{I}) \cap H^2(\mathcal{I})$. ω comes from the Wiener probability space $(\Omega, \mathcal{F}, \mathcal{P})$ introduced in Section 2.3.1.

To define an NRDS for system (3.12), let us define a group $\{\theta_s\}_{s\in\mathbb{R}}$ acting on $\mathbb{R}$ by

$$\theta_s\tau = \tau + s \quad \text{for all } s, \tau \in \mathbb{R}. \tag{3.14}$$

Then $\{\theta_s\}_{s\in\mathbb{R}}$ is a base flow on $\mathbb{R}$, (see Section 2.2.1, taking $\Sigma = \mathbb{R}$).

Consider the one-dimensional Ornstein-Uhlenbeck equation

$$\mathrm{d}y - \delta y \mathrm{d}t = \mathrm{d}\omega, \tag{3.15}$$

of which a stationary solution is provided by

$$y(t) = y(\vartheta_t\omega) \equiv -\delta \int_0^\infty e^{-\delta\tau}(\vartheta_t\omega)(\tau)\mathrm{d}\tau, \quad t \in \mathbb{R}.$$

It is known that there exists a ϑ_t-invariant set $\tilde{\Omega} \subseteq \Omega$ with $\mathcal{P}(\tilde{\Omega}) = 1$ such that $y(\vartheta_t\omega)$ is continuous in t for every $\omega \in \tilde{\Omega}$, and the random variable $|y(\vartheta_t\omega)|$ is tempered (see Lemma 2.3.1 and also, e.g., [83, 84, 1, 33]). Hereafter we will not distinguish $\widetilde{\Omega}$ from Ω. By [1, Proposition 4.3.3] (see also [82, 93, 90]), for any $\gamma > 0$ there exists a tempered variable $r(\omega) > 0$ such that

$$|y(\omega)|^6 \leqslant r(\omega), \tag{3.16}$$

with $r(\omega)$ satisfying

$$r(\vartheta_t\omega) \leqslant e^{\frac{\gamma}{2}|t|}r(\omega), \quad t \in \mathbb{R}. \tag{3.17}$$

Let $z(\vartheta_t\omega) = hy(\vartheta_t\omega)$, $f(s) = |s|^2 s$ and

$$v(t,\tau,\omega,v_0) = u(t,\tau,\omega,u_0) - z(\vartheta_t\omega) \quad \text{with } v_0 = u_0 - z(\vartheta_\tau\omega). \tag{3.18}$$

Then if $u(t)$ solves (3.12)-(3.13), $v(t)$ should satisfy, by (3.15), (3.18) and $h(x) \in H_0^1(\mathcal{I}) \cap H^2(\mathcal{I})$,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = (\lambda + \mathrm{i}\alpha(t))\triangle v - (\kappa + \mathrm{i}\beta(t))f(v + z(\vartheta_t\omega)) + \delta v + g(x,t) + (\lambda + \mathrm{i}\alpha(t))\triangle z(\vartheta_t\omega), \tag{3.19}$$

with conditions

$$v(\tau,\tau,\omega,v_0) = v_0 = u_0 - z(\vartheta_\tau\omega), \tag{3.20}$$

$$v(t,\tau,\omega,v_0)|_{\partial\mathcal{I}} = 0, \tag{3.21}$$

for all $t \geqslant \tau \in \mathbb{R}$ and $x \in \mathcal{I} = (0,1)$. Since (3.19)-(3.21) is a deterministic problem, by the 'standard' Galërkin method as in [26] or similar arguments of [81] (see also [79, 89, 29] for autonomous G.-L. equations), we have the following well-possessedness result. We write $H = \big(L^2(\mathcal{I}), \|\cdot\|\big)$ and $V = \big(H_0^1(\mathcal{I}), \|\cdot\|_V\big)$.

Lemma 3.3.1. *Assume that*

(i) $\lambda \in \mathbb{R}^+$, $\kappa \in \mathbb{R}^+$, $\delta \in \mathbb{R}^+$, $h(x) \in H_0^1(\mathcal{I};\mathbb{C}) \cap H^2(\mathcal{I};\mathbb{C})$*;*

(ii) $\beta(t) \in C(\mathbb{R};\mathbb{R})$ *and* $\sup_{t\in\mathbb{R}} |\beta(t)| \leqslant \sqrt{3}\kappa$*;*

(iii) $\alpha(t) \in C(\mathbb{R};\mathbb{R})$ *and* $g(x,t) \in L^2_{loc}(\mathbb{R};H)$ *are such that*

$$\int_{-\infty}^{\tau} e^{\frac{\gamma}{2}s}\big(|\alpha(t)|^4 + \|g(x,s)\|^2\big)\mathrm{d}s < +\infty \quad \textit{for every } \tau \in \mathbb{R}.$$

Then, for each $v_0 \in H$*, the initial-boundary value problem* (3.19)-(3.21) *has a unique weak solution*

$$v(t,\tau,\omega,v_0) \in C([\tau,\infty);H) \cap L^2_{loc}(\tau,\infty;V) \cap L^4_{loc}(\tau,\infty;L^4(\mathcal{I})).$$

Besides, $v(t,\tau,\omega,v_0)$ *is* $(\mathcal{F},\mathcal{B}(H))$*-measurable in* $\omega \in \Omega$ *and continuous in* v_0 *in* H *for each* $t \geqslant \tau$*.*

Let

$$u(t,\tau,\omega,u_0)=v(t,\tau,\omega,v_0)+z(\vartheta_t\omega)\quad \text{with } u_0=v_0+z(\vartheta_\tau\omega). \tag{3.22}$$

Then under assumptions of Lemma 3.3.1 it is evident that u solves problem (3.12)-(3.13), and is $(\mathcal{F},\mathcal{B}(H))$-measurable in $\omega\in\Omega$, continuous in both $t\geqslant\tau$ and $u_0\in H$. Consider the mapping $\phi:\mathbb{R}^+\times\mathbb{R}\times\Omega\times H\to H$ with

$$\phi(t,\tau,\omega,u_0)=u(t+\tau,\tau,\vartheta_{-\tau}\omega,u_0)=v(t+\tau,\tau,\vartheta_{-\tau}\omega,v_0)+z(\vartheta_t\omega), \tag{3.23}$$

where $v_0=u_0-z(\omega)$. By the property of solution trajectories of well-possessed non-autonomous dynamical systems one can readily check that (3.23) defines a continuous cocycle ϕ for problem (3.12)-(3.13) on H with base flows $\{\theta_t\}_{t\in\mathbb{R}}$ and $\{\vartheta_t\}_{t\in\mathbb{R}}$ acting on $\mathbb{R}$ and Ω, respectively.

Now, for an arbitrarily fixed $\gamma>0$, define

$$\hat{\mathcal{D}}_H=\Big\{\hat{D}:\hat{D}\text{ is a non-autonomous random set in }H\text{, satisfying}$$
$$\lim_{t\to\infty}e^{-\gamma t}\|D(\theta_{-t}\tau,\vartheta_{-t}\omega)\|_H^2=0\text{ for each }\tau,\omega\text{ fixed}\Big\},$$

and

$$\hat{\mathcal{D}}_V=\Big\{\hat{D}:\hat{D}\text{ is a non-autonomous random set in }V\text{, satisfying}$$
$$\lim_{t\to\infty}e^{-\gamma t}\|D(\theta_{-t}\tau,\vartheta_{-t}\omega)\|_V^2=0\text{ for each }\tau,\omega\text{ fixed}\Big\}.$$

Then $\hat{\mathcal{D}}_H$ and $\hat{\mathcal{D}}_V$ are inclusion- and neighborhood-closed universes.

In the following, for the non-autonomous stochastic Ginzburg-Landau system (3.12) we study the (H,V)-cocycle attractor belonging to $\hat{\mathcal{D}}_V$ with attraction universe $\hat{\mathcal{D}}_H$. We begin with uniform estimates of solutions.

3.3.1 Uniform estimates of solutions

In this section we estimate the solution of problem (3.12)-(3.13). First, by Young's inequality and Gagliardo-Nirenberg's inequality (see Appendix), we write the following lemma.

Lemma 3.3.2. *Let $\mathcal{I}\subseteq\mathbb{R}$. Then it holds for every well-defined function n defined on $\mathcal{I}$ that*

$$\|n\|_4^4\leqslant C\|n\|^3\|\nabla n\|,\quad \|n\|_6^6\leqslant C\|n\|^4\|\nabla n\|^2,\quad \|\nabla n\|^2\leqslant C\|\triangle n\|^{\frac{6}{7}}\|n\|_4^{\frac{8}{7}}. \tag{3.24}$$

Lemma 3.3.3. *Let assumptions of Lemma 3.3.1 hold. Then for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $\hat{D}\in\hat{\mathcal{D}}_H$, there exists a $T=T(\tau,\omega,\hat{D})\geqslant 1$ and a positive constant L, which depends on γ but is independent of τ, ω and $\hat{D}$, such that the solution $v(t,\tau,\omega,v_0)$ with $v_0\in\hat{D}(\theta_{-t}\tau,\vartheta_{-\tau}\omega)$ of (3.19)-(3.21) satisfies, for all $t\geqslant T$, that*

$$\|v(\tau,\theta_{-t}\tau,\vartheta_{-t}\omega,v_0)\|^2\leqslant Le^{-\gamma\tau}\int_{-\infty}^{\tau}e^{\gamma s}\big(|\alpha(t)|^4+\|g(x,s)\|^2\big)\mathrm{d}s+Lr(\omega)+L, \tag{3.25}$$

$$\begin{aligned}\int_{\theta_{-t}\tau}^{\tau} e^{\gamma s}\|\nabla v(s,\theta_{-t}\tau,\vartheta_{-\tau}\omega,v_0)\|^2\,\mathrm{d}s+\int_{\theta_{-t}\tau}^{\tau} e^{\gamma s}\|v(s,\theta_{-t}\tau,\vartheta_{-\tau}\omega,v_0)\|_4^4\,\mathrm{d}s\\ \leqslant L\int_{-\infty}^{\tau} e^{\gamma s}\big(|\alpha(t)|^4+\|g(x,s)\|^2\big)\mathrm{d}s+Le^{\gamma\tau}r(\omega)+L,\end{aligned}\tag{3.26}$$

where $r(\omega)$ is the tempered random variable given by (3.16) and (3.17).

Proof. Taking the inner product of (3.19) with v in H and taking the real part, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2=&-\lambda\|\nabla v\|^2-\mathrm{Re}\big((\kappa+\mathrm{i}\beta(t))f(v+z(\vartheta_t\omega)),v\big)+\delta\|v\|^2\\ &+\mathrm{Re}(g(t),v)+\mathrm{Re}\big((\lambda+\mathrm{i}\alpha)\triangle z(\vartheta_t\omega),v\big).\end{aligned}\tag{3.27}$$

By conditions $f(s)=|s|^2s$, $|\beta(t)|\leqslant\sqrt{3}\kappa$ and $z(\vartheta_t\omega)=hy(\vartheta_t\omega)$ we derive that

$$\begin{aligned}&-\mathrm{Re}\big((\kappa+\mathrm{i}\beta(t))f(v+z(\vartheta_t\omega)),v\big)\\ &=-\mathrm{Re}\big((\kappa+\mathrm{i}\beta(t))|v+z(\vartheta_t\omega)|^2(v+z(\vartheta_t\omega)),v\big)\\ &=-\mathrm{Re}\Big((\kappa+\mathrm{i}\beta(t))\big(|v|^2v+2v|z|^2+v^2\bar{z}+|z|^2z+\bar{v}z^2+2|v|^2z\big),v\Big)\\ &\leqslant-\kappa\|v\|_4^4+\Big|\int_{\mathcal{I}}(\kappa+\mathrm{i}\beta(t))\Big(v|v|^2\bar{z}+|z|^2z\bar{v}+\bar{v}^2z^2+2|v|^2z\bar{v}\Big)\mathrm{d}x\Big|\\ &\leqslant-\kappa\|v\|_4^4+2\kappa\int_{\mathcal{I}}\big(3|v|^3|z|+|z|^3|v|+|v|^2|z|^2\big)\mathrm{d}x\\ &\leqslant-\kappa\|v\|_4^4+\frac{\kappa}{2}\|v\|_4^4+c\|z\|_4^4\quad\big(\text{by Lemma A.1.1}\big)\\ &\leqslant-\frac{\kappa}{2}\|v\|_4^4+c|y(\vartheta_t\omega)|^4\quad\big(\text{by (3.24) and } h\in H_0^1(\mathcal{I})\big),\end{aligned}\tag{3.28}$$

where $c=c(\kappa,\|h\|,\|\nabla h\|)$. Since similarly we have

$$\begin{aligned}&\mathrm{Re}(g(t),v)+\mathrm{Re}\big((\lambda+\mathrm{i}\alpha(t))\triangle z(\vartheta_t\omega),v\big)\\ &\leqslant\frac{1}{2}\|g(x,t)\|^2+\frac{1}{2}\|v\|^2+\frac{\lambda}{2}\|\nabla v\|^2+c|\lambda+\mathrm{i}\alpha(t)|^2|y(\vartheta_t\omega)|^2\\ &\leqslant\frac{1}{2}\|g(x,t)\|^2+\frac{1}{2}\|v\|^2+\frac{\lambda}{2}\|\nabla v\|^2+c\Big(|\alpha(t)|^4+|y(\vartheta_t\omega)|^4+|y(\vartheta_t\omega)|^2\Big),\end{aligned}\tag{3.29}$$

where $c=c(\lambda,\|\nabla h\|)$, from (3.27)-(3.29) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\lambda\|\nabla v\|^2+\kappa\|v\|_4^4\leqslant(2\delta+1)\|v\|^2+\|g(x,t)\|^2+c\big(|\alpha(t)|^4+|y(\vartheta_t\omega)|^4+|y(\vartheta_t\omega)|^2\big),$$

which implies that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\gamma\|v\|^2+\lambda\|\nabla v\|^2+\frac{\kappa}{2}\|v\|_4^4\\ &\leqslant-\frac{\kappa}{2}\int_{\mathcal{I}}\Big(|v|^2-\frac{2\delta+1+\gamma}{\kappa}\Big)^2\mathrm{d}x+\frac{\kappa}{2}\int_{\mathcal{I}}\Big(\frac{2\delta+1+\gamma}{\kappa}\Big)^2\mathrm{d}x\\ &\quad+\|g(x,t)\|^2+c\big(|\alpha(t)|^4+|y(\vartheta_t\omega)|^4+|y(\vartheta_t\omega)|^2\big)\\ &\leqslant\|g(x,t)\|^2+c|\alpha(t)|^4+c|y(\vartheta_t\omega)|^4+c,\end{aligned}\tag{3.30}$$

where $c = c(\gamma, \kappa, \delta)$. Multiply (3.30) by $e^{\gamma t}$ and integrate over $(\tau - t, \tau)$, $t \in \mathbb{R}^+$, to get, for each $\omega \in \Omega$,

$$\begin{aligned} &\|v(\tau, \theta_{-t}\tau, \omega, v_0)\|^2 + \lambda \int_{\theta_{-t}\tau}^{\tau} e^{\gamma(s-\tau)} \|\nabla v(s, \theta_{-t}\tau, \omega, v_0)\|^2 \, \mathrm{d}s \\ &\quad + \frac{\kappa}{2} \int_{\theta_{-t}\tau}^{\tau} e^{\gamma(s-\tau)} \|v(s, \theta_{-t}\tau, \omega, v_0)\|_4^4 \, \mathrm{d}s \\ &\leqslant \int_{\theta_{-t}\tau}^{\tau} e^{\gamma(s-\tau)} \big(c|\alpha(s)|^4 + \|g(x,s)\|^2\big) \mathrm{d}s \\ &\quad + c \int_{\theta_{-t}\tau}^{\tau} e^{\gamma(s-\tau)} (|y(\vartheta_s \omega)|^4 + 1) \, \mathrm{d}s + e^{-\gamma t} \|v_0\|^2. \end{aligned} \tag{3.31}$$

Notice that $|y|^4 \leqslant |y|^6 + c$. Therefore, replacing ω in (3.31) with $\vartheta_{-\tau}\omega$ and by (3.16)-(3.17) we obtain

$$\begin{aligned} &\|v(\tau, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 + \lambda e^{-\gamma\tau} \int_{\theta_{-t}\tau}^{\tau} e^{\gamma s} \|\nabla v(s, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 \, \mathrm{d}s \\ &\quad + \frac{\kappa}{2} e^{-\gamma\tau} \int_{\theta_{-t}\tau}^{\tau} e^{\gamma s} \|v(s, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|_4^4 \, \mathrm{d}s \\ &\leqslant \int_{\theta_{-t}\tau}^{\tau} e^{\gamma(s-\tau)} \big(c|\alpha(s)|^4 + \|g(x,s)\|^2\big) \mathrm{d}s \\ &\quad + c \int_{\theta_{-t}\tau}^{\tau} e^{\gamma(s-\tau)} (|y(\vartheta_{s-\tau} \omega)|^6 + 1) \, \mathrm{d}s + e^{-\gamma t} \|v_0\|^2 \\ &\leqslant c e^{-\gamma\tau} \int_{-\infty}^{\tau} e^{\gamma s} \big(|\alpha(s)|^4 + \|g(x,s)\|^2\big) \mathrm{d}s \\ &\quad + c \int_{-\infty}^{0} e^{\gamma s} r(\vartheta_s \omega) \, \mathrm{d}s + c + e^{-\gamma t} \|D(\theta_{-t}\tau, \vartheta_{-t}\omega)\|^2, \end{aligned} \tag{3.32}$$

where c is a positive constant depending on γ but independent of τ, ω and $\hat{D}$. Since $\hat{D} \in \hat{\mathcal{D}}_H$, there exists a $T = T(\tau, \omega, \hat{D}) \geqslant 1$ such that

$$e^{-\gamma t} \|\hat{D}(\theta_{-t}\tau, \vartheta_{-t}\omega)\|^2 \leqslant 1 \quad \text{for all } t \geqslant T,$$

which along with (3.32) and (3.16)-(3.17) completes the proof. □

Lemma 3.3.4. *Let assumptions of Lemma 3.3.1 hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\hat{D} \in \hat{\mathcal{D}}_H$, there exists a $T = T(\tau, \omega, \hat{D}) \geqslant 1$ and a positive constant C, which depends on γ but is independent of τ, ω and $\hat{D}$, such that the solution $v(t, \tau, \omega, v_0)$ with $v_0 \in \hat{D}(\theta_{-t}\tau, \vartheta_{-\tau}\omega)$ of (3.19)-(3.21) satisfies, for all $t \geqslant T$,*

$$\begin{aligned} &\int_{\tau-1}^{\tau} \|\nabla v(s, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 \, \mathrm{d}s + \int_{\tau-1}^{\tau} \|v(s, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|_4^4 \, \mathrm{d}s \\ &\quad \leqslant C e^{-\gamma\tau} \int_{-\infty}^{\tau} e^{\gamma s} \big(|\alpha(s)|^4 + \|g(x,s)\|^2\big) \mathrm{d}s + C r(\omega) + C, \end{aligned} \tag{3.33}$$

where $r(\omega)$ is the tempered random variable given by (3.16) and (3.17).

Proof. Notice that $e^{\gamma s} \geqslant e^{\gamma(\tau-1)}$ for all $s \in (\tau-1, \tau)$. Hence, by (3.26) we have

$$\begin{aligned}\int_{\tau-1}^{\tau} &\|\nabla v(s, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 \,\mathrm{d}s + \int_{\tau-1}^{\tau} \|v(s, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|_4^4 \,\mathrm{d}s \\ &\leqslant e^{\gamma(1-\tau)} \int_{\theta_{-t}\tau}^{\tau} e^{\gamma s}\|\nabla v(s, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 \,\mathrm{d}s \\ &\quad + e^{\gamma(1-\tau)} \int_{\theta_{-t}\tau}^{\tau} e^{\gamma s}\|v(s, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|_4^4 \,\mathrm{d}s \\ &\leqslant ce^{-\gamma\tau} \int_{-\infty}^{\tau} e^{\gamma s}\big(|\alpha(s)|^4 + \|g(x,s)\|^2\big)\mathrm{d}s + cr(\omega) + c,\end{aligned} \tag{3.34}$$

for all $t \geqslant T \geqslant 1$, which concludes the lemma. □

Lemma 3.3.5. *Let assumptions of Lemma 3.3.1 hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\hat{D} \in \hat{\mathcal{D}}_H$, there exists a $T = T(\tau, \omega, \hat{D}) \geqslant 1$ and a positive constant K, which depends on γ but is independent of τ, ω and $\hat{D}$, such that the solution $v(t, \tau, \omega, v_0)$ with $v_0 \in \hat{D}(\theta_{-t}\tau, \vartheta_{-\tau}\omega)$ of (3.19)-(3.21) satisfies, for all $t \geqslant T$,*

$$\|\nabla v(\tau, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 \leqslant K\big(r(\omega)+1\big)e^{-\gamma\tau}\int_{-\infty}^{\tau} e^{\gamma s}\big(|\alpha(s)|^4 + \|g(x,s)\|^2\big)\mathrm{d}s + K\big(r(\omega)+1\big)^2,$$

where $r(\omega)$ is the tempered random variable given by (3.16) and (3.17).

Proof. Taking the inner product of (3.19) with $-\triangle v$ in H and taking the real part, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2 + \lambda\|\triangle v\|^2 &= \mathrm{Re}\Big((\kappa + \mathrm{i}\beta(t))f(v + z(\vartheta_t\omega)), \triangle v\Big) + \delta\|\nabla v\|^2 \\ &\quad - \mathrm{Re}(g(t), \triangle v) - \mathrm{Re}\big((\lambda + \mathrm{i}\alpha(t))\triangle z(\vartheta_t\omega), \triangle v\big).\end{aligned} \tag{3.35}$$

Estimate the first term in the right hand side of (3.35) to get (3.41). Since $f(s) = |s|^2 s$, we have

$$\begin{aligned}&\mathrm{Re}\Big((\kappa + \mathrm{i}\beta(t))f(v + z(\vartheta_t\omega)), \triangle v\Big) \\ &\quad = \mathrm{Re}\Big((\kappa + \mathrm{i}\beta(t))\big(|v|^2 v, \triangle v\big)\Big) + \mathrm{Re}\Big((\kappa + \mathrm{i}\beta(t))\big(v^2\bar{z} + 2|v|^2 z\big), \triangle v\Big) \\ &\qquad + \mathrm{Re}\Big((\kappa + \mathrm{i}\beta(t))\big(2v|z|^2 + |z|^2 z + \bar{v}z^2\big), \triangle v\Big).\end{aligned} \tag{3.36}$$

By the condition $|\beta(t)| \leqslant \sqrt{3}\kappa$, $\forall t \in \mathbb{R}$, we have

$$\begin{aligned}&\mathrm{Re}\Big((\kappa + \mathrm{i}\beta(t))(|v|^2 v, \triangle v)\Big) \\ &\quad = -\mathrm{Re}\Big((\kappa + \mathrm{i}\beta(t))\int_{\mathcal{I}}\big(|v|^2|\nabla v|^2 + v\nabla|v|^2\nabla\bar{v}\big)\mathrm{d}x\Big) \\ &\quad = -\kappa\int_{\mathcal{I}}|v|^2|\nabla v|^2\mathrm{d}x - \kappa\int_{\mathcal{I}}\mathrm{Re}(v\nabla\bar{v})\nabla|v|^2\mathrm{d}x + \beta(t)\int_{\mathcal{I}}\mathrm{Im}(v\nabla\bar{v})\nabla|v|^2\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= -\kappa \int_{\mathcal{I}} |v|^2 |\nabla v|^2 \mathrm{d}x - \frac{\kappa}{2} \int_{\mathcal{I}} (\nabla |v|^2)^2 \mathrm{d}x - \frac{\mathrm{i}\beta(t)}{2} \int_{\mathcal{I}} (v\nabla \bar{v} - \bar{v}\nabla v)\nabla |v|^2 \mathrm{d}x \\
&= -\frac{1}{4} \int_{\mathcal{I}} \Big(3\kappa (\nabla |v|^2)^2 + 2\mathrm{i}\beta(t)(v\nabla\bar{v} - \bar{v}\nabla v)\nabla |v|^2 + \kappa |v\nabla\bar{v} - \bar{v}\nabla v|^2 \Big) \mathrm{d}x \\
&= -\frac{1}{4} \int_{\mathcal{I}} \eta \mathbf{M} \eta^{\top} \mathrm{d}x \leqslant 0, \quad \forall t \in \mathbb{R},
\end{aligned} \tag{3.37}
$$

where $\eta^{\top}$ denotes the conjugate transpose of matrix η and

$$
\eta = \Big[\nabla |v|^2, \ v\nabla\bar{v} - \bar{v}\nabla v \Big], \qquad \mathbf{M} = \begin{bmatrix} 3\kappa & -\mathrm{i}\beta \\ \mathrm{i}\beta & \kappa \end{bmatrix}.
$$

Recall the Agmon inequality that

$$
\|z(x)\|_{\infty} \leqslant c\|z\|^{\frac{1}{2}} \|\nabla z\|^{\frac{1}{2}}, \quad x \in \mathcal{I} \subseteq \mathbb{R}. \tag{3.38}
$$

By Lemma A.1.1, $|\beta(t)| \leqslant \sqrt{3}\kappa$ and (3.38) we estimate the second term in the right hand side of (3.36) to obtain

$$
\begin{aligned}
\mathrm{Re}\Big((\kappa + \mathrm{i}\beta(t)) \big(v^2 \bar{z} + 2|v|^2 z \big), \triangle v \Big) &\leqslant |\kappa + \mathrm{i}\beta(t)| \int_{\mathcal{I}} 3|v|^2 |z| |\triangle v| \, \mathrm{d}x \\
&\leqslant \frac{\lambda}{4} \|\triangle v\|^2 + c\|z(\vartheta_t\omega)\|_{\infty}^2 \|v\|_4^4 \\
&\leqslant \frac{\lambda}{4} \|\triangle v\|^2 + c|y(\vartheta_t\omega)|^2 \|v\|_4^4,
\end{aligned} \tag{3.39}
$$

where $c = c(\lambda, \kappa, \|h\|, \|\nabla h\|)$. Similarly, for the last term of (3.36) we have

$$
\begin{aligned}
&\mathrm{Re}\Big((\kappa + \mathrm{i}\beta(t)) \big(2v|z|^2 + |z|^2 z + \bar{v} z^2 \big), \triangle v \Big) \\
&\leqslant 2\kappa \int_{\mathcal{I}} \Big(3|v||z|^2 |\triangle v| + |z|^3 |\triangle v| \Big) \mathrm{d}x \\
&\leqslant \frac{\lambda}{4} \|\triangle v\|^2 + c\big(\|z\|_{\infty}^4 \|v\|_4^4 + \|z\|_{\infty}^4 + \|z(\vartheta_t\omega)\|^6 \big) \\
&\leqslant \frac{\lambda}{4} \|\triangle v\|^2 + c\big(|y(\vartheta_t\omega)|^4 \|v\|_4^4 + |y(\vartheta_t\omega)|^4 + |y(\vartheta_t\omega)|^6 \big).
\end{aligned} \tag{3.40}
$$

From (3.36)-(3.40) and Lemma A.1.1 it follows that

$$
\mathrm{Re}\Big((\kappa + \mathrm{i}\beta(t)) f(v + z(\vartheta_t\omega)), \triangle v \Big) \leqslant \frac{\lambda}{2} \|\triangle v\|^2 + c\big(|y(\vartheta_t\omega)|^6 + 1 \big) \|v\|_4^4 + c|y(\vartheta_t\omega)|^6 + c. \tag{3.41}
$$

For the last term of (3.35), by Lemma A.1.1 again we get

$$
\begin{aligned}
-\mathrm{Re}\big((\lambda + \mathrm{i}\alpha(t)) \triangle z(\vartheta_t\omega), \triangle v \big) &\leqslant \frac{\lambda}{4} \|\triangle v\|^2 + c|\lambda + \mathrm{i}\alpha(t)|^2 \|\triangle z(\vartheta_t\omega)\|^2 \\
&\leqslant \frac{\lambda}{4} \|\triangle v\|^2 + c|\alpha(t)|^2 \|\triangle z(\vartheta_t\omega)\|^2 + c\lambda^2 \|\triangle z(\vartheta_t\omega)\|^2 \\
&\leqslant \frac{\lambda}{4} \|\triangle v\|^2 + c|\alpha(t)|^4 + c|y(\vartheta_t\omega)|^6 + c,
\end{aligned} \tag{3.42}
$$

where $c = c(\lambda, \|\triangle h\|)$. Since, by (3.24) and Lemma A.1.1 again,

$$\delta\|\nabla v\|^2 \leqslant \frac{\lambda}{8}\|\triangle v\|^2 + c\|v\|_4^4 + c, \tag{3.43}$$

$$-\mathrm{Re}(g(t), \triangle v) \leqslant \frac{4}{\lambda}\|g(x,t)\|^2 + \frac{\lambda}{8}\|\triangle v\|^2, \tag{3.44}$$

from (3.35), (3.41) and (3.42) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2 \leqslant c\big(|y(\vartheta_t\omega)|^6 + 1\big)\big(\|v\|_4^4 + 1\big) + c\big(\|g(x,t)\|^2 + |\alpha(t)|^4\big), \tag{3.45}$$

where c is a positive constant independent of τ, ω and $\hat{D}$. Given $t \geqslant 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $s \in (\tau-1, \tau)$, integrating (3.45) over (s,τ) and by (3.16) we find that

$$\begin{aligned}
\|\nabla v(\tau, \theta_{-t}\tau, \omega, v_0)\|^2 &\leqslant \|\nabla v(s, \theta_{-t}\tau, \omega, v_0)\|^2 + c\int_s^\tau \big(|y(\vartheta_\xi\omega)|^6 + 1\big)\big(\|v(\xi)\|_4^4 + 1\big)\,\mathrm{d}\xi \\
&\quad + c\int_s^\tau \big(\|g(x,\xi)\|^2 + |\alpha(\xi)|^4\big)\mathrm{d}\xi \\
&\leqslant \|\nabla v(s, \theta_{-t}\tau, \omega, v_0)\|^2 + c\int_{\tau-1}^\tau \big(r(\vartheta_\xi\omega) + 1\big)\|v(\xi)\|_4^4\,\mathrm{d}\xi \\
&\quad + c\int_{\tau-1}^\tau \big(r(\vartheta_\xi\omega) + 1\big)\,\mathrm{d}\xi + c\int_{\tau-1}^\tau \big(\|g(x,\xi)\|^2 + |\alpha(\xi)|^4\big)\mathrm{d}\xi,
\end{aligned} \tag{3.46}$$

where c is a positive constant independent of τ, ω and $\hat{D}$. Integrating (3.46) with respect to s over $(\tau-1,\tau)$ and replacing ω with $\vartheta_{-\tau}\omega$, by (3.17) we derive that

$$\begin{aligned}
\|\nabla v(\tau, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 &\leqslant \int_{\tau-1}^\tau \|\nabla v(\xi, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2\,\mathrm{d}\xi \\
&\quad + \big(cr(\omega) + c\big)\int_{\tau-1}^\tau \|v(\xi, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|_4^4\,\mathrm{d}\xi \\
&\quad + cr(\omega) + c + c\int_{\tau-1}^\tau \big(\|g(x,\xi)\|^2 + |\alpha(\xi)|^4\big)\mathrm{d}\xi,
\end{aligned} \tag{3.47}$$

where c depends on γ but is independent of τ, ω and $\hat{D}$. Let $T = T(\tau, \omega, \hat{D}) \geqslant 1$ be the same as in Lemma 3.3.4. Then from (3.47) and (3.33) it follows that

$$\begin{aligned}
&\|\nabla v(\tau, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 \\
&\leqslant \big(cr(\omega) + c\big)e^{-\gamma\tau}\int_{-\infty}^\tau e^{\gamma s}\big(|\alpha(s)|^4 + \|g(x,s)\|^2\big)\mathrm{d}s \\
&\quad + \big(cr(\omega) + c\big)^2 + c\int_{\tau-1}^\tau \big(\|g(x,\xi)\|^2 + |\alpha(\xi)|^4\big)\mathrm{d}\xi \\
&\leqslant \big(cr(\omega) + c\big)e^{-\gamma\tau}\int_{-\infty}^\tau e^{\gamma s}\big(|\alpha(s)|^4 + \|g(x,s)\|^2\big)\mathrm{d}s + \big(cr(\omega) + c\big)^2,
\end{aligned} \tag{3.48}$$

which completes the proof. □

To derive uniform estimates on the solutions u of (3.12)-(3.13), recall from (3.23) that

$$u(\tau, \theta_{-t}\tau, \vartheta_{-\tau}\omega, u_0) = v(\tau, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0) + z(\omega), \tag{3.49}$$

where $v_0 = u_0 - z(\vartheta_{-t}\omega)$. Hence, for $X = H$ or V, we have

$$\begin{aligned}\|u(\tau, \theta_{-t}\tau, \vartheta_{-\tau}\omega, u_0)\|_X^2 &\leqslant 2\|v(\tau, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|_X^2 + 2\|z(\omega)\|_X^2 \\ &\leqslant 2\|v(\tau, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|_X^2 + 2\|h(x)\|_X^2\big(r(\omega) + 1\big).\end{aligned}$$

Moreover, by the tempered property of $r(\omega)$ it is evident that v_0 comes from a non-autonomous random set in $\hat{\mathcal{D}}_H$ provided so does u_0. Therefore, Lemma 3.3.3 and Lemma 3.3.5 imply the following lemma.

