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Aging in the one-dimensional Ising model with Glauber dynamics

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Abstract. – We study the zero-temperature limit of the one-dimensional Ising model with nearest-neighbor interactions and Glauber dynamics. An exact evolution equation is derived for the spin-spin two-time correlation functions following an instantaneous quench from equilibrium at low temperature. In the limit of long waiting times the correlations become independent of the distance and reduce to the autocorrelation function, which exhibits aging, *i.e.* it decays over a time which scale with the waiting time.

The dynamics of glasses exhibits a very rich phenomenology characterized by slow relaxation and aging effects. The term aging is used in this context to indicate that the relaxation of the system depends on its history. More precisely, it refers to the property that two-time correlation functions $C(t,t')$ are not invariant under time translation even in the limit of large age t' of the system, *i.e.* they depend both on the time interval $\tau = t - t'$ and t'. The time t' is usually referred to as the waiting time and denoted by t_ω. Experimentally, aging phenomena have been extensively observed in the context of spin glasses below the transition temperature [1]. Also, most of the phenomenological theories [2], [3] and simple models [4]-[9] proposed trying to understand the physical origin of aging deal with spin glasses, although very recently the phenomenon has also been studied in non-disordered systems [10], [11]. For some mean-field models, analytical expressions for the correlation functions showing the presence of aging have been derived [4], [11]. This is also the case for some simple domain-growth problems reviewed in ref. [12].

The purpose of this paper is to study the existence of aging in the one-dimensional Ising model with nearest-neighbor interactions and Glauber dynamics [13]. This is a model with short-range interactions and, nevertheless, simple enough to allow an exact analytical treatment in many situations. In particular, we will see that it is possible to obtain the asymptotic behavior which is relevant to identify aging effects, providing information about the role played by spatial correlations in aging in simple lattice systems. Besides, it has already been shown that the system presents many of the characteristic features of glassy dynamics, both when relaxing at constant temperature and when submitted to thermal cycles [14]. Finally, let us mention that the dynamics of some more complex systems is related to that of Ising models. For instance, it has been proved that a one-dimensional chain of particles with anharmonic and competing interactions reduces to the one-dimensional Ising model with Glauber dynamics in the low-temperature region under certain conditions [15]. In summary, the Ising model provides the simplest system to study glassy relaxation beyond the mean-field approximation.

Let us mention that Koper and Hilhorst [16] have studied an Ising chain with randomly temperature-dependent couplings, from which the Ising model is obtained by an appropriate transformation of the spin variables. At low temperatures they found a time regime where the behaviour of the system was identified by the authors as showing aging affects. Nevertheless, not real aging effects are present in the Ising model an any temperature $T > 0$. For long waiting times, one-time quantities reach their equilibrium values and two-time correlations reduce to equilibrium correlations, where time translation invariance holds.

Since the one-dimensional Ising model does not present any transition except in the limit of zero temperature, we will focus our aging analysis on the behavior of time correlations at that temperature. A similar situation is found in some of the mean-field models which have been studied [11]. At $T = 0$ the all-up and all-down spin states are absorbing states. Nevertheless, the system does not present *true* ergodicity breaking, since both absorbing states are accessible from any initial configuration. It must be noticed that aging is associated to *weak* ergodicity breaking (WEB), *i.e.* to the property that the system needs an infinite time to explore the accessible region of phase space [3]. Nevertheless, it is important to note that it is not necessary to have activated mechanisms as the one used in [3] to have WEB [8], [17].

To get explicit expressions for the two-time correlation functions, we have to specify the initial state of the system. We will consider that the system is in equilibrium at a low temperature when it is instantaneously quenched to $T = 0$ at $t_\omega = 0$. It is then possible to introduce a continuum space description for both the time-dependent spatial correlations and the two-time correlations. The former are needed as initial conditions for solving the equation for the latter. Both set of functions obey, over a slow time scale, pure diffusion equations, which can be solved analytically. In the limit of large age of the system, the spatial dependence of the two-time spin-spin correlation functions disappears, so that all of them collapse into the spin autocorrelation function. Furthermore, they depend on time only through the ratio τ/t_ω and aging effects show up.

