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Angular distribution in two-particle emission induced by neutrinos and electrons

I. Ruiz Simo,[1] C. Albertus,[1] J. E. Amaro,[1] M. B. Barbaro,[2] J. A. Caballero,[3] and T. W. Donnelly[4]

[1]*Departamento de Física Atómica, Molecular y Nuclear, and Instituto de Física Teórica y Computacional Carlos I, Universidad de Granada, Granada 18071, Spain*

[2]*Dipartimento di Fisica, Università di Torino and INFN, Sezione di Torino, Via P. Giuria 1, 10125 Torino, Italy*

[3]*Departamento de Física Atómica, Molecular y Nuclear, Universidad de Sevilla, Apartado 1065, 41080 Sevilla, Spain*

[4]*Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

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The angular distribution of the phase space arising in two-particle emission reactions induced by electrons and neutrinos is computed in the laboratory (Lab) system by boosting the isotropic distribution in the center of mass (CM) system used in Monte Carlo generators. The Lab distribution has a singularity for some angular values, coming from the Jacobian of the angular transformation between CM and Lab systems. We recover the formula we obtained in a previous calculation for the Lab angular distribution. This is in accordance with the Monte Carlo method used to generate two-particle events for neutrino scattering [J. T. Sobczyk, Phys. Rev. C 86, 015504 (2012)]. Inversely, by performing the transformation to the CM system, it can be shown that the phase-space function, which is proportional to the two-particle-two-hole (2p-2h) hadronic tensor for a constant current operator, can be computed analytically in the frozen nucleon approximation, if Pauli blocking is absent. The results in the CM frame confirm our previous work done using an alternative approach in the Lab frame. The possibilities of using this method to compute the hadronic tensor by a boost to the CM system are analyzed.

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I. INTRODUCTION

Multinucleon emission by electroweak probes is of much interest nowadays [1–4]. Evidence of its presence in the quasielastic (QE) peak region has been emphasized in the analysis of recent neutrino and antineutrino scattering experiments [5–8]. The role of theoretical calculations is crucial for these analyses; they have first suggested the importance of multinucleon emission in quasielastic and inclusive neutrino-nucleus cross sections [9–12], including in the dynamics various nuclear effects such as meson-exchange currents (MEC) with and without Δ-isobar excitations, final-state interactions (FSI), short-range correlations (SRC), the random-phase approximation (RPA), effective interactions, etc. These ingredients lead to discrepancies between the theoretical predictions, and these need to be clarified in order to reduce the systematic uncertainties in neutrino data analyses [13–16].

The implementation of two-nucleon ejection in Monte Carlo (MC) neutrino event generators requires an algorithm to generate events of two-nucleon final states from given values of momentum and energy transfer. The standard way to proceed, followed in [17–19], is to select two nucleons from the Fermi sea, invoke energy-momentum conservation and compute the four-momentum of the final two-nucleon state (selecting two nucleon momenta in the final state). In the CM frame one assumes that the two final nucleons move back-to-back with the same given energy and opposite momentum. The emission angles are chosen assuming an isotropic distribution in the CM. Once the final momenta are given, a boost is performed to the Lab system to obtain the momenta of the two ejected nucleons in this frame; these are then further propagated in the MC cascade model.

We have recently studied the angular distribution in the Lab frame corresponding to two-particle (2p) emission in the frozen nucleon approximation [20], where the two nucleons are initially at rest. This distribution appears in the phase-space integration of the inclusive hadronic tensor in the 2p-2h channel. We found that the angular distribution has singularities coming from the Jacobian obtained by integration of the Dirac delta function of energy conservation, where a denominator appears that can be zero for some angles. This behavior is due to the fact that for a fixed pair of hole momenta $\mathbf{h}_1, \mathbf{h}_2$, and for given momentum transfer, q, and emission angle θ'_1 of the first particle, there are two solutions for the momentum of the ejected nucleon p'_1 that are compatible with energy conservation. For a given value of the energy transfer ω, these two solutions collapse into only one for the maximum allowed emission angle. For this angle there is a minimum in the 2p-2h excitation energy, E_{ex}, as a function of p'_1, and therefore the derivative that appears in the denominator of the Jacobian is zero: $dE_{ex}/dp'_1 = 0$.

In [20] we showed that the divergence of the angular distribution in the Lab system is of the type $\int_0^1 f(x)dx/\sqrt{x}$. Hence it is integrable around zero, and we gave an analytic formula for the integral around the divergence. The interest of the detailed study of the angular integral was to reduce the CPU time in the calculation of the hadronic tensor for inclusive neutrino scattering. Here a 7D integral appears that has to be computed in a reasonable time in order to use it to predict flux integrated neutrino cross sections, where one additional integration is needed.