Lemma 3.3.6. *Let assumptions of Lemma 3.3.1 hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\hat{D} \in \hat{\mathcal{D}}_H$, there exists a $T = T(\tau, \omega, \hat{D}) \geqslant 1$ and a positive constant M, which depends on γ but is independent of τ, ω and D, such that the solution $u(t, \tau, \omega, u_0)$ with $u_0 \in \hat{D}(\theta_{-t}\tau, \vartheta_{-\tau}\omega)$ of (3.12)-(3.13) satisfies, for all $t \geqslant T$, that*

$$\|u(\tau, \theta_{-t}\tau, \vartheta_{-t}\omega, u_0)\|^2 \leqslant Me^{-\gamma\tau}\int_{-\infty}^{\tau} e^{\gamma s}\big(|\alpha(t)|^4 + \|g(x,s)\|^2\big)\mathrm{d}s + Mr(\omega) + M, \tag{3.50}$$

and that

$$\|u(\tau, \theta_{-t}\tau, \vartheta_{-\tau}\omega, u_0)\|_V^2 \leqslant M\big(r(\omega)+1\big)e^{-\gamma\tau}\int_{-\infty}^{\tau} e^{\gamma s}\big(|\alpha(s)|^4 + \|g\|^2\big)\mathrm{d}s + M\big(r(\omega)+1\big)^2, \tag{3.51}$$

where $r(\omega)$ is the tempered random variable given by (3.16) and (3.17).

3.3.2 Existence of the cocycle attractor in H

In this part, we establish the existence of the $\hat{\mathcal{D}}_H$-cocycle attractor $\hat{A}$ in H for system (3.12).

Consider the non-autonomous random set $\hat{E} = \{\hat{E}_\tau(\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ in V given by

$$\hat{E}_\tau(\omega) := \big\{u \in V : \|u\|_V^2 \leqslant \mathcal{J}(\tau, \omega)\big\} \tag{3.52}$$

with

$$\mathcal{J}(\tau,\omega) := M\big(r(\omega)+1\big)e^{-\gamma\tau}\int_{-\infty}^{\tau} e^{\gamma s}\big(|\alpha(s)|^4 + \|g\|^2\big)\mathrm{d}s + M\big(r(\omega)+1\big)^2,$$

where M is the constant found out by Lemma 3.3.6. It is evident that $\mathcal{J}(\tau, \cdot) : \omega \to \mathbb{R}^+$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for each $\tau \in \mathbb{R}$. Moreover, by the temperance of $r(\omega)$ and assumption (iii) of Lemma 3.3.1 one can readily verify that $\hat{E} \in \hat{\mathcal{D}}_V$.

Theorem 3.3.7. *Let assumptions of Lemma 3.3.1 hold. Then the NRDS ϕ associated to problem (3.12)-(3.13) has a unique $\hat{\mathcal{D}}_H$-cocycle attractor $\hat{A} = \{\hat{A}_\tau(\omega)\}_{\tau,\omega}$ in H, where*

$$\hat{A}_\tau(\omega) = \mathcal{W}(\omega, \tau, \hat{E}). \tag{3.53}$$

Proof. By Sobolev compactness embedding it is clear that the non-autonomous random set $\hat{E}$ is a compact non-autonomous random set in H, and is $\hat{\mathcal{D}}_H$-absorbing under ϕ by Lemma 3.3.6. Hence, the result follows from Theorem 3.2.3. □

3.3.3 Existence of (H, V)-cocycle attractor

In this part, we shall show that the cocycle attractor $\hat{A}$ in H of the Ginzburg-Landau equation is in fact (H, V) bi-spatial, i.e., having the regularity in V and pullback attracting in the topology of V.

Consider the operator $-\triangle$. It is well known (see [74]) that there exists a complete orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of H consisted of eigenvectors of $-\triangle$ who has countable spectrum λ_j, $j = 1, 2, \cdots$, such that

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty \text{ and } -\triangle e_j = \lambda_j e_j.$$

We denote by $V_m = span\{e_1, e_2, \cdots, e_m\} \subset V$ and $V_m^{\perp}$ its orthogonal complement such that $V = V_m \oplus V_m^{\perp}$. Therefore, for each $v \in V$ there exists a unique decomposition

$$v = v_m + v_m^{\perp},$$

where $v_m \in V_m$ and $v_m^{\perp} \in V_m^{\perp}$. Denote the orthogonal projector from V to V_m by $P_m : v \mapsto v_m$.

The following lemma implies the flattening property of the system, see Section 3.2.2.

Lemma 3.3.8. *Let assumptions of Lemma 3.3.1 hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\hat{D} \in \hat{\mathcal{D}}_H$, there exists a $T' = T'(\tau, \omega, \hat{D}) \geqslant 1$ such that for every $\eta > 0$ we can find an $M = M(\eta, \tau, \omega) \in \mathbb{N}$ satisfying*

$$\|(I - P_m)\nabla v(\tau, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 < \eta$$

uniformly in $t \geqslant T'$ and $v_0 \in \hat{D}$ for all $m > M$.

Proof. Step 1. Complementary uniform estimates in tails. Taking the inner product of (3.19) with v in H and taking the real part, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 = &-\lambda\|\nabla v\|^2 - \mathrm{Re}\big((\kappa + i\beta(t))f(v + z(\vartheta_t\omega)), v\big) + \delta\|v\|^2 \\ &+ \mathrm{Re}(g(t), v) + \mathrm{Re}\big((\lambda + i\alpha)\triangle z(\vartheta_t\omega), v\big).\end{aligned}$$

After similar calculations as in Lemma 3.3.3 we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \gamma\|v\|^2 + \lambda\|\nabla v\|^2 + \frac{\kappa}{2}\|v\|_4^4 \leqslant \|g(x,t)\|^2 + c|\alpha(t)|^4 + c|y(\vartheta_t\omega)|^4 + c,$$

and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \gamma\|v\|^2 \leqslant \|g(x,t)\|^2 + c|\alpha(t)|^4 + c|y(\vartheta_t\omega)|^4 + c =: H_1(t, \vartheta_t\omega), \tag{3.54}$$

where $c = c(\gamma, \kappa, \delta)$. Now for any $\varrho \in [\tau - 1, \tau]$, we apply Gronwall techniques to (3.54) over $(\theta_{-t}\tau, \varrho)$ for $t \geqslant 2$ to get

$$\begin{aligned}\|v(\varrho, \theta_{-t}\tau, \omega, v_0)\|^2 &\leqslant e^{-\gamma\varrho}\left(\frac{1}{\varrho - (\theta_{-t}\tau)}\int_{\theta_{-t}\tau}^{\varrho}\|v(s, \theta_{-t}\tau, \omega, v_0)\|^2 e^{\gamma s} + \int_{\theta_{-t}\tau}^{\varrho} H_1(s, \vartheta_s\omega)e^{\gamma s}\right) \\ &\leqslant ce^{-\gamma\tau}\left(\int_{\theta_{-t}\tau}^{\tau}\|v(s, \theta_{-t}\tau, \omega, v_0)\|^2 e^{\gamma s}\,\mathrm{d}s + \int_{\theta_{-t}\tau}^{\tau} H_1(s, \vartheta_s\omega)e^{\gamma s}\,\mathrm{d}s\right).\end{aligned}$$

Thus by Lemma 3.3.3, for all $t > T + 2$ we have

$$\begin{aligned}
\|v(\varrho, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 &\leqslant ce^{-\gamma\tau} \int_{\theta_{-t}\tau}^{\tau} e^{\gamma s} \|v(s, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 \mathrm{d}s \\
&\quad + ce^{-\gamma\tau} \int_{\tau-t}^{\tau} H_1(s, \vartheta_{s-\tau}\omega) e^{\gamma s} \mathrm{d}s \\
&\leqslant ce^{-\gamma\tau} \int_{-\infty}^{\tau} e^{\gamma s} \big(|\alpha(t)|^4 + \|g(x,s)\|^2\big) \mathrm{d}s + cr(\omega) + ce^{-\gamma\tau} \\
&\quad + ce^{-\gamma\tau} \int_{-\infty}^{\tau} H_1(s, \vartheta_{s-\tau}\omega) e^{\gamma s} \,\mathrm{d}s \\
&=: R_1(\tau, \omega), \quad \forall \varrho \in [\tau - 1, \tau].
\end{aligned} \tag{3.55}$$

Clearly, $R_1(\tau, \omega)$ is a random variable bounded for each $(\tau, \omega) \in \mathbb{R} \times \Omega$.

Taking the inner product of (3.19) with $-\triangle v$ in H and taking the real part, we get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2 + \lambda \|\triangle v\|^2 = \mathrm{Re}\Big((\kappa + i\beta(t)) f(v + z(\vartheta_t\omega)), \triangle v\Big) - \mathrm{Re}(g(t), \triangle v) \\
- \mathrm{Re}\big((\lambda + i\alpha(t))\triangle z(\vartheta_t\omega), \triangle v\big).
\end{aligned}$$

Recall from (3.45) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2 \leqslant c\big(|y(\vartheta_t\omega)|^6 + 1\big)\big(\|v\|_4^4 + 1\big) + c\big(\|g(x,t)\|^2 + |\alpha(t)|^4\big) =: H_2(t, \vartheta_t\omega).$$

For any $s \in [\tau - 1, \tau]$, we integrate the above relation over (ξ, s) with $\xi \in (s-1, s)$ to find that, for $t > 2$,

$$\begin{aligned}
\|\nabla v(s, \theta_{-t}\tau, \omega, v_0)\|^2 &\leqslant \|\nabla v(\xi, \theta_{-t}\tau, \omega, v_0)\|^2 + \int_{\xi}^{s} H_2(\varrho, \vartheta_\varrho\omega) \,\mathrm{d}\varrho \\
&\leqslant \|\nabla v(\xi, \theta_{-t}\tau, \omega, v_0)\|^2 + \int_{\tau-2}^{\tau} H_2(\varrho, \vartheta_\varrho\omega) \,\mathrm{d}\varrho.
\end{aligned}$$

Integrating the above relation with respect to ξ over $(s-1, s)$, we get

$$\begin{aligned}
\|\nabla v(s, \theta_{-t}\tau, \omega, v_0)\|^2 &\leqslant \int_{s-1}^{s} \|\nabla v(\xi, \theta_{-t}\tau, \omega, v_0)\|^2 \,\mathrm{d}\xi + \int_{\tau-2}^{\tau} H_2(\varrho, \vartheta_\varrho\omega) \,\mathrm{d}\varrho \\
&\leqslant \int_{\tau-2}^{\tau} \|\nabla v(\xi, \theta_{-t}\tau, \omega, v_0)\|^2 \,\mathrm{d}\xi + \int_{\tau-2}^{\tau} H_2(\varrho, \vartheta_\varrho\omega) \,\mathrm{d}\varrho, \ \forall s \in [\tau - 1, \tau].
\end{aligned}$$

Replacing ω with $\vartheta_{-\tau}\omega$, we have

$$\begin{aligned}
\|\nabla v(s, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 &\leqslant \int_{\tau-2}^{\tau} \|\nabla v(\xi, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 \mathrm{d}\xi + \int_{\tau-2}^{\tau} H_2(\varrho, \vartheta_{\varrho-\tau}\omega) \mathrm{d}\varrho \\
&= \int_{\tau-2}^{\tau} \|\nabla v(\xi, \theta_{-t}\tau, \vartheta_{-\tau}\omega, v_0)\|^2 \mathrm{d}\xi \\
&\quad + c\int_{\tau-2}^{\tau} \big(|y(\vartheta_{\xi-\tau}\omega)|^6 + 1\big) \|v(\xi)\|_4^4 \,\mathrm{d}\xi \\
&\quad + c\int_{\tau-2}^{\tau} \Big(|y(\vartheta_{\xi-\tau}\omega)|^6 + 1 + \|g(x,t)\|^2 + |\alpha(t)|^4\Big) \mathrm{d}\xi.
\end{aligned}$$

Let $r'(\omega) = \sup_{-2<s<0} |y(\vartheta_s\omega)|^6 + 1 \geqslant 1$, then $r'(\omega)$ is a tempered random variable. And moreover, we have

$$\begin{aligned}
\|\nabla v(s,\theta_{-t}\tau,\vartheta_{-\tau}\omega,v_0)\|^2 \leqslant & \int_{\tau-2}^{\tau} \|\nabla v(\xi,\theta_{-t}\tau,\vartheta_{-\tau}\omega,v_0)\|^2 \mathrm{d}\xi + cr'(\omega)\int_{\tau-2}^{\tau} \|v(\xi)\|_4^4 \,\mathrm{d}\xi \\
& + c\int_{\tau-2}^{\tau} \Big(\|g(x,t)\|^2 + |\alpha(t)|^4\Big)\mathrm{d}\xi + cr'(\omega) \\
\leqslant & \, cr'(\omega)\Big[\int_{\tau-2}^{\tau} \Big(\|\nabla v(\xi,\theta_{-t}\tau,\vartheta_{-\tau}\omega,v_0)\|^2 + \|v(\xi)\|_4^4\Big)\mathrm{d}\xi \\
& + \int_{\tau-2}^{\tau} (\|g(x,t)\|^2 + |\alpha(t)|^4)\mathrm{d}\xi + 1\Big] \\
\leqslant & \, cr'(\omega)e^{\gamma(2-\tau)}\Big[\int_{\tau-2}^{\tau} e^{\gamma\xi}\Big(\|\nabla v(\xi,\theta_{-t}\tau,\vartheta_{-\tau}\omega,v_0)\|^2 + \|v(\xi)\|_4^4\Big)\mathrm{d}\xi \\
& + \int_{\tau-2}^{\tau} e^{\gamma\xi}(\|g(x,t)\|^2 + |\alpha(t)|^4)\,\mathrm{d}\xi + e^{\gamma(\tau-2)}\Big].
\end{aligned}$$

Thus by Lemma 3.3.3 again, we have, for all $t \geqslant T+2$,

$$\begin{aligned}
&\|\nabla v(s,\theta_{-t}\tau,\vartheta_{-\tau}\omega,v_0)\|^2 \\
&\quad \leqslant cr'(\omega)e^{-\gamma\tau}\Big(\int_{-\infty}^{\tau} e^{\gamma\xi}(\|g(x,t)\|^2 + |\alpha(t)|^4)\mathrm{d}\xi + e^{\gamma\tau}r(\omega) + 1 + e^{\gamma\tau}\Big) \\
&\quad =: R_2(\tau,\omega), \quad \forall s \in [\tau-1,\tau],
\end{aligned} \tag{3.56}$$

where $R_2(\tau,\omega)$ is defined a random variable bounded for each $(\tau,\omega) \in \mathbb{R}\times\Omega$.

Step 2. Estimates for large scales. Multiply (3.19) by $A\bar{v}_m^{\perp}$, integrate over $\mathcal{I}$ and take the real part to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v_m^{\perp}\|^2 + \lambda\|Av_m^{\perp}\|^2 - \delta\|\nabla v_m^{\perp}\|^2 \\
&\quad \leqslant \Big|Re\int_I (\kappa + i\beta(t))f(u)\triangle\bar{v}_m^{\perp}\mathrm{d}x\Big| \\
&\qquad + c\Big|Re\int_I (\lambda + i\alpha(t))\triangle z(\vartheta_t\omega)\triangle\bar{v}_m^{\perp}\mathrm{d}x\Big| + \Big|\int_I g(t)\triangle\bar{v}_m\,\mathrm{d}x\Big| \\
&\quad =: G_1 + G_2 + G_3.
\end{aligned}$$

Note that $f(u) = |u|^2u$. Therefore, by interpolation inequalities,

$$\begin{aligned}
G_1 &\leqslant \int_I c\Big|f(u)\triangle\bar{v}_m^{\perp}\Big|\mathrm{d}x \leqslant c\|u\|_6^6 + \frac{\lambda}{4}\|\triangle v_m^{\perp}\|^2 \leqslant c(\|\nabla u\|^4 + \|u\|^8) + \frac{\lambda}{4}\|\triangle v_m^{\perp}\|^2, \\
G_2 &\leqslant c|\lambda + \alpha(t)|^4 + c|y(\vartheta_t\omega)|^4 + \frac{\lambda}{8}\|\triangle v_m^{\perp}\|^2, \\
G_3 &\leqslant c\|g(t)\|^2 + \frac{\lambda}{8}\|\triangle v_m^{\perp}\|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v_m^\perp\|^2+\lambda\|Av_m^\perp\|^2-2\delta\|\nabla v_m^\perp\|^2\\&\quad\leqslant c(\|\nabla u\|^4+\|u\|^8)+c|\lambda+\alpha(t)|^4+c|y(\vartheta_t\omega)|^4+c\|g(t)\|^2.\end{aligned}\tag{3.57}$$

By $\|Av_m^\perp\|^2\geqslant\lambda_m\|\nabla v_m^\perp\|^2$ and $v(t,\tau,\omega,v_0)=u(t,\tau,\omega,u_0)-z(\vartheta_t\omega)$, it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v_m^\perp\|^2+(\lambda\lambda_m-2\delta)\|\nabla v_m^\perp\|^2\leqslant c(\|\nabla v\|^4+\|v\|^8)+c|\lambda+\alpha(t)|^4+cr(\vartheta_t\omega)+c\|g(t)\|^2.$$

By Gronwall's techniques again, we obtain

$$\begin{aligned}&\|\nabla v_m^\perp(\tau,\theta_{-t}\tau,\vartheta_{-\tau}\omega,v_0)\|^2-\int_{\tau-1}^{\tau}e^{(\lambda\lambda_m-2\delta)(s-\tau)}\|\nabla v_m^\perp(s,\theta_{-t}\tau,\vartheta_{-\tau}\omega,v_0)\|^2\,\mathrm{d}s\\&\quad\leqslant c\int_{\tau-1}^{\tau}e^{(\lambda\lambda_m-2\delta)(s-\tau)}\Big(\|\nabla v(s,\theta_{-t}\tau,\vartheta_{-\tau}\omega,v_0)\|^4+\|v(s,\theta_{-t}\tau,\vartheta_{-\tau}\omega,v_0)\|^8\Big)\mathrm{d}s\\&\qquad+c\int_{\tau-1}^{\tau}e^{(\lambda\lambda_m-2\delta)(s-\tau)}\Big(|\lambda+\alpha(s)|^4+r(\vartheta_{s-\tau}\omega)+\|g(s)\|^2\Big)\mathrm{d}s.\end{aligned}$$

Thus by (3.55) and (3.56) we have

$$\begin{aligned}\|\nabla v_m^\perp(\tau,\theta_{-t}\tau,\vartheta_{-\tau}\omega,v_0)\|^2&\leqslant c\int_{\tau-1}^{\tau}e^{(\lambda\lambda_m-2\delta)(s-\tau)}\Big(R_2^2(\tau,\omega)+R_1^4(\tau,\omega)+1\Big)\mathrm{d}s\\&\quad+c\int_{\tau-1}^{\tau}e^{(\lambda\lambda_m-2\delta)(s-\tau)}\Big(|\lambda+\alpha(s)|^4+r(\vartheta_{s-\tau}\omega)+\|g(s)\|^2\Big)\mathrm{d}s\\&\leqslant\frac{cR_2^2(\tau,\omega)+cR_1^4(\tau,\omega)+cH_3(\tau,\omega)}{\lambda\lambda_m-2\delta}\end{aligned}$$

for all $t\geqslant T+2$ and large m, where

$$H_3(\tau,\omega)=\sup_{-1<s<0}\Big(|\lambda+\alpha(s+\tau)|^4+r(\vartheta_s\omega)+\|g(s+\tau)\|^2\Big)+1.$$

Since $\lambda_m\to\infty$, the proof is completed. □

Now we are ready to prove the existence of the bi-spatial cocycle attractor, applying Theorem 3.2.11.

Theorem 3.3.9. *Let assumptions of Lemma 3.3.1 hold. Then the non-autonomous random set $\hat{A}\in\hat{\mathcal{D}}_V$ defined by (3.53) is the (L^2,H_0^1)-cocycle attractor with attraction universe $\hat{\mathcal{D}}_H$ for the NRDS ϕ associated to (3.19).*

Proof. We prove by Theorem 3.2.11. First, notice that the NRDS generated by the Ginzburg-Landau equation (3.12) is (H,V)-dissipative, which can be seen in the same way as for reaction-diffusion equation discussed in Section 2.3.2. Second, the non-autonomous random set $\hat{E}$ given by (3.52) is clearly a $\hat{\mathcal{D}}_V$-pullback absorbing set belonging to $\hat{\mathcal{D}}_V$. Third, Lemmas 3.3.8 and 3.3.6 show that the NRDS ϕ is $\hat{\mathcal{D}}_V$-pullback flattening, which along with Theorem 3.2.7 implies that the omega-limit set $\mathcal{W}(\cdot,\cdot,\hat{E})$ of $\hat{E}$ is a compact $\hat{\mathcal{D}}_V$-pullback attracting set in V. Hence, the result follows from Theorem 3.2.11. □

Chapter 4

Cocycle attractors for non-autonomous random dynamical systems II: autonomous attraction universe case

In this chapter we study non-autonomous cocycle attractors with autonomous attraction universes. The results indicate the differences of attractors caused by different attraction universes, and also contribute to the study of uniform attractors latter in Chapter 5.

Note that though we consider continuous NRDS in this chapter, the continuity can be weakened to quasi-S2W continuity as in Chapter 3. We shall not argue on this continuity issue, as the difference on attraction universes will not cause any difficulties. For the same reason we shall not discuss the bi-spatial case as well.

4.1 Preliminaries

We follow the notations in Chapter 3. Suppose $(X, \|\cdot\|_X)$ is a separable Banach space, and let ϕ be an NRDS with base flows $\{\theta_t\}_{t\in\mathbb{R}}$ and $\{\vartheta_t\}_{t\in\mathbb{R}}$ acting on topological space Σ and probability space $(\Omega, \mathcal{F}, \mathbb{P})$, respectively. Most generally, we do not require compactness or boundedness (under some metric) on Σ unless otherwise stated. Denote by $\hat{\mathcal{D}}_X$ some neighborhood-closed and inclusion-closed class of non-autonomous random sets in X, and denote by $\mathcal{D}_X$ some class of autonomous random sets in X which is neighborhood-closed and inclusion-closed as well, satisfying $\mathcal{D}_X \subset \hat{\mathcal{D}}_X$.

The following lemma on measurability is useful.

Lemma 4.1.1. *[32, Chapter 2] (1) Given a set-valued mapping $D : \Omega \mapsto 2^X$, not necessarily closed or open, D is measurable if and only if its closure $\omega \mapsto \overline{D(\omega)}$ is a closed random set.*

(2) A closed-valued mapping $D : \Omega \mapsto 2^X$ is measurable if and only if either of the following holds

- *for each $\delta > 0$ Graph(D^δ) is a measurable subset of $X \times \Omega$, where Graph(D^δ) denotes the graph of the (open) δ-neighborhood $\omega \mapsto D^\delta(\omega)$ of D;*

- *Graph(D) is a measurable subset of $X \times \Omega$.*

Definition 4.1.2. Given an NRDS ϕ, a non-autonomous random set $\hat{K} = \{\hat{K}_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is called $\mathcal{D}_X$-pullback attracting under ϕ if for each $D \in \mathcal{D}_X$,

$$\lim_{t\to\infty} \operatorname{dist}\Big(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)), \hat{K}_\sigma(\omega)\Big) = 0, \quad \forall \sigma \in \Sigma, \omega \in \Omega,$$

while it is called $\mathcal{D}_X$-pullback absorbing if, for each $D \in \mathcal{D}_X$, $\sigma \in \Sigma$ and $\omega \in \Omega$, there exists a time $T_D(\omega, \sigma)$ such that

$$\bigcup_{t\geqslant T_D(\omega,\sigma)} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)) \subset \hat{K}_\sigma(\omega).$$

For each non-empty non-autonomous random set $\hat{D}$ and $\sigma \in \Sigma$, the random omega-limit set $\mathcal{W}(\cdot, \sigma, \hat{D})$ of $\hat{D}$ driven by σ is defined by

$$\mathcal{W}(\omega, \sigma, \hat{D}) = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \hat{D}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega))}, \quad \forall \omega \in \Omega.$$

In particular, for an autonomous random set $D \in \mathcal{D}_X$ and $\sigma \in \Sigma$,

$$\mathcal{W}(\omega, \sigma, D) = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega))}, \quad \forall \omega \in \Omega.$$

Omega-limit sets are important in attractor theory. It is straightforward to have the following lemma.

Lemma 4.1.3. *For each $\omega \in \Omega$, $\sigma \in \Sigma$ and autonomous random set D, $y \in \mathcal{W}(\omega, \sigma, D)$ if and only if there exit sequences $t_n \to \infty$ and $x_n \in \phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, D(\vartheta_{-t_n}\omega))$ such that $x_n \to y$.*

Now we give the definition of cocycle attractor with autonomous attraction universe.

Definition 4.1.4. A non-autonomous random set $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is called a *$\mathcal{D}_X$-(random) cocycle attractor* for an NRDS ϕ, if

(1) A is compact;

(2) A is $\mathcal{D}_X$-pullback attracting;

(3) A is invariant under ϕ, that is,

$$\phi(t, \omega, \sigma, A_\sigma(\omega)) = A_{\theta_t\sigma}(\vartheta_t\omega), \quad \forall t \in \mathbb{R}^+, \omega \in \Omega, \sigma \in \Sigma;$$

(4) A is the minimal among all the closed non-autonomous random sets satisfying (2).

Moreover, the attractor A is said to be *uniformly compact* if $\cup_{\sigma\in\Sigma} A_\sigma(\omega)$ is compact for each $\omega \in \Omega$.

Note that, unlike the $\hat{\mathcal{D}}_X$-attractors discussed in Chapter 3, a $\mathcal{D}_X$-attractor does not belong to its attraction universe $\mathcal{D}_X$ in general, and thereby the minimal condition is required to ensure the uniqueness.

4.2 Comparison to non-autonomous attraction universe case

In this section we compare random cocycle attractors with autonomous and non-autonomous attraction universes $\mathcal{D}_X$ and $\hat{\mathcal{D}}_X$, respectively. Such a subject was studied in [70, 43] for pullback attractors of deterministic non-autonomous dynamical systems. Note that for autonomous RDS, [31] compared cocycle attractors with deterministic and random attraction universes.

Now we study the relationship between $\mathcal{D}_X$- and $\hat{\mathcal{D}}_X$-random cocycle attractors. Without additional assumptions, the two attractors are not identical even for deterministic non-autonomous dynamical systems, see [70, Example 11].

Proposition 4.2.1. *Suppose that A and $\hat{A}$ are respectively $\mathcal{D}_X$- and $\hat{\mathcal{D}}_X$-random cocycle attractors of an NRDS ϕ. If $A \in \hat{\mathcal{D}}_X$, then*

$$A_\sigma(\omega) \subset \hat{A}_\sigma(\omega), \quad \forall \sigma \in \Sigma, \omega \in \Omega.$$

If, moreover, there exists some (autonomous) random set $E \in \mathcal{D}_X$ such that for each $\sigma \in \Sigma$ and $\omega \in \Omega$ there exists a time $T_\star = T_\star(\sigma, \omega) \in \mathbb{R}$ with

$$\hat{A}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega) \subset E(\vartheta_{-t}\omega), \quad \forall t \geqslant T_\star, \tag{4.1}$$

then

$$A_\sigma(\omega) = \hat{A}_\sigma(\omega), \quad \forall \sigma \in \Sigma, \omega \in \Omega.$$

Proof. Since $A \in \hat{\mathcal{D}}_X$ is attracted by $\hat{A}$, by the invariance of A we have

$$\operatorname{dist}(A_\sigma(\omega), \hat{A}_\sigma(\omega)) = \operatorname{dist}(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, A_{\theta_{-t}\sigma}(\vartheta_{-t}\omega), \hat{A}_\sigma(\omega)) \to 0, \quad \text{as } t \to \infty.$$

Hence, $A_\sigma(\omega) \subset \hat{A}_\sigma(\omega)$.

Since $\mathcal{D}_X \subset \hat{\mathcal{D}}_X$, $\hat{A}$ is a $\mathcal{D}_X$-pullback attracting set as well. Theorem 4.3.3 (which will not cause recurrent proof) indicates

$$A_\sigma(\omega) = \mathcal{W}(\omega, \sigma, E), \quad \forall \sigma \in \Sigma, \omega \in \Omega.$$

On the other hand, by Theorem 3.2.3 and (4.1) we have

$$\hat{A}_\sigma(\omega) = \mathcal{W}(\omega, \sigma, \hat{A}) \subset \mathcal{W}(\omega, \sigma, E) = A_\sigma(\omega), \quad \forall \sigma \in \Sigma, \omega \in \Omega.$$

Therefore, $A_\sigma(\omega) = \hat{A}_\sigma(\omega)$ as desired. □

Remark 4.2.2. *Proposition 4.2.1 indicates that $\hat{\mathcal{D}}_X$-random cocycle attractor can be determined by attracting autonomous random sets in $\mathcal{D}_X$.*

4.3 Existence criteria and characterization

In this section we establish some existence criteria and characterization for $\mathcal{D}_X$-attractors. Though analogous study for $\hat{\mathcal{D}}_X$-random cocycle attractors was done in [83, 84], $\mathcal{D}_X$-attractors worth an independent study since we have seen that a $\mathcal{D}_X$-attractor is not a particular $\hat{\mathcal{D}}_X$-attractor, though $\mathcal{D}_X \subset \hat{\mathcal{D}}_X$.

4.3.1 Existence criteria

The following proposition indicates the importance of omega-limit sets in the study of attractors. Note that the invariance result improves [83, Lemma 2.17], where the random set D is required to absorb itself. Inspired by [31], we remove this requirement here.

Proposition 4.3.1. *Suppose that non-autonomous random set $\hat{K}$ is a compact $\mathcal{D}_X$-pullback attracting set of a continuous NRDS ϕ. Then for each $D \in \mathcal{D}_X$, every $\mathcal{W}(\omega, \sigma, D)$ for ω, σ fixed is a non-empty compact set, and pullback attracts D, i.e.,*

$$\lim_{t\to\infty} \operatorname{dist}(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)), \mathcal{W}(\omega, \sigma, D)) = 0.$$

Moreover, the omega-limit set is invariant under ϕ in the sense that

$$\mathcal{W}(\vartheta_t\omega, \theta_t\sigma, D) = \phi(t, \omega, \sigma, \mathcal{W}(\omega, \sigma, D)), \quad \forall t \geqslant 0, \omega \in \Omega, \sigma \in \Sigma.$$

Also, for each closed non-autonomous random set $\hat{B}$ pullback attracting D it holds

$$\mathcal{W}(\omega, \sigma, D) \subset \hat{B}(\omega, \sigma), \quad \forall \omega \in \Omega, \sigma \in \Sigma.$$

Proof. The non-empty, compact and pullback attracting properties are similar to [83, Lemma 2.17]. We now prove the invariance property. Firstly, we have

$$\begin{aligned}
\phi(t, \omega, \sigma, \mathcal{W}(\omega, \sigma, D)) &= \phi\big(t, \omega, \sigma, \cap_{s\geqslant 0}\overline{\cup_{\eta\geqslant s}\phi(\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, D(\vartheta_{-\eta}\omega))}\big) \\
&\subset \cap_{s\geqslant 0}\phi\big(t, \omega, \sigma, \overline{\cup_{\eta\geqslant s}\phi(\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, D(\vartheta_{-\eta}\omega))}\big) \\
&\subset \cap_{s\geqslant 0}\overline{\cup_{\eta\geqslant s}\phi\big(t, \omega, \sigma, \phi(\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, D(\vartheta_{-\eta}\omega))\big)} \\
&= \cap_{s\geqslant 0}\overline{\cup_{\eta\geqslant s}\phi(t+\eta, \vartheta_{-\eta}\omega, \theta_{-\eta}\sigma, D(\vartheta_{-\eta}\omega))} \\
&= \cap_{s\geqslant 0}\overline{\cup_{\eta\geqslant s}\phi(t+\eta, \vartheta_{-(t+\eta)}\circ\vartheta_t\omega, \theta_{-(t+\eta)}\circ\theta_t\sigma, D(\vartheta_{-(t+\eta)}\circ\vartheta_t\omega))} \\
&= \cap_{s\geqslant t}\overline{\cup_{\eta\geqslant s}\phi(\eta, \vartheta_{-\eta}\circ\vartheta_t\omega, \theta_{-\eta}\circ\theta_t\sigma, D(\vartheta_{-\eta}\circ\vartheta_t\omega))} \\
&= \mathcal{W}(\vartheta_t\omega, \theta_t\sigma, D), \quad \forall t \in \mathbb{R}^+, \omega \in \Omega, \sigma \in \Sigma,
\end{aligned}$$

where we have used $f(\cap_\alpha A_\alpha) \subset \cap_\alpha f(A_\alpha)$ for arbitrary f, and $f(\bar{A}) \subset \overline{f(A)}$ for f continuous. To see the inverse inclusion, let $y \in \mathcal{W}(\vartheta_t\omega, \theta_t\sigma, D)$. Then by Lemma 4.1.3 there exists a sequence

$$\begin{aligned}
x_n &\in \phi(t_n, \vartheta_{-t_n}\vartheta_t\omega, \theta_{-t_n}\theta_t\sigma, D(\vartheta_{t-t_n}\omega)) \\
&= \phi(t, \omega, \sigma, \phi(t_n - t, \vartheta_{t-t_n}\omega, \theta_{t-t_n}\sigma, D(\vartheta_{t-t_n}\omega))
\end{aligned}$$

with $t < t_n \to \infty$ such that $x_n \to y$. Suppose $z_n \in \phi(t_n - t, \vartheta_{t-t_n}\omega, \theta_{t-t_n}\sigma, D(\vartheta_{t-t_n}\omega))$ is such that $x_n = \phi(t, \omega, \sigma, z_n)$. Then by the pullback attracting and compact properties of $\hat{K}$, there exists a $z \in \hat{K}$ such that $z_n \to z$ up to a subsequence, which implies that $z \in \mathcal{W}(\omega, \sigma, D))$ by Lemma 4.1.3. Hence, by the continuity of ϕ we have

$$x_n = \phi(t, \omega, \sigma, z_n) \to \phi(t, \omega, \sigma, z).$$

Thus, by the uniqueness of a limit, we have $y = \phi(t,\omega,\sigma,z) \in \phi(t,\omega,\sigma,\mathcal{W}(\omega,\sigma,D))$. The invariance is clear.