The energy of the one-dimensional Ising model is given by

$$\mathcal{H}(\sigma) = -J \sum_i \sigma_i \sigma_{i+1}, \tag{1}$$

where $\sigma = \{\sigma_i\}$, $\sigma_i = \pm 1$ is the spin at site i, and J is a positive coupling constant. The time evolution of the system is assumed to be described by a master equation with Glauber's dynamics [13]. The conditional probability $p_{1|1}(\sigma, t|\sigma', t')$ of finding the system in state σ at time t, given it was in state σ' at time $t' \leq t$, obeys the equation

$$\frac{\partial}{\partial t} p_{1|1}(\sigma, t|\sigma', t') = \sum_{i=-\infty}^{\infty} \left[W_i(R_i\sigma) p_{1|1}(R_i\sigma, t|\sigma', t') - W_i(\sigma) p_{1|1}(\sigma, t|\sigma', t') \right]. \tag{2}$$

Here $R_i\sigma$ denotes the configuration obtained from σ by flipping the i-th spin, and $W_i(\sigma)$ is the transition rate for that flip,

$$W_i(\sigma) = \frac{\alpha}{2}\left[1 - \frac{\gamma}{2}\sigma_i(\sigma_{i-1} + \sigma_{i+1})\right], \tag{3}$$

where α is a positive constant defining the natural time scale of the evolution, and $\gamma = \tanh(2J/k_{\rm B}T)$, $k_{\rm B}$ being the Boltzmann constant. Equation (2) has to be solved with the initial condition $p_{1|1}(\sigma, t'|\sigma', t') = \delta_{\sigma,\sigma'}$. In the following we will be interested in the time evolution of the system at $T = 0$, for which eq. (3) becomes

$$W_i^{(0)} = \frac{1}{2}\left[1 - \frac{1}{2}\sigma_i(\sigma_{i-1} + \sigma_{i+1})\right]. \tag{4}$$

We have defined the unit of time by $\alpha = 1$. Then, the transition rates do not vanish in the limit $T \to 0$, indicating that the system can evolve in this zero-temperature limit. On the other hand, $W_i^{(0)}$ vanishes for those spins i which are parallel to both of their nearest neighbours, *i.e.* transitions leading to an increase of the energy are forbidden.

The quantities we will focus on are the spin-spin two-time correlation functions $A_{i,j}(t, t')$ defined by

$$A_{i,j}(t, t') \equiv \sum_{\sigma}\sum_{\sigma'} \sigma_i\sigma_j p_{1|1}(\sigma, t|\sigma', t')p(\sigma', t'), \tag{5}$$

for $t \geq t'$. In the above expression we have introduced the one-time probability distribution $p(\sigma, t)$, which also obeys eq. (2), but the initial condition $p(\sigma, 0)$ must now be specified. For $t = t'$, eq. (5) reduces to

$$A_{i,j}(t, t) = \sum \sigma_i\sigma_j p(\sigma, t) \equiv B_{i,j}(t), \tag{6}$$

where the last equality defines the spatial correlations $B_{i,j}(t)$.

At any temperature $T \neq 0$, the distribution $p(\sigma, t)$ tends to the equilibrium form in the limit of large t, and the correlations $A_{i,j}(t, t')$ depend on time only through the difference $\tau = t - t'$ for $t' \to \infty$. The age of the system is given by the waiting time $t_\omega \equiv t'$ passed before starting to measure the two-time correlations. A hierarchy of equations for $A_{i,j}(\tau|t_\omega) \equiv A_{i,j}(t_\omega + \tau, t_\omega)$ at $T = 0$ is obtained from the master equation,

$$\frac{\partial}{\partial \tau}A_{i,j}(\tau|t_\omega) = -A_{i,j}(\tau|t_\omega) + \frac{1}{2}A_{i-1,j}(\tau|t_\omega) + \frac{1}{2}A_{i+1,j}(\tau|t_\omega). \tag{7}$$

To solve this equation we need the initial condition $A_{i,j}(0|t_\omega) = B_{i,j}(t_\omega)$. Again, a hierarchy of equations for these correlations is obtained by taking moments in the master equation. For $i \neq j$ one gets

$$\frac{\partial}{\partial t_\omega}B_{i,j}(t_\omega) = -2B_{i,j} + \frac{1}{2}(B_{i-1,j} + B_{i+1,j} + B_{i,j-1} + B_{i,j+1}). \tag{8}$$