In this paper we show that the isotropic angular distribution in the CM frame, as the one used in Monte Carlo generators [21], corresponds exactly to the angular distribution obtained by us in the Lab system after integration of the Dirac delta function of energy. Although this correspondence seems to be evident, in practice it is not so obvious because in Monte Carlo generators no integration of a delta function of energy is explicitly performed, or at least no Jacobian is present in the algorithm to select the emission angle [17]. That means that the phase-space angular distribution in the Monte Carlo codes is known except for a normalization factor. Besides it was not evident earlier why the divergence in the angular distribution appears in the Lab system from a constant distribution in the CM and how it can be handled by the Monte Carlo procedure.

Furthermore, we also show that upon performing the phase-space integral in the CM system one finds that the result is analytic if there is no Pauli blocking, and we give a simple formula for it in the frozen nucleon approximation. This integration method in the CM frame provides an alternative way to compute the hadronic tensor in neutrino and electron scattering.

The interest of the present study is directly linked to the reliability of the frozen nucleon approximation to get sensible results for intermediate to high momentum and energy transfers. This was already applied to a preliminary evaluation of the hadronic tensor in the case of the seagull current. Moreover the frozen nucleon approximation is the leading term if the current is expanded in powers of (h1, h2) around (0, 0). An integral over the emission angle remains to be performed. Under the assumption that the dependence of the elementary hadronic tensor on the emission angle is soft, one could factorize it out of the integral, evaluating it for some average angle, say $(\theta_{\rm Max} + \theta_{\rm Min})/2$, times the phase-space integral. In fact, the strong dependence of the electroweak matrix elements comes from the (q, ω) dependence of the electroweak form factor and not from the angular dependence for fixed (q, ω). The validity of these assumptions will be verified in a coming paper where the angular dependence of the elementary hadronic tensor will be studied.

In Sec. II we present a detailed study of the general formalism with explicit evaluation of the phase space and discussions on how to perform explicitly the boost between the two reference frames, Lab and CM. We introduce all of the variables required to analyze the 2p-2h problem and make contact with the frozen nucleon approximation where the calculations can be done in a straightforward way. Importantly, we show that these ideas can be incorporated into fully relativistic 2p-2h analyses of neutrino reactions. In Sec. III we summarize our basic findings and point out the main issues to be considered in future work, i.e., in any approach that attempts to take into account two-nucleon ejection effects in lepton scattering reactions.

II. FORMALISM

A. Lab frame

The starting point is the 2p-2h hadronic tensor for neutrino and electron scattering in the Lab system, given in the Fermi gas by

$$W^{\mu\nu}_{2p-2h} = \frac{V}{(2\pi)^9}\int d^3p'_1 d^3h_1 d^3h_2 \frac{m_N^4}{E_1E_2E'_1E'_2} \times r^{\mu\nu}(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{h}_1, \mathbf{h}_2)\delta(E'_1 + E'_2 - E_1 - E_2 - \omega) \times \Theta(p'_1, p'_2, h_1, h_2), \tag{1}$$

where $Q^\mu = (\omega, \mathbf{q})$ is the four-momentum transfer, m_N is the nucleon mass, and V is the volume of the system. The four-momenta of the final particles and holes are $P'_i = (E'_i, \mathbf{p}'_i)$ and $H_i = (E_i, \mathbf{h}_i)$, respectively. Momentum conservation implies $\mathbf{p'_2} = \mathbf{h_1} + \mathbf{h_2} + \mathbf{q} - \mathbf{p'_1}$. The initial Fermi gas ground state and Pauli blocking imply that $h_i < k_F$, and $p'_i > k_F$. These conditions are included in the Θ function, defined as the product of step functions

$$\Theta(p'_1, p'_2, h_1, h_2) = \theta(p'_2 - k_F)\theta(p'_1 - k_F) \times \theta(k_F - h_1)\theta(k_F - h_2). \tag{2}$$

The function $r^{\mu\nu}(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{h}_1, \mathbf{h}_2)$ is the hadronic tensor for the elementary transition of a nucleon pair with the given initial and final momenta, summed over spin and isospin [20].