Suppose there is another closed non-autonomous random set $\hat{B}$ pullback attracting D. Then for each $y \in \mathcal{W}(\omega,\sigma,D)$, $\forall\omega \in \Omega$, $\sigma \in \Sigma$, by Lemma 4.1.3, there exist a sequence $t_n \to \infty$ and

$$x_n \in \phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, D(\vartheta_{-t_n}\omega))$$

such that $x_n \to y$. On the other hand, by the pullback attraction of $\hat{B}$, we have

$$\operatorname{dist}(x_n, \hat{B}_\sigma(\omega)) \leqslant \operatorname{dist}(\phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma, D(\vartheta_{-t_n}\omega)), \hat{B}_\sigma(\omega)) \to 0,$$

which implies $y \in \hat{B}_\sigma(\omega)$ as $\hat{B}$ is closed. Thus, $\mathcal{W}(\omega,\sigma,D) \subset \hat{B}_\sigma(\omega)$ and the proof is complete. □

Now we are able to establish the following existence result for $\mathcal{D}_X$-cocycle attractors.

Theorem 4.3.2. *Suppose that a continuous NRDS ϕ has a compact $\mathcal{D}_X$-pullback attracting set $\hat{K}$ and a closed $\mathcal{D}_X$-pullback absorbing set $\hat{B}$. If there exists some (autonomous) random set $E \in \mathcal{D}_X$ such that for each $\sigma \in \Sigma$ and $\omega \in \Omega$ there exists a time $T_\star = T_\star(\sigma,\omega) \in \mathbb{R}$ with*

$$\hat{B}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega) \subset E(\vartheta_{-t}\omega), \quad \forall t \geqslant T_\star, \tag{4.2}$$

then ϕ has a unique $\mathcal{D}_X$-random cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ given by

$$A_\sigma(\omega) = \mathcal{W}(\omega,\sigma,E), \quad \forall\sigma \in \Sigma, \omega \in \Omega. \tag{4.3}$$

Proof. It is clear that the non-autonomous random set A defined by (4.3) is nonempty, compact and invariant.

To see the measurability, in view of [84, Theorem 2.14] we only need to prove that

$$A_\sigma(\omega) = \bigcap_{s\in\mathbb{N}} \overline{\bigcup_{n=s}^{\infty} \phi(n, \vartheta_{-n}\omega, \theta_{-n}\sigma, E(\vartheta_{-n}\omega))}, \quad \forall\omega \in \Omega. \tag{4.4}$$

The $\supseteq$ direction is clearly true. Now we prove the opposite. Since $\hat{B}$ is a closed $\mathcal{D}_X$-pullback absorbing set, from the invariance of A, Proposition 4.3.1 and (4.2) it follows that

$$\begin{aligned} A_\sigma(\omega) &= \phi(n, \vartheta_{-n}\omega, \theta_{-n}\sigma, A_{\theta_{-n}\sigma}(\vartheta_{-n}\omega)) \\ &\subset \phi(n, \vartheta_{-n}\omega, \theta_{-n}\sigma, \hat{B}_{\theta_{-n}\sigma}(\vartheta_{-n}\omega)) \\ &\subset \phi(n, \vartheta_{-n}\omega, \theta_{-n}\sigma, E(\vartheta_{-n}\omega)), \quad \forall n \geqslant T_\star. \end{aligned}$$

This implies the $\subseteq$ relation and hence (4.4) holds.

Next we prove that A pullback attracts each $D \in \mathcal{D}_X$. Since $\hat{B}$ is $\mathcal{D}$-pullback absorbing, for each $D \in \mathcal{D}_X$ and $\omega \in \Omega$, $\sigma \in \Sigma$ there is a time $T_D(\omega,\sigma) \geqslant T_\star$ such that

$$\bigcup_{t\geqslant T_D(\omega,\sigma)} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D_{\theta_{-t}\sigma}(\vartheta_{-t}\omega)) \subset \hat{B}_\sigma(\omega).$$

On the other hand, since A pullback attracts E by Proposition 4.3.1, for each $\varepsilon > 0$, $\sigma \in \Sigma$ and $\omega \in \Omega$ fixed, there is a time $\bar{s} = \bar{s}_D(\omega, \sigma) \geqslant T_D(\omega, \sigma) \geqslant T_\star$ such that

$$\operatorname{dist}\Big(\phi(\bar{s}, \vartheta_{-\bar{s}}\omega, \theta_{-\bar{s}}\sigma, E(\vartheta_{-\bar{s}}\omega)), A_\sigma(\omega)\Big) < \varepsilon.$$

Hence,

$$\begin{aligned}
&\operatorname{dist}\Big(\phi(t+\bar{s}, \vartheta_{-t-\bar{s}}\omega, \theta_{-t-\bar{s}}\sigma, D(\vartheta_{-t-\bar{s}}\omega)), A_\sigma(\omega)\Big)\\
&\quad = \operatorname{dist}\Big(\phi(\bar{s}, \vartheta_{-\bar{s}}\omega, \theta_{-\bar{s}}\sigma, \phi(t, \vartheta_{-t}\vartheta_{-\bar{s}}\omega, \theta_{-t}\theta_{-\bar{s}}\sigma, D(\vartheta_{-t}\vartheta_{-\bar{s}}\omega)), A_\sigma(\omega)\Big)\\
&\quad \leqslant \operatorname{dist}\Big(\phi(\bar{s}, \vartheta_{-\bar{s}}\omega, \theta_{-\bar{s}}\sigma, \hat{B}_{\theta_{-\bar{s}}\sigma}(\vartheta_{-\bar{s}}\omega)), A_\sigma(\omega)\Big)\\
&\quad \leqslant \operatorname{dist}\Big(\phi(\bar{s}, \vartheta_{-\bar{s}}\omega, \theta_{-\bar{s}}\sigma, E(\vartheta_{-\bar{s}}\omega)), A_\sigma(\omega)\Big)\\
&\quad < \varepsilon, \quad \forall t \geqslant T_D(\vartheta_{-\bar{s}}\omega, \theta_{-\bar{s}}\sigma),
\end{aligned}$$

which indicates that A pullback attracts D. The minimality property follows from Proposition 4.3.1. □

The following theorem indicates that the $\mathcal{D}_X$-absorbing set is not essential to ensure the existence of a $\mathcal{D}_X$-attractor. The readers could compare with Theorem 3.2.3 for $\hat{\mathcal{D}}_X$-attractors.

Theorem 4.3.3. *Suppose that a continuous NRDS ϕ has a compact $\mathcal{D}_X$-pullback attracting set $\hat{K}$. If, moreover, there exists some (autonomous) random set $E \in \mathcal{D}_X$ such that for each $\sigma \in \Sigma$ and $\omega \in \Omega$ there exists a time $T_\star = T_\star(\sigma, \omega) \in \mathbb{R}$ with*

$$\hat{K}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega) \subset E(\vartheta_{-t}\omega), \quad \forall t \geqslant T_\star, \tag{4.5}$$

then ϕ has a unique $\mathcal{D}_X$-random cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ given by

$$A_\sigma(\omega) = \mathcal{W}(\omega, \sigma, E), \quad \forall \sigma \in \Sigma, \omega \in \Omega. \tag{4.6}$$

Proof. Since $\mathcal{D}_X$ and $\hat{\mathcal{D}}_X$ are both neighborhood-closed and inclusion-closed, there exists an $\varepsilon > 0$ such that the closed ε-neighborhood $\mathcal{N}_\varepsilon(E)$ of E is a closed random set in $\mathcal{D}_X$ and that $\mathcal{N}_\varepsilon(\hat{K})$ of $\hat{K}$ belongs to $\hat{\mathcal{D}}_X$. Moreover, it holds (4.5) for the $\mathcal{N}_\varepsilon(\hat{K})$ that

$$\mathcal{N}_\varepsilon((\hat{K}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega)) \subset \mathcal{N}_\varepsilon(E(\vartheta_{-t}\omega)), \quad \forall t \geqslant T_\star.$$

Hence, $\mathcal{N}_\varepsilon(\hat{K}) \in \hat{\mathcal{D}}_X$ is a $\mathcal{D}_X$-pullback absorbing set and by Theorem 4.3.2 we know that ϕ has a unique $\mathcal{D}_X$-random cocycle attractor A given by

$$A_\sigma(\omega) = \mathcal{W}(\omega, \sigma, \mathcal{N}_\varepsilon(E)).$$

Now we prove $A_\sigma(\omega) = \mathcal{W}(\omega, \sigma, E)$. Clearly, $A_\sigma(\omega) \supseteq \mathcal{W}(\omega, \sigma, E)$ since $E_\varepsilon \supseteq E$. Thus it suffices to prove $A_\sigma(\omega) \subseteq \mathcal{W}(\omega, \sigma, E)$. Notice that since, by Proposition 4.3.1, A is the minimal closed non-autonomous set pullback attracting $\mathcal{N}_\varepsilon(E)$, we have $A \subset \hat{K}$. Hence, $\mathcal{W}(\omega, \sigma, A) \subseteq \mathcal{W}(\omega, \sigma, \hat{K}) \subseteq$

$\mathcal{W}(\omega,\sigma,E)$, where the second inclusion is due to the property (4.5) of $\hat{K}$. Therefore, by the invariance of A, we have

$$\begin{aligned} A_\sigma(\omega) &= \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s} A_\sigma(\omega)} = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,A_{\theta_{-t}\sigma}(\vartheta_{-t}\omega))} \\ &= \mathcal{W}(\omega,\sigma,A)\subseteq\mathcal{W}(\omega,\sigma,E), \end{aligned}$$

which completes the proof. □

Remark 4.3.4. Theorem 4.3.3 shows a direct relationship between attractors and compact attracting sets (see also Theorem 3.2.3). However, the existence of a compact attracting set is often the open problem. Hence, there are other dynamical compactnesses in the literature, such as asymptotic compactness, flattening and squeezing properties [68, 53, 38, 30], which ensures the omega-limit set of a $\mathcal{D}_X$-absorbing set is a compact $\mathcal{D}_X$-attracting set. We refer the readers to Section 3.2.2, and we shall not repeat the discussions here.

4.3.2 Characterization by complete trajectories

In this part, we characterize $\mathcal{D}_X$-cocycle attractors by $\mathcal{D}_X$-complete trajectories. See [83] for an analogous study for $\hat{\mathcal{D}}_X$-cocycle attractors.

Definition 4.3.5. Given any a $\sigma\in\Sigma$, a mapping $\xi\colon\ \Omega\times\mathbb{R}\to X$ is called a σ-driven complete trajectory of ϕ if $\xi(\vartheta_t\omega,t)=\phi(t-s,\vartheta_s\omega,\theta_s\sigma,\xi(\vartheta_s\omega,s))$ for all $t\geqslant s$ and $\omega\in\Omega$. If there exists a random set $B\in\mathcal{D}_X$ such that $\cup_{t\in\mathbb{R}}\xi(\cdot,t)\subset B(\cdot)$, then ξ is called a σ-driven $\mathcal{D}_X$-complete trajectory of ϕ.

The following result shows that $\mathcal{D}_X$-complete trajectories are included in $\mathcal{D}_X$-cocycle attractors.

Theorem 4.3.6. *Suppose that $A=\{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is the $\mathcal{D}_X$-cocycle attractor for an NRDS ϕ, then*

$$\Big\{\xi(\omega,0):\xi \text{ is a } \sigma\text{-driven } \mathcal{D}_X\text{-complete trajectory of } \phi\Big\}\subseteq A_\sigma(\omega),\ \forall\sigma\in\Sigma,\omega\in\Omega.$$

Proof. Given a σ-driven $\mathcal{D}_X$-complete trajectory ξ of NRDS ϕ, suppose $B\in\mathcal{D}_X$ is a random set such that $\cup_{t\in\mathbb{R}}\xi(\cdot,t)\subset B(\cdot)$. Then by the pullback attraction of A we have

$$\begin{aligned} \operatorname{dist}(\xi(\omega,0),A_\sigma(\omega)) &= \operatorname{dist}\Big(\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\xi(\vartheta_{-t}\omega,-t)),A_\sigma(\omega)\Big) \\ &\leqslant \operatorname{dist}\Big(\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\cup_{s\in\mathbb{R}}\xi(\vartheta_{-t}\omega,s)),A_\sigma(\omega)\Big) \\ &\leqslant \operatorname{dist}\Big(\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,B(\vartheta_{-t}\omega)),A_\sigma(\omega)\Big)\to 0,\quad \text{as } t\to\infty. \end{aligned}$$

Hence, $\xi(\omega,0)\in A_\sigma(\omega)$ and the proof is complete. □

Lemma 4.3.7. *Suppose that $A=\{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is the $\mathcal{D}_X$-cocycle attractor for an NRDS ϕ, then*

$$A_\sigma(\omega)\subseteq\Big\{\xi(\omega,0):\xi \text{ is a } \sigma\text{-driven complete trajectory of } \phi\Big\},\ \forall\sigma\in\Sigma,\omega\in\Omega.$$

Proof. Let $y \in A_\sigma(\omega)$, and we establish a σ-driven complete trajectory ξ of ϕ such that $\xi(\omega, 0) = y$. By the invariance of A that

$$A_\sigma(\omega) = \phi(1, \vartheta_{-1}\omega, \theta_{-1}\sigma, A_{\theta_{-1}\sigma}(\vartheta_{-1}\omega)),$$

there exists a $z_1 \in A_{\theta_{-1}\sigma}(\vartheta_{-1}\omega)$ such that

$$y = \phi(1, \vartheta_{-1}\omega, \theta_{-1}\sigma, z_1),$$

and, moreover,

$$\begin{aligned}\phi(t, \vartheta_{-1}\omega, \theta_{-1}\sigma, z_1) &= \phi(t-1, \omega, \sigma, \phi(1, \vartheta_{-1}\omega, \theta_{-1}\sigma, z_1)) \\ &= \phi(t-1, \omega, \sigma, y), \quad \forall t \geqslant 1.\end{aligned}$$

Also, by the invariance of A we know $\phi(t, \vartheta_{-1}\omega, \theta_{-1}\sigma, z_1) \in A_{\theta_{t-1}\sigma}(\vartheta_{t-1}\omega)$.

In the same way we consider next the invariance

$$A_{\theta_{-1}\sigma}(\vartheta_{-1}\omega) = \phi(t, \vartheta_{-t-1}\omega, \theta_{-t-1}\sigma, A_{\theta_{-t-1}\sigma}(\vartheta_{-t-1}\omega)), \quad \forall t \geqslant 0.$$

For the $z_1 \in A_{\theta_{-1}\sigma}(\vartheta_{-1}\omega)$, there exists a $z_2 \in A_{\theta_{-2}\sigma}(\vartheta_{-2}\omega)$ such that

$$z_1 = \phi(1, \vartheta_{-2}\omega, \theta_{-2}\sigma, z_2),$$

and, moreover, that

$$\phi(t, \vartheta_{-2}\omega, \theta_{-2}\sigma, z_2) = \phi(t-1, \vartheta_{-1}\omega, \theta_{-1}\sigma, z_1), \quad \forall t \geqslant 1.$$

By the invariance of A again we have $\phi(t, \vartheta_{-2}\omega, \theta_{-2}\sigma, z_2) \in A_{\theta_{t-2}\sigma}(\vartheta_{t-2}\omega)$.

Continue this process and then we obtain a sequence $\{z_n\}_{n\in\mathbb{N}}$ with $z_n \in A_{\theta_{-n}\sigma}(\vartheta_{-n}\omega)$ such that

$$\begin{gathered} y = \phi(n, \vartheta_{-n}\omega, \theta_{-n}\sigma, z_n), \quad \forall n \in \mathbb{N}, \\ \phi(t, \vartheta_{-n}\omega, \theta_{-n}\sigma, z_n) \in A_{\theta_{t-n}\sigma}(\vartheta_{t-n}\omega), \quad \forall t \geqslant 0, \end{gathered}$$

and that

$$\phi(t, \vartheta_{-n}\omega, \theta_{-n}\sigma, z_n) = \phi(t-1, \vartheta_{1-n}\omega, \theta_{1-n}\sigma, z_{n-1}), \quad \forall t \geqslant 1.$$

Define a mapping $\xi(\vartheta.\omega, \cdot) : \mathbb{R} \to X$ such that, for every $t \in \mathbb{R}$, $\xi(\vartheta_t\omega, t)$ is the common value of $\phi(t+n, \vartheta_{-n}\omega, \theta_{-n}\sigma, z_n)$ for all $n \geqslant -t$. Then $\xi(\omega, 0) = y$, $\xi(\vartheta_t\omega, t) \in A_{\theta_t\sigma}(\vartheta_t\omega)$ and, moreover, ξ is a σ-driven complete trajectory. Indeed, for any $t \geqslant s \in \mathbb{R}$ and $n \geqslant -s \geqslant -t$,

$$\begin{aligned}\phi(t-s, \vartheta_s\omega, \theta_s\sigma, \xi(\vartheta_s\omega, s)) &= \phi(t-s, \vartheta_s\omega, \theta_s\sigma, \phi(s+n, \vartheta_{-n}\omega, \theta_{-n}\sigma, z_n)) \\ &= \phi(t+n, \vartheta_{-n}\omega, \theta_{-n}\sigma, z_n) = \xi(\vartheta_t\omega, t).\end{aligned}$$

The proof is complete. □

Corollary 4.3.8. *Suppose that $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is the $\mathcal{D}_X$-cocycle attractor for a continuous NRDS ϕ, and, moreover, there is a random set $B \in \mathcal{D}_X$ such that $\cup_{\sigma\in\Sigma} A_\sigma(\cdot) \subset B(\cdot)$. Then*

$$A_\sigma(\omega) = \Big\{\xi(\omega, 0) : \xi \textit{ is a } \sigma\textit{-driven } \mathcal{D}_X\textit{-complete trajectory of } \phi\Big\}, \ \forall \sigma \in \Sigma, \omega \in \Omega.$$

Proof. For each $y \in A_\sigma(\omega)$, by Lemma 4.3.7 there exists a σ-driven complete trajectory ξ of ϕ such that $\xi(\omega, 0) = y$ and $\xi(\vartheta_t\omega, t) \in A_{\theta_t\sigma}(\vartheta_t\omega)$. Since

$$\bigcup_{t\in\mathbb{R}} \xi(\omega, t) = \bigcup_{t\in\mathbb{R}} \xi(\vartheta_t\vartheta_{-t}\omega, t) \subseteq \bigcup_{t\in\mathbb{R}} A_{\theta_t\sigma}(\omega) \subset B(\omega),$$

ξ is a σ-driven $\mathcal{D}_X$-complete trajectory and hence the $\subseteq$ relation holds. The inverse inclusion follows from Theorem 4.3.6. □

In fact, we have proved the following slightly stronger conclusion.

Theorem 4.3.9. *Suppose $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is the $\mathcal{D}_X$-random cocycle attractor for a continuous NRDS ϕ, and, moreover, there is a random set $B \in \mathcal{D}_X$ such that $\cup_{\sigma\in\Sigma} A_\sigma(\cdot) \subset B(\cdot)$. Then*

$$A_{\theta_t\sigma}(\vartheta_t\omega) = \Big\{\xi(\vartheta_t\omega, t) : \xi \textit{ is a } \sigma\textit{-driven } \mathcal{D}_X\textit{-complete trajectory of } \phi\Big\}, \ \forall t \in \mathbb{R}, \sigma \in \Sigma, \omega \in \Omega.$$

Proof. For each $T \in \mathbb{R}$, $\sigma \in \Sigma$ and $\omega \in \Omega$ fixed, let $\sigma' = \theta_T\sigma$ and $\omega' = \vartheta_T\omega$. Then by Corollary 4.3.8,

$$A_{\theta_T\sigma}(\vartheta_T\omega) = A_{\sigma'}(\omega') = \big\{\xi'(\omega', 0) : \xi' \text{ is a } \sigma'\text{-driven } \mathcal{D}_X\text{-complete trajectory of } \phi\big\}.$$

Since ξ' is a σ'-driven $\mathcal{D}_X$-complete trajectory of ϕ if and only if ξ defined by

$$\xi(\omega, t) := \xi'(\vartheta_{-T}\omega, t - T), \quad \forall \omega \in \Omega, t \in \mathbb{R},$$

is a σ-driven $\mathcal{D}_X$-complete trajectory of ϕ, we have

$$A_{\theta_T\sigma}(\vartheta_T\omega) = \big\{\xi(\vartheta_T\omega, T) : \xi \text{ is a } \sigma\text{-driven } \mathcal{D}_X\text{-complete trajectory of } \phi\big\}.$$

The proof is complete. □

4.4 Upper semi-continuity in symbols

The non-autonomous feature of a cocycle attractor for an NRDS is represented by the σ-dependence. In this part, we study the upper semi-continuity of the mapping $\sigma \mapsto A_\sigma(\omega)$ for each $\omega \in \Omega$ fixed.

Definition 4.4.1. A $\mathcal{D}_X$-random cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is said to be *upper semi-continuous (in symbols)* if, for each $\omega \in \Omega$,

$$\text{dist}(A_\sigma(\omega), A_{\sigma_0}(\omega)) \to 0, \quad \text{whenever } \sigma \to \sigma_0 \text{ in } \Sigma.$$

Theorem 4.4.2. *Suppose that $A = \{A_\sigma(\omega)\}$ is a $\mathcal{D}_X$-random cocycle attractor of an NRDS ϕ. Then*

(1) if the mapping $\phi(t, \omega, \cdot, \cdot)$ is jointly continuous for each t and ω fixed, and there is a compact random set $D \in \mathcal{D}_X$ such that $\cup_{\sigma\in\Sigma} A_\sigma(\cdot) \subseteq D(\cdot)$, then A is upper semi-continuous;

(2) if A is upper semi-continuous and Σ is compact, then $\cup_{\sigma\in\Sigma}A_\sigma(\omega)$ is compact for each ω.

Proof. (1) We prove by contradiction. If the mapping $\sigma\to A_\sigma(\omega)$ is not upper semi-continuous at some σ, there must be a $\delta>0$ and $\sigma_n\to\sigma$ such that

$$\mathrm{dist}(A_{\sigma_n}(\omega),A_\sigma(\omega))\geqslant\delta,\quad \forall n\in\mathbb{N}.$$

By the compactness of attractors, there is a sequence $x_n\in A_{\sigma_n}(\omega)$ such that

$$\mathrm{dist}(x_n,A_\sigma(\omega))=\mathrm{dist}(A_{\sigma_n}(\omega),A_\sigma(\omega))\geqslant\delta,\quad \forall n\in\mathbb{N}. \tag{4.7}$$

On the other hand, by the pullback attraction of A, there is a $T\geqslant 0$ such that

$$\mathrm{dist}\Big(\phi\big(T,\vartheta_{-T}\omega,\theta_{-T}\sigma,D(\vartheta_{-T}\omega)\big),A_\sigma(\omega)\Big)\leqslant\delta/3.$$

Moreover, by the invariance of A we have a sequence $y_n\in A_{\theta_{-T}\sigma_n}(\vartheta_{-T}\omega)\subset D(\vartheta_{-T}\omega)$ such that

$$x_n=\phi(T,\vartheta_{-T}\omega,\theta_{-T}\sigma_n,y_n),\quad \forall n\in\mathbb{N},$$

and $y_n\to y$ for some $y\in D(\vartheta_{-T}\omega)$ up to a subsequence as D is compact. Hence, since the mappings $(\sigma,x)\mapsto\phi(T,\omega,\sigma,x)$ and $\sigma\mapsto\theta_{-T}\sigma$ are both continuous, there is an N large enough such that

$$\mathrm{dist}\Big(\phi\big(T,\vartheta_{-T}\omega,\theta_{-T}\sigma_n,y_n\big),\phi\big(T,\vartheta_{-T}\omega,\theta_{-T}\sigma,y\big)\Big)\leqslant\delta/3,\quad \forall n\geqslant N.$$

Hence,

$$\begin{aligned}\mathrm{dist}(x_n,A_\sigma(\omega))&=\mathrm{dist}\Big(\phi\big(T,\vartheta_{-T}\omega,\theta_{-T}\sigma_n,y_n\big),A_\sigma(\omega)\Big)\\&\leqslant\mathrm{dist}\Big(\phi\big(T,\vartheta_{-T}\omega,\theta_{-T}\sigma_n,y_n\big),\phi\big(T,\vartheta_{-T}\omega,\theta_{-T}\sigma,y\big)\Big)\\&\quad+\mathrm{dist}\Big(\phi\big(T,\vartheta_{-T}\omega,\theta_{-T}\sigma,D(\vartheta_{-T}\omega)\big),A_\sigma(\omega)\Big)\\&\leqslant 2\delta/3,\quad \forall n\geqslant N,\end{aligned}$$

which contradicts (4.7).

(2) In order to prove the compactness of $\cup_{\sigma\in\Sigma}A_\sigma(\omega)$, take arbitrarily a sequence $\{x_n\}_{n\in\mathbb{N}}\subset\cup_{\sigma\in\Sigma}A_\sigma(\omega)$ and, without loss of generality, let $x_n\in A_{\sigma_n}(\omega)$. Since Σ is compact, there is a $\sigma\in\Sigma$ such that $\sigma_n\to\sigma$ in a subsequence sense. Therefore, by the upper semi-continuity of A we have

$$\mathrm{dist}(x_n,A_\sigma(\omega))\leqslant\mathrm{dist}(A_{\sigma_n}(\omega),A_\sigma(\omega))\to 0,\quad \text{as } n\to\infty.$$

Since A_σ is compact, there is an $x\in A_\sigma(\omega)$ such that $x_n\to x$ up to a subsequence. Hence, $\cup_{\sigma\in\Sigma}A_\sigma(\omega)$ is compact and the proof is complete. □

Remark 4.4.3. It is important to note that, though under the upper semi-continuity of A we have proved the compactness of $\cup_{\sigma\in\Sigma}A_\sigma(\omega)$, we cannot call $\omega\mapsto\cup_{\sigma\in\Sigma}A_\sigma(\omega)$ a compact random set since the $(\mathcal{F},\mathcal{B}(X))$-measurability is not clear yet.

When Ω is a singleton, then the NRDS ϕ reduces to a deterministic non-autonomous dynamical system. For this deterministic case, we have the following corollary which slightly improves [55, Theorem 3.34].

Corollary 4.4.4. *Suppose that Σ is compact, Ω is a singleton and $A=\{A_\sigma\}_{\sigma\in\Sigma}$ is the $\mathcal{D}_X$-cocycle attractor of a continuous (deterministic) non-autonomous dynamical system ϕ which is also continuous in symbols. Then A is upper semi-continuous in σ if and only if $\cup_{\sigma\in\Sigma}A_\sigma$ is compact.*

4.5 Applications to a 2D stochastic Navier-Stokes equation

In this section we study the cocycle attractor for a stochastic 2D Navier-Stokes equation. In order to illustrate the theoretical results in this chapter, we endow the equation with translation bounded external forcing. Let us recall briefly some properties for translation bounded functions, see [26] or Section 5.5.2 later on.

Let $\mathcal{O}$ be an open bounded set of $\mathbb{R}^2$ with smooth boundary Γ. Take $g(t,x) \in L^2_{loc}(\mathbb{R}; (L^2(\mathcal{O}))^2)$ and define the symbol space Σ as the closed hull $\mathcal{H}(g)$ of g in $L^2_{loc}(\mathbb{R}; (L^2(\mathcal{O}))^2)$ under the local weak convergence topology. Define a flow $\{\theta_s\}_{s\in\mathbb{R}}$ on Σ by

$$\theta_s\sigma(\cdot) := \sigma(\cdot + s).$$

Definition 4.5.1. A function $g(t,x) \in L^2_{loc}(\mathbb{R}; (L^2(\mathcal{O}))^2)$ is called translation bounded if

$$\eta(g) := \sup_{\tau\in\mathbb{R}} \int_{\tau-1}^{\tau} |g(s)|^2 \,\mathrm{d}s < \infty. \tag{4.8}$$

For translation bounded functions we have the following proposition.

Proposition 4.5.2. *Suppose $g(t,x) \in L^2_{loc}(\mathbb{R}; (L^2(\mathcal{O}))^2)$ is translation bounded. Then*

(i) g is translation compact, i.e., the symbol space Σ defined above is compact;

(ii) Σ is invariant under θ_t;

(iii) any function $\sigma \in \mathcal{H}(g)$ is translation bounded, and $\eta(\sigma) \leqslant \eta(g)$;

(iv) for any positive constant ε it holds

$$\sup_{\sigma\in\Sigma} \int_{-\infty}^{0} e^{\varepsilon s}|\sigma(s)|^2 \,\mathrm{d}s \leqslant \frac{\eta(g)}{1-e^{-\varepsilon}}. \tag{4.9}$$

Proof. Similar to Propositions 5.5.2 and 5.5.3. □

Now we shall study the two-dimensional stochastic Navier-Stokes equation on $\mathcal{O}$ with translation bounded external forcing and scalar additive noise. This equation reads

$$\begin{cases} \mathrm{d}u + (-\nu\Delta u + (u\cdot\nabla)u)\mathrm{d}t = \sigma(t)\mathrm{d}t + \psi \mathrm{d}\omega(t), \\ \nabla\cdot u = 0, \end{cases} \tag{4.10}$$

endowed with initial-boundary value condition

$$\begin{cases} u(t,x)|_{t=0} = u_0(x), \\ u(t,x)|_{\Gamma} = 0, \end{cases} \tag{4.11}$$

where $\nu > 0$ is a constant, $\sigma \in \Sigma$ and Σ is the previously defined symbol space defined as the hull of a translation bounded function g. The term ω is a scalar Brownian motion from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ specified in Section 2.3.1. Similar to Proposition 5.5.1 one can see that Σ is in fact a compact Polish metric space, and the mapping $t \mapsto \theta_t \sigma$ is $(\mathbb{R}, \Sigma)$-continuous.

Set Banach spaces $(H, |\cdot|)$ and $(V, \|\cdot\|)$ by

$$H = \left\{ \varphi \in (L^2(\mathcal{O}))^2 : \nabla \cdot \varphi = 0,\, n \cdot \varphi = 0 \text{ on } \Gamma \right\},$$
$$V = \left\{ \varphi \in (H_0^1(\mathcal{O}))^2 : \nabla \cdot \varphi = 0 \right\},$$

respectively, where n is the outward normal.

Define the Stokes operator $A : D(A) \subset H \to H$ as $Au = -P\Delta u$, where P is the orthogonal projection in $(L^2(\mathcal{O}))^2$ over H and $D(A) = (H^2(\mathcal{O}))^2 \cap V$. Moreover, define the bilinear operator B as

$$\begin{aligned} \langle B(u,v), w \rangle &= \int_{\mathcal{O}} w(x) \cdot (u \cdot \nabla) v \, \mathrm{d}x \\ &= \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j \, \mathrm{d}x, \quad \forall u \in H, v \in V, w \in H. \end{aligned}$$

By the incompressibility condition we have

$$\langle B(u,v), v \rangle = 0, \quad \langle B(u,v), w \rangle = -\langle B(u,w), v \rangle.$$

Let $\psi \in (W^{1,\infty}(\mathcal{O}))^2 \cap D(A)$. With these preliminaries, equation (4.10) is written in the following abstract form

$$\mathrm{d}u + (\nu A u + B(u,u))\mathrm{d}t = \sigma(t)\mathrm{d}t + \psi \mathrm{d}\omega(t). \tag{4.12}$$

Inspired by [33], for some $\alpha > 0$ (specified later by (4.25)), we consider

$$z(\omega) = -\int_{-\infty}^{0} e^{\alpha \tau} \mathrm{d}\omega(\tau), \quad \forall \omega \in \Omega.$$

Then from Section 2.3.1 we know $z(\omega)$ is a stationary solution of the one-dimensional Ornstein-Uhlenbeck equation

$$\mathrm{d}z(\vartheta_t \omega) + \alpha z(\vartheta_t \omega)\mathrm{d}t = \mathrm{d}\omega(t). \tag{4.13}$$

Moreover, there exists a ϑ-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that $z(\vartheta_t \omega)$ is continuous in t for every $\omega \in \tilde{\Omega}$ and the random variable $|z(\cdot)|$ is tempered (see Section 2.3.1), namely, for each $\varepsilon > 0$ it holds

$$\lim_{t \to \infty} e^{-\varepsilon t} |z(\vartheta_{-t} \omega)| = 0, \quad \forall \omega \in \Omega. \tag{4.14}$$

Hereafter, we will not distinguish $\tilde{\Omega}$ and Ω.