Of course, $B_{i,j}$ must verify the boundary condition $B_{i,i}(t_\omega) = 1$ for all i and t_ω, while the initial condition will follow from the initial distribution $p(\sigma, 0)$. As discussed above, we suppose that the system is in equilibrium at a temperature T before being instantaneously quenched to $T = 0$ at $t_\omega = 0$. Therefore, the initial spatial correlations are those of equilibrium at a temperature T, namely

$$B_{i,j}(0) = \eta^{|i-j|}, \tag{9}$$

where $\eta = \tanh(J/k_{\rm B}T) \leq 1$. From the form of eqs. (7)-(8) and the above initial condition it follows that both $B_{i,j}$ and $A_{i,j}$ depend on i and j only through the distance $|i - j|$. Then, it is convenient to define $f_n(\tau|t_\omega) = A_{j+n,j}(\tau|t_\omega)$ and $g_n(t_\omega) = B_{j+n,j}(t_\omega)$. Now, we will use that T lies in the low-temperature region, defined by the condition $L \gg 1$, where $L^{-1} \equiv -\ln\eta \geq 0$ is the equilibrium correlation length. In this case, $g_n(0) = \exp[-|n|/L]$ is a very smooth function of n and it is useful to introduce a scaled length x as

$$x = \frac{n}{L}. \tag{10}$$

Then, expansion of eq. (8) neglecting terms of order L^{-4} yields

$$\frac{\partial}{\partial s} g(x,s) = \frac{\partial^2}{\partial x^2} g(x,s), \tag{11}$$

where s is a slow time scale defined by $s = L^{-2} t_\omega$ and $g(x,s) = g_n(t_\omega)$. The initial and boundary conditions for eq. (11) are

$$g(x,0) = e^{-|x|}, \;\; g(0,s) = 1. \tag{12}$$

The above continuum space limit for the equations of the spatial correlations implies a similar limit for the two-time correlation functions. For large L it is found that

$$\frac{\partial}{\partial \zeta} f(x,\zeta|s) = \frac{1}{2} \frac{\partial^2}{\partial x^2} f(x,\zeta|s), \tag{13}$$

with $\zeta = L^{-2}\tau$ and $f(x,\zeta|s) = f_n(\tau|t_\omega)$. The initial condition for this equation is

$$f(x,0|s) = g(x,s). \tag{14}$$

A continuous space equation for the equilibrium spin-spin time correlation function in the one-dimensional Ising model with Glauber dynamics at low temperatures has been previously derived [18]. The main difference between eq. (13) and the one obtained in ref. [18] is that the latter contains, in addition to the diffusion term, a purely relaxational contribution. This contribution does not appear here since we are considering the time evolution of the correlations at $T = 0$, and the elementary processes which are responsible for the relaxational term have zero probability at this temperature.

The solution of eq. (11) can be easily found for any of the standard procedures and it reads

$$g(x,s) = 1 - \frac{2}{\pi} \int_0^\infty \mathrm{d}k \, \frac{\sin k|x|}{k(1+k^2)} \exp\left[-sk^2\right]. \tag{15}$$

Next, eq. (13) is solved with this initial condition. Again, it is straightforward to obtain

$$f(x,\zeta|s) = 1 - \left(\frac{2}{\zeta\pi^3}\right)^{1/2} \int_{-\infty}^\infty \mathrm{d}y \int_0^\infty \mathrm{d}k \, \frac{\sin k|x|}{k(1+k^2)} \exp\left[-sk^2 - \frac{(x-y)^2}{2\zeta}\right]. \tag{16}$$

This is an exact equation for the spin-spin two-time correlation function at $T = 0$ following an instantaneous quench from equilibrium at a very low temperature. In order to identify aging effects, we take the limit of large waiting times, *i.e.* we formally consider $s \to \infty$. At the same time, we keep $z \equiv \zeta/s$ finite. In this limit, eq. (16) leads to

$$f(x,\zeta|s) \to \phi(z) = 1 - \left(\frac{2}{\pi^3}\right)^{1/2} \int_\infty^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v \, \frac{\sin v|u|}{v} \exp\left[-\frac{v^2}{z} - \frac{u^2}{2}\right]. \tag{17}$$