We choose the $\mathbf{q}$ direction to be along the z axis. Then the above integral is reduced to 7 dimensions. First there is a global rotational symmetry over one of the azimuthal angles. We choose $\phi'_1 = 0$ and multiply by a factor 2π. Furthermore, the energy delta function enables an analytic integration over p'_1. This 7D integral has to be performed numerically [22,23]. Under some approximations [24–27] the number of dimensions can be further reduced, but this cannot be done in the fully relativistic calculation.

In a previous paper [20] we compared different methods to evaluate the above integral numerically. In particular we studied the special case of the phase-space function

$F(q,\omega)$, obtained by using a constant elementary tensor $r^{\mu\nu}=1$ (independent of the kinematics), defined, except for a factor $V/(2\pi)^9$, as

$$F(q,\omega)\equiv\int d^3p'_1 d^3h_1 d^3h_2\frac{m_N^4}{E_1E_2E'_1E'_2}\times\delta(E'_1+E'_2-E_1-E_2-\omega)\Theta(p'_1,p'_2,h_1,h_2) \tag{3}$$

with $\mathbf{p}'_2=\mathbf{h_1}+\mathbf{h_2}+\mathbf{q}-\mathbf{p}'_1$.

For fixed hole momenta, the energy of the two final particles is

$$E'=E'_1+E'_2=\sqrt{p_1'^2+m_N^2}+\sqrt{(\mathbf{p}'-\mathbf{p}'_1)^2+m_N^2}, \tag{4}$$

where

$$\mathbf{p}'=\mathbf{h}_1+\mathbf{h}_2+\mathbf{q} \tag{5}$$

is the final momentum of the pair. For fixed emission angle θ'_1, we integrate over p'_1 changing to the variable E'. By differentiation we arrive at the following Jacobian [note that the Jacobian of [12] agrees with Eq. (6)]

$$\left|\frac{dp'_1}{dE'}\right|=\left|\frac{p'_1}{E'_1}-\frac{\mathbf{p}'_2\cdot\hat{\mathbf{p}}'_1}{E'_2}\right|^{-1} \tag{6}$$

with $\hat{\mathbf{p}}'_1\equiv\mathbf{p}'_1/p'_1$. Now integration of the Dirac delta function of energy gives $E'=E_1+E_2+\omega$ and the phase-space function becomes

$$F(q,\omega)=2\pi\int d^3h_1d^3h_2d\theta'_1\sin\theta'_1\frac{m_N^4}{E_1E_2}\times\sum_{\alpha=\pm}\frac{p_1'^2}{\left|\frac{p'_1}{E'_1}-\frac{\mathbf{p}'_2\cdot\hat{\mathbf{p}}'_1}{E'_2}\right|}\frac{\Theta(p'_1,p'_2,h_1,h_2)}{E'_1E'_2}\Bigg|_{p'_1=p_1'^{(\alpha)}}, \tag{7}$$

where the sum inside the integral runs over the two solutions $p_1'^{(\pm)}$ of the energy conservation equation which is quadratic in p'_1. The explicit expressions of the two solutions are given in [20].

In this paper we are interested in the angular dependence of the integrand. We define the angular distribution function for fixed values of $(q,\omega,\mathbf{h}_1,\mathbf{h}_2)$ as

$$\begin{aligned}\Phi(\theta'_1)&=\sin\theta'_1\int p_1'^2dp'_1\delta(E_1+E_2+\omega-E'_1-E'_2)\times\Theta(p'_1,p'_2,h_1,h_2)\frac{m_N^4}{E_1E_2E'_1E'_2}\\&=\sum_{\alpha=\pm}\frac{m_N^4\sin\theta'_1p_1'^2\Theta(p'_1,p'_2,h_1,h_2)}{E_1E_2E'_1E'_2\left|\frac{p'_1}{E'_1}-\frac{\mathbf{p}'_2\cdot\hat{\mathbf{p}}'_1}{E'_2}\right|}\Bigg|_{p'_1=p_1'^{(\alpha)}}\\&\equiv\Phi_+(\theta'_1)+\Phi_-(\theta'_1),\end{aligned} \tag{8}$$

where $\Phi_\pm(\theta'_1)$ correspond to the two terms of the sum. Once more $\mathbf{p}'_2=\mathbf{h}_1+\mathbf{h}_2+\mathbf{q}-\mathbf{p}'_1$. The function $\Phi(\theta'_1)$ thus measures the distribution of final nucleons as a function of the angle θ'_1. Note that this function is computed analytically in the Lab system, given as a sum over the two solutions of the energy conservation condition. Thus there are really two distributions corresponding to the two possible energies of final particles for a given emission angle. The angular distribution is referred to the first particle. The second one is determined by energy-momentum conservation.