Consider the following deterministic problem with random coefficients

$$\begin{cases} \frac{dv}{dt} + \nu A v + B(v + z(\vartheta_t \omega)\psi, v + z(\vartheta_t \omega)\psi) = \sigma(t) + \alpha z(\vartheta_t \omega)\psi - \nu z(\vartheta_t \omega) A \psi, \\ \nabla \cdot v = 0, \end{cases} \tag{4.15}$$

with the initial-boundary condition

$$\begin{cases} v(t,x)|_{t=0} = v_0(x), \\ v(t,x)|_\Gamma = 0. \end{cases} \tag{4.16}$$

The following result is standard, see [26, Chapter VI.1] and references therein.

Lemma 4.5.3. *For each $\omega \in \Omega$, $\sigma \in \Sigma$, and $v_0 \in H$, problem (4.15) and (4.16) has a unique solution $v(t,\omega,\sigma,v_0) \in C(\mathbb{R}^+; H) \cap L^2_{loc}(\mathbb{R}^+; V)$ and $\partial_t v \in L^2_{loc}(\mathbb{R}^+; V')$. Moreover, the mapping $v(t,\omega,\cdot,\cdot)$ is $(\Sigma \times H, H)$-continuous.*

Now define an NRDS $\phi : \mathbb{R}^+ \times \Omega \times \Sigma \times H \to H$ for the stochastic problem (4.10). Given $t \geqslant 0$, $\omega \in \Omega$ and $\sigma \in \Sigma$ and $u_0 \in H$, set

$$\phi(t,\omega,\sigma,u_0) = u(t,\omega,\sigma,u_0) := v(t,\omega,\sigma,u_0 - z(\omega)) + z(\vartheta_t\omega)\psi, \tag{4.17}$$

where v is the solution of (4.15) and (4.16). Then by (4.13) and $\psi \in D(A)$ we see that $u(t)$ is the solution of (4.10) and (4.11). Moreover, the mapping ϕ defines a jointly continuous NRDS.

In the following, we study tempered cocycle attractors for the Navier-Stokes equation. Recall that a non-autonomous random set $\hat{D}$ in H is said to be tempered if it is a bounded random set such that for any $\varepsilon > 0$ it holds

$$\lim_{t\to\infty} e^{-\varepsilon t} |\hat{D}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega)|^2 = 0, \quad \forall \sigma \in \Sigma, \omega \in \Omega. \tag{4.18}$$

Similarly, for a non-autonomous random variable r, namely, a real-valued non-autonomous random set mapping from $\Omega \times \Sigma$ to $\mathbb{R}$, is said to be tempered if, for any $\varepsilon > 0$,

$$\lim_{t\to\infty} e^{-\varepsilon t} |r_{\theta_{-t}\sigma}(\vartheta_{-t}\omega)| = 0, \quad \forall \sigma \in \Sigma, \omega \in \Omega.$$

Autonomous tempered random sets and random variables are defined analogously. Let

$$\mathcal{D}_H = \Big\{ D : D \text{ is a tempered autonomous random set in } H \Big\},$$
$$\hat{\mathcal{D}}_H = \Big\{ \hat{D} : \hat{D} \text{ is a tempered non-autonomous random set in } H \Big\}.$$

Then the two universes $\mathcal{D}_H$ and $\hat{\mathcal{D}}_H$ are both neighborhood- and inclusion-closed.

4.5.1 Uniform estimates of solutions

In this part we establish uniform estimates for solutions of the NS equation. Note that, as $\mathcal{D}_H \subset \hat{\mathcal{D}}_H$, all the estimates in this section holds for solutions with initial data in $\mathcal{D}_H$, though we work only on $\hat{\mathcal{D}}_H$.

Lemma 4.5.4. *For each $\hat{D} \in \hat{\mathcal{D}}_H$ and $\omega \in \Omega$, there exists a time $T = T(\hat{D}, \omega) > 0$ such that, for every $\sigma \in \Sigma$, the estimate*

$$|v(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, v_0)|^2 \leqslant 1 + c\int_{-\infty}^{0} e^{\int_s^0(-\nu\lambda_1+c_0|z(\vartheta_r\omega)|)dr}(|z(\vartheta_s\omega)|^4 + |\sigma(s)|^2 + 1)\mathrm{d}s \\ =: R_1(\omega, \sigma) \tag{4.19}$$

holds uniformly in $v_0 \in \hat{D}$ and $t \geqslant T$, where $c_0 = (|\psi| + 1)\|\nabla\psi\|_{(L^\infty)^2}$ and c is a positive constant.

Proof. It follows from (4.15) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \nu\|v\|^2 \leqslant |\langle B(v + z(\vartheta_t\omega)\psi, v + z(\vartheta_t\omega)\psi), v\rangle| + |\sigma(t)||v| \\ + \alpha|z(\vartheta_t\omega)||\psi||v| + \nu|z(\vartheta_t\omega)|\|\psi\|\|v\|.$$

Since $\langle B(\xi, \eta), \eta\rangle = 0$ and $\psi \in (W^{1,\infty}(\mathcal{O}))^2$, we have

$$|\langle B(v + z(\vartheta_t\omega)\psi, v + z(\vartheta_t\omega)\psi), v\rangle| = |\langle B(v + z(\vartheta_t\omega)\psi, z(\vartheta_t\omega)\psi), v\rangle| \\ \leqslant \|\nabla\psi\|_{(L^\infty)^2}|z(\vartheta_t\omega)||v|^2 + |\psi|\|\nabla\psi\|_{(L^\infty)^2}|z(\vartheta_t\omega)|^2|v|.$$

Denote by $c_0 = (|\psi| + 1)\|\nabla\psi\|_{(L^\infty)^2}$. Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \nu\|v\|^2 \leqslant c_0|z(\vartheta_t\omega)||v|^2 + c_0|z(\vartheta_t\omega)|^2|v| + |\sigma(t)||v| \\ + \alpha c|z(\vartheta_t\omega)||v| + c|z(\vartheta_t\omega)|\|v\|, \tag{4.20}$$

where and hereafter c denotes a positive constant which depends only on ψ and ν and may change its value when necessary. By Young's inequality and Poincaré's inequality $\|v\| \geqslant \lambda_1|v|$ for some $\lambda_1 > 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \nu\|v\|^2 \leqslant c_0|z(\vartheta_t\omega)||v|^2 + c|z(\vartheta_t\omega)|^4 + c|\sigma(t)|^2 + \alpha^4 + c, \tag{4.21}$$

and then

$$\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \nu\lambda_1|v|^2 \leqslant c_0|z(\vartheta_t\omega)||v|^2 + c|z(\vartheta_t\omega)|^4 + c|\sigma(t)|^2 + \alpha^4 + c. \tag{4.22}$$

Applying Gronwall's technique to (4.22) we have

$$|v(t, \omega, \sigma, v_0)|^2 \leqslant e^{\int_0^t(-\nu\lambda_1+c_0|z(\vartheta_r\omega)|)dr}|v_0|^2 \\ + c\int_0^t e^{\int_s^t(-\nu\lambda_1+c_0|z(\vartheta_r\omega)|)dr}(|z(\vartheta_s\omega)|^4 + |\sigma(s)|^2 + \alpha^4 + 1)\mathrm{d}s.$$

Replacing ω by $\vartheta_{-t}\omega$ and σ by $\theta_{-t}\sigma$, respectively, we have

$$|v(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, v_0)|^2 \leqslant e^{\int_0^t(-\nu\lambda_1+c_0|z(\vartheta_{r-t}\omega)|)dr}|v_0|^2 \\ + c\int_0^t e^{\int_s^t(-\nu\lambda_1+c_0|z(\vartheta_{r-t}\omega)|)dr}(|z(\vartheta_{s-t}\omega)|^4 + |\sigma(s-t)|^2 + \alpha^4 + 1)\mathrm{d}s \\ \leqslant e^{\int_{-t}^0(-\nu\lambda_1+c_0|z(\vartheta_r\omega)|)dr}|v_0|^2 \\ + c\int_{-\infty}^0 e^{\int_s^0(-\nu\lambda_1+c_0|z(\vartheta_r\omega)|)dr}(|z(\vartheta_s\omega)|^4 + |\sigma(s)|^2 + \alpha^4 + 1)\,\mathrm{d}s. \tag{4.23}$$

Since the process $|z(\theta_t\omega)|$ is stationary and ergodic, by the ergodic theorem we see that

$$\frac{1}{s}\int_{-s}^{0}|z(\theta_t\omega)|\,\mathrm{d}t \to E(|z(\omega)|) \quad \text{as } s\to\infty.$$

This means that there exists an $s_0(\omega)>0$ such that

$$\frac{1}{s}\int_{-s}^{0}|z(\theta_t\omega)|\,\mathrm{d}t \leqslant E(|z(\omega)|)+1 \leqslant \frac{1}{\sqrt{2\alpha}}+1, \quad \forall s \geqslant s_0(\omega). \tag{4.24}$$

Now and hereafter we fix $\alpha>0$ such that

$$\frac{1}{\sqrt{2\alpha}}+1=\frac{\nu\lambda_1}{2c_0}, \tag{4.25}$$

so that (4.24) implies

$$e^{\int_{-s}^{0}(-\nu\lambda_1+c_0|z(\vartheta_r\omega)|)dr} \leqslant e^{-\frac{\nu\lambda_1}{2}s}, \quad \forall s\geqslant s_0(\omega). \tag{4.26}$$

Hence, since $v_0\in\hat{D}_{\theta_{-t}\sigma}(\vartheta_{-t}\omega)$ is tempered, it follows from (4.26), (4.18) and (4.23) that there exists a $T=T(\omega,\hat{D})>0$ such that

$$|v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)|^2 \leqslant 1+c\int_{-\infty}^{0} e^{\int_s^0(-\nu\lambda_1+c_0|z(\vartheta_r\omega)|)dr}(|z(\vartheta_s\omega)|^4+|\sigma(s)|^2+1)\mathrm{d}s.$$

The proof is complete. □

Lemma 4.5.5. *For each $\hat{D}\in\hat{\mathcal{D}}_H$ and $\omega\in\Omega$, there exists a time $T=T(\hat{D},\omega)>0$ given by Lemma 4.5.4 such that, for every $\sigma\in\Sigma$, the estimates*

$$\int_{t-1}^{t}\|v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2\,\mathrm{d}s \leqslant cR_2(\omega,\sigma) \tag{4.27}$$

holds uniformly in $v_0\in\hat{D}$ and $t\geqslant T+1$, where c is a positive constant and $R_2(\omega,\sigma)$ is a non-autonomous random variable given by (4.28).

Proof. Integrating (4.21) over $(t-1,t)$ we obtain

$$\int_{t-1}^{t}\|v(s,\omega,\sigma,v_0)\|^2\,\mathrm{d}s \leqslant c\int_{t-1}^{t}|z(\vartheta_s\omega)||v(s,\omega,\sigma,v_0)|^2\,\mathrm{d}s + c\int_{t-1}^{t}(|z(\vartheta_s\omega)|^4+|\sigma(s)|^2+1)\,\mathrm{d}s+|v(t-1,\omega,\sigma,v_0)|^2.$$

Replacing ω by $\vartheta_{-t}\omega$ and σ by $\theta_{-t}\sigma$, we have that

$$\int_{t-1}^{t}\|v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2\,\mathrm{d}s \leqslant c\int_{t-1}^{t}|z(\vartheta_{s-t}\omega)||v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)|^2\,\mathrm{d}s + c\int_{t-1}^{t}(|z(\vartheta_{s-t}\omega)|^4+|\sigma(s-t)|^2)\,\mathrm{d}s + |v(t-1,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)|^2+c.$$

For any $t \geqslant T+1$, by Lemma 4.5.4, we conclude that

$$\int_{t-1}^{t} \|v(s, \vartheta_{-t}\omega, \theta_{-t}\sigma, v_0)\|^2 \,\mathrm{d}s \leqslant cR_2(\omega, \sigma)$$

with

$$\begin{aligned} R_2(\omega,\sigma) := &\int_{-1}^{0} |z(\vartheta_s\omega)| R_1(\vartheta_s\omega, \theta_s\sigma)\,\mathrm{d}s \\ &+ \int_{-1}^{0} (|z(\vartheta_s\omega)|^4 + |\sigma(s)|^2)\,\mathrm{d}s + R_1(\vartheta_{-1}\omega, \theta_{-1}\sigma). \end{aligned} \tag{4.28}$$

The proof is complete. □

Lemma 4.5.6. *For each $\hat{D} \in \hat{\mathcal{D}}_H$ and $\omega \in \Omega$, there exist a $T = T(\hat{D}, \omega) > 0$ given by Lemma 4.5.4 and a non-autonomous random variable $R_3(\omega, \sigma)$ such that, for every $\sigma \in \Sigma$,*

$$\|v(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, v_0)\|^2 \leqslant R_3(\omega, \sigma) \tag{4.29}$$

holds uniformly in $v_0 \in \hat{D}$ and $t \geqslant T+1$, where c is a positive constant.

Proof. Taking the inner product of (4.15) with Av in H, by Young's inequalities we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \nu|Av|^2 &= (\sigma(t), Av) + (\alpha z(\vartheta_t(\omega))\psi, Av) - \nu(z(\vartheta_t(\omega))A\psi, Av) \\ &\quad - (B(v + z(\vartheta_t\omega)\psi, v + z(\vartheta_t\omega)\psi), Av) \\ &\leqslant \frac{\nu}{2}|Av|^2 + c|\sigma(t)|^2 + c|z(\vartheta_t(\omega))\psi|^2 + c|z(\vartheta_t(\omega))A\psi|^2 \\ &\quad - (B(v + z(\vartheta_t\omega)\psi, v + z(\vartheta_t\omega)\psi), Av). \end{aligned}$$

Since $|\langle B(\eta, \xi, \varrho)\rangle \leqslant |\eta|^{1/2}|A\eta|^{1/2}\|\xi\||\varrho|$ for each $\eta \in D(A), \xi \in V$ and $\varrho \in H$, see [79, p. 106], we have

$$\begin{aligned} &|(B(v + z(\vartheta_t\omega)\psi, v + z(\vartheta_t\omega)\psi), Av)| \\ &\quad \leqslant |u|^{\frac{1}{2}}|Au|^{\frac{1}{2}}|\nabla u||Av| \leqslant \frac{\nu}{8}|Av|^2 + \frac{\nu}{8}|Au|^2 + c|u|^2|\nabla u|^4 \\ &\quad \leqslant \frac{\nu}{2}|Av|^2 + c|z(\vartheta_t\omega)A\psi|^2 + c|u|^2|\nabla v|^4 + c|u|^2|z(\vartheta_t\omega)\nabla\psi|^4 \\ &\quad \leqslant \frac{\nu}{2}|Av|^2 + c|z(\vartheta_t\omega)|^2 + c|v + z(\vartheta_t\omega)\psi|^2|\nabla v|^4 + c|v|^2|z(\vartheta_t\omega)|^4 + c|z(\vartheta_t\omega)|^6. \end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t,\omega,\sigma,v_0)\|^2 \leqslant c\mathcal{M}(t,\omega,\sigma,v_0) + c\mathcal{N}(t,\omega,\sigma,v_0)\|v\|^2,$$

where

$$\begin{aligned} \mathcal{M}(t,\omega,\sigma,v_0) &= |v(t,\omega,\sigma,v_0)|^2|z(\vartheta_t\omega)|^4 + |\sigma(t)|^2 + |z(\vartheta_t\omega)|^6 + 1, \\ \mathcal{N}(t,\omega,\sigma,v_0) &= |v(t,\omega,\sigma,v_0) + z(\vartheta_t\omega)\psi|^2\|v(t,\omega,\sigma,v_0)\|^2. \end{aligned} \tag{4.30}$$

By Gronwall's inequality for $s \in (t-1,t)$, we obtain

$$\|v(t,\omega,\sigma,v_0)\|^2 \leqslant c e^{\int_{t-1}^{t} c\mathcal{N}(r,\omega,\sigma,v_0)\,\mathrm{d}r} \left(\|v(s,\omega,\sigma,v_0)\|^2 + \int_{t-1}^{t} \mathcal{M}(r,\omega,\sigma,v_0)\,\mathrm{d}r \right). \tag{4.31}$$

Now integrating (4.31) with respect to s over $(t-1,t)$ and replacing ω and σ by $\vartheta_{-t}\omega$ and $\theta_{-t}\sigma$, respectively, we have

$$\begin{aligned} &\|v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 \leqslant c e^{\int_{t-1}^{t} c\mathcal{N}(r,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\,\mathrm{d}r} \times \\ &\quad \times \left(\int_{t-1}^{t} \|v(s,\vartheta_{s-t}\omega,\theta_{s-t}\sigma,v_0)\|^2\,\mathrm{d}s + \int_{t-1}^{t} \mathcal{M}(r,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\,\mathrm{d}r \right). \end{aligned} \tag{4.32}$$

Notice that by Lemmas 4.5.4 and 4.5.5, for all $t \geqslant T+1$ we have

$$\begin{aligned} &\int_{t-1}^{t} \mathcal{N}(r,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\,\mathrm{d}r \\ &\quad \leqslant c \int_{t-1}^{t} (|R_1(\vartheta_{s-t}\omega,\theta_{s-t}\sigma)|^2 + |z(\vartheta_{s-t}\omega)|^2)\|v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2\,\mathrm{d}s \\ &\quad \leqslant c \sup_{s\in(-1,0)} (|R_1(\vartheta_s\omega,\theta_s\sigma)|^2 + |z(\vartheta_s\omega)|^2) R_2(\omega,\sigma), \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \int_{t-1}^{t} \mathcal{M}(r,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\,\mathrm{d}r \leqslant c \sup_{s\in(-1,0)} &\left(|R_1(\vartheta_s\omega,\theta_s\sigma)|^2 |z(\vartheta_s\omega)|^4 + |z(\vartheta_s\omega)|^6 \right) \\ &+ \int_{-1}^{0} |\sigma(s)|^2\,\mathrm{d}s + c. \end{aligned} \tag{4.34}$$

Therefore, from (4.32)-(4.34) it follows that for all $t \geqslant T+1$,

$$\|v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 \leqslant R_3(\omega,\sigma) \tag{4.35}$$

with $R_3(\omega,\sigma)$ given by

$$\begin{aligned} R_3(\omega,\sigma) := c e^{c\sup_{s\in(-1,0)}(|R_1(\vartheta_s\omega,\theta_s\sigma)|^2 + |z(\vartheta_s\omega)|^2)R_2(\omega,\sigma)} &\Bigg(R_2(\omega,\sigma) \\ + \sup_{s\in(-1,0)} \left(|R_1(\vartheta_s\omega,\theta_s\sigma)|^2 |z(\vartheta_s\omega)|^4 + |z(\vartheta_s\omega)|^6 \right) &+ \int_{-1}^{0} |\sigma(s)|^2\,\mathrm{d}s + 1 \Bigg). \end{aligned} \tag{4.36}$$

The proof is complete. □

By (4.17) and Lemmas 4.5.4 and 4.5.6 we have the following estimate for solutions (4.17).

Corollary 4.5.7. *For each $\hat{D} \in \hat{\mathcal{D}}_H$ and $\omega \in \Omega$, there exists a time $T = T(\hat{D}, \omega) > 1$ such that, for every $\sigma \in \Sigma$, the solution u of stochastic Navier-stokes equation (4.12) satisfies*

$$|u(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, u_0)|^2 \leqslant cR_1(\omega, \sigma) + c|z(\omega)|^2,$$
$$\|u(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, u_0)\|^2 \leqslant cR_3(\omega, \sigma) + c|z(\omega)|^2,$$

uniformly in $u_0 \in \hat{D}$ and $t \geqslant T$, where $R_1(\omega, \sigma)$, given by (4.19), i.e.

$$R_1(\omega, \sigma) := 1 + c\int_{-\infty}^{0} e^{\int_s^0 (-\nu\lambda_1 + c_0|z(\vartheta_r\omega)|)\mathrm{d}r} (|z(\vartheta_s\omega)|^4 + |\sigma(s)|^2 + 1)\,\mathrm{d}s, \tag{4.37}$$

is a tempered non-autonomous random variable, $R_3(\omega, \sigma)$ is the non-autonomous random variable given by (4.36) and c is a positive constant.

4.5.2 Cocycle attractor with non-autonomous attraction universe $\hat{\mathcal{D}}_H$

For each $\omega \in \Omega$ and $\sigma \in \Sigma$, define

$$\begin{aligned} \hat{B}_\sigma(\omega) &= \{u \in H : |u|^2 \leqslant cR_1(\omega, \sigma) + c|z(\omega)|^2\}, \\ \hat{K}_\sigma(\omega) &= \{u \in V : \|u\|^2 \leqslant cR_3(\omega, \sigma) + c|z(\omega)|^2\}, \end{aligned} \tag{4.38}$$

where non-autonomous random variables $R_1(\omega, \sigma)$ and $R_3(\omega, \sigma)$ and the positive constant c are specified in Corollary 4.5.7. Clearly, $\hat{B} = \{\hat{B}_\sigma(\omega)\}$ is a tempered non-autonomous random set belonging to $\hat{\mathcal{D}}_H$, and $\hat{K} = \{\hat{K}_\sigma(\omega)\}$ is a compact non-autonomous random set in H. Corollary 4.5.7 indicates that $\hat{B}$ and $\hat{K}$ are both $\hat{\mathcal{D}}_H$-pullback absorbing sets. But it is unclear whether or not R_3 is tempered, and thus $\hat{K}$ is possibly not in $\hat{\mathcal{D}}_H$. This fact makes it complex to analyze the $\mathcal{D}_H$-cocycle attractor for the NS equation, and we put the study in next section.

For $\hat{\mathcal{D}}_H$-cocycle attractor, it is straightforward to have the following existence result by Theorem 3.2.2.

Theorem 4.5.8. *The NRDS ϕ generated by the NS equation (4.17) with translation bounded forcing has a $\hat{\mathcal{D}}_H$-random cocycle attractor $\hat{A} = \{\hat{A}_\sigma(\cdot)\}_{\sigma\in\Sigma}$ in H given by*

$$\hat{A}_\sigma(\omega) = \mathcal{W}(\omega, \sigma, \hat{B}), \quad \forall \sigma \in \Sigma,\ \omega \in \Omega. \tag{4.39}$$

4.5.3 Cocycle attractor with autonomous attraction universe $\mathcal{D}_H$

To see the existence of a $\mathcal{D}_H$-cocycle attractor, consider the σ-dependent term involved in $R_1(\omega, \sigma)$ given by (4.37). This term reads

$$\int_{-\infty}^{0} e^{\int_s^0 (-\nu\lambda_1 + c_0|z(\vartheta_r\omega)|)\mathrm{d}r} |\sigma(s)|^2\,\mathrm{d}s = \int_{-s_0}^{0} + \int_{-\infty}^{-s_0},$$

which has been split into two parts at $s_0 = s_0(\omega) > 0$ given by (4.24). By (4.26) and Proposition 4.5.2 (iv), the two parts are respectively bounded by

$$\begin{aligned}\int_{-s_0}^{0} e^{\int_s^0(-\nu\lambda_1+c_0|z(\vartheta_r\omega)|)\mathrm{d}r}|\sigma(s)|^2\,\mathrm{d}s &\leqslant e^{\int_{-s_0}^0(-\frac{3\nu\lambda_1}{4}+c_0|z(\vartheta_r\omega)|)\mathrm{d}r}\int_{-s_0}^{0} e^{-\int_s^0\frac{\nu\lambda_1}{4}\mathrm{d}r}|\sigma(s)|^2\,\mathrm{d}s \\ &\leqslant e^{\int_{-\infty}^0(-\frac{3\nu\lambda_1}{4}+c_0|z(\vartheta_r\omega)|)\mathrm{d}r}\int_{-\infty}^{0} e^{\frac{\nu\lambda_1}{4}s}|\sigma(s)|^2\,\mathrm{d}s \\ &\leqslant e^{\int_{-\infty}^0(-\frac{3\nu\lambda_1}{4}+c_0|z(\vartheta_r\omega)|)\mathrm{d}r}\frac{\eta(g)}{1-e^{-\frac{\nu\lambda_1}{4}}},\end{aligned}$$

and

$$\int_{-\infty}^{-s_0} e^{\int_s^0(-\nu\lambda_1+c_0|z(\vartheta_r\omega)|)\mathrm{d}r}|\sigma(s)|^2\,\mathrm{d}s \leqslant \int_{-\infty}^{-s_0} e^{\frac{\nu\lambda_1}{2}s}|\sigma(s)|^2\,\mathrm{d}s \leqslant \frac{\eta(g)}{1-e^{-\frac{\nu\lambda_1}{2}}},$$

where $\eta(g)$ is a positive constant given by (4.8). Hence, if we set

$$G(\omega) = e^{\int_{-\infty}^0(-\frac{3\nu\lambda_1}{4}+c_0|z(\vartheta_r\omega)|)\mathrm{d}r}\frac{\eta(g)}{1-e^{-\frac{\nu\lambda_1}{4}}} + \frac{\eta(g)}{1-e^{-\frac{\nu\lambda_1}{2}}}, \tag{4.40}$$

then G is a tempered (autonomous) random variable such that

$$\sup_{\sigma\in\Sigma}\int_{-\infty}^{0} e^{\int_s^0(-\nu\lambda_1+c_0|z(\vartheta_r\omega)|)\mathrm{d}r}|\sigma(s)|^2\,\mathrm{d}s \leqslant G(\omega).$$

Therefore, for the non-autonomous random variable R_1 defined by (4.37), it holds

$$\sup_{\sigma\in\Sigma} R_1(\omega,\sigma) \leqslant cG(\omega) + c\int_{-\infty}^{0} e^{\int_s^0(-\nu\lambda_1+c_0|z(\vartheta_r\omega)|)\mathrm{d}r}(|z(\vartheta_s\omega)|^4+1)\mathrm{d}s =: R(\omega).$$

This means that the autonomous random set E defined by

$$E(\omega) := \left\{u\in H : |u|^2 \leqslant R(\omega) + c|z(\omega)|^2\right\} \tag{4.41}$$

is tempered and thereby belongs to $\mathcal{D}_H$, and is such that $\cup_{\sigma\in\Sigma}\hat{B}_\sigma(\omega) \subset E(\omega)$.

Hereby, we are able to prove the following result on $\mathcal{D}_H$-cocycle attractors for the NS equation.

Theorem 4.5.9. *The NRDS ϕ generated by the NS equation (4.17) with translation bounded forcing has a $\mathcal{D}_H$-random cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ in H given by*

$$\begin{aligned} A_\sigma(\omega) &= \mathcal{W}(\omega,\sigma,E) && (4.42)\\ &= \left\{\xi(\omega,0) : \xi \text{ is a } \sigma\text{-driven } \mathcal{D}_H\text{-complete trajectory of } \phi\right\},\ \forall\sigma\in\Sigma,\omega\in\Omega. && (4.43)\end{aligned}$$

Moreover, the $\mathcal{D}_H$-attractor A and the $\hat{\mathcal{D}}_H$-attractor $\hat{A}$ given by Theorem 4.5.8 are identical, i.e.,

$$A_\sigma(\omega) = \hat{A}_\sigma(\omega), \quad \forall\sigma\in\Sigma,\ \omega\in\Omega.$$

Proof. Since we have shown that $E(\omega)$ is a closed tempered random set belonging to $\mathcal{D}_H$, from Theorem 3.11 and $\cup_{\sigma\in\Sigma}\hat{B}_\sigma(\cdot) \subset E(\cdot)$ it follows that the NRDS ϕ generated by equation (4.17) has a $\mathcal{D}_H$-random cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ with characterization (4.42). Characterization (4.43) follows from Corollary 4.3.8. Note that, as the $\hat{\mathcal{D}}_H$-random cocycle attractor $\hat{A}$ is smaller than the $\hat{\mathcal{D}}_H$-pullback absorbing set $\hat{B}$, $\hat{A}$ is such that $\cup_{\sigma\in\Sigma}\hat{A}_\sigma(\cdot) \subset \cup_{\sigma\in\Sigma}\hat{B}_\sigma(\cdot) \subset E(\cdot)$, and thereby the two attractors are identical by Proposition 4.2.1. □

Remark 4.5.10. The result $A = \hat{A}$ indicates that the tempered cocycle attractor for the Navier-Stokes equation is fully determined by pullback attracting autonomous tempered random sets.

Now we prove further properties for the $\mathcal{D}_H$-cocycle attractor A.

Observe that the tempered random set E defined by (4.41) is a uniformly $\mathcal{D}_H$-absorbing set, i.e., for any $\omega \in \Omega$ and $D \in \mathcal{D}_H$ there exists a positive $T = T(\omega, D)$ (here can be chosen as the one given in Lemma 4.5.4) such that

$$\bigcup_{\sigma\in\Sigma} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)) \subset E(\omega), \quad \forall t \geqslant T.$$

Note also that since $\sup_{\sigma\in\Sigma} R_3(\omega,\sigma)$ is bounded as so are R_1 and R_2, it is easy to see that there exists a bounded (autonomous) random set K in V containing $\cup_{\sigma\in\Sigma}\hat{K}_\sigma(\omega)$ (where $\hat{K}$ is given by (4.38)). Moreover, by Sobolev compact embedding, K is a compact random set in H (but not tempered). Hence, from Theorem 4.4.2 it follows the following result.

Theorem 4.5.11. *The $\mathcal{D}_H$-random cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ of the Navier-Stokes equation* (4.17) *given in Theorem 4.5.9 is such that*

(i) for each $\omega \in \Omega$, $\cup_{\sigma\in\Sigma}A_\sigma(\omega)$ is compact in H;

(ii) A is upper semi-continuous in $\sigma \in \Sigma$, i.e., for each $\omega \in \Omega$,

$$\mathrm{dist}(A_\sigma(\omega), A_{\sigma_0}(\omega)) \to 0, \quad \textit{whenever } \sigma \to \sigma_0 \textit{ in } \Sigma.$$

Up to now, we have shown what we can get by previous theoretical results. Nevertheless, we would like to write more properties for the cocycle attractor A of the 2D Navier-Stokes equation, making use of random uniform attractor theory established in the next chapter, Chapter 5.

Note that, though we have shown in Theorem 4.5.11 the compactness of each $\cup_{\sigma\in\Sigma}A_\sigma(\omega)$, we cannot say the mapping $\omega \mapsto \cup_{\sigma\in\Sigma}A_\sigma(\omega)$ is a compact random set in H as we have not seen the measurability of it yet. However, this measurability holds in this case. In fact, in view of Theorems 5.2.5 and 5.3.13 established latter, the mapping $\omega \mapsto \cup_{\sigma\in\Sigma}A_\sigma(\omega)$ is the $\mathcal{D}_H$-random uniform attractor (cf. Definition 5.1.4) of the the 2D Navier-Stokes equation. To sum up, we have

Theorem 4.5.12. *The cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ given in Theorem 4.5.9 for the NRDS ϕ generated by the stochastic Navier-Stokes equation* (4.17) *with translation bounded forcing has the following properties:*

(i) the mapping $\omega \mapsto \cup_{\sigma\in\Sigma}A_\sigma(\omega)$ is measurable, and is a compact random set in H;

(ii) the mapping $\omega \mapsto \cup_{\sigma\in\Sigma} A_\sigma(\omega)$ *is uniformly* $\mathcal{D}_H$*-pullback attracting, see Definition 5.1.3.*

Remark 4.5.13. Properties of $\mathcal{D}_H$-cocycle attractor A in Theorem 4.5.11 and Theorem 4.5.12 hold for $\hat{\mathcal{D}}_H$-cocycle attractor $\hat{A}$ given by (4.39) as well, since they are identical by Theorem 4.5.9.

Remark 4.5.14. By random uniform attractor theory we prove (see next chapter) the measurability and compactness of the union $\cup_{\sigma\in\Sigma} A_\sigma(\cdot)$ of all the sections of a cocycle attractor, which seems new in the literature. Actually, since Σ is Polish, these properties would be straightforward if we had the lower semi-continuity of $\sigma \mapsto A_\sigma(\omega)$. However, though in [36] an equi-attracting condition was shown equivalent to such lower semi-continuity, it seems still hard to verify in applications.

Part III

Random uniform attractors

Chapter 5

Uniform attractors for non-autonomous random dynamical systems

Uniform attractor is a useful tool to study non-autonomous dynamical systems, see for instance [26, 27, 67] and references therein. In this chapter, we establish a uniform attractor theory for NRDS. As have been introduced in Introduction, uniform attractor for NRDS defined later coincides with usual uniform attractors when the NRDS reduces to a nonrandom non-autonomous system, and has its own advantages over cocycle attractor in depicting the long time dynamics of the NRDS. For example, it is shown that, though uniform attractor is defined uniformly attracting in pullback sense, it is forward uniformly attracting in probability, while this forward attraction property does not hold for random cocycle attractors studied in Chapters 3 and 4. For more summary of properties of random uniform attractors see Introduction.