The integrals in this expression can be evaluated [19], yielding

$$\phi(z) = \frac{2}{\pi} \arcsin\left(\frac{2}{2+z}\right)^{1/2}. \tag{18}$$

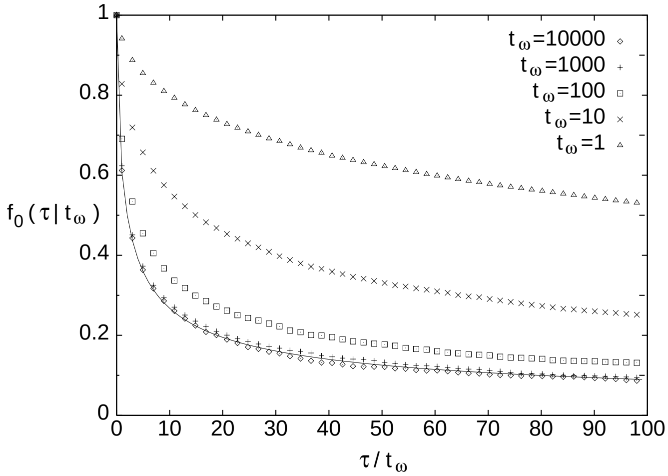

Fig. 1. – Spin time autocorrelation function $f_0(\tau|t_\omega)$ as a function of τ/t_ω for different values of the waiting time t_ω at zero temperature. The solid line is the asymptotic result in the infinite waiting time limit, eq. (18).

Therefore, for large t_ω the correlation function goes, after a short transient time, as $\phi(z = \tau/t_\omega)$, *i.e.* its decay rate is inversely proportional to t_ω. This is the signature of aging effects, as discussed at the beginning of this paper. Similar scaling behaviours in the large t_ω limit have been obtained in some mean-field models [3], [5], although some significant deviations have also been observed [4], [7]. Moreover, the fact that the right-hand side of eq. (18) does not depend on x indicates that all the spin-spin two-time correlations have collapsed onto the spin autocorrelation function. That means that spatial correlations play no role at all in the observed aging effects. Besides, as a function of z, $\phi(z)$ presents an algebraic long tail, namely,

$$\phi(z) = \frac{2^{3/2}}{\pi} z^{-1/2}, \tag{19}$$

for $z \gg 1$, which indicates the presence of weak long-term memory [4], [20] in the system.

As a test of the above results, we have carried out Monte Carlo simulation of the Ising model at $T = 0$, using the Bortz, Kalos, and Lebowitz algorithm [21]. In fig. 1 the spin time autocorrelation function $f_0(\tau|t_\omega)$ is plotted as a function of τ/t_ω for different values of the waiting time, $t_\omega = 10^n$ with $0 \leq n \leq 4$. The number of spins in the simulation was $N = 5000$ and the shown curves have been averaged over 1000 runs. The temperature at which the system was equilibrated before doing the quench to $T = 0$ is $k_B T/J = 2/3$, which corresponds to a correlation length $L \simeq 10$. Therefore, the curves correspond to scaled waiting times s between 10^{-2} and 10^2. Also plotted is the theoretical prediction given by eq. (18). It is seen that the agreement is already very good for $t_\omega = 1000$, *i.e.* $s = 10$.

A simple asymptotic analysis shows that in the limit of z small, eq. (18) can be accurately approximated by the stretched exponential

$$\phi(z) \simeq \exp\left[-\frac{\sqrt{2}}{\pi} z^{1/2}\right]. \tag{20}$$

This expression is characteristic of slow relaxation. Although the description of the relaxation provided by eqs. (19) and (20) reminds the two-regime picture proposed by Bouchaud [3] in the context of disordered systems, the relationship between the parameters describing the initial and final part of the relaxation is different in both models. Also, it is important to realize the range of validity of eq. (20) as derived here. It holds in the limits $t_\omega \to \infty$, $\tau \to \infty$, $\tau/t_\omega \to 0$.

To summarize, the relaxation of the one-dimensional Ising model with Glauber dynamics at $T = 0$ has been exactly solved. Relaxation takes place through purely diffusive processes which are responsible for slow relaxation and aging. Finally, let us stress that the Ising model as formulated here does not have energy barriers separating the states (which could be introduced in the factor α appearing in eq. (3) [14]). Therefore, glassy behaviour is due to the entropic contribution to the free-energy barriers. From this point of view, the Ising model is similar to the mean-field model introduced by Ritort [11].

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