In [20] it was shown that the angular distribution in Eq. (8) has divergences for some angles where the denominator coming from the Jacobian is zero. Examples were given in the frozen nucleon approximation. It was also shown that the divergence is integrable, and an analytic formula was given for the integral over θ'_1 around the divergence. The integral in the remaining intervals was performed numerically.

B. Boost from the CM frame

In Monte Carlo event generators the angular distribution is obtained from an isotropic distribution in the CM frame, and then transformed back to the Lab system. Here we show that our distribution is recovered except for a normalization constant that we determine.

First we fix the kinematics of $(q,\omega,\mathbf{h}_1,\mathbf{h}_2)$. To simplify our formalism, we consider the particular case of the frozen nucleon approximation, i.e., $h_1=h_2=0$. The general case can be done similarly. The frozen nucleon approximation has the advantage that the total final momentum is equal to $\mathbf{p}'=\mathbf{q}$ and hence the CM frame moves in the z direction ("upwards"). Therefore, the x,y components are invariant under the boost from the CM to the Lab frames. In [20] it was shown that the frozen nucleon approximation gives an accurate representation of the total phase-space function, so one expects the angular distribution in the frozen nucleon approximation to be representative of the general case.

Doubly primed variables refer to the CM system. The total final momentum is

$$\mathbf{p}''=\mathbf{p}''_1+\mathbf{p}''_2=0, \tag{9}$$

and the total final energy E'' is determined by invariance of the squared four-momentum

$$E'' = \sqrt{E'^2 - p'^2}, \tag{10}$$

where $(E', \mathbf{p}') = (2m_N + \omega, \mathbf{q})$ are the final energy and momentum in the Lab frame.

In the CM frame the two final nucleons are assumed to go back-to-back with the same momentum and with the same energy

$$E_1'' = E_2'' = \frac{E''}{2} = \frac{1}{2}\sqrt{E'^2 - p'^2}. \tag{11}$$

The condition $E_1'' > m_N$ restricts the allowed (ω, q) region where the two-nucleon emission is possible.

Let θ_1'' be the emission angle corresponding to the first particle. To obtain the nucleon momentum in the Lab system we perform a boost of the four vector $(P_1'')^\mu = (E_1'', \mathbf{p}_1'')$ back to the Lab frame, that is moving downward along the z axis with dimensionless velocity v, where this is the velocity of the CM system with respect to the Lab system, given by

$$v = \frac{p'}{E'}. \tag{12}$$

The boost transformation of the $(0, z)$ four-vector components is given by a 2×2 Lorentz matrix equation

$$\begin{pmatrix} E_1' \\ p_{1z}' \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} E_1'' \\ p_{1z}'' \end{pmatrix}, \tag{13}$$

where $\gamma \equiv 1/\sqrt{1 - v^2}$. From here we get

$$E_1' = \gamma(E_1'' + v p_1'' \cos\theta_1'') \tag{14}$$

$$p_1' \cos\theta_1' = \gamma(v E_1'' + p_1'' \cos\theta_1''). \tag{15}$$

Therefore the momentum and angle in the Lab system are

$$p_1' = \sqrt{\gamma^2 (E_1'' + v p_1'' \cos\theta_1'')^2 - m_N^2} \tag{16}$$

$$\cos\theta_1' = \frac{\gamma(v E_1'' + p_1'' \cos\theta_1'')}{\sqrt{\gamma^2 (E_1'' + v p_1'' \cos\theta_1'')^2 - m_N^2}}. \tag{17}$$

In Fig. 1 we show the Lab emission angle as a function of the CM angle for momentum and energy transfers: $q = 3$ GeV/c and $\omega = 2$ GeV. We choose in this case a high value of the momentum transfer to avoid effects linked to Pauli blocking. The ω value is close to the QE peak, $\omega_{QE} = \sqrt{q^2 + m_N^2} - m_N$, and below it. As the CM angle runs from 0 to 180 degrees, for this kinematics the Lab angle starts growing, reaches a maximum and then decreases. Therefore, for a given emission angle in the Lab system, θ_1', there correspond two angles in the CM, that we denote $(\theta_1'')^+$ and $(\theta_1'')^-$. They differ in the value of the Lab momentum p_1', that is plotted in the lower panel of Fig. 1. Hence there are two different values of p_1' for a given Lab angle. These two p_1' values obviously correspond to the two solutions, $(p_1')^\pm$ of energy conservation, appearing in the sum of the phase-space function in Eqs. (7), (8). The momentum of the second nucleon, p_2', could be obtained by changing $\cos\theta_1''$ by $(-\cos\theta_1'')$ in Eq. (16). Therefore the range of values it takes is the same as p_1'.