5.1 Uniform attractors and the uniform attraction

5.1.1 Preliminaries

In this chapter we shall work on an NRDS ϕ on separable Banach space $(X, \|\cdot\|_X)$, with base flows $\{\theta_t\}_{t\in\mathbb{R}}$ and $\{\vartheta_t\}_{t\in\mathbb{R}}$ acting on *compact* topological space Σ and probability space $(\Omega, \mathcal{F}, \mathbb{P})$, respectively. Note that the compactness of the symbol space Σ is crucial in this chapter, nevertheless, with efforts paid on the dynamics inside Σ it is also possible to develop analogous results for non-compact Σ. For the sake of simplicity, we restrict ourselves on compact Σ only.

We follow the notations in Chapter 4. Denote by $\mathcal{D}_X$ some inclusion- and neighborhood-closed universe of autonomous random sets in X. An example is the collection of all the bounded autonomous random sets in X.

In the following, "random set" means autonomous random set. We will frequently make use of the two continuities defined below.

Definition 5.1.1. An NRDS ϕ is said to be *continuous in symbols*, if for each $t \in \mathbb{R}^+$, $\omega \in \Omega$ and $x \in X$, the mapping $\sigma \mapsto \phi(t, \omega, \sigma, x)$ is (Σ, X)-continuous.

Definition 5.1.2. An NRDS ϕ is said to be *jointly continuous*, if for each $t \in \mathbb{R}^+$ and $\omega \in \Omega$, the mapping $(\sigma, x) \mapsto \phi(t, \omega, \sigma, x)$ is $(\Sigma \times X, X)$-continuous.

To study the uniform attractor for the NRDS ϕ and for simplicity, we often write

$$\phi(t, \omega, \Xi, B) := \bigcup_{\sigma \in \Xi} \bigcup_{x \in B} \phi(t, \omega, \sigma, x) \tag{5.1}$$

for each $t \in \mathbb{R}^+$, $\omega \in \Omega$, $\Xi \in 2^\Sigma \setminus \emptyset$ and $B \in 2^X \setminus \emptyset$. In fact, for the case $\Xi = \Sigma$, the mapping Φ given by

$$\Phi(t, \omega, x) := \phi(t, \omega, \Sigma, x), \quad \forall t \in \mathbb{R}^+, \omega \in \Omega, x \in X,$$

is a continuous multi-valued RDS, provided that ϕ is jointly continuous, see section 5.4.

5.1.2 Uniform attractors and the uniform attraction

In this section we shall give the definition and study the uniformly attracting property of random uniform attractors.

Definition 5.1.3. Given two random sets D and B, it is said that D *uniformly (pullback) attracts* B under the NRDS ϕ if

$$\lim_{t \to \infty} \operatorname{dist}(\phi(t, \vartheta_{-t}\omega, \Sigma, B(\vartheta_{-t}\omega)), D(\omega)) = 0, \quad \forall \omega \in \Omega. \tag{5.2}$$

If D uniformly attracts every element in $\mathcal{D}_X$, i.e., (5.2) holds for each $B \in \mathcal{D}_X$, then D is said to be uniformly $\mathcal{D}_X$-(pullback) attracting.

Definition 5.1.4. A random set $\mathscr{A}$ is called the $\mathcal{D}_X$*-(random) uniform attractor* of NRDS ϕ, if $\mathscr{A}$ belongs to $\mathcal{D}_X$ and is the minimal compact uniformly $\mathcal{D}_X$-(pullback) attracting set.

Notice that uniformly $\mathcal{D}_X$-pullback attracting/absorbing sets need not belong to $\mathcal{D}_X$.

Definition 5.1.5. A random set D in X is said to be uniformly $\mathcal{D}_X$-(pullback) absorbing if for each $\omega \in \Omega$ and $B \in \mathcal{D}_X$ there exists a time $T = (\omega, B) > 0$ such that

$$\phi(t, \vartheta_{-t}\omega, \Sigma, B(\vartheta_{-t}\omega)) \subset D(\omega), \quad \text{for each } t \geqslant T. \tag{5.3}$$

In the following, the attraction universe $\mathcal{D}_X$ is often omitted when no confusion occurs.
Note that by the invariance of Σ under θ, (5.2) is equivalent to each of the following

$$\sup_{\sigma \in \Sigma} \operatorname{dist}(\phi(t, \vartheta_{-t}\omega, \sigma, B(\vartheta_{-t}\omega)), D(\omega)) \xrightarrow{t \to \infty} 0; \tag{5.4}$$

$$\sup_{\sigma \in \Sigma} \operatorname{dist}\big(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, B(\vartheta_{-t}\omega)), D(\omega)\big) \xrightarrow{t \to \infty} 0. \tag{5.5}$$

By (5.5) we see that the uniform attraction is in the pullback sense, and, unlike nonrandom uniform attractors, this is not equivalent to the forward uniform attraction

$$\sup_{\sigma \in \Sigma} \operatorname{dist}(\phi(t, \omega, \sigma, B(\vartheta_{-t}\omega)), D(\vartheta_t\omega)) \xrightarrow{t \to \infty} 0.$$

This non-equivalence is the usual case for attractors of RDS, see [1, 34, 33, 42]. But we shall show in next subsection that this pullback uniform attraction implies a forward uniform attraction in probability.

Even though random uniform attractor is defined to uniformly attract random sets in $\mathcal{D}_X$, we shall prove that it is uniquely determined (with full probability) by attracting deterministic compact sets (of course the attraction universe $\mathcal{D}_X$ should include all the nonrandom compact sets when talking about this property), see Proposition 5.4.11.

In applications, if g is the non-autonomous symbol of the system, the symbol space Σ often serves as the closure $\mathcal{H}(g)$ of $\mathring{\mathcal{H}}(g)$, the translations of the symbol, just like the case in Section 5.5.2. Next proposition indicates that, when the NRDS is continuous in symbols, the uniform attractor is actually fully determined by $\mathring{\mathcal{H}}(g)$ without closure. Since the space $\mathring{\mathcal{H}}(g)$ has an equivalence relationship with $\mathbb{R}$ (see a discussion in [36]), where $\mathbb{R}$ servers as the space of initial time, this result shows that the uniform attractor gives and is determined by pullback attraction uniformly in initial time, only provided that ϕ is continuous in symbols.

Proposition 5.1.6. *Let ϕ be an NRDS which is continuous in symbols, and Ξ a dense subset of Σ. Then a random set is $\mathcal{D}_X$-pullback attracting uniformly in Σ if and only if it is $\mathcal{D}_X$-pullback attracting uniformly in Ξ.*

Proof. Given any a random set K, we need to prove that, for each random set $B \in \mathcal{D}_X$,

$$\lim_{t\to\infty} \operatorname{dist}(\phi(t, \vartheta_{-t}\omega, \Sigma, B(\vartheta_{-t}\omega)), K(\omega)) = 0 \tag{5.6}$$

holds if and only if

$$\lim_{t\to\infty} \operatorname{dist}(\phi(t, \vartheta_{-t}\omega, \Xi, B(\vartheta_{-t}\omega)), K(\omega)) = 0. \tag{5.7}$$

Only the sufficient condition needs a proof. Suppose that (5.6) does not hold under (5.7), then there exist $t_n \to \infty$, $\sigma_n \in \Sigma$ and $x_n \in B(\vartheta_{-t_n}\omega)$ such that

$$\operatorname{dist}(\phi(t_n, \vartheta_{-t_n}\omega, \sigma_n, x_n), K(\omega)) \geqslant \delta \tag{5.8}$$

for some $\delta > 0$. On the other hand, since Ξ is dense in Σ and ϕ is continuous on Σ, for each $n \in \mathbb{N}$ there is a $\sigma_n' \in \Xi$ such that

$$\operatorname{dist}(\phi(t_n, \vartheta_{-t_n}\omega, \sigma_n, x_n), \phi(t_n, \vartheta_{-t_n}\omega, \sigma_n', x_n)) < 1/n,$$

which, thanks to (5.7), implies that

$$\begin{aligned}&\operatorname{dist}(\phi(t_n, \vartheta_{-t_n}\omega, \sigma_n, x_n), K(\omega)) \\ &\quad \leqslant 1/n + \operatorname{dist}(\phi(t_n, \vartheta_{-t_n}\omega, \sigma_n', x_n), K(\omega)) \to 0.\end{aligned}$$

This contradicts (5.8). The proof is complete. □

a). Forward uniform attraction in probability

As have been previously stated, the uniform attracting property of random uniform attractors is only in a *pullback* sense, while that of deterministic uniform attractors is in both forward and pullback senses (indeed, forward and pullback uniform attractions are equivalent in deterministic cases). The next proposition indicates that the pullback uniform attraction implies a forward uniform attraction in probability.

Proposition 5.1.7. *Suppose that a random set $\mathscr{A}$ is uniformly $\mathcal{D}_X$-pullback attracting under an NRDS ϕ, then it is forward uniformly attracting in probability in the sense that*

$$\lim_{t\to\infty} \mathcal{P}\left\{\omega\in\Omega : \sup_{\sigma\in\Sigma} \operatorname{dist}(\phi(t,\omega,\sigma,B(\omega)),\mathscr{A}(\vartheta_t\omega)) > \varepsilon\right\} = 0, \quad \forall \varepsilon>0,\ B\in\mathcal{D}_X. \tag{5.9}$$

Proof. Given any $\varepsilon > 0$ and $B \in \mathcal{D}_X$, by the uniform $\mathcal{D}_X$-pullback attraction of $\mathscr{A}$ we have

$$\lim_{t\to\infty} \mathcal{P}\left\{\omega\in\Omega : \sup_{\sigma\in\Sigma} \operatorname{dist}(\phi(t,\vartheta_{-t}\omega,\sigma,B(\vartheta_{-t}\omega)),\mathscr{A}(\omega)) > \varepsilon\right\} = 0.$$

Since ϑ is $\mathcal{P}$-preserving and Ω is invariant under ϑ, for each $t > 0$ it holds

$$\begin{aligned}
&\mathcal{P}\left\{\omega\in\Omega : \sup_{\sigma\in\Sigma} \operatorname{dist}(\phi(t,\omega,\sigma,B(\omega)),\mathscr{A}(\vartheta_t\omega)) > \varepsilon\right\} \\
&\quad = \mathcal{P}\left\{\vartheta_{-t}\omega\in\Omega : \sup_{\sigma\in\Sigma} \operatorname{dist}(\phi(t,\omega,\sigma,B(\omega)),\mathscr{A}(\vartheta_t\omega)) > \varepsilon\right\} \\
&\quad = \mathcal{P}\left\{\omega\in\Omega : \sup_{\sigma\in\Sigma} \operatorname{dist}(\phi(t,\vartheta_{-t}\omega,\sigma,B(\vartheta_{-t}\omega)),\mathscr{A}(\omega)) > \varepsilon\right\}.
\end{aligned}$$

Hence, we have the result. □

b). Almost uniform attraction for discrete time

Notice that the attracting property of a random uniform attractor is expected for each (almost every) fixed sample ω. This is the usual case in the study of random cocycle attractors [1]. Now we show that the usual pullback attraction implies an almost uniform (w.r.t. $\omega\in\Omega$) pullback attracting property.

The following lemma is a generalization of Egoroff's theorem (see [46, p88]).

Lemma 5.1.8. *Suppose $\{D_n\}_{n=0}^\infty$ is a sequence of bounded random sets such that*

$$\lim_{n\to\infty} \operatorname{dist}(D_n, D_0) = 0, \quad \mathcal{P}\text{-a.s.} \tag{5.10}$$

Then for each $\varepsilon > 0$, there exists an $F\in\mathcal{F}$ with $\mathcal{P}(F) < \varepsilon$ such that

$$\lim_{n\to\infty} \operatorname{dist}(D_n(\omega), D_0(\omega)) = 0 \quad \textit{holds uniformly for all } \omega\in\Omega\setminus F.$$

Proof. Up to a full measure subset, we let (5.10) holds for all $\omega \in \Omega$. Define

$$\Omega_n^m = \bigcap_{i=n}^{\infty} \{\omega \in \Omega : \mathrm{dist}(D_i(\omega), D_0(\omega)) < 1/m\}.$$

Then it is clear that Ω_n^m is non-decreasing in n and, by (5.10),

$$\Omega \subset \lim_{n\to\infty} \Omega_n^m, \quad \forall m \in \mathbb{N}.$$

Hence, $\lim_{n\to\infty} \mathcal{P}(\Omega \setminus \Omega_n^m) = 0$, and thereby there exists a positive $n_0(m) \in \mathbb{N}$ such that

$$\mathcal{P}(\Omega \setminus \Omega_{n_0(m)}^m) < \frac{\varepsilon}{2^m}.$$

Set

$$F = \bigcup_{m=1}^{\infty} \Big(\Omega \setminus \Omega_{n_0(m)}^m\Big).$$

Then F is measurable and

$$\mathcal{P}(F) \leqslant \sum_{m=1}^{\infty} \mathcal{P}(\Omega \setminus \Omega_{n_0(m)}^m) < \varepsilon.$$

Since $\Omega \setminus F = \Omega \cap (\cap_{m=1}^{\infty} \Omega_{n_0(m)}^m)$, for each $m \in \mathbb{N}$ and all $\omega \in \Omega \setminus F$ (then $\omega \in \Omega_{n_0(m)}^m$) it holds

$$\mathrm{dist}(D_n(\omega), D_0(\omega)) < 1/m, \quad \forall n \geqslant n_0(m),$$

i.e., the limit holds uniformly in $\Omega \setminus F$. □

Proposition 5.1.9. *Suppose that ϕ is an NRDS and $\mathscr{A}$ is the $\mathcal{D}_X$-uniform attractor. Then for each $t_n \to \infty$ and any $\varepsilon > 0$ there exists an $F \in \mathcal{F}$ (depending on $\{t_n\}_{n\in\mathbb{N}}$ and ε) with $\mathcal{P}(F) < \varepsilon$ such that, for any $D \in \mathcal{D}_X$,*

$$\sup_{\sigma\in\Sigma} \mathrm{dist}(\phi(t_n, \vartheta_{-t_n}\omega, \sigma, D(\vartheta_{-t_n}\omega)), \mathscr{A}(\omega)) \xrightarrow{n\to\infty} 0, \quad \textit{uniformly for all } \omega \in \Omega \setminus F.$$

Proof. Since $\mathcal{D}_X$ is neighborhood-closed, there exists a $\delta > 0$ such that $\mathcal{N}_\delta(\mathscr{A}) \in \mathcal{D}_X$. Hence, since $\mathscr{A}$ uniformly attracts $\mathcal{N}_\delta(\mathscr{A})$ for each $\omega \in \Omega$, by Lemma 5.1.8 we know that for each $t_n \to \infty$ and any $\varepsilon > 0$ there exists an $F \in \mathcal{F}$ with $\mathcal{P}(F) < \varepsilon$ such that

$$\sup_{\sigma\in\Sigma} \mathrm{dist}(\phi(t_n, \vartheta_{-t_n}\omega, \sigma, \mathcal{N}_\delta(\mathscr{A}(\vartheta_{-t_n}\omega))), \mathscr{A}(\omega)) \xrightarrow{n\to\infty} 0, \quad \text{uniformly for all } \omega \in \Omega \setminus F.$$

Noticing that $\mathcal{N}_\delta(\mathscr{A})$ is in fact a uniformly $\mathcal{D}_X$-absorbing set, we have completed the proof. □

5.2 Existence of uniform attractors

This section is aimed to establish some criteria for the existence of uniform attractors for NRDS. To ensure the measurability of the random uniform attractor, which is considerably the most important feature of random attractors compared with deterministic ones, the symbol space Σ is required to be Polish. Note that such a Polish condition is satisfied by the hull of translation-bounded functions, see Section 5.5.2.

Omega-limit sets are always important in the study of attractors. For any $\Xi \subset \Sigma$ and $\omega \in \Omega$, the omega limit set of each $B \in \mathcal{D}_X$ under NRDS ϕ is defined by

$$\mathcal{W}(\omega, \Xi, B) = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t, \vartheta_{-t}\omega, \theta_{-t}\Xi, B(\vartheta_{-t}\omega))}. \tag{5.11}$$

It is important to note that $\mathcal{W}(\omega, \Xi, B) \neq \cup_{\sigma\in\Xi}\mathcal{W}(\omega, \sigma, B)$ generally. Nevertheless, we have

$$\begin{aligned}\overline{\bigcup_{\sigma\in\Xi}\mathcal{W}(\omega, \sigma, B)} &= \overline{\bigcup_{\sigma\in\Xi}\bigcap_{s\in\mathbb{N}}\overline{\bigcup_{t\geqslant s}\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, B)}} \subseteq \bigcap_{s\in\mathbb{N}}\overline{\bigcup_{\sigma\in\Xi}\overline{\bigcup_{t\geqslant s}\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, B)}} \\ &\subseteq \bigcap_{s\in\mathbb{N}}\overline{\overline{\bigcup_{\sigma\in\Xi}\bigcup_{t\geqslant s}\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, B)}} = \bigcap_{s\in\mathbb{N}}\overline{\overline{\bigcup_{t\geqslant s}\phi(t, \vartheta_{-t}\omega, \theta_{-t}\Xi, B)}} \\ &= \bigcap_{s\in\mathbb{N}}\overline{\bigcup_{t\geqslant s}\phi(t, \vartheta_{-t}\omega, \theta_{-t}\Xi, B)} = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi(t, \vartheta_{-t}\omega, \theta_{-t}\Xi, B)} \\ &= \mathcal{W}(\omega, \Xi, B).\end{aligned} \tag{5.12}$$

It is standard to have the following characterization of omega-limit sets.

Proposition 5.2.1. *For any $\Xi \subset \Sigma$ and $\omega \in \Omega$, $y \in \mathcal{W}(\omega, \Xi, B)$ if and only if there exist sequences $t_n \to \infty$ and $\sigma_n \in \Xi$ such that for some sequence*

$$x_n \in \phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma_n, B(\vartheta_{-t_n}\omega))$$

it holds $x_n \to y$.

The following lemma is crucial to prove the measurability of a uniform attractor.

Lemma 5.2.2. *Suppose that ϕ is an NRDS continuous in symbols. If $\Xi \subset \Sigma$ densely, then for each $\omega \in \Omega$,*

$$\mathcal{W}(\omega, \Xi, B) = \mathcal{W}(\omega, \Sigma, B), \quad \forall B \in \mathcal{D}_X.$$

Proof. For fixed ω and $B \in \mathcal{D}_X$, note that

$$\mathcal{W}(\omega, \beth, B) = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\overline{\phi(t, \vartheta_{-t}\omega, \theta_{-t}\beth, B(\vartheta_{-t}\omega))}}, \quad \forall \beth \subset \Sigma.$$

It suffices to prove

$$\overline{\phi(t, \vartheta_{-t}\omega, \theta_{-t}\Sigma, B(\vartheta_{-t}\omega))} = \overline{\phi(t, \vartheta_{-t}\omega, \theta_{-t}\Xi, B(\vartheta_{-t}\omega))}, \quad \forall t \in \mathbb{R}^+. \tag{5.13}$$

Only the $\subset$ inclusion needs a proof. Let $y \in \overline{\phi(t, \vartheta_{-t}\omega, \theta_{-t}\Sigma, B(\vartheta_{-t}\omega))}$. Then there are sequences $\sigma_n \in \Sigma$ and $x_n \in B(\vartheta_{-t}\omega)$ such that

$$\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma_n, x_n) \to y, \quad \text{as } n \to \infty.$$

As $\sigma \mapsto \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, x)$ is continuous, by the density of Ξ in Σ there is a sequence $\sigma'_n \in \Xi$ such that

$$\mathrm{dist}\Big(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma'_n, x_n), \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma_n, x_n)\Big) \leqslant 1/n, \quad \forall n \in \mathbb{N},$$

which implies $\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma'_n, x_n) \xrightarrow{n\to\infty} y$. Hence, $y \in \overline{\phi(t, \vartheta_{-t}\omega, \theta_{-t}\Xi, B(\vartheta_{-t}\omega))}$ as desired. □

5.2.1 First results based on compact uniformly attracting sets

Lemma 5.2.3. *Suppose that ϕ is a jointly continuous NRDS with a compact uniformly $\mathcal{D}_X$-attracting random set K. Then, for any closed random set $B \in \mathcal{D}_X$, $\mathcal{W}(\cdot, \Sigma, B)$ is non-empty, compact and negatively semi-invariant in the sense that*

$$\mathcal{W}(\vartheta_t\omega, \Sigma, B) \subset \phi(t, \omega, \Sigma, \mathcal{W}(\omega, \Sigma, B)), \quad \forall t \geqslant 0,\ \omega \in \Omega. \tag{5.14}$$

Moreover, for any closed random set D which uniformly attracts B it holds

$$\mathcal{W}(\omega, \Sigma, B) \subset D(\omega), \quad \forall \omega \in \Omega. \tag{5.15}$$

Proof. Non-empty. Take a sequence $x_n \in \phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma_n, B(\vartheta_{-t_n}\omega))$ with $t_n \to \infty$ and $\sigma_n \in \Sigma$. Then, since K uniformly attracts B,

$$\mathrm{dist}(x_n, K(\omega)) \leqslant \mathrm{dist}(\phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma_n, B(\vartheta_{-t_n}\omega)), K(\omega)) \to 0, \quad \text{as } n \to \infty.$$

Since $K(\omega)$ is compact, there exists a $y \in K(\omega)$ such that $x_n \to y$ in a subsequence sense. Therefore, by Proposition 5.2.1 we have $y \in \mathcal{W}(\omega, \Sigma, B)$ and thereby the nonempty is clear.

Compactness. For any $y \in \mathcal{W}(\omega, \Sigma, B)$, Proposition 5.2.1 indicates that there exist sequences $t_n \to \infty$, $\sigma_n \in \Sigma$ and $z_n \in B(\vartheta_{-t_n}\omega)$ such that

$$\phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma_n, z_n) \to y. \tag{5.16}$$

On the other hand, since B is uniformly attracted by K and K is compact, $y \in K(\omega)$. Hence, $\mathcal{W}(\omega, \Sigma, B) \subset K(\omega)$ is compact, as a closed subset of a compact set is compact.

Negative semi-invariance. Let $t \in \mathbb{R}^+$ be fixed, and take $y \in \mathcal{W}(\vartheta_t\omega, \Sigma, B)$. Then by Proposition 5.2.1 there exist sequences $t < t_n \to \infty$, $\sigma_n \in \Sigma$ and $z_n \in B(\vartheta_{-t_n}\vartheta_t\omega)$ such that

$$\phi(t_n, \vartheta_{-t_n}\vartheta_t\omega, \theta_{-t_n}\theta_t\sigma_n, z_n) \to y. \tag{5.17}$$

Hence, by the invariance of ϕ we have

$$\phi(t_n, \vartheta_{-t_n}\vartheta_t\omega, \theta_{-t_n}\theta_t\sigma_n, z_n) = \phi(t, \omega, \sigma_n) \circ \phi(t_n - t, \vartheta_{t-t_n}\omega, \theta_{t-t_n}\sigma_n, z_n) \to y. \tag{5.18}$$

On the other hand, since K is compact and uniformly pullback attracts B, there exists a $k \in K(\omega)$ such that

$$\phi(t_n - t, \vartheta_{t-t_n}\omega, \theta_{t-t_n}\sigma_n, z_n) \to k \tag{5.19}$$

in a subsequence sense. This implies $k \in \mathcal{W}(\omega, \Sigma, B)$ by Proposition 5.2.1. Moreover, by the compactness of Σ, σ_n converges to some σ up to a subsequence. Hence, by the joint continuity of ϕ we have

$$\phi(t, \omega, \sigma_n) \circ \phi(t_n - t, \vartheta_{t-t_n}\omega, \theta_{t-t_n}\sigma_n, z_n) \to \phi(t, \omega, \sigma, k)$$

which along with (5.18) implies $y = \phi(t, \omega, \sigma, k) \in \phi(t, \omega, \sigma, \mathcal{W}(\omega, \Sigma, B))$; negative semi-invariance is clear.

To prove (5.15), suppose we are given another closed random set D uniformly attracting B. Then for any $y \in \mathcal{W}(\omega, \Sigma, B)$ we have a sequence $x_n \in \phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\Sigma, B(\vartheta_{-t_n}\omega))$ with $t_n \to \infty$ such that $x_n \to y$. By the uniformly attraction and the closedness of D we know $y \in D(\omega)$. Therefore, $\mathcal{W}(\omega, \Sigma, B) \subset D(\omega)$ and (5.15) follows. □

Lemma 5.2.4. *Suppose that ϕ is a jointly continuous NRDS with a compact uniformly $\mathcal{D}_X$-attracting random set K. If a closed random set $B \in \mathcal{D}_X$ uniformly attracts itself, then $\mathcal{W}(\cdot, \Sigma, B)$ is a compact and negatively semi-invariant random set, and is the minimal closed random set uniformly attracting B.*

Proof. By Lemma 5.2.3 we know that for each $\omega \in \Omega$, $\mathcal{W}(\omega, \Sigma, B)$ is non-empty and compact. Now we prove the measurability. First, let us show that

$$\mathcal{W}(\omega, \Sigma, B) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m=n}^{\infty} \phi(m, \vartheta_{-m}\omega, \Sigma, B(\vartheta_{-m}\omega))}, \quad \forall \omega \in \Omega. \tag{5.20}$$

Since B uniformly attracts itself, by (5.15) we have $\mathcal{W}(\cdot, \Sigma, B) \subset B(\cdot)$. Hence, by the negative invariance of $\mathcal{W}(\omega, \Sigma, B)$, we have

$$\begin{aligned}\mathcal{W}(\omega, \Sigma, B) &\subset \phi(m, \vartheta_{-m}\omega, \Sigma, \mathcal{W}(\vartheta_{-m}\omega, \Sigma, B)) \\ &\subset \phi(m, \vartheta_{-m}\omega, \Sigma, B(\vartheta_{-m}\omega)), \quad \forall m \in \mathbb{N}.\end{aligned}$$

Therefore,

$$\mathcal{W}(\omega, \Sigma, B) \subset \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m=n}^{\infty} \phi(m, \vartheta_{-m}\omega, \Sigma, B(\vartheta_{-m}\omega))}, \quad \forall \omega \in \Omega,$$

and thereby (5.20) holds, as the inverse inclusion is straightforward.

Since Σ is Polish, suppose $\Xi = \{\sigma_i\}_{i\in\mathbb{N}}$ is a dense subset of Σ. Denote by

$$D_n(\omega) = \overline{\bigcup_{m=n}^{\infty} \bigcup_{i\in\mathbb{N}} \phi(m, \vartheta_{-m}\omega, \theta_{-m}\sigma_i, B(\vartheta_{-m}\omega))}, \quad \forall n \in \mathbb{N},\ \omega \in \Omega.$$

Then $\mathcal{W}(\omega, \Sigma, B) = \cap_{n\in\mathbb{N}} \overline{D_n(\omega)}$ in view of (5.20) and (5.13). Now we first prove that each D_n is measurable.

Since B is a non-empty closed random set, by Lemma 2.2.4 (II) there exists a sequence $\{f_j\}_{j\in\mathbb{N}}$ of measurable functions such that $B(\vartheta_{-m}\omega) = \overline{\cup_{j\in\mathbb{N}} f_j(\vartheta_{-m}\omega)}$, which makes

$$\overline{\phi(m,\vartheta_{-m}\omega,\theta_{-m}\sigma_i,B(\vartheta_{-m}\omega))} = \overline{\bigcup_{j\in\mathbb{N}} \phi(m,\vartheta_{-m}\omega,\theta_{-m}\sigma_i,f_j(\omega))}$$

as $x \to \phi(m,\vartheta_{-m}\omega,\theta_{-m}\sigma_i,x)$ is continuous. Since $\phi(m,\vartheta_{-m}\omega,\theta_{-m}\sigma_i,x)$ is $(\mathcal{F},\mathcal{B}(X))$-measurable in ω, it is measurable in the sense of Definition 2.2.2 as well since it is single-valued. Hence the right-hand side term of the above identity is measurable and then so is the left-hand side term. Therefore, by Lemma 2.2.4 (I) we know D_n is measurable.

On the other hand, clearly, D_n is decreasing and every sequence $\{x_n\}$ inside $\mathcal{W}(\omega,\Sigma,B)$ is pre-compact since $\mathcal{W}(\omega,\Xi,B)$ is compact itself, by Lemma 2.2.4 (I) we know $\mathcal{W}(\omega,\Sigma,B) = \cap_{n\in\mathbb{N}}\overline{D_n(\omega)}$ is measurable.

Now we prove by contradiction that $\mathcal{W}(\cdot,\Sigma,B)$ uniformly attracts B. Suppose it is not true, then there exist a $\delta > 0$ and a sequence $x_n \in \phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma_n,B(\vartheta_{-t_n}\omega))$ with $t_n \to \infty$ and $\sigma_n \in \Sigma$ such that

$$\mathrm{dist}(x_n,\mathcal{W}(\omega,\Sigma,B)) > \delta, \quad \forall n \in \mathbb{N}. \tag{5.21}$$

However, by the uniformly attracting property and the compactness of K again, there is a $y \in \mathcal{W}(\omega,\Sigma,B)$ such that $x_n \to y$, which contradicts (5.21).

The minimal property follows from Lemma 5.2.3. The proof is complete. □

Now we give a sufficient condition for the existence of random uniform attractors. Note that, since the attraction universe is inclusion-closed, the necessary statement of the following result holds true as well.

Theorem 5.2.5. *Suppose that ϕ is an NRDS continuous in both Σ and X, and Ξ is any a dense subset of Σ. If ϕ has a compact uniformly $\mathcal{D}_X$-attracting set K and a closed uniformly $\mathcal{D}_X$-absorbing set $B \in \mathcal{D}_X$, then it has a unique $\mathcal{D}_X$-uniform attractor $\mathscr{A} \in \mathcal{D}_X$ given by*

$$\mathscr{A}(\omega) = \mathcal{W}(\omega,\Sigma,B) = \mathcal{W}(\omega,\Xi,B), \quad \forall \omega \in \Omega.$$

Moreover, the uniform attractor $\mathscr{A}$ is negatively semi-invariant

$$\mathscr{A}(\vartheta_t\omega) \subseteq \phi(t,\omega,\Sigma,\mathscr{A}(\omega)), \quad \forall t \geqslant 0,\ \omega \in \Omega.$$

Proof. The non-empty, compactness and measurability properties, along with minimal and negative semi-invariant properties, are proved by Lemma 5.2.4. We now prove the uniformly $\mathcal{D}_X$-attracting property. Since $\mathscr{A}$ uniformly attracts B by Lemma 5.2.4, for each $\varepsilon > 0$ and $\sigma \in \Sigma$, $\omega \in \Omega$ fixed, there is a time $T > 0$ such that

$$\mathrm{dist}\Big(\phi(t,\vartheta_{-t}\omega,\theta_{-t}\Sigma,B(\vartheta_{-t}\omega)),\mathscr{A}(\omega)\Big) < \varepsilon, \quad \forall t \geqslant T.$$

On the other hand, for each $D \in \mathcal{D}_X$ and $\omega \in \Omega$, there is a time $T_D(\omega) > 0$ such that

$$\bigcup_{t\geqslant T_D(\omega)} \phi(t,\vartheta_{-t}\omega,\theta_{-t}\Sigma,D(\vartheta_{-t}\omega)) \subset B(\omega)$$

as B is a uniformly $\mathcal{D}_X$-absorbing set. Hence,

$$\begin{aligned}
&\operatorname{dist}\Big(\phi(t+T,\vartheta_{-t-T}\omega,\theta_{-t-T}\Sigma,D(\vartheta_{-t-T}\omega)),\mathscr{A}(\omega)\Big)\\
&\quad=\operatorname{dist}\Big(\phi(T,\vartheta_{-T}\omega,\theta_{-T}\Sigma,\phi(t,\vartheta_{-t}\vartheta_{-T}\omega,\theta_{-t}\theta_{-T}\Sigma,D(\vartheta_{-t}\vartheta_{-T}\omega)),\mathscr{A}(\omega)\Big)\\
&\quad\leqslant\operatorname{dist}\Big(\phi(T,\vartheta_{-T}\omega,\theta_{-T}\Sigma,B(\vartheta_{-T}\omega)),\mathscr{A}(\omega)\Big)<\varepsilon,\quad \forall t\geqslant T_D(\vartheta_{-T}\omega),
\end{aligned}$$

which indicates that $\mathscr{A}$ uniformly pullback attracts D. Since $B\in\mathcal{D}_X$, $\mathscr{A}$ belongs to $\mathcal{D}_X$ as well since $\mathscr{A}\subset B$ and $\mathcal{D}_X$ is inclusion-closed. The proof is complete. □

Remark 5.2.6. Note that, in Theorem 5.2.5, K unnecessarily belongs to $\mathcal{D}_X$. In Theorem 5.3.14 we will prove that a compact uniformly $\mathcal{D}_X$-attracting set $K\in\mathcal{D}_X$ alone is a sufficient (and also necessary) condition for the existence of the uniform attractor.

Remark 5.2.7. In applications, the symbol space Σ is often defined as the hull of the non-autonomous forcing. Theorem 5.2.5 (and also Theorem 5.3.14) indicate that, under the required conditions, the uniform attractor is actually determined by the hull without the closure. See also the discussion before Proposition 5.1.6.