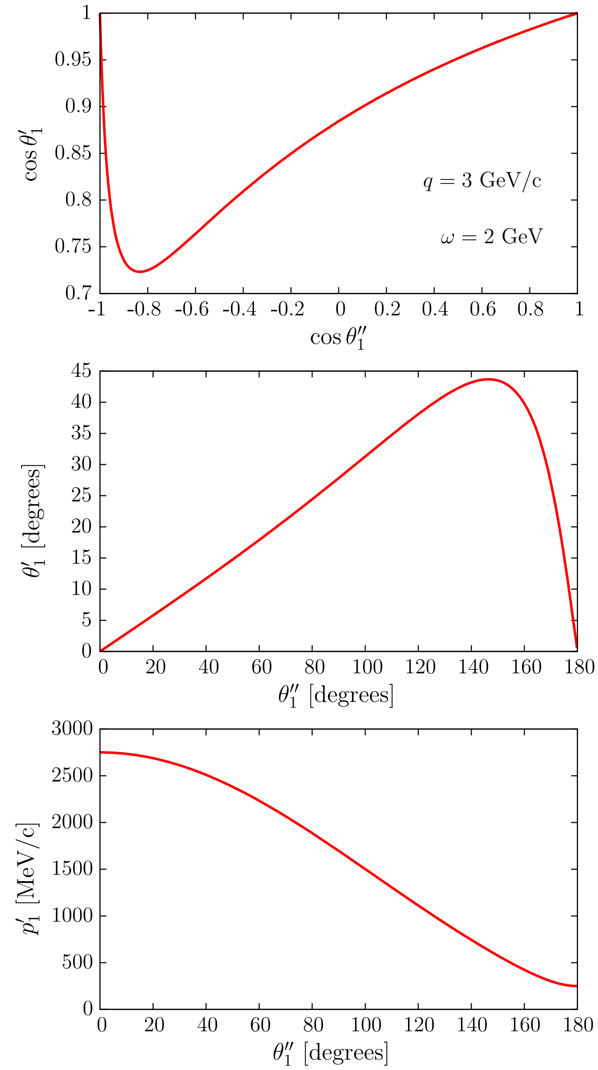

FIG. 1 (color online). Lab magnitudes as a function of CM magnitudes. The momentum and energy transfer are $q = 3$ GeV/c and $\omega = 2$ GeV. Top panel: $\cos\theta_1'$ versus $\cos\theta_1''$. Middle panel: θ_1' versus θ_1''. Bottom panel: p_1' versus θ_1''.

C. Transformation of the angular distribution

We assume that the angular distribution in the CM frame is independent of the emission angle, except for Pauli blocking restrictions,

$$n''(\theta_1'') = C\Theta(p_1', p_2', 0, 0), \tag{18}$$

where C is a constant that is determined below. The step function ensures Pauli blocking. The angular distribution in the Lab system, $n'(\theta'_1)$, is obtained by imposing conservation of the number of particles emitted within two corresponding solid angles $d\Omega'_1$ and $d\Omega''_1$, in the Lab and the CM systems

$$n'(\theta'_1)d\Omega'_1 = n''(\theta''_1)d\Omega''_1. \tag{19}$$

Since the boost conserves the azimuthal angle $d\phi''_1 = d\phi'_1$, we get the well-known transformation expression:

$$n'(\theta'_1) = \frac{C\Theta(p'_1, p'_2, 0, 0)}{|\frac{d\cos\theta'_1}{d\cos\theta''_1}|}. \tag{20}$$

The derivative in the Jacobian is computed by differentiation of Eq. (17) with respect to $\cos\theta''_1$, and can be written in the form

$$\frac{d\cos\theta'_1}{d\cos\theta''_1} = \gamma p''_1 \frac{p'_1 - vE'_1\cos\theta'_1}{(p'_1)^2} \tag{21}$$

Writing γ in the form

$$\gamma = \frac{E'}{\sqrt{E'^2 - p'^2}} = \frac{E'}{2E''_1} \tag{22}$$

we arrive at the following formula for the angular distribution in the Lab frame

$$n'(\theta'_1) = \frac{2E''_1}{E'p''_1}\frac{(p'_1)^2}{|p'_1 - vE'_1\cos\theta'_1|}C\Theta(p'_1, p'_2, 0, 0). \tag{23}$$

Note that this distribution is not unique, because, as shown in Fig. 1, there may be two different CM angles, and two different values of p'_1 corresponding to the same Lab angle θ'_1. Therefore there are two possible angular distributions, and the total distribution is given by the sum of the two,

$$n'(\theta'_1) = n'_+(\theta'_1) + n'_-(\theta'_1), \tag{24}$$

where each partial distribution $n'_\pm(\theta'_1)$ corresponds to Eq. (23) using the $(p'_1)^\pm$ values, respectively.

D. Equivalence of Lab distributions

The next step is to compare the functions $n'_\pm(\theta'_1)\sin\theta'_1$ with the angular distribution $\Phi_\pm(\theta'_1)$ computed for nucleons at rest, $h_1 = h_2 = 0$, given by Eq. (8)

$$\Phi_\pm(\theta'_1) = \sin\theta'_1 \frac{m_N^2(p'_1)^2\Theta(p'_1, p'_2, 0, 0)}{|E'_2 p'_1 - E'_1\mathbf{p}'_2\cdot\hat{\mathbf{p}}'_1|}, \tag{25}$$

where $p'_1 = (p'_1)^\pm$. Using

$$\mathbf{p}'_2\cdot\hat{\mathbf{p}}'_1 = q\cos\theta'_1 - p'_1 \tag{26}$$

the denominator in Eq. (25) can be written as

$$\begin{aligned} E'_2 p'_1 - E'_1\mathbf{p}'_2\cdot\hat{\mathbf{p}}'_1 &= E'p'_1 - E'_1 q\cos\theta'_1 \\ &= E'(p'_1 - E'_1 v\cos\theta'_1). \end{aligned} \tag{27}$$

Substituting in Eq. (25) we obtain

$$\Phi_\pm(\theta'_1) = \sin\theta'_1 \frac{m_N^2(p'_1)^2\Theta(p'_1, p'_2, 0, 0)}{E'|p'_1 - E'_1 v\cos\theta'_1|}. \tag{28}$$

Comparing with Eq. (23), it follows that

$$n'_\pm(\theta'_1)\sin\theta'_1 = \Phi_\pm(\theta'_1) \tag{29}$$

provided that

$$C = \frac{m_N^2}{2}\frac{p''_1}{E''_1}. \tag{30}$$

In Fig. 2 we show the two angular distributions $\Phi_\pm(\theta'_1)$ for $q = 3$ GeV/c and three values of ω. We can see that both distributions are zero above a maximum allowed angle in the Lab system. Both distributions present a divergence (they are infinite) at that precise maximum angle, because the derivative in the denominator of Eq. (20) is zero at that point. This is in agreement with our previous work [20] where we also demonstrated that the divergence is integrable. The results of Fig. 2 for the total distribution agree with the findings of [20]. In Fig. 2 we have not included Pauli blocking in the plots of $\Phi_\pm$, but it is included in the total distribution. We see that Pauli blocking only is effective in the last case, $\omega = 2200$ MeV, killing the divergence.

E. Integration in the CM

The method of the previous section can be reversed by making the inverse boost from Lab to CM. This allows us to perform the integral over θ'_1 in Eq. (7) using the CM emission angle, by changing variables $\theta'_1 \to \theta''_1$. Since this is the inverse transformation applied in the previous sections, the Jacobian cancels the denominator in Eq. (7).

We start by fixing $\mathbf{h}_1$ and $\mathbf{h}_2$ and define the phase-space integral over the final momenta

$$\begin{aligned} G(\mathbf{h}_1, \mathbf{h}_2, q, \omega) \equiv \int d^3p'_1 d^3p'_2 \frac{m_N^2}{E'_1 E'_2}\Theta(p'_1, p'_2, h_1, h_2) \\ \times \delta^4(H_1 + H_2 + Q - P'_1 - P'_2), \end{aligned} \tag{31}$$

such that

$$F(q, \omega) = \int d^3h_1 d^3h_2 \frac{m_N^2}{E_1 E_2} G(\mathbf{h}_1, \mathbf{h}_2, q, \omega). \tag{32}$$

We recall from special relativity that the integral measure $\int d^3p/E$ is Lorentz invariant because of the result,