5.2.2 Alternative dynamical compactnesses

Theorem 5.2.5 implies a direct relationship between uniform attracting sets and uniform attractors. However, the existence of a compact uniformly attracting set K is often nontrivial to establish. Therefore, several dynamical compactnesses have been introduced in attractor theory, such as asymptotic compactness, pullback omega-limit compactness, flattening and squeezing properties [68, 53, 38, 30]. Now we introduce analogous concepts in the context of uniform attractors, and show that these dynamical compactness of an NRDS will ensure the omega-limit set of a uniformly $\mathcal{D}_X$-absorbing set to be a compact uniformly $\mathcal{D}_X$-attracting set. For analogous discussions for cocycle attractors see Section 3.2.2.

Definition 5.2.8. An NRDS ϕ on a Banach space $(X,\|\cdot\|)$ is called uniformly $\mathcal{D}_X$-(pullback) flattening if for each $D\in\mathcal{D}_X$, $\varepsilon>0$, $\omega\in\Omega$ there exist a $T_0=T_0(D,\varepsilon,\omega)>0$ and a finite-dimensional subspace X_ε of X such that

(i) $\cup_{t\geqslant T_0}P_\varepsilon\phi(t,\vartheta_{-t}\omega,\Sigma,D(\vartheta_{-t}\omega))$ is bounded, and

(ii) $\|(I-P_\varepsilon)\cup_{t\geqslant T_0}\phi(t,\vartheta_{-t}\omega,\Sigma,D(\vartheta_{-t}\omega))\|<\varepsilon$,

where $P_\varepsilon: X\mapsto X_\varepsilon$ is a bounded projection.

Definition 5.2.9. An NRDS ϕ on a Banach space $(X,\|\cdot\|)$ is called uniformly $\mathcal{D}_X$-(pullback) omega-limit compact if for each $D\in\mathcal{D}_X$, $\varepsilon>0$, $\omega\in\Omega$ there exists a $T_1=T_1(D,\varepsilon,\omega)>0$ such that

$$\kappa\left(\bigcup_{t\geqslant T_1}\phi(t,\vartheta_{-t}\omega,\Sigma,D(\vartheta_{-t}\omega))\right)<\varepsilon,$$

where κ denotes the Kuratowski measure [68] of noncompactness of sets defined as

$$\kappa(B) = \inf \big\{ \delta : B \text{ has a finite cover by balls of } X \text{ of diameter less than } \delta \big\}, \quad \forall B \subset X.$$

Definition 5.2.10. An NRDS ϕ on a Banach space $(X, \|\cdot\|)$ is called uniformly $\mathcal{D}_X$-(pullback) asymptotically compact if for each $D \in \mathcal{D}_X$, $\omega \in \Omega$ and any sequences $0 < t_n \to \infty$ and $x_k \in D(\vartheta_{-t_n}\omega)$ the set $\{\phi(t_n, \vartheta_{-t_n}\omega, \Sigma, x_k)\}$ is precompact in X.

Theorem 5.2.11. *Suppose that X is a uniformly convex Banach space (particularly, a Hilbert space). The following dynamical compactness properties of an NRDS ϕ on X are equivalent:*

(i) uniformly $\mathcal{D}_X$-(pullback) flattening;

(ii) uniformly $\mathcal{D}_X$-(pullback) omega-limit compactness;

(iii) uniformly $\mathcal{D}_X$-(pullback) asymptotically compactness,

where the uniformly convex property of X is only for the relation (iii)⇒(i). Moreover, each of these dynamical compactnesses implies that the omega-limit set $\mathcal{W}(\cdot, \Sigma, B)$ of a uniformly $\mathcal{D}_X$-absorbing set $B \in \mathcal{D}_X$ is a compact $\mathcal{D}_X$-uniformly attracting random set.

Proof. Similar to, e.g., [53, Theorems 4.5 & 4.6], or [30, Section 2]. □

The above theorem implies that these dynamical compactnesses could replace the requirement of a compact uniformly $\mathcal{D}_X$-attracting set in Theorem 5.2.5, since they are stronger under suitable conditions.

5.3 Relationship between different random attractors

In this section we study the relationship between uniform and cocycle attractors for NRDS. To this end, we first introduce a dynamical system named skew-product cocycle generated by an NRDS and define a proper cocycle attractor with proper attraction universe for the skew-product cocycle as a bridge.

5.3.1 Proper cocycle attractor for skew-product cocycle on extended phase space

Denote by $(\mathbb{X}, d_{\mathbb{X}})$ the extended phase space $\Sigma \times X$, that is, $\chi \in \mathbb{X}$ if and only if it has the form $\chi = \{\sigma\} \times \{x\}$ for some $\sigma \in \Sigma$ and $x \in X$, endowed with the skew-product metric given by

$$d_{\mathbb{X}}(\chi_1, \chi_2) = \rho(\sigma_1, \sigma_2) + d_X(x_1, x_2), \quad \forall \chi_j = \{\sigma_j\} \times \{x_j\} \in \mathbb{X}.$$

Clearly, any set $\mathbb{B}$ in $\mathbb{X}$ has the form $\mathbb{B} = \cup_{\sigma \in \Sigma} \{\sigma\} \times B(\sigma)$, where each $B(\sigma)$ is a (possibly empty) subset of X called the σ-section of $\mathbb{B}$. Denote by P_σ the mapping from each $\mathbb{B} \subset \mathbb{X}$ to its σ-section, i.e., $P_\sigma \mathbb{B} := B(\sigma)$, and let

$$P_X \mathbb{B} = \bigcup_{\sigma \in \Sigma} P_\sigma \mathbb{B} = \Big\{ x \in X : \text{there is some } \sigma \in \Sigma \text{ such that } \{\sigma\} \times \{x\} \in \mathbb{B} \Big\}.$$

Then P_X is the projection from $\mathbb{X}$ to X. Denote by P_Σ the projection from $\mathbb{X}$ to Σ.

Given an NRDS ϕ, define a mapping π: $\mathbb{R}^+ \times \Omega \times \mathbb{X} \to \mathbb{X}$ by

$$\pi\big(t, \omega, \{\sigma\} \times \{x\}\big) = \{\theta_t\sigma\} \times \{\phi(t, \omega, \sigma, x)\}. \tag{5.22}$$

Then the mapping π is a (random) cocycle, namely, satisfying

- π is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(\mathbb{X}), \mathcal{B}(\mathbb{X}))$-measurable;
- $\pi(0, \omega, \chi) = \chi, \quad \forall \omega \in \Omega, \chi \in \mathbb{X}$;
- the cocycle property $\pi(t+s, \omega, \chi) = \pi(t, \vartheta_s\omega, \pi(s, \omega, \chi))$, $\forall t, s \in \mathbb{R}^+, \omega \in \Omega, \chi \in \mathbb{X}$.

The cocycle π is called the skew-product cocycle (with base flow $\{\theta_t\}_{t\in\mathbb{R}}$) generated by ϕ. Note that π is continuous, namely, the mapping $\chi \mapsto \pi(\cdot, \cdot, \chi)$ is continuous in $\mathbb{X}$, if and only if ϕ is jointly continuous. Very often, we write $\pi(t, \omega, \chi)$ as $\pi(t, \omega)\chi$ for convenience.

a). Proper random sets and proper cocycle attractor

For the cocycle π, a particular (autonomous) RDS, it is sensible to study its long-time behavior in terms of cocycle attractors. In order to serve uniform attractors, we define proper random sets in $\mathbb{X}$ and then define a proper cocycle attractor for π.

Definition 5.3.1. A set-valued mapping $\mathbb{B}(\cdot) : \Omega \mapsto 2^{\mathbb{X}} \setminus \emptyset$ is called a random set in $\mathbb{X}$ if for each $\chi \in \mathbb{X}$ the mapping $\omega \mapsto \mathrm{dist}_{\mathbb{X}}(\chi, \mathbb{B}(\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}^+))$-measurable. If, moreover, $\mathbb{B}$ satisfies that

$$P_\sigma(\mathbb{B}(\omega)) \neq \emptyset \quad \text{for each } \sigma \in \Sigma,\ \omega \in \Omega, \tag{5.23}$$

and that

$$P_X(\mathbb{B}) \in \mathcal{D}_X, \tag{5.24}$$

then it is said to be *proper*.

Note that condition (5.23) is equivalent to that

$$P_\Sigma(B(\omega)) \equiv \Sigma \quad \text{for all } \omega \in \Omega, \tag{5.25}$$

which implies that stochastic perturbation happens only to the X-component. Let

$$\mathcal{D}_{\mathbb{X}} = \Big\{ B : B \text{ is a proper random set in } \mathbb{X} \Big\}.$$

Example elements of $\mathcal{D}_{\mathbb{X}}$ are random sets in the form $\Sigma \times D = \{\Sigma \times D(\omega)\}_{\omega\in\Omega}$ with $D \in \mathcal{D}_X$.

Now we define the proper cocycle attractor, which pullback attracts proper random sets in $\mathbb{X}$.

Definition 5.3.2. A random set $\mathbb{A} \in \mathcal{D}_{\mathbb{X}}$ is called a $\mathcal{D}_{\mathbb{X}}$-cocycle attractor of the skew product cocycle π if

(1) $\mathbb{A}$ is compact;

(2) $\mathbb{A}$ is $\mathcal{D}_{\mathbb{X}}$-pullback attracting, that is,

$$\lim_{t\to\infty} \mathrm{dist}_{\mathbb{X}}(\pi(t,\vartheta_{-t}\omega,\mathbb{D}(\vartheta_{-t}\omega)),\mathbb{A}(\omega)) = 0, \quad \forall \omega\in\Omega,\ \mathbb{D}\in\mathcal{D}_{\mathbb{X}};$$

(3) $\mathbb{A}$ is invariant under π, that is,

$$\pi(t,\omega,\mathbb{A}(\omega)) = \mathbb{A}(\vartheta_t\omega), \quad \forall t\in\mathbb{R}^+,\ \omega\in\Omega.$$

Note that, since $\mathbb{A}\in\mathcal{D}_{\mathbb{X}}$, $\mathbb{A}$ attracts itself by definition. Moreover, by the invariance property, it is the minimal compact random set in $\mathbb{X}$ satisfying (2).

Cocycle attractors for (autonomous) RDS have been relatively much studied during recent years, see for instance [34, 42, 4]. But due to the special setting of the attraction universe $\mathcal{D}_{\mathbb{X}}$, no results in the literature could be directly applied in our case, since the collection $\mathcal{D}_{\mathbb{X}}$ is even not inclusion-closed due to the requirement (5.23). Hence, in the following we must be careful when proving the existence of a $\mathcal{D}_{\mathbb{X}}$-cocycle attractor.

b). Relationship between uniform attractor for ϕ and proper cocycle attractor for π

Now we are interested in the relationship between the uniform attractor $\mathscr{A}$ of an NRDS ϕ and the $\mathcal{D}_{\mathbb{X}}$-cocycle attractor $\mathbb{A}$ of the corresponding skew-product cocycle π. This helps to understand the random uniform attractor as implied by the following results.

Proposition 5.3.3. *If the NRDS ϕ is jointly continuous and has a $\mathcal{D}$-uniform attractor $\mathscr{A}$, then the (continuous) skew product cocycle π generated by ϕ has a $\mathcal{D}_{\mathbb{X}}$-cocycle attractor $\mathbb{A}$, defined by*

$$\mathbb{A}(\omega) = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\pi(t,\vartheta_{-t}\omega,\mathbb{K}(\vartheta_{-t}\omega))}, \quad \forall\omega\in\Omega,$$

with $\mathbb{K}=\Sigma\times\mathcal{N}_\varepsilon(\mathscr{A})$.

Proof. We first claim that the extended set $\Sigma\times\mathscr{A}$ is a compact $\mathcal{D}_{\mathbb{X}}$-pullback attracting set of π belonging to $\mathcal{D}_{\mathbb{X}}$. Indeed, it is clearly a random set in $\mathbb{X}$ and $\Sigma\times\mathscr{A}\in\mathcal{D}_{\mathbb{X}}$ as $P_\sigma(\Sigma\times\mathscr{A})\equiv\mathscr{A}\in\mathcal{D}_X$ for each $\sigma\in\Sigma$; the compactness follows from that of $\mathscr{A}$ directly; for any $\mathbb{D}\in\mathcal{D}_{\mathbb{X}}$, by the uniform attraction of $\mathscr{A}$ we have

$$\begin{aligned}
&\mathrm{dist}_{\mathbb{X}}\Big(\pi(t,\vartheta_{-t}\omega)\mathbb{D}(\vartheta_{-t}\omega),\Sigma\times\mathscr{A}(\omega)\Big)\\
&\quad=\mathrm{dist}_{\mathbb{X}}\Big(\pi(t,\vartheta_{-t}\omega)\bigcup_{\sigma\in\Sigma}\{\sigma\}\times P_\sigma\mathbb{D}(\vartheta_{-t}\omega),\Sigma\times\mathscr{A}(\omega)\Big)\\
&\quad=\mathrm{dist}_{\mathbb{X}}\Big(\bigcup_{\sigma\in\Sigma}\{\theta_t\sigma\}\times\phi(t,\vartheta_{-t}\omega,\sigma,P_\sigma\mathbb{D}(\vartheta_{-t}\omega)),\Sigma\times\mathscr{A}(\omega)\Big)\\
&\quad\leqslant\mathrm{dist}_{\mathbb{X}}\Big(\Sigma\times(\cup_{\sigma\in\Sigma}\phi(t,\vartheta_{-t}\omega,\sigma,P_X\mathbb{D}(\vartheta_{-t}\omega)),\Sigma\times\mathscr{A}(\omega)\Big)\\
&\quad=\sup_{\sigma\in\Sigma}\mathrm{dist}_X\Big(\phi(t,\vartheta_{-t}\omega,\sigma,P_X\mathbb{D}(\vartheta_{-t}\omega)),\mathscr{A}(\omega)\Big)\to 0,\quad \text{as } t\to\infty,
\end{aligned}\tag{5.26}$$

and thereby $\Sigma \times \mathscr{A}$ is $\mathcal{D}_{\mathbb{X}}$-pullback attracting.

Now we construct the $\mathcal{D}_{\mathbb{X}}$-cocycle attractor. Since $\mathcal{D}_X$ is neighborhood closed, there exists an $\varepsilon > 0$ such that the closed ε-neighborhood $\mathcal{N}_\varepsilon(\mathscr{A})$ of $\mathscr{A}$ belongs to $\mathcal{D}_X$. Set $\mathbb{K} = \Sigma \times \mathcal{N}_\varepsilon(\mathscr{A})$. Then $\mathbb{K} \in \mathcal{D}_{\mathbb{X}}$ is a closed $\mathcal{D}_{\mathbb{X}}$-absorbing set of π. Consider the omega-limit set of K (under π) given by

$$\Omega_{\mathbb{K}}(\omega) = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} \pi(t, \vartheta_{-t}\omega, \mathbb{K}(\vartheta_{-t}\omega))}, \quad \forall \omega \in \Omega. \tag{5.27}$$

Following the proof of [83, Lemmas 2.17 & 2.21] and [84, Theorem 2.14] we know that $\Omega_{\mathbb{K}}$ is a (non-empty) compact random set in $\mathbb{X}$ which is invariant and $\mathcal{D}_{\mathbb{X}}$-pullback attracting (under π). In order to show that $\Omega_{\mathbb{K}}$ is the $\mathcal{D}_{\mathbb{X}}$-cocycle attractor of π, we need to prove further that (5.23) and (5.24) hold for $\Omega_{\mathbb{K}}$.

For each $\sigma \in \Sigma$ and $\omega \in \Omega$, since the uniform attractor $\mathscr{A}$ pullback attracts $P_X(\mathbb{K})$, we have

$$\operatorname{dist}(\phi(n, \vartheta_{-n}\omega, \theta_{-n}\sigma, P_X(\mathbb{K}(\vartheta_{-n}\omega))), \mathscr{A}(\omega)) \to 0, \quad \text{as } n \to \infty.$$

Hence, take any sequence $x_n \in P_{\theta_{-n}\sigma}\mathbb{K}(\vartheta_{-n}\omega)$, by the compactness of $\mathscr{A}(\omega)$ there exists a $y \in \mathscr{A}(\omega)$ such that

$$\phi(n, \vartheta_{-n}\omega, \theta_{-n}\sigma, x_n) \xrightarrow{n\to\infty} y$$

holds in a subsequence sense. This means that

$$\pi(n, \vartheta_{-n}\omega, \{\theta_{-n}\sigma\} \times \{x_n\}) \to \{\sigma\} \times \{y\}, \quad \text{as } n \to \infty,$$

and thereby $\{\sigma\} \times \{y\} \in \Omega_{\mathbb{K}}(\omega)$. Hence, $y \in P_\sigma(\Omega_{\mathbb{K}}(\omega))$ and (5.23) holds.

Take an arbitrarily $y \in X$, and recall that $\mathbb{K} = \Sigma \times \mathcal{N}_\varepsilon(\mathscr{A})$. Then for each ω fixed, $y \in P_X(\Omega_{\mathbb{K}}(\omega))$, i.e. $\{\sigma\}\times\{y\} \in \Omega_{\mathbb{K}}(\omega)$ for some $\sigma \in \Sigma$, if and only if there exist sequences $0 < t_n \to \infty$ and $\sigma_n \in \Sigma$, $x_n \in \mathcal{N}_\varepsilon(\mathscr{A}(\vartheta_{-t_n}\omega))$ such that $\pi(t_n, \vartheta_{-t_n}\omega, \{\theta_{-t_n}\sigma_n\} \times \{x_n\})) \to \{\sigma\} \times \{y\}$, or in other words by (5.22), that $\sigma_n \to \sigma$ and $\phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma_n, x_n)) \to y$, which is equivalent to that $y \in \mathcal{W}(\omega, \Sigma, \mathcal{N}_\varepsilon(\mathscr{A}))$ by Proposition 5.2.1. Hence, we have

$$P_X(\Omega_{\mathbb{K}}(\omega)) = \mathcal{W}(\omega, \Sigma, \mathcal{N}_\varepsilon(\mathscr{A})), \quad \forall \omega \in \Omega.$$

Since $\mathcal{W}(\cdot, \Sigma, \mathcal{N}_\varepsilon(\mathscr{A}))$ is a compact random set in X proved by Lemma 5.2.4 and $\mathcal{W}(\cdot, \Sigma, \mathcal{N}_\varepsilon(\mathscr{A}))$ is smaller than $\mathscr{A}$ (as it is the minimal closed random set uniformly attracting $\mathcal{N}_\varepsilon(\mathscr{A})$ by Lemma 5.2.4), we conclude that $P_X(\Omega_{\mathbb{K}}(\cdot)) = \mathcal{W}(\cdot, \Sigma, \mathcal{N}_\varepsilon(\mathscr{A})) \in \mathcal{D}_X$ by the inclusion-closed property of $\mathcal{D}_X$. Hence, (5.24) holds. The proof is complete. □

Proposition 5.3.4. *Let ϕ be an NRDS. If the random cocycle π generated by ϕ has a $\mathcal{D}_{\mathbb{X}}$-cocycle attractor $\mathbb{A} \in \mathcal{D}_{\mathbb{X}}$, then the random set $\mathscr{A} := P_X\mathbb{A}$ is the $\mathcal{D}_X$-random uniform attractor for ϕ.*

Proof. The compactness and measurability of $\mathscr{A}$ follows from $\mathbb{A}$ directly.

Let us prove uniformly attracting property of $\mathscr{A}$. Notice that, for each $x \in X$, $\sigma \in \Sigma$, $\omega \in \Omega$ and $t \geqslant 0$, we have

$$\begin{aligned}
\operatorname{dist}_X(x, \mathscr{A}(\omega)) &= \inf_{\sigma'\in\Sigma} \operatorname{dist}_X(x, P_{\sigma'}\mathbb{A}(\omega)) \\
&\leqslant \inf_{\sigma'\in\Sigma} \Big(\operatorname{dist}_X(x, P_{\sigma'}\mathbb{A}(\omega)) + \rho(\sigma, \sigma')\Big) \\
&= \operatorname{dist}_{\mathbb{X}}\Big(\{\sigma\} \times \{x\}, \cup_{\sigma'\in\Sigma}\{\sigma'\} \times P_{\sigma'}\mathbb{A}(\omega)\Big).
\end{aligned}$$

Hence, for any $D \in \mathcal{D}_X$, since $\mathbb{D}$ with $\mathbb{D}(\omega) := \Sigma \times D(\omega)$ belongs to $\mathcal{D}_{\mathbb{X}}$ and thereby attracted by $\mathbb{A}$, we have

$$\begin{aligned}
&\sup_{\sigma\in\Sigma} \operatorname{dist}_X \Big(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)), \mathscr{A}(\omega) \Big) \\
&\quad \leqslant \sup_{\sigma\in\Sigma} \operatorname{dist}_{\mathcal{X}} \Big(\{\sigma\} \times \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)), \cup_{\sigma'\in\Sigma}\{\sigma'\} \times P_{\sigma'}\mathbb{A}(\omega) \Big) \\
&\quad = \sup_{\sigma\in\Sigma} \operatorname{dist}_{\mathcal{X}} \Big(\pi\big(t, \vartheta_{-t}\omega, \{\theta_{-t}\sigma\} \times D(\vartheta_{-t}\omega)\big), \mathbb{A}(\omega) \Big) \\
&\quad = \operatorname{dist}_{\mathcal{X}} \big(\pi\big(t, \vartheta_{-t}\omega, \mathbb{D}(\vartheta_{-t}\omega)\big), \mathbb{A}(\omega) \big) \to 0, \quad \text{as } t \to \infty,
\end{aligned}$$

where the uniform attraction of $\mathscr{A}$ follows.

To see the minimal property, we assume $\mathscr{A}'$ is another closed uniformly attracting set of ϕ. Then, in view of (5.26), it is clear that $\Sigma \times \mathscr{A}'$ is a closed pullback attracting set of π. On the other hand, the cocycle attractor $\mathbb{A}$ is the minimal closed pullback attracting set of π since $\mathbb{A} \in \mathcal{D}_{\mathbb{X}}$ is invariant and pullback attracts itself. Therefore, it holds $\mathbb{A} \subset \Sigma \times \mathscr{A}'$, and thereby $\mathscr{A} = P_X\mathbb{A} \subset \mathscr{A}'$; the minimal property follows. □

Corollary 5.3.5. *Suppose that ϕ is a jointly continuous NRDS. Then the $\mathcal{D}$-random uniform attractor $\mathscr{A}$ of the NRDS exists if and only if so does the $\mathcal{D}_{\mathbb{X}}$-cocycle attractor $\mathbb{A}$ of the associated skew product cocycle. Moreover, it holds that*

$$\mathscr{A}(\omega) = P_X\mathbb{A}(\omega), \quad \forall \omega \in \Omega.$$

5.3.2 Cocycle attractors for NRDS

In this section we first study the relationship between cocycle attractor for any NRDS and proper cocycle attractor for the generated skew-product cocycle, and then we conclude the relationship between uniform and cocycle attractors.

a). Cocycle attractors with autonomous attraction universes

The following definitions and known results for cocycle attractors with autonomous attraction universes were established in Chapter 4. We put here for convenience.

Definition 5.3.6. A non-autonomous random set $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ is called a $\mathcal{D}_X$-random cocycle attractor of the NRDS ϕ if

(i) every $A_\sigma(\cdot)$ is a compact random set in X;

(ii) A is $\mathcal{D}_X$-pullback attracting;

(iii) A is invariant under ϕ, that is, $\phi(t, \omega, \sigma, A_\sigma(\omega)) = A_{\theta_t\sigma}(\vartheta_t\omega)$ for all $t \in \mathbb{R}^+$;

(iv) A is the minimal compact non-autonomous random set in X satisfying (2).

Lemma 5.3.7. *Suppose that ϕ is a continuous NRDS. If there exists a compact uniformly $\mathcal{D}_X$-pullback attracting set $K \in \mathcal{D}_X$, then ϕ has a unique $\mathcal{D}_X$-cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ given by*

$$A_\sigma(\cdot) = \mathcal{W}(\cdot, \sigma, K), \quad \forall \sigma \in \Sigma.$$

Lemma 5.3.8. *Suppose that $A = \{A_\sigma(\omega)\}$ is the $\mathcal{D}_X$-random cocycle attractor of an NRDS ϕ. Then*

(i) if ϕ is jointly continuous and there is a compact random set $D \in \mathcal{D}_X$ such that

$$\cup_{\sigma\in\Sigma} A_\sigma(\cdot) \subseteq D(\cdot),$$

then A is upper semi-continuous in Σ, namely,

$$\operatorname{dist}(A_\sigma(\omega), A_{\sigma_0}(\omega)) \to 0, \quad \textit{whenever } \sigma \to \sigma_0 \textit{ in } \Sigma;$$

(ii) if A is upper semi-continuous in symbols, then, since we have assumed the compactness of Σ throughout this chapter, we have $\cup_{\sigma\in\Sigma} A_\sigma(\omega)$ is compact for each $\omega \in \Omega$.

In Lemma 5.3.8 (ii), it is unclear whether the mapping $\omega \mapsto \cup_{\sigma\in\Sigma} A_\sigma(\omega)$ is measurable or not. However, it will be shown that, if the NRDS ϕ is jointly continuous and has a $\mathcal{D}_X$-random uniform attractor, then $\cup_{\sigma\in\Sigma} A_\sigma(\cdot)$ is measurable since it is actually the uniform attractor itself.

Definition 5.3.9. A mapping $\xi\colon \Omega \times \mathbb{R} \to X$ is called a (σ-driven) complete trajectory of an NRDS ϕ if $\xi(\vartheta_t\omega, t) = \phi(t - s, \vartheta_s\omega, \theta_s\sigma, \xi(\vartheta_s\omega, s))$ for each $t \geqslant s$ and $\omega \in \Omega$. If there exists a random set $B \in \mathcal{D}_X$ such that $\cup_{t\in\mathbb{R}}\xi(\cdot, t) \subset B(\cdot)$, then ξ is called a $\mathcal{D}_X$-complete trajectory.

Lemma 5.3.10. *Suppose that $A = \{A_\sigma(\omega)\}$ is the $\mathcal{D}_X$-cocycle attractor for NRDS ϕ and, moreover, there is a random set $B \in \mathcal{D}_X$ such that $\cup_{\sigma\in\Sigma} A_\sigma(\cdot) \subset B(\cdot)$. Then*

$$A_{\theta_t\sigma}(\vartheta_t\omega) = \Big\{\xi(\vartheta_t\omega, t) : \xi \textit{ is a } \sigma\textit{-driven } \mathcal{D}_X\textit{-complete trajectory of } \phi\Big\}.$$

b). Relationship between cocycle and uniform attractors

To see the relationship between $\mathcal{D}_X$-random cocycle attractors and $\mathcal{D}_X$-random uniform attractors, let us show first the relationship between $\mathcal{D}_X$-random cocycle attractors and $\mathcal{D}_{\mathbb{X}}$-cocycle attractors of the corresponding skew product cocycle π. For any proper random set $\mathbb{A}$ in $\mathbb{X}$, i.e., $\mathbb{A} \in \mathcal{D}_{\mathbb{X}}$, we write $\{P_\sigma(\mathbb{A}(\omega))\}_{\sigma\in\Sigma,\omega\in\Omega}$ as $\{P_\sigma(\mathbb{A})\}_{\sigma\in\Sigma}$. Note that it is unclear whether each $\omega \mapsto P_\sigma(\mathbb{A}(\omega))$ is a random mapping or not, but Proposition 5.3.12 will indicate that it is a random mapping if $\mathbb{A}$ is the $\mathcal{D}_{\mathbb{X}}$-cocycle attractor of the skew-product cocycle π, provided ϕ is continuous.

Proposition 5.3.11. *Given an NRDS ϕ, suppose that the skew-product cocycle π generated by ϕ has the $\mathcal{D}_{\mathbb{X}}$-cocycle attractor $\mathbb{A}$. Then the set $\{P_\sigma(\mathbb{A})\}_{\sigma\in\Sigma}$ is invariant under the NRDS ϕ, i.e.*

$$P_{\theta_t\sigma}(\mathbb{A}(\vartheta_t\omega)) = \phi(t, \omega, \sigma, P_\sigma(\mathbb{A}(\omega))), \quad \textit{for all } t \geqslant 0, \sigma \in \Sigma, \omega \in \Omega,$$

and is $\mathcal{D}_X$-pullback attracting under ϕ in the sense that

$$\lim_{t\to\infty} \operatorname{dist}(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, D(\vartheta_{-t}\omega)), P_\sigma(\mathbb{A}(\omega))) = 0, \quad \forall \omega \in \Omega, \sigma \in \Sigma, D \in \mathcal{D}_X. \tag{5.28}$$

Proof. We first prove the invariance. Let $\sigma \in \Sigma$, $\omega \in \Omega$ and $t \in \mathbb{R}^+$ be arbitrarily fixed, and $y \in P_{\theta_t\sigma}(\mathbb{A}(\vartheta_t\omega))$, which implies that $\{\theta_t\sigma\} \times \{y\} \in \mathbb{A}(\vartheta_t\omega)$. Then by the invariance of $\mathbb{A}$ that $\mathbb{A}(\vartheta_t\omega) = \pi(t,\omega)\mathbb{A}(\omega)$, there exists a $\{\sigma'\} \times \{x\} \in \mathbb{A}(\omega)$ such that

$$\{\theta_t\sigma\} \times \{y\} = \pi(t,\omega)\{\sigma'\} \times \{x\} = \{\theta_t\sigma'\} \times \{\phi(t,\omega,\sigma',x)\},$$

which indicates that $\sigma' = \sigma$ and $y = \phi(t,\omega,\sigma,x)$ with $x \in P_\sigma(\mathbb{A}(\omega))$. Hence,

$$P_{\theta_t\sigma}(\mathbb{A}(\vartheta_t\omega)) \subset \phi(t,\omega,\sigma,P_\sigma(\mathbb{A}(\omega))).$$

On the other hand, since

$$\begin{aligned}\{\theta_t\sigma\} \times \phi(t,\omega,\sigma,P_\sigma(\mathbb{A}(\omega))) &= \pi(t,\omega,\{\sigma\} \times P_\sigma A(\omega)) \\ &\subset \pi(t,\omega)\mathbb{A}(\omega) = \mathbb{A}(\vartheta_t\omega) = \bigcup_{\sigma'\in\Sigma}\{\sigma'\} \times P_{\sigma'}(\mathbb{A}(\vartheta_t\omega)),\end{aligned}$$

we have $\phi(t,\omega,\sigma,P_\sigma(\mathbb{A}(\omega))) \subset P_{\theta_t\sigma}(\mathbb{A}(\vartheta_t\omega))$. Hence, the invariance is clear.

Next, we prove (5.28) by contradiction. If not, then there are $\delta > 0$, $\omega \in \Omega$, $D \in \mathcal{D}$ and sequences $t_n \to \infty$ and $x_n \in \phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma,D(\vartheta_{-t_n}\omega))$ such that

$$\operatorname{dist}(x_n,P_\sigma(\mathbb{A}(\omega))) \geqslant \delta, \quad \forall n \in \mathbb{N}. \tag{5.29}$$

Denote by $\chi_n := \{\sigma\} \times x_n$ and $\mathbb{D} := \Sigma \times D \in \mathcal{D}_{\mathbb{X}}$. Then by the pullback attraction of $\mathbb{A}$ under π we have

$$\begin{aligned}\operatorname{dist}_{\mathbb{X}}(\chi_n,\mathbb{A}(\omega)) &\leqslant \operatorname{dist}_{\mathbb{X}}\big(\{\sigma\} \times \phi(t_n,\vartheta_{-t_n}\omega,\theta_{-t_n}\sigma,D(\vartheta_{-t_n}\omega)),\mathbb{A}(\omega)\big) \\ &= \operatorname{dist}_{\mathbb{X}}\big(\pi(t_n,\vartheta_{-t_n}\omega)\{\theta_{-t_n}\sigma\} \times D(\vartheta_{-t_n}\omega),\mathbb{A}(\omega)\big) \\ &\leqslant \operatorname{dist}_{\mathbb{X}}\big(\pi(t_n,\vartheta_{-t_n}\omega)\mathbb{D}(\vartheta_{-t_n}\omega),\mathbb{A}(\omega)\big) \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Therefore, as $\mathbb{A}(\omega) = \cup_{\sigma\in\Sigma}\{\sigma\} \times P_\sigma(\mathbb{A}(\omega))$ is compact, there is a $\bar\chi \in \{\sigma'\} \times P_{\sigma'}(\mathbb{A}(\omega))$ for some $\sigma' \in \Sigma$ such that, up to a subsequence,

$$\chi_n = \{\sigma\} \times \{x_n\} \to \bar\chi,$$

which implies that $\sigma' = \sigma$ and $x_n \to P_\sigma\bar\chi \in P_\sigma(\mathbb{A}(\omega))$, a contradiction. □

Proposition 5.3.12. *Suppose ϕ is a continuous NRDS. If the cocycle π generated by ϕ has a $\mathcal{D}_{\mathbb{X}}$-cocycle attractor $\mathbb{A}$, then ϕ has a $\mathcal{D}_X$-cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$. Moreover, they have the relation*

$$A_\sigma(\omega) = P_\sigma\mathbb{A}(\omega), \quad \forall\omega \in \Omega,\ \sigma \in \Sigma. \tag{5.30}$$

Proof. Proposition 5.3.4 indicates that the NRDS ϕ has a $\mathcal{D}_X$-uniform attractor $\mathscr{A} = P_X\mathbb{A}$, and also a $\mathcal{D}_X$-cocycle attractor A by Lemma 5.3.7. Now we prove the relation (5.30).