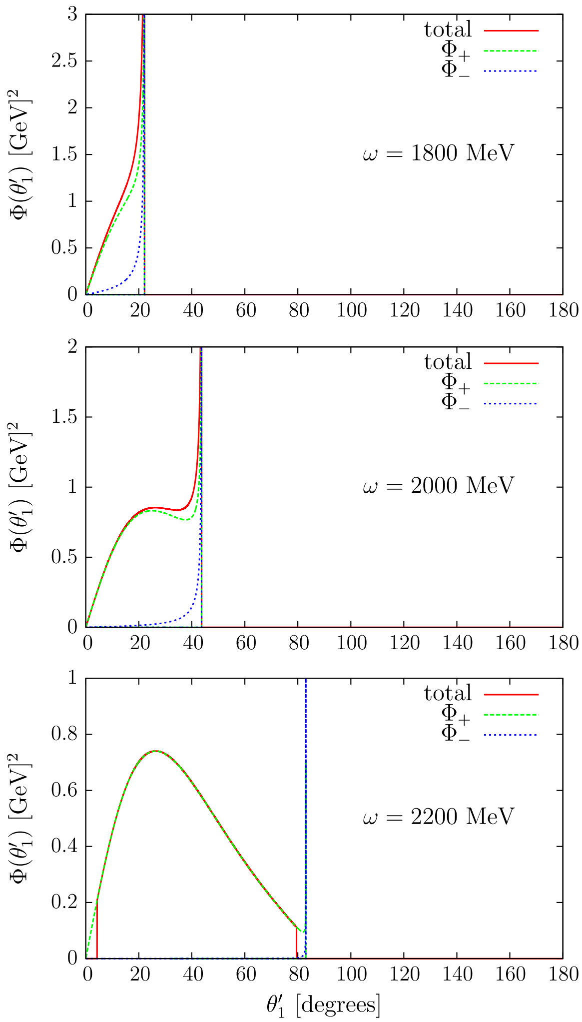

FIG. 2 (color online). The two angular distributions $\Phi_\pm$ and the total, in the Lab system, for two-nucleon emission in the frozen nucleon approximation. The momentum transfer is $q = 3$ GeV/c and three values of $\omega = 1800, 2000$ and 2200 GeV are considered.

$$\int \frac{d^3p}{2E(p)} = \int d^4p\delta(p^\mu p_\mu - m_N^2)\theta(p^0). \qquad (33)$$

Then we can write

$$G(\mathbf{h}_1, \mathbf{h}_2, q, \omega) = \int d^3p''_1 d^3p''_2 \frac{m_N^2}{E''_1 E''_2}\Theta(p'_1, p'_2, h_1, h_2) \times \delta^4(H''_1 + H''_2 + Q'' - P''_1 - P''_2), \qquad (34)$$

where the doubly primed variables refer to the momenta in the CM frame. The CM is defined by $\mathbf{p}'' = (\mathbf{h}_1 + \mathbf{h}_2 + \mathbf{q})'' = 0$. The step functions, which are not invariant, must be computed in the Lab system, i.e., the momenta inside the integral have to be transformed back to the Lab system to compute the argument of the step function. Integrating over $\mathbf{p}''_2$ we obtain

$$G(\mathbf{h}_1, \mathbf{h}_2, q, \omega) = \int d^3p''_1 \delta(E'' - E''_1 - E''_2) \times \frac{m_N^2}{E''_1 E''_2}\Theta(p'_1, p'_2, h_1, h_2) \qquad (35)$$

with $\mathbf{p}''_2 = -\mathbf{p}''_1$. Therefore, the CM energies satisfy the relationship $E''_1 = E''_2$, and we can write

$$G(\mathbf{h}_1, \mathbf{h}_2, q, \omega) = \int d^3p''_1 \delta(E'' - 2E''_1) \times \frac{m_N^2}{(E''_1)^2}\Theta(p'_1, p'_2, h_1, h_2). \qquad (36)$$

Now we change variables $p''_1 \to E''_1$, and integrate over E''_1 using $p''_1 dp''_1 = E''_1 dE''_1$,

$$G(\mathbf{h}_1, \mathbf{h}_2, q, \omega) = \frac{m_N^2}{2}\frac{p''_1}{E''_1}\int d\Omega''_1 \Theta(p'_1, p'_2, h_1, h_2). \qquad (37)$$

The remaining integral of the step function over the emission angles is in general nontrivial and has to be performed numerically. If there is no Pauli blocking, the above integral takes its maximum value:

$$G(\mathbf{h}_1, \mathbf{h}_2, q, \omega)_{n.p.b} = 4\pi \frac{m_N^2}{2}\frac{p''_1}{E''_1}. \qquad (38)$$