By the invariance of $\{P_\sigma(\mathbb{A})\}_{\sigma\in\Sigma}$ established in Proposition 5.3.11 we have

$$\begin{aligned}\operatorname{dist}(P_\sigma(\mathbb{A}(\omega)),A_\sigma(\omega)) &= \operatorname{dist}\Big(\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,P_{\theta_{-t}\sigma}(\mathbb{A}(\vartheta_{-t}\omega))),A_\sigma(\omega)\Big) \\ &\leqslant \operatorname{dist}\Big(\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,\mathscr{A}(\vartheta_{-t}\omega)),A_\sigma(\omega)\Big) \\ &\to 0, \quad \text{as } t \to \infty,\ \forall\omega \in \Omega,\end{aligned}$$

since A pullback attracts $\mathscr{A}$. Hence, $P_\sigma(\mathbb{A}) \subset A_\sigma$ for each $\sigma \in \Sigma$.

Let us prove the converse inclusion. Since A is a $\mathcal{D}_X$-cocycle attractor, for each $\sigma \in \Sigma$, A_σ is a compact random set in X. Hence, the mapping $\omega \mapsto \overline{\cup_{j\in\mathbb{N}} A_{\theta_{-j}\sigma}(\omega)}$ is a closed random set in X by Lemma 2.2.4. Moreover, by the minimal property of A, $\overline{\cup_{j\in\mathbb{N}} A_{\theta_{-j}\sigma}}$ is smaller than $\mathscr{A}$ and thereby belongs to $\mathcal{D}_X$. Hence,

$$\begin{aligned}\operatorname{dist}(A_\sigma(\omega), P_\sigma(\mathbb{A}(\omega))) &= \operatorname{dist}\Big(\phi(n, \vartheta_{-n}\omega, \theta_{-n}\sigma, A_{\theta_{-n}\sigma}(\vartheta_{-n}\omega)), P_\sigma(\mathbb{A}(\omega))\Big)\\ &\leqslant \operatorname{dist}\Big(\phi\big(n, \vartheta_{-n}\omega, \theta_{-n}\sigma, \overline{\cup_{j\in\mathbb{N}} A_{\theta_{-j}\sigma}(\vartheta_{-n}\omega)}\big), P_\sigma(\mathbb{A}(\omega))\Big)\\ &\to 0, \quad \text{as } n\to\infty,\ \forall \omega\in\Omega,\end{aligned}$$

where we have used (5.28). Thus, it follows that $A_\sigma \subset P_\sigma(\mathbb{A})$ for each $\sigma \in \Sigma$. The proof is complete. □

Since Proposition 5.3.12 shows the relationship between the $\mathcal{D}_{\mathbb{X}}$-cocycle attractor $\mathbb{A}$ of π and the cocycle attractor A of ϕ, while Corollary 5.3.5 indicates the relationship between the $\mathcal{D}_{\mathbb{X}}$-cocycle attractor $\mathbb{A}$ of π and the uniform attractor $\mathscr{A}$ of ϕ, we are able to give the relationship between uniform and cocycle attractors of ϕ as follows.

Theorem 5.3.13. *Suppose that ϕ is a jointly continuous NRDS. If ϕ has a $\mathcal{D}_X$-uniform attractor $\mathscr{A}$, then it also has a $\mathcal{D}_X$-cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$. Moreover, the two attractors admit the relation*

$$\mathscr{A}(\omega) = \bigcup_{\sigma\in\Sigma} A_\sigma(\omega), \quad \forall \omega\in\Omega. \tag{5.31}$$

Proof. The existence of the $\mathcal{D}_X$-uniform attractor $\mathscr{A}$ implies that of the $\mathcal{D}_X$-cocycle attractor A by Lemma 5.3.7, and implies that of the $\mathcal{D}_{\mathbb{X}}$-cocycle attractor $\mathbb{A}$ of the skew-product cocycle generated by ϕ by Proposition 5.3.3. Then from Corollary 5.3.5 and Proposition 5.3.12 it follows that

$$\mathscr{A}(\omega) = P_X\mathbb{A}(\omega) = \bigcup_{\sigma\in\Sigma} P_\sigma\mathbb{A}(\omega) = \bigcup_{\sigma\in\Sigma} A_\sigma(\omega), \quad \forall\omega\in\Omega.$$

The proof is complete. □

5.3.3 More about uniform attractors

Now we present several important properties of uniform attractors, making use of the relation (5.31) given by Theorem 5.3.13. We first strengthen the existence Theorem 5.2.5 to the following one, in which we characterize the uniform attractor by the omega limit set of an arbitrarily compact uniformly attracting set K instead of that of uniformly absorbing sets.

Theorem 5.3.14. *Let Ξ be any a dense subset of Σ and ϕ a jointly continuous NRDS. If $K \in \mathcal{D}_X$ is a compact uniformly $\mathcal{D}_X$-attracting set, then ϕ has a unique $\mathcal{D}_X$-uniform attractor $\mathscr{A} \in \mathcal{D}_X$ given by*

$$\mathscr{A}(\omega) = \mathcal{W}(\omega, \Sigma, K) = \mathcal{W}(\omega, \Xi, K), \quad \forall\omega\in\Omega.$$

Proof. Since $\mathcal{D}_X$ is neighborhood-closed, there exists an $\varepsilon > 0$ such that the closed ε-neighborhood $\mathcal{N}_\varepsilon(K)$ of K belongs to $\mathcal{D}_X$. Then since K is a uniformly attracting set, $\mathcal{N}_\varepsilon(K)$ is a closed uniformly absorbing set. By Theorem 5.2.5 we have

$$\mathscr{A}(\omega) = \mathcal{W}(\omega, \Sigma, \mathcal{N}_\varepsilon(K)).$$

Clearly, $\mathscr{A}(\omega) \supseteq \mathcal{W}(\omega, \Sigma, K)$.

Now we prove $\mathscr{A}(\omega) \subset \mathcal{W}(\omega, \Sigma, K)$. Notice that

$$\mathcal{W}(\omega, \Sigma, \mathscr{A}) = \mathscr{A}(\omega), \quad \forall \omega \in \Omega.$$

Indeed, since, by Lemma 5.2.4, $\mathcal{W}(\cdot, \Sigma, \mathscr{A})$ is the minimal closed random set which uniformly pullback attracts $\mathscr{A}$, it holds $\mathcal{W}(\omega, \Sigma, \mathscr{A}) \subset \mathscr{A}(\omega)$ as $\mathscr{A}$ uniformly attracts itself as well; the inverse inclusion follows from

$$\begin{aligned}
\mathcal{W}(\omega, \Sigma, \mathscr{A}) &= \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \bigcup_{\sigma \in \Sigma} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \mathscr{A}(\vartheta_{-t}\omega))} \\
&= \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \bigcup_{\sigma \in \Sigma} \phi\big(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, \cup_{\sigma' \in \Sigma} A_{\sigma'}(\vartheta_{-t}\omega)\big)} \\
&\supseteq \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \bigcup_{\sigma \in \Sigma} \phi\big(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, A_{\theta_{-t}\sigma}(\vartheta_{-t}\omega)\big)} \\
&= \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \bigcup_{\sigma \in \Sigma} A_\sigma(\omega)} = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \mathscr{A}(\omega)} = \mathscr{A}(\omega),
\end{aligned}$$

where the relation (5.31) and the invariance of the cocycle attractor are employed. Hence, for K a compact random uniformly attracting set, $\mathscr{A}(\omega) = \mathcal{W}(\omega, \Sigma, \mathscr{A}) \subset \mathcal{W}(\omega, \Sigma, K)$ as $\mathscr{A}(\omega) \subset K(\omega)$ by the minimality of $\mathscr{A}$. The proof is complete. □

Proposition 5.3.15. *Let ϕ be a jointly continuous NRDS and have a compact uniformly $\mathcal{D}_X$-pullback attracting set $K \in \mathcal{D}_X$ (hence ϕ has a $\mathcal{D}_X$-cocycle attractor A by Lemma 5.3.7). Suppose that $y \in X$ is such that*

$$\phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma_n, x_n) \to y$$

for some sequences $t_n \to \infty$, $\sigma_n \in \Sigma$ and $x_n \in D(\vartheta_{-t_n}\omega)$. Then

$$y \in A_\sigma(\omega),$$

where σ is such that there exists a subsequence $\sigma_{n_k} \xrightarrow{k \to \infty} \sigma$.

Proof. By Theorem 5.3.14 and Proposition 5.3.3 we know that the existence of a compact uniformly $\mathcal{D}_X$-pullback attracting set implies the existence of the $\mathcal{D}_X$-uniform attractor $\mathscr{A}$ for ϕ and that of the $\mathcal{D}_{\mathbb{X}}$-cocycle attractor $\mathbb{A}$ for the generated semi-product cocycle π.

Denote by $\hat{D} := \Sigma \times D$. Then $\hat{D}$ belongs to $\mathcal{D}_{\mathbb{X}}$ and is attracted by $\mathbb{A}$ under π. Hence, by (5.22) and Proposition 5.3.12 we have

$$\begin{aligned}
&\mathrm{dist}_{\mathbb{X}}\Big(\{\sigma_{n_k}\} \times \phi\big(t_{n_k}, \vartheta_{-t_{n_k}}\omega, \theta_{-t_{n_k}}\sigma_{n_k}, x_{n_k}\big), \cup_{\sigma'\in\Sigma}\{\sigma'\} \times A_{\sigma'}(\omega)\Big)\\
&\quad = \mathrm{dist}_{\mathbb{X}}\Big(\pi\big(t_{n_k}, \vartheta_{-t_{n_k}}\omega, \{\theta_{-t_{n_k}}\sigma_{n_k}\} \times \{x_{n_k}\}\big), \mathbb{A}(\omega)\Big)\\
&\quad \leqslant \mathrm{dist}_{\mathbb{X}}\Big(\pi(t_{n_k}, \vartheta_{-t_{n_k}}\omega, \hat{D}(\vartheta_{-t_{n_k}}\omega)), \mathbb{A}(\omega)\Big) \xrightarrow{k\to\infty} 0.
\end{aligned}$$

Therefore, from $\sigma_{n_k} \xrightarrow{k\to\infty} \sigma$ and $\phi(t_n, \vartheta_{-t_n}\omega, \theta_{-t_n}\sigma_n, x_n) \to y$ it follows that $y \in A_\sigma(\omega)$. □

Proposition 5.3.16. *Suppose that ϕ is a jointly continuous NRDS. If the NRDS ϕ has a $\mathcal{D}_X$-uniform attractor $\mathscr{A}$, then it also has a $\mathcal{D}_X$-cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ which is upper semi-continuous in symbols and satisfies $\mathscr{A}(\omega) = \cup_{\sigma\in\Sigma}A_\sigma(\omega)$ for each $\omega \in \Omega$.*

Proof. The proof is concluded by Theorem 5.3.13 and Lemma 5.3.8. □

Proposition 5.3.17. *Suppose that ϕ is a jointly continuous NRDS. If ϕ has a $\mathcal{D}_X$-random uniform attractor $\mathscr{A}$, then*

$$\mathscr{A}(\vartheta_t\omega) = \Big\{\xi(\vartheta_t\omega, t) : \xi \text{ is a } \mathcal{D}_X\text{-complete trajectory of } \phi\Big\}, \quad \forall t \in \mathbb{R}, \omega \in \Omega.$$

Proof. The proof is concluded by (5.31) and Lemma 5.3.10. □

We are now able to prove neatly the negative semi-invariance of uniform attractors, see Proposition 5.2.5.

Proposition 5.3.18. *Suppose that ϕ is a jointly continuous NRDS. Then the uniform attractor $\mathscr{A}$ of ϕ, if exists, is negatively semi-invariant, that is,*

$$\mathscr{A}(\vartheta_t\omega) \subseteq \Phi(t, \omega, \mathscr{A}(\omega)), \quad \forall t \in \mathbb{R}^+,\ \omega \in \Omega,$$

where $\Phi(t, \omega, x) := \cup_{\sigma\in\Sigma}\phi(t, \omega, \sigma, x)$. (Note that Φ is actually a multi-valued RDS, see Proposition 5.4.4.)

Proof. By the relation (5.31) and the invariance of the cocycle attractor,

$$\begin{aligned}
\mathscr{A}(\vartheta_t\omega) &= \bigcup_{\sigma\in\Sigma} A_{\theta_t\sigma}(\vartheta_t\omega) = \bigcup_{\sigma\in\Sigma} \phi(t, \omega, \sigma, A_\sigma(\omega))\\
&\subseteq \bigcup_{\sigma\in\Sigma} \phi(t, \omega, \sigma, \cup_{\sigma'\in\Sigma}A_{\sigma'}(\omega)) = \Phi(t, \omega, \mathscr{A}(\omega)).
\end{aligned}$$

□

5.4 Uniform attractor as multi-valued cocycle attractor

It is known that the theory of multi-valued dynamical systems is often used to deal with differential equations without the uniqueness of solutions, see [3, 72, 80, 39] and references therein. In this section we shall show that it is also useful for the study of uniform attractors, as a uniform attractor for an NRDS could be regarded as the cocycle attractor of a corresponding multi-valued RDS. The relationship between uniform attractors for single-valued dynamical systems and cocycle attractors for multi-valued dynamical systems seems new in the literature.

Dynamical systems mentioned in this section are single-valued, except otherwise clearly stated. This is the case in all the other parts of this thesis.

Let $C(X)$ be the collection of non-empty closed sets in X.

Definition 5.4.1. A set-valued mapping $\Phi : \mathbb{R}^+ \times \Omega \times X \mapsto C(X)$ is called a multi-valued RDS on X, if

(1) Φ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(2) $\Phi(0, \omega, x) = x, \quad \forall \omega \in \Omega, x \in X;$

(3) it holds the negative invariance property

$$\Phi(t + s, \omega, x) \subset \Phi(t, \vartheta_s\omega, \Phi(s, \omega, x)), \quad \forall t, s \in \mathbb{R}^+, \omega \in \Omega, x \in X.$$

If it holds the identity in (3), then Φ is called a strict multi-valued RDS. Moreover, the multi-valued RDS Φ is said to be upper (or lower) semi-continuous if the mapping $\Phi(t, \omega, \cdot)$ is upper (or lower) semi-continuous for each fixed t, ω. If it is both upper and lower semi-continuous, then it is said *continuous*.

Definition 5.4.2. A random set $\mathscr{A}(\cdot) \in \mathcal{D}_X$ is called a $\mathcal{D}_X$-cocycle attractor for a multi-valued RDS Φ, if

(1) $\mathscr{A}$ is compact;

(2) $\mathscr{A}$ is $\mathcal{D}_X$-(pullback) attracting under Φ, namely, for any $B \in \mathcal{D}_X$ it holds that

$$\operatorname{dist}(\Phi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega)), \mathscr{A}(\omega)) \to 0, \quad \text{as } t \to \infty;$$

(3) $\mathscr{A}$ is negatively invariant in the sense that

$$\mathscr{A}(\vartheta_t\omega) \subset \Phi(t, \omega, \mathscr{A}(\omega)), \quad \forall t \in \mathbb{R}^+, \omega \in \Omega.$$

Remark 5.4.3. Note that the negative invariance of a $\mathcal{D}_X$-cocycle attractor implies the minimal property that for any closed random set $\mathscr{A}'$ which is $\mathcal{D}_X$-pullback attracting under the multi-valued RDS Φ it holds $\mathscr{A} \subset \mathscr{A}'$. This can be seen from

$$\operatorname{dist}(\mathscr{A}(\omega), \mathscr{A}'(\omega)) \leqslant \operatorname{dist}(\Phi(t, \vartheta_{-t}\omega, \mathscr{A}(\vartheta_{-t}\omega)), \mathscr{A}'(\omega)) \to 0, \quad \text{as } t \to \infty.$$

Hence, cocycle attractors for multi-valued RDS must be unique.

Several papers can be found on the study of attractors for multi-valued RDS, e.g., [15, 14, 87, 85]. Here we show an equivalence relationship between uniform attractors and multi-valued attractors.

5.4.1 Uniform attractor for an NRDS is the cocycle attractor for associated multi-valued RDS

First we show that any jointly continuous NRDS generates a continuous multi-valued RDS.

Proposition 5.4.4. *Suppose that ϕ is a jointly continuous NRDS on X. Then the mapping Φ on $\mathbb{R}^+ \times \Omega \times X$ defined by*

$$\Phi(t, \omega, x) = \bigcup_{\sigma \in \Sigma} \phi(t, \omega, \sigma, x) \tag{5.32}$$

is a continuous multi-valued RDS, which is said generated by the NRDS ϕ.

Proof. Since the NRDS ϕ is continuous in Σ and Σ is compact, Φ, defined by (5.32), takes values in $C(X)$, and hence

$$\Phi(t, \omega, x) = \overline{\bigcup_{\sigma \in \Sigma} \phi(t, \omega, \sigma, x)} = \overline{\bigcup_{\sigma \in \Xi} \phi(t, \omega, \sigma, x)}, \tag{5.33}$$

where Ξ is an arbitrary countable dense set of Σ. Thus, in view of Lemma 2.2.4, the measurability of Φ is clear.

The negative invariance of Φ follows from that of ϕ and the invariance of Σ. Indeed,

$$\begin{aligned}
\Phi(t+s, \omega, x) &= \bigcup_{\sigma \in \Sigma} \phi(t+s, \omega, \sigma, x) \\
&\subset \bigcup_{\sigma \in \Sigma} \phi(t, \vartheta_s \omega, \theta_s \sigma, \phi(s, \omega, \sigma, x)) \\
&\subset \bigcup_{\sigma \in \Sigma} \phi\big(t, \vartheta_s \omega, \theta_s \sigma, \cup_{\sigma' \in \Sigma} \phi(s, \omega, \sigma', x)\big) \\
&= \Phi(t, \vartheta_s \omega, \Phi(s, \omega, x)), \quad \forall t, s \in \mathbb{R}^+, \omega \in \Omega, x \in X.
\end{aligned}$$

Now we prove the continuity by proving the upper and lower semi-continuity, respectively. Let $x_n \to x$. First we prove upper semi-continuity by contradiction. Suppose for some $\varepsilon > 0$ it holds that

$$\operatorname{dist}(\Phi(t, \omega, x_n), \Phi(t, \omega, x)) > \varepsilon, \quad \forall n \in \mathbb{N}.$$

Then by (5.32) we have a sequence $\sigma_n \in \Sigma$ such that

$$\operatorname{dist}(\phi(t, \omega, \sigma_n, x_n), \Phi(t, \omega, x)) > \varepsilon, \quad \forall n \in \mathbb{N}.$$

Noticing that $\sigma_n \to \sigma$ for some $\sigma \in \Sigma$ by the compactness of Σ, up to a subsequence we have

$$\phi(t, \omega, \sigma_n, x_n) \to \phi(t, \omega, \sigma, x) \in \Phi(t, \omega, x)$$

as ϕ is continuous in both Σ and X; a contradiction. To see the lower semi-continuity, it suffices to notice that for each $y \in \Phi(t, \omega, x)$, there is a $\sigma \in \Sigma$ such that $y = \phi(t, \omega, \sigma, x)$ which is approximated by $\phi(t, \omega, \sigma, x_n)$ which belongs to $\Phi(t, \omega, x_n)$. The proof is complete. □

Theorem 5.4.5. *Suppose that ϕ is a jointly continuous NRDS on X, and that Φ is the continuous multi-valued RDS generated by ϕ. Then the random set $\mathscr{A}$ is the $\mathcal{D}_X$-uniform attractor of ϕ if and only if it is the $\mathcal{D}_X$-cocycle attractor of Φ.*

Proof. First it is trivial to observe that, for any $B \in \mathcal{D}_X$ the pullback attraction

$$\lim_{t\to\infty} \operatorname{dist}(\Phi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega)), \mathscr{A}(\omega)) = 0$$

is equivalent to

$$\lim_{t\to\infty} \operatorname{dist}(\phi(t, \vartheta_{-t}\omega, \Sigma, B(\vartheta_{-t}\omega)), \mathscr{A}(\omega)) = 0.$$

Hence, $\mathscr{A}$ is $\mathcal{D}_X$-pullback attracting under Φ if and only it is uniformly $\mathcal{D}_X$-attracting under ϕ.

If $\mathscr{A}$ is the $\mathcal{D}_X$-uniform attractor of ϕ, then Theorem 5.2.5 implies the negative invariance of $\mathscr{A}$ under Φ, and hence it is the $\mathcal{D}_X$-cocycle attractor of Φ. Conversely, if $\mathscr{A}$ is the $\mathcal{D}_X$-cocycle attractor of Φ, since by Remark 5.4.3 we have shown the minimal property of $\mathscr{A}$, it is clearly the $\mathcal{D}_X$-uniform attractor of ϕ. □

5.4.2 Random uniform attractor is determined by uniformly attracting non-random compact sets

In this section we show that uniform attractor for any jointly continuous NRDS ϕ is determined by uniformly attracting nonrandom compact sets. This could be regarded as a non-autonomous generalization of analogous results in [31] for (autonomous) RDS.

Let Φ be the continuous multi-valued RDS generated by ϕ. The omega-limit set of a compact set $D \subset X$ is defined as

$$\Omega_D(\omega) = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} \Phi(t, \vartheta_{-t}\omega, D)}, \quad \forall \omega \in \Omega.$$

Note that the omega-limit set Ω_D of a compact set D for a general multi-valued RDS is not measurable in general. But for the continuous multi-valued RDS Φ generated by NRDS ϕ, in view of [34, Theorem 3.11] it is clear that the omega-limit set is measurable at least with respect to the $\mathcal{P}$-completion of $\mathcal{F}$, since Σ and X are Polish. In this part we shall work in this $\mathcal{P}$-completion sense.

Hereafter, for any random sets I and D, we write $\mathcal{P}\{\omega \in \Omega : I(\omega) \subset D(\omega)\}$ simply as $\mathcal{P}(I \subset D)$. For the ease of description, let

$$\mathfrak{D} = \Big\{ K : K \text{ is a (nonrandom) compact subset of } X \Big\} \subset \mathcal{D}_X.$$

Now we define the $\mathfrak{D}$-cocycle attractor, which pullback attracts nonrandom compact sets.

Definition 5.4.6. A random set $\mathfrak{A}$ is said to be a $\mathfrak{D}$-cocycle attractor of Φ if

(1) $\mathfrak{A}$ is compact;

(2) $\mathfrak{A}$ is $\mathfrak{D}$-(pullback) attracting under Φ, namely, for any $K \in \mathfrak{D}$ it holds that

$$\operatorname{dist}(\Phi(t, \vartheta_{-t}\omega, K), \mathfrak{A}(\omega)) \to 0, \quad \text{as } t \to \infty;$$

(3) $\mathfrak{A}$ is negatively invariant under Φ, i.e.,

$$\mathfrak{A}(\vartheta_t\omega) \subset \Phi(t, \omega, \mathfrak{A}(\omega)), \quad \forall t \in \mathbb{R}^+, \omega \in \Omega.$$

The following lemma is a multi-valued generalization of [31, Proposition 5.2].

Lemma 5.4.7. *Suppose that I is a random set which is negatively invariant under Φ. Then for each compact set $D \subset X$ we have*

$$\mathcal{P}(I \subset D) \leqslant \mathcal{P}(I \subset \Omega_D).$$

Proof. The proof is achieved similarly to [31, Proposition 5.2] using Poincaré recurrence theorem. □

Lemma 5.4.8. *Suppose that I is a compact random set which is negatively invariant under Φ. Then for each $\varepsilon > 0$ there exists a compact nonrandom set $K \subset X$ such that*

$$\mathcal{P}(I \subset \Omega_K) \geqslant 1 - \varepsilon.$$

Proof. Similar to [31, Corollary 5.4]. □

Corollary 5.4.9. *Suppose that I is a compact random set which is negatively invariant under Φ. If $\mathfrak{A}$ is a $\mathfrak{D}$-cocycle attractor of Φ, then*

$$\mathcal{P}(I \subset \mathfrak{A}) = 1.$$

Proof. By Lemma 5.4.8 we see that for each $\varepsilon > 0$ there exists a compact set $K \in \mathfrak{D}$ such that $\mathcal{P}(I \subset \Omega_K) \geqslant 1 - \varepsilon$. On the other hand, since $\mathfrak{A}$ is the $\mathfrak{D}$-cocycle attractor, $\Omega_K \subset \mathfrak{A}$ for any $K \in \mathfrak{D}$. Hence,

$$\mathcal{P}(I \subset \mathfrak{A}) \geqslant \mathcal{P}(I \subset \Omega_K) \geqslant 1 - \varepsilon, \quad \forall \varepsilon > 0,$$

which completes the proof. □

The above corollary implies that $\mathfrak{D}$-cocycle attractor is $\mathcal{P}$-almost surely unique, as any $\mathfrak{D}$-cocycle attractor is negatively invariant. Noticing also that any $\mathcal{D}_X$-cocycle attractor is also a $\mathfrak{D}$-cocycle attractor, we have the following result.

Proposition 5.4.10. *Suppose that $\mathscr{A}$ is the $\mathcal{D}_X$-cocycle attractor of Φ and $\mathfrak{A}$ is the $\mathfrak{D}$-cocycle attractor of Φ, then*

$$\mathcal{P}(\mathscr{A} = \mathfrak{A}) = 1.$$

Proof. Firstly, by Corollary 5.4.9 we have $\mathcal{P}(\mathscr{A} \subset \mathfrak{A}) = 1$. On the other hand, by Lemma 5.4.8 we know that for any $\varepsilon > 0$ there exists a compact set $K \in \mathfrak{D}$ such that $\mathcal{P}(\mathfrak{A} \subset \Omega_K) \geqslant 1 - \varepsilon$. Note that, since $\mathfrak{D} \subset \mathcal{D}_X$, $\mathscr{A}$ is $\mathfrak{D}$-pullback attracting as well, which indicates that $\Omega_K \subset \mathscr{A}$ for any $K \in \mathfrak{D}$. Hence, $\mathcal{P}(\mathfrak{A} \subset \mathscr{A}) \geqslant 1 - \varepsilon$ for each $\varepsilon > 0$ and thereby $\mathcal{P}(\mathfrak{A} \subset \mathscr{A}) = 1$. The proof is complete. □

Now we rewrite Proposition 5.4.10 to indicate that uniform attractors for jointly continuous NRDS are determined by uniformly attracting nonrandom compact sets.

Proposition 5.4.11. *Suppose that ϕ is a jointly continuous NRDS with a $\mathcal{D}_X$-random uniform attractor $\mathscr{A}$ and a $\mathfrak{D}$-random uniform attractor $\mathfrak{A}$. Then*

$$\mathcal{P}(\mathscr{A} = \mathfrak{A}) = 1.$$

5.5 A class of Polish spaces and translation-bounded functions

In this section we give a general example for the crucial Polish symbol space, and then recall and study the so-called translation-bounded functions. This makes our previous theoretical analysis convenient to apply.

5.5.1 A class of Polish spaces

Let $\mathcal{O} \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ be a domain. Recall that for each $p, r > 1$, the space $L^p_{loc}(\mathbb{R}; L^r(\mathcal{O}))$ is defined as the space of functions g such that for any bounded interval $(T_1, T_2) \subset \mathbb{R}$

$$\int_{T_1}^{T_2} \|g(s)\|^p_{L^r(\mathcal{O})} \mathrm{d}s < \infty.$$

A sequence g_n is said converging to some g in $L^p_{loc}(\mathbb{R}; L^r(\mathcal{O}))$ if $\int_{T_1}^{T_2} \|g_n(s) - g(s)\|^p_{L^r(\mathcal{O})} \mathrm{d}s \to 0$ for any bounded interval $(T_1, T_2) \subset \mathbb{R}$. Standardly, let us define the group $\{\theta_t\}_{t\in\mathbb{R}}$ of translation operators on $L^p_{loc}(\mathbb{R}; L^r(\mathcal{O}))$ by

$$\theta_t g(\cdot) := g(\cdot + t), \quad \forall t \in \mathbb{R}, g \in L^p_{loc}(\mathbb{R}; L^r(\mathcal{O})),$$

and define the hull $\mathcal{H}(g)$ of $g \in L^p_{loc}(\mathbb{R}; L^r(\mathcal{O}))$ as $\mathcal{H}(g) := \overline{\{\theta_t g(\cdot) : t \in \mathbb{R}\}}$.

The following proposition shows that the mapping $t \mapsto \theta_t g$ is $\big(\mathbb{R}, L^p_{loc}(\mathbb{R}; L^r(\mathcal{O}))\big)$-continuous for every $g \in L^p_{loc}(\mathbb{R}; L^r(\mathcal{O}))$ with $p, r > 1$, which ensures the hull of g to be Polish.

Proposition 5.5.1. *Let $g \in L^p_{loc}(\mathbb{R}; L^r(\mathcal{O}))$ with $p, r > 1$. Then for any bounded interval $(T_1, T_2) \subset \mathbb{R}$ we have*

$$\int_{T_1}^{T_2} \|g(\cdot + \varepsilon) - g(\cdot)\|^p_{L^r(\mathcal{O})} \mathrm{d}s \to 0, \quad \textit{as } \varepsilon \to 0.$$

Proof. Without loss of generality, let $\varepsilon \in (-1, 1)$. Denote by $X = L^p(T_1, T_2; L^r(\mathcal{O}))$. Then we need to prove that $g(\cdot + \varepsilon)$ converges to g strongly in X. Let (p, q) and (r, s) be conjugate indices.

Firstly, for any $v \in C^\infty(T_1 - 1, T_2 + 1; L^s(\mathcal{O}))$ we have

$$\begin{aligned}
&\left|\int_{T_1}^{T_2}\int_{\mathcal{O}} (g(s+\varepsilon) - g(s))v(s)\, \mathrm{d}x\mathrm{d}s\right| \\
&\leqslant \left|\int_{T_1}^{T_2}\int_{\mathcal{O}} g(s+\varepsilon)\big(v(s) - v(s+\varepsilon)\big)\mathrm{d}x\mathrm{d}s\right| + \left|\int_{T_1}^{T_2}\int_{\mathcal{O}} g(s+\varepsilon)v(s+\varepsilon) - g(s)v(s)\mathrm{d}x\mathrm{d}s\right| \\
&= |\varepsilon|\left|\int_{T_1}^{T_2}\int_{\mathcal{O}} g(s+\varepsilon)v'(\xi_s)\, \mathrm{d}x\mathrm{d}s\right| + \left|\int_{T_1}^{T_2}\int_{\mathcal{O}} g(s+\varepsilon)v(s+\varepsilon) - g(s)v(s)\, \mathrm{d}x\mathrm{d}s\right| \\
&\leqslant k|\varepsilon|\|g(s)\|_{L^p_{loc}(\mathbb{R};L^r)} + \left|\int_{T_1+\varepsilon}^{T_2+\varepsilon}\int_{\mathcal{O}} - \int_{T_1}^{T_2}\int_{\mathcal{O}}\right| \xrightarrow{\varepsilon\to 0} 0,
\end{aligned}$$

where the last inequality is due to Hölder's inequality and k is a constant such that

$$\sup_{\xi\in(T_1-1,T_2+1)} \|v'(\xi)\|_{L^s(\mathcal{O})} \leqslant k.$$

Hence, $g(\cdot+\varepsilon)$ converges weakly to g in X as $C^\infty(T_1,T_2;L^s(\mathcal{O}))$ is dense in $X^*=L^q(T_1,T_2;L^s(\mathcal{O}))$ (the dual of X). Therefore, by

$$\begin{aligned}\|g(\cdot+\varepsilon)\|_X &= \left(\int_{T_1}^{T_2}\|g(s+\varepsilon)\|_{L^r(\mathcal{O})}^p \mathrm{d}s\right)^{1/p}\\ &= \left(\int_{T_1+\varepsilon}^{T_2+\varepsilon}\|g(s)\|_{L^r(\mathcal{O})}^p \mathrm{d}s\right)^{1/p} \xrightarrow{\varepsilon\to 0} \|g\|_X\end{aligned}$$

and Lemma A.1.4 we know $g(\cdot+\varepsilon)$ indeed converges to $g(\cdot)$ strongly in X as desired. □

5.5.2 Translation bounded functions

For any domain $\mathcal{O}\subset\mathbb{R}^N$ ($N\in\mathbb{N}$), let $H=\big(L^2(\mathcal{O}),\|\cdot\|\big)$. Consider the space $L^2_{loc}(\mathbb{R};H)$, endowed with the two-power mean convergence topology on any bounded segment of $\mathbb{R}$, i.e., $g_n\to g$ in $L^2_{loc}(\mathbb{R};H)$ means $\int_{t_1}^{t_2}\|g_n(s)-g(s)\|^2\,\mathrm{d}s\to 0$ for any bounded $[t_1,t_2]\subset\mathbb{R}$. A function $g\in L^2_{loc}(\mathbb{R};H)$ is said to be *translation compact in* $L^{2,w}_{loc}(\mathbb{R};H)$ if its hull

$$\mathcal{H}(g):=\overline{\{\theta_t g(\cdot):t\in\mathbb{R}\}}^W$$

is compact in $L^{2,w}_{loc}(\mathbb{R};H)$, where

$$\theta_t g(\cdot):=g(\cdot+t),\quad \forall t\in\mathbb{R}, g\in L^2_{loc}(\mathbb{R};H), \tag{5.34}$$

and the closure is in the local weak convergence topology sense, i.e., $\sigma_n\to\sigma$ in $\mathcal{H}(g)$ if and only if

$$\int_{t_1}^{t_2}\big\langle v(s),\sigma_n(s)-\sigma(s)\big\rangle\,\mathrm{d}s\to 0$$

for each bounded $(t_1,t_2)\subset\mathbb{R}$ and $v\in L^2_{loc}(\mathbb{R};H)$. Note that g is translation compact in $L^{2,w}_{loc}(\mathbb{R};H)$ if and only if it is translation bounded in $L^2_{loc}(\mathbb{R};H)$, as indicated by the following result.