What remains to be performed is the integral over $\mathbf{h}_1, \mathbf{h}_2$, that in general should be evaluated numerically. However, in the frozen nucleon approximation one assumes that the integrand depends very mildly on $\mathbf{h}_1, \mathbf{h}_2$, and therefore one can employ this fact to fix the kinematics to the frozen nucleon value, $h_1 = h_2 = 0$. The phase-space integral in this case is trivial, and takes on the value

$$F(q, \omega)_{n.p.b} = 4\pi \left(\frac{4}{3}\pi k_F^3\right)^2 \frac{m_N^2}{2}\frac{p''_1}{E''_1}, \qquad (39)$$

where the ratio p''_1/E''_1 in the frozen nucleon approximation is given by

$$\frac{p''_1}{E''_1} = \sqrt{1 - \frac{4m_N^2}{(2m_N + \omega)^2 - q^2}}. \qquad (40)$$

Note that in the asymptotic limit $\omega \to \infty$, a constant value is obtained,

$$F(q, \infty) = 4\pi \left(\frac{4}{3}\pi k_F^3\right)^2 \frac{m_N^2}{2}. \qquad (41)$$

This asymptotic limit is in agreement with the one obtained in [20] by integration in the Lab system.

As an example, we show in Fig. 3 the phase-space function $F(q, \omega)$ for $q = 3$ GeV/c, computed using the

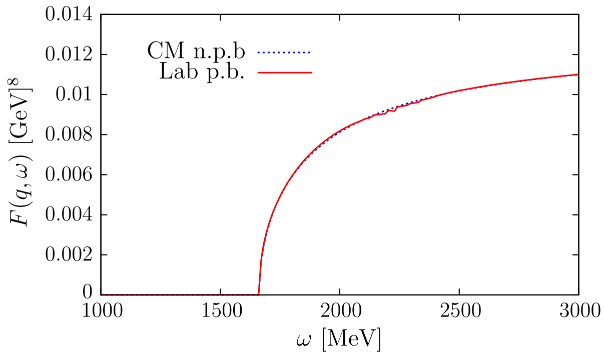

FIG. 3 (color online). Phase-space function in the frozen nucleon approximation for $q = 3$ GeV/c, computed in the CM using the analytic formula without Pauli blocking (n.p.b.), and computed numerically in the Lab system including Pauli blocking (p.b.). The value of the Fermi momentum is $k_F = 225$ MeV/c.

analytic formula without Pauli blocking, Eq. (39), and by numerical integration in the Lab frame using the method of [20] with Pauli blocking. Both results agree except in the small region around the quasielastic peak, where Pauli blocking produces the very small difference seen between the two results; there the Pauli-blocked function $F(q,\omega)$ is slightly below the analytic result.

III. CONCLUSIONS AND PERSPECTIVES

In this work we have analyzed the angular distribution of 2p-2h final states in the relativistic Fermi gas, finding the connections between the CM and Lab systems. Theoretical calculations of many-particle emission in neutrino and electron scattering usually rely on the Lab frame to be the most appropriate to perform the calculations, since the Fermi gas state description is simpler, mainly because Pauli blocking necessarily has to be checked in the Lab system where the initial nucleons are below the Fermi surface. However the description of the 2p angular distribution is simpler in the CM frame, where the angular dependence is isotropic, if no Pauli blocking is assumed.

On the contrary, the phase-space integral in the Lab system has the difficulty that the angular distribution has a singularity at the maximum allowed angle. The integration of this singularity in the Lab system was made in our previous work [20]. Here we have studied the alternative method of performing the angular integral in the CM frame, where the angular dependence is trivial. We show that such an integral can be solved analytically in the absence of Pauli blocking.

Of interest for the neutrino scattering data analysis, we have shown that the algorithms used in Monte Carlo event generators produce 2p angular distributions that are in agreement with the theoretical calculations in the Lab system if the nuclear current is disregarded.

We have considered the angular distribution coming from phase space alone. In a complete calculation one is involved with the interaction between the two nucleons and the lepton that introduces an additional angular dependence which needs to be evaluated to correctly describe the events. A proper model of 2p-2h emission requires at least the introduction of meson-exchange currents, or nuclear correlations [22,23]. Work along these lines is in progress.

Finally, the integration method proposed here could also be used to compute the 2p-2h hadronic tensor in Eq. (1) as an alternative procedure to the common Lab frame calculations. Comparisons of the two methods would be of interest because neither of them presents clear numerical advantages. Although angular integration in the CM frame allows one to avoid