Proposition 5.5.2. *[26, Proposition 4.2] If $g\in L^2_{loc}(\mathbb{R};H)$ is translation compact in $L^{2,w}_{loc}(\mathbb{R};H)$, then*

(1) for all $t\in\mathbb{R}$, the translation operator θ_t defined by (5.34) is continuous on $\mathcal{H}(g)$ in $L^{2,w}_{loc}(\mathbb{R};H)$;

(2) the hull of g is translation invariant $\mathcal{H}(g)=\theta_t\mathcal{H}(g)$, $\forall t\in\mathbb{R}$;

(3) any function $\sigma\in\mathcal{H}(g)$ is translation compact in $L^{2,w}_{loc}(\mathbb{R};H)$ and $\mathcal{H}(\sigma)\subseteq\mathcal{H}(g)$;

(4) equivalently, g is translation bounded in $L^2_{loc}(\mathbb{R};H)$, i.e.,

$$\eta(g):=\sup_{\tau\in\mathbb{R}}\int_{\tau-1}^{\tau}\|g(s)\|^2\,\mathrm{d}s<\infty; \tag{5.35}$$

(5) for any $\sigma\in\mathcal{H}(g)$, $\eta(\sigma)\leqslant\eta(g)$.

We will need the following bound.

Proposition 5.5.3. *Let $g \in L^2_{loc}(\mathbb{R}; H)$ be translation bounded in $L^2_{loc}(\mathbb{R}; H)$. Then*

$$\sup_{\sigma \in \mathcal{H}(g)} \int_{-\infty}^{0} e^{\lambda s} \|\sigma(s)\|^2 \, \mathrm{d}s \leqslant \frac{\eta(g)}{1 - e^{-\lambda}}, \quad \forall \lambda > 0, \tag{5.36}$$

where $\eta(g)$ is the constant given by (5.35).

Proof. For each $\sigma \in \mathcal{H}(g)$, by Proposition 5.5.2 (5) and (5.35) we have

$$\begin{aligned} \int_{-\infty}^{0} e^{\lambda s} \|\sigma(s)\|^2 \, \mathrm{d}s &= \sum_{n \in \mathbb{N}} \int_{-n}^{-(n-1)} e^{\lambda s} \|\sigma(s)\|^2 \, \mathrm{d}s \\ &\leqslant \sum_{n \in \mathbb{N}} e^{-\lambda(n-1)} \int_{-n}^{-(n-1)} \|\sigma(s)\|^2 \, \mathrm{d}s \\ &\leqslant \sum_{n \in \mathbb{N}} e^{-\lambda(n-1)} \eta(\sigma) \\ &\leqslant \sum_{n \in \mathbb{N}} e^{-\lambda(n-1)} \eta(g) = \frac{\eta(g)}{1 - e^{-\lambda}}. \end{aligned}$$

□

5.6 Application to a stochastic reaction-diffusion equation

In this section we take a reaction-diffusion equation with both translation-bounded non-autonomous forcing and additive white noise as an example to illustrate our theoretical analysis.

5.6.1 NRDS generated by the reaction-diffusion equation

Suppose $\mathcal{O} \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ is a bounded domain with smooth boundary, and denote by $H = \big(L^2(\mathcal{O}), \|\cdot\|\big)$. Let $g(t,x) \in L^2_{loc}(\mathbb{R}; H)$ be a translation bounded mapping, namely, satisfying

$$\eta(g) := \sup_{\tau \in \mathbb{R}} \int_{\tau-1}^{\tau} \|g(s)\|^2 \, \mathrm{d}s < \infty. \tag{5.37}$$

Then according to disscussions in Section 5.5.2 we see that the hull $\mathcal{H}(g) := \overline{\{\theta_t g(\cdot) : t \in \mathbb{R}\}}$ of g is compact and Polish in $L^{2,w}_{loc}(\mathbb{R}; H)$, where the closure is in the local weak convergence topology sense and

$$\theta_t \sigma(\cdot) := \sigma(\cdot + t), \quad \forall t \in \mathbb{R}, \sigma \in L^2_{loc}(\mathbb{R}; H). \tag{5.38}$$

Moreover, $\{\theta_t\}_{t \in \mathbb{R}}$ forms a group of translation operators acting on $\mathcal{H}(g)$. In this section, we let $\Sigma = \mathcal{H}(g)$ be the symbol space, endowed with the local weak convergence topology.

Consider the following stochastic reaction-diffusion equation with additive scalar white noise

$$\mathrm{d}u + (\lambda u - \triangle u)\mathrm{d}t = f(x,u)\mathrm{d}t + \sigma(x,t)\mathrm{d}t + h(x)\mathrm{d}\omega, \quad x \in \mathcal{O},\ t \geqslant 0, \tag{5.39}$$

endowed with the initial and boundary conditions

$$\begin{aligned} u(x,t)|_{t=0} &= u_0(x), \\ u(x,t)|_{\partial\mathcal{O}} &= 0, \end{aligned} \tag{5.40}$$

where $\lambda > 0$ is a constant, $\sigma \in \Sigma$, ω comes from Wiener probability space $(\Omega, \mathcal{F}, \mathcal{P})$ (see Section 2.3.1), and $h(x) \in W^{2,p}(\mathcal{O})$ for some $p \geqslant 2$ is the perturbation intensity. The nonlinear term $f(x,u)$ is assumed to satisfy the following standard conditions

$$f(x,s)s \leqslant -\alpha_1|s|^p + \psi_1(x), \tag{5.41}$$

$$|f(x,s)| \leqslant \alpha_2|s|^{p-1} + \psi_2(x), \tag{5.42}$$

$$\frac{\partial f}{\partial s}(x,s) \leqslant \alpha_3, \tag{5.43}$$

$$\left|\frac{\partial f}{\partial x}(x,s)\right| \leqslant \psi_3(x), \tag{5.44}$$

where α_j are positive constants, $\psi_1 \in L^1(\mathcal{O})$ and $\psi_2, \psi_3 \in H := (L^2(\mathcal{O}), \|\cdot\|)$.

Consider

$$z(\omega) := -\lambda \int_{-\infty}^{0} e^{\lambda\tau}\omega(\tau)\,\mathrm{d}\tau, \quad \forall \omega \in \Omega. \tag{5.45}$$

By discussions in Section 2.3.1 we know that $z(\omega)$ is a stationary solution of the one-dimensional Ornstein-Uhlenbeck equation

$$\mathrm{d}z(\vartheta_t\omega) + \lambda z(\vartheta_t\omega)\mathrm{d}t = \mathrm{d}\omega.$$

Moreover, $z(\vartheta_t\omega)$ is continuous in t for every $\omega \in \Omega$ and the random variable $|z(\cdot)|$ is tempered satisfying, for each $\varepsilon > 0$,

$$\lim_{t\to\infty} e^{-\varepsilon t}|z(\vartheta_{-t}\omega)| = 0, \quad \forall \omega \in \Omega. \tag{5.46}$$

Consider the following conjugate deterministic problem with random coefficients

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} + \lambda v - \triangle v = f(x, v + hz(\vartheta_t\omega)) + \sigma(x,t) + z(\vartheta_t\omega)\triangle h(x), \\ v(x,t)|_{t=0} = v_0(x), \\ v(x,t)|_{\partial\mathcal{O}} = 0. \end{cases} \tag{5.47}$$

Following a standard method of [79, 26] we know, for each initial data $v_0 \in H$, (5.47) has a unique solution $v(\cdot,\omega,\sigma,v_0) \in C([0,\infty); H) \cap L^2_{loc}((0,\infty); V)$ with $v(0,\omega,\sigma,v_0) = v_0$. Moreover, v is $(\mathcal{F}, \mathcal{B}(H))$-measurable in ω and continuous in σ and v_0.

For each $t \geqslant 0$, $\omega \in \Omega$, $\sigma \in \Sigma$ and $u_0 \in H$, set

$$\phi(t,\omega,\sigma,u_0) = v(t,\omega,\sigma,u_0 - hz(\omega)) + hz(\vartheta_t\omega). \tag{5.48}$$

Then $\phi(t, \omega, \sigma, u_0)$ is the solution of (5.39) at time t with initial data u_0 (at time $t = 0$) satisfying Definition 3.1.1. Hence, (5.22) defines a jointly continuous NRDS on H, with base flows $\{\theta_t\}_{t\in\mathbb{R}}$ and $\{\vartheta_t\}_{t\in\mathbb{R}}$ acting on Σ and Ω, respectively.

For the equation (5.39) we study the tempered uniform and cocycle attractors. Take the universe of tempered random sets in H as the attraction universe $\mathcal{D}_H$, i.e.,

$$\mathcal{D}_H = \Big\{ D : D \text{ is a bounded random set in } H \text{ with } \lim_{t\to\infty} e^{-\lambda t}\|D(\vartheta_{-t}\omega)\|^2 = 0,\ \forall \omega \in \Omega \Big\}.$$

Clearly, the (autonomous) universe $\mathcal{D}_H$ is both inclusion-closed and neighborhood-closed.

5.6.2 Uniform estimates of solutions

Now we establish uniform estimates which are always essential in particular studies. The calculation techniques are standard in spirit. But noticing that most, if not all, recent publications on NRDS took the real line as the symbol space, we prove in details since our symbol space is the hull $\mathcal{H}(g)$.

Lemma 5.6.1. *For each $D \in \mathcal{D}_H$ and $\omega \in \Omega$, there exists a time $T = T(D, \omega) > 1$ such that, for every $\sigma \in \Sigma$,*

$$\begin{aligned} &\|v(t, \vartheta_{-t}\omega, \theta_{-t}\sigma, v_0)\|^2 + \int_0^t e^{\lambda(s-t)}\|v(s, \vartheta_{-t}\omega, \theta_{-t}\sigma, v_0)\|^2\, \mathrm{d}s \\ &+ \int_0^t e^{\lambda(s-t)}\|\nabla v(s)\|^2\, \mathrm{d}s + \int_0^t e^{\lambda(s-t)}\|v\|_p^p\, \mathrm{d}s \\ &\leqslant c\int_{-\infty}^0 e^{\lambda s}\|\sigma(s)\|^2\, \mathrm{d}s + c\int_{-\infty}^0 e^{\lambda s}|z(\vartheta_s\omega)|^p\, \mathrm{d}s + c \end{aligned}$$

holds uniformly in $v_0 \in D$ and $t \geqslant T$, where c is an absolute positive constant.

Proof. Take the inner product of (5.47) with v in H to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t, \omega, \sigma, v_0)\|^2 + \lambda\|v\|^2 + \|\nabla v\|^2 = \int v\Big(f(x, v + hz(\vartheta_t\omega)) + \sigma(t) + z(\vartheta_t\omega)\triangle h\Big)\mathrm{d}x. \quad (5.49)$$

By (5.41), (5.42) and Young's inequality we have

$$\begin{aligned} \int v f(x, v + hz(\vartheta_t\omega))\mathrm{d}x &= \int (v + hz(\vartheta_t\omega)) f(x, v + hz(\vartheta_t\omega))\mathrm{d}x - \int hz(\vartheta_t\omega) f(x, v + hz(\vartheta_t\omega))\mathrm{d}x \\ &\leqslant -\alpha_1\|v + hz(\vartheta_t\omega)\|_p^p + \int \psi_1(x)\mathrm{d}x \\ &\quad + \alpha_2\int |hz(\vartheta_t\omega)||v + hz(\omega)|^{p-1}\mathrm{d}x + \int |hz(\vartheta_t\omega)||\psi_2|\mathrm{d}x \\ &\leqslant -\frac{\alpha_1}{2}\|v\|_p^p + c(|z(\vartheta_t\omega)|^p + 1), \end{aligned} \quad (5.50)$$

where and hereafter c denotes an absolute positive constant which may change its value. As

$$\int v\big(\sigma(t) + z(\vartheta_t\omega)\triangle h\big)\mathrm{d}x \leqslant \frac{\lambda}{4}\|v\|^2 + \frac{4}{\lambda}\|\sigma\|^2 + c(|z(\vartheta_t\omega)|^p + 1), \quad (5.51)$$

by (5.49)-(5.51) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\frac{3}{2}\lambda\|v\|^2+\|\nabla v\|^2+\alpha_1\|v\|_p^p\leqslant\frac{8}{\lambda}\|\sigma\|^2+c(|z(\vartheta_t\omega)|^p+1). \tag{5.52}$$

Multiply (5.52) by $e^{\lambda t}$ and then integrate over $(0,t)$ to obtain

$$\begin{aligned}&\|v(t,\omega,\sigma,v_0)\|^2+\frac{\lambda}{2}\int_0^t e^{\lambda(s-t)}\|v(s,\omega,\sigma,v_0)\|^2\,\mathrm{d}s\\&+\int_0^t e^{\lambda(s-t)}\|\nabla v(s)\|^2\,\mathrm{d}s+\alpha_1\int_0^t e^{\lambda(s-t)}\|v\|_p^p\,\mathrm{d}s\\&\leqslant e^{-\lambda t}\|v_0\|^2+\frac{8}{\lambda}\int_0^t e^{\lambda(s-t)}\|\sigma(s)\|^2\,\mathrm{d}s+c\int_0^t e^{\lambda(s-t)}(|z(\vartheta_s\omega)|^p+1)\,\mathrm{d}s.\end{aligned} \tag{5.53}$$

Replacing ω and σ with $\vartheta_{-t}\omega$ and $\theta_{-t}\sigma$, respectively, we have

$$\begin{aligned}&\|v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2+\frac{\lambda}{2}\int_0^t e^{\lambda(s-t)}\|v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2\,\mathrm{d}s\\&+\int_0^t e^{\lambda(s-t)}\|\nabla v(s)\|^2\,\mathrm{d}s+\alpha_1\int_0^t e^{\lambda(s-t)}\|v\|_p^p\,\mathrm{d}s\\&\leqslant e^{-\lambda t}\|v_0\|^2+\frac{8}{\lambda}\int_0^t e^{\lambda(s-t)}\|\sigma(s-t)\|^2\,\mathrm{d}s+c\int_0^t e^{\lambda(s-t)}(|z(\vartheta_{s-t}\omega)|^p+1)\,\mathrm{d}s.\end{aligned}$$

Since $v_0\in D(\vartheta_{-t}\omega)$, by the tempered condition of D there exists a $T=T(\omega,D)>1$ such that

$$\begin{aligned}&\|v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2+\frac{\lambda}{2}\int_0^t e^{\lambda(s-t)}\|v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2\,\mathrm{d}s\\&+\int_0^t e^{\lambda(s-t)}\|\nabla v(s)\|^2\,\mathrm{d}s+\alpha_1\int_0^t e^{\lambda(s-t)}\|v\|_p^p\,\mathrm{d}s\\&\leqslant 1+\frac{8}{\lambda}\int_0^t e^{\lambda(s-t)}\|\sigma(s-t)\|^2\mathrm{d}s+c\int_0^t e^{\lambda(s-t)}(|z(\vartheta_{s-t}\omega)|^p+1)\,\mathrm{d}s\\&\leqslant\frac{8}{\lambda}\int_{-\infty}^0 e^{\lambda s}\|\sigma(s)\|^2\,\mathrm{d}s+c\int_{-\infty}^0 e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s+c,\quad\forall t\geqslant T,\end{aligned}$$

by which the proof is completed. □

Lemma 5.6.2. *For each $D\in\mathcal{D}_H$ and $\omega\in\Omega$, there exists a time $T>1$ given by Lemma 5.6.1 such that, for every $\sigma\in\Sigma$,*

$$\begin{aligned}&\int_{t-1}^t\|\nabla v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2\,\mathrm{d}s+\int_{t-1}^t\|v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|_p^p\,\mathrm{d}s\\&\leqslant c\int_{-\infty}^0 e^{\lambda s}\|\sigma(s)\|^2\,\mathrm{d}s+c\int_{-\infty}^0 e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s+c\end{aligned}$$

holds uniformly in $v_0\in D$ and $t\geqslant T$, where c is an absolute positive constant.

Proof. For any $t \geqslant T$, by Lemma 5.6.1,

$$\begin{aligned}
&\int_{t-1}^{t} \|\nabla v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 \,\mathrm{d}s + \int_{t-1}^{t} \|v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|_p^p \,\mathrm{d}s \\
&\quad \leqslant e^{\lambda}\int_{t-1}^{t} e^{\lambda(s-t)}\|\nabla v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 \mathrm{d}s + e^{\lambda}\int_{t-1}^{t} e^{\lambda(s-t)}\|v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|_p^p \mathrm{d}s \\
&\quad \leqslant e^{\lambda}\int_{0}^{t} e^{\lambda(s-t)}\|\nabla v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 \,\mathrm{d}s + e^{\lambda}\int_{0}^{t} e^{\lambda(s-t)}\|v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|_p^p \,\mathrm{d}s \\
&\quad \leqslant c\int_{-\infty}^{0} e^{\lambda s}\|\sigma(s)\|^2 \,\mathrm{d}s + c\int_{-\infty}^{0} e^{\lambda s}|z(\vartheta_s\omega)|^p \,\mathrm{d}s + c.
\end{aligned}$$

□

Lemma 5.6.3. *For each $D \in \mathcal{D}_H$ and $\omega \in \Omega$, there exists a time $T > 1$ (given by Lemma 5.6.1) such that, for every $\sigma \in \Sigma$,*

$$\|\nabla v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 \leqslant c\int_{-\infty}^{0} e^{\lambda s}\|\sigma(s)\|^2 \,\mathrm{d}s + c\int_{-\infty}^{0} e^{\lambda s}|z(\vartheta_s\omega)|^p \,\mathrm{d}s + c$$

holds uniformly in $v_0 \in D$ and $t \geqslant T$, where c is an absolute positive constant.

Proof. Multiply (5.47) by $-\triangle v$ and then integrate over $\mathcal{O}$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v(t,\omega,\sigma,v_0)\|^2 + \lambda\|\nabla v\|^2 + \|\triangle v\|^2 \\
&\quad = -\int \triangle v\Big(f(x,v+hz(\vartheta_t\omega)) + \sigma(t) + z(\vartheta_t\omega)\triangle h\Big)\mathrm{d}x.
\end{aligned} \tag{5.54}$$

By (5.42)-(5.44) we have

$$\begin{aligned}
&-\int \triangle v f(x,v+hz(\vartheta_t\omega))\mathrm{d}x \\
&\quad = -\int \triangle u f(x,u)\,\mathrm{d}x + \int \triangle h z(\vartheta_t\omega) f(x,u)\,\mathrm{d}x \\
&\quad = \int \Big(\nabla u\frac{\partial f}{\partial x}(x,u) + |\nabla u|^2\frac{\partial f}{\partial u}(x,u)\Big)\mathrm{d}x + \int \triangle h z(\vartheta_t\omega) f(x,u)\mathrm{d}x \\
&\quad \leqslant \|\nabla u\|\|\psi_3\| + \alpha_3\|\nabla u\|^2 + \int |\triangle h z(\vartheta_t\omega)|\Big(\alpha_2|u|^{p-1} + \psi_2\Big)\mathrm{d}x \\
&\quad \leqslant c\|\nabla v\|^2 + c\|v\|_p^p + c|z(\vartheta_t\omega)|^p + c,
\end{aligned} \tag{5.55}$$

where we have used the notation $u = v + hz(\vartheta_t\omega)$. Since

$$-\int \triangle v\big(\sigma(t) + z(\vartheta_t\omega)\triangle h\big)\mathrm{d}x \leqslant \|\triangle v\|^2 + \|\sigma\|^2 + c|z(\vartheta_t\omega)|^2, \tag{5.56}$$

from (5.54)-(5.56) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v(t,\omega,\sigma,v_0)\|^2 \leqslant c\|\nabla v\|^2 + c\|v\|_p^p + c|z(\vartheta_t\omega)|^p + 2\|\sigma\|^2 + c. \tag{5.57}$$

For each $t \geqslant T$, let $s \in (t-1,t)$. Integrate (5.57) over (s,t) to obtain

$$\begin{aligned}\|\nabla v(t,\omega,\sigma,v_0)\|^2 - \|\nabla v(s,\omega,\sigma,v_0)\|^2 \leqslant c\int_{t-1}^{t}\Big(\|\nabla v(\tau)\|^2 + \|v(\tau)\|_p^p\Big)\mathrm{d}\tau \\ + c\int_{t-1}^{t}\Big(|z(\vartheta_\tau\omega)|^p + \|\sigma(\tau)\|^2\Big)\mathrm{d}\tau + c.\end{aligned} \tag{5.58}$$

Then integrating (5.58) with respect to s over $(t-1,t)$ and replacing ω and σ by $\vartheta_{-t}\omega$ and $\theta_{-t}\sigma$, respectively, we have

$$\begin{aligned}\|\nabla v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 \leqslant c\int_{t-1}^{t}\|\nabla v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2\,\mathrm{d}s + c\int_{t-1}^{t}\|v(s,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|_p^p\,\mathrm{d}s \\ + c\int_{-1}^{0}\Big(|z(\vartheta_\tau\omega)|^p + \|\sigma(\tau)\|^2\Big)\mathrm{d}\tau + c.\end{aligned}$$

Hence, by Lemma 5.6.2 we conclude that

$$\|\nabla v(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,v_0)\|^2 \leqslant c\int_{-\infty}^{0} e^{\lambda s}\|\sigma(s)\|^2\,\mathrm{d}s + c\int_{-\infty}^{0} e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s + c,$$

where we have used the relation $\int_{-1}^{0}\|\sigma(s)\|^2\mathrm{d}s \leqslant e^{\lambda}\int_{-1}^{0}e^{\lambda s}\|\sigma(s)\|^2\mathrm{d}s$. □

By (5.22) and Lemma 5.6.3 we have the following uniform estimate for solutions of (5.39).

Corollary 5.6.4. *For each $D \in \mathcal{D}_H$ and $\omega \in \Omega$, there exist a time $T = T(D,\omega) > 1$ and an absolute positive constant L such that, for every $\sigma \in \Sigma$,*

$$\|\nabla\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma,u_0)\|^2 \leqslant L\int_{-\infty}^{0} e^{\lambda s}\|\sigma(s)\|^2\,\mathrm{d}s + L\int_{-\infty}^{0} e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s + L|z(\omega)| + L$$

holds uniformly in $u_0 \in D$ and $t \geqslant T$.

5.6.3 Cocycle and uniform attractors

For each $\omega \in \Omega$ and $\sigma \in \Sigma$, let us define $E = \{E(\omega)\}_{\omega\in\Omega}$ by

$$E(\omega) = \left\{u \in V : \|\nabla u\|^2 \leqslant \frac{L\eta(g)}{1-e^{-\lambda}} + L\int_{-\infty}^{0} e^{\lambda s}|z(\vartheta_s\omega)|^p\,\mathrm{d}s + L|z(\omega)| + L\right\}, \tag{5.59}$$

where L and $\eta(g)$ are positive constants determined by Corollary 5.6.4 and g (5.37), respectively, and $|z(\cdot)|$ is the tempered random variable given by (5.45). Then by Sobolev compactness embeddings, E

is a compact random set in H belonging to $\mathcal{D}_H$, and, furthermore, Corollary 5.6.4 and (5.36) indicate that it is a uniformly $\mathcal{D}_H$-pullback absorbing set for the NRDS ϕ.

Now we show the existence of $\mathcal{D}_H$-cocycle and $\mathcal{D}_H$-uniform attractors, and also more properties discussed in theoretical parts. First, by results in Chapter 4 we have the following result on $\mathcal{D}_H$-cocycle attractor.

Theorem 5.6.5. *The NRDS ϕ generated by the stochastic reaction-diffusion equation (5.39) has a unique $\mathcal{D}_H$-random cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma\in\Sigma}$ given by*

$$A_\sigma(\omega) = \mathcal{W}(\omega, \sigma, E), \quad \forall \sigma \in \Sigma,\ \omega \in \Omega, \tag{5.60}$$

where E is the random set defined by (5.59). Moreover, the cocycle attractor A has the following properties:

(I) it is upper semi-continuous in symbols, i.e., for each $\omega \in \Omega$,

$$\mathrm{dist}_H(A_\sigma(\omega), A_{\sigma_0}(\omega)) \to 0, \quad \textit{whenever } \sigma \to \sigma_0 \textit{ in } \Sigma; \tag{5.61}$$

(II) it is uniformly compact, i.e., for each $\omega \in \Omega$, the set $\cup_{\sigma\in\Sigma} A_\sigma(\omega)$ is compact in H;

(III) it is characterized by $\mathcal{D}_H$-complete trajectories of ϕ, i.e.,

$$A_\sigma(\omega) = \Big\{\xi(\omega, 0) : \xi \textit{ is a } \sigma\textit{-driven } \mathcal{D}_H\textit{-complete trajectory of } \phi\Big\},\ \forall \sigma \in \Sigma, \omega \in \Omega.$$

Proof. As $E \in \mathcal{D}_H$ is a compact uniformly $\mathcal{D}_H$-pullback absorbing set for the NRDS ϕ, Lemma 5.3.7 indicates that the NRDS ϕ has a unique $\mathcal{D}_H$-random cocycle attractor A with characterizations (5.60). Since $\cup_{\sigma\in\Sigma} A_\sigma(\omega) \subset E(\omega)$ and $E(\omega)$ is compact in H, A is upper semi-continuous and $\cup_{\sigma\in\Sigma} A_\sigma(\omega)$ is compact in H by Lemma 5.3.8. Property (III) follows from Lemma 5.3.10. The proof is complete. □

Now, thanks to Theorem 5.2.5, Theorem 5.3.13, Proposition 5.1.7 and Proposition 5.4.11, we are able to strengthen Theorem 5.6.5 to the following result.

Theorem 5.6.6. *The NRDS ϕ generated by the stochastic reaction-diffusion equation (5.39) has a $\mathcal{D}_H$-uniform attractor $\mathscr{A} \in \mathcal{D}_H$ and a $\mathcal{D}_H$-cocycle attractor A satisfying*

$$\begin{aligned}\mathscr{A}(\omega) &= \mathcal{W}(\omega, \Sigma, E) \\ &= \bigcup_{\sigma\in\Sigma} A_\sigma(\omega) \\ &= \Big\{\xi(\omega, 0) : \xi \textit{ is a } \mathcal{D}_H\textit{-complete trajectory of } \phi\Big\},\ \forall \omega \in \Omega,\end{aligned} \tag{5.62}$$

where E is the random set defined by (5.59). Moreover, the uniform attractor $\mathscr{A}$ is forward-attracting in probability and is determined by uniformly attracting nonrandom compact sets in H, and the cocycle attractor A is upper semi-continuous in symbols satisfying (5.61).

Remark 5.6.7. (i) Observe that (5.62) indicates that the mapping $\omega \to \cup_{\sigma\in\Sigma} A_\sigma(\omega)$ is a compact random set as it is in fact the uniform attractor, which improves Theorem 5.6.5 (II) where the measurability is not proved.

(ii) For E given by (5.59), from (5.60) and (5.62) it follows that

$$\mathcal{W}(\omega, \Sigma, E) = \bigcup_{\sigma\in\Sigma} \mathcal{W}(\omega, \sigma, E).$$

But note that for a general random set we only have the $\supset$ inclusion, see (5.12).

Appendix A

Some useful lemmas from functional analysis

In this chapter, we recall and prove some useful results in functional analysis.

Lemma A.1.1. *(Young's inequality.) Let* $a,\ b > 0$*. Then for every* p, q *satisfying* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$*, it holds*

$$a \cdot b \leqslant \varepsilon \frac{a^p}{p} + \varepsilon^{-\frac{q}{p}} \frac{b^q}{q} \quad \text{for all } \varepsilon > 0.$$

Lemma A.1.2. *(Gagliardo-Nirenberg's inequality.) Let* $u \in L^q(\mathbb{R}^n),\ D^m u \in L^r(\mathbb{R}^n)$, $1 \leqslant q, r \leqslant \infty$*. Then for* $0 \leqslant j < m$, $\frac{j}{m} \leqslant a < 1$*, there exists a constant* C *such that*

$$\|D^j u\|_p \leqslant C \|D^m u\|_r^a \cdot \|u\|_q^{1-a},$$

where $\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}$.

Lemma A.1.3 (Ehrling's Lemma)**.** *[75, Lemma 8.2] Suppose that* X, H *and* Y *are three Banach spaces with* $X \subset\subset H \subset Y$*. Then for each* $\eta > 0$ *there exists a constant* $c_\eta > 0$ *such that*

$$\|u\|_H \leqslant \eta \|u\|_X + c_\eta \|u\|_Y, \quad \forall u \in X.$$

Lemma A.1.4. *Suppose* X *is a uniformly convex Banach space (particularly, a Hilbert space) and* $\{x_n\}_{n \in \mathbb{N}}$ *is a sequence in* X*. If* $x_n \rightharpoonup x_0$ *and* $\|x_n\| \to \|x_0\|$*, then* $x_n \to x_0$.

Lemma A.1.5. *Suppose* $\mathcal{O}$ *is an open subset of* $\mathbb{R}^N$ *with* $N \in \mathbb{N}$*. Given* $p, q \geqslant 1$*, set*

$$Z := L^p(\mathcal{O}) \cap L^q(\mathcal{O}),$$

with $\|\cdot\|_Z = \|\cdot\|_p + \|\cdot\|_q$*. If* $\{g_k\}_{k \in \mathbb{N}}$ *is a sequence in* Z *such that* $\sum_{k=1}^{\infty} \|g_k\|_Z < \infty$*, then there exists a function* $f \in Z$ *such that*

$$\sum_{k=1}^{\infty} g_k = f,$$

where the sum converges pointwise a.e. and in Z.

Proof. Define functions $h_n, h : Z \to \mathbb{R}^+$ by

$$h_n = \sum_{k=1}^{n} |g_k|, \quad h = \sum_{k=1}^{\infty} |g_k|.$$

Then $\{h_n\}$ is an increasing sequence of functions that converges pointwise to h, so the monotone convergence theorem implies that

$$\int (h^p + h^q)\, \mathrm{d}x = \lim_{n\to\infty} \int (h_n^p + h_n^q)\, \mathrm{d}x.$$

By Minkowski's inequality, for each $n \in \mathbb{N}$ we have

$$\|h_n\|_Z \leqslant \sum_{k=1}^{n} \|g_k\|_Z \leqslant M := \sum_{k=1}^{\infty} \|g_k\|_Z.$$

It follows that $h \in Z$ with $\|h\|_Z \leqslant M$, and in particular that h is finite pointwise a.e. Moreover, the sum $\sum_{k=1}^{\infty} g_k$ is absolutely convergent pointwise a.e., so it converges pointwise a.e. to a function $f \in Z$ with $|f| \leqslant h$. Since

$$\begin{aligned} \left| f - \sum_{k=1}^{n} g_k \right|^p + \left| f - \sum_{k=1}^{n} g_k \right|^q &\leqslant \left(|f| + \sum_{k=1}^{n} |g_k| \right)^p + \left(|f| + \sum_{k=1}^{n} |g_k| \right)^q \\ &\leqslant (2h)^p + (2h)^q \in L^1(\mathcal{O}), \end{aligned}$$

the dominated convergence theorem implies that

$$\int \left(\left| f - \sum_{k=1}^{n} g_k \right|^p + \left| f - \sum_{k=1}^{n} g_k \right|^q \right) \mathrm{d}x \to 0, \quad \text{as } n \to \infty,$$

meaning that $\sum_{k=1}^{\infty} g_k$ converges to f in Z. □

The main idea of the following lemma is similar to Riesz-Fischer theorem.

Lemma A.1.6. *The space $Z := L^p(\mathcal{O}) \cap L^q(\mathcal{O})$, defined in Lemma A.1.5 with $p, q \geqslant 1$, is a Banach space densely embedded into $L^p(\mathcal{O})$.*

Proof. To see Z is a Banach space, we need to prove that Z is complete. If $\{f_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in Z, then we can choose a subsequence $\{f_{k_j}\}_{j\in\mathbb{N}}$ such that

$$\|f_{k_{j+1}} - f_{k_j}\|_Z \leqslant \frac{1}{2^j}.$$

Writing $g_j = f_{k_{j+1}} - f_{k_j}$, we have $\sum_{j=1}^{\infty} \|g_j\|_Z < \infty$, and then by Lemma A.1.5, the sum $f_{k_1} + \sum_{j=1}^{\infty} g_j$ converges pointwise a.e. and in Z to a function $f \in Z$. Hence, the limit

$$\lim_{j\to\infty} f_{k_j} = f_{k_1} + \sum_{j=1}^{\infty} g_j = f$$

exists in Z. Since the original sequence is Cauchy, it follows that $f_k \to f$ in Z and thereby Z is complete as desired. The dense embedding follows from the fact that $C_c^0(\mathcal{O})$, the space of continuous functions with compact support, is dense in both $L^p(\mathcal{O})$ and $L^q(\mathcal{O})$. □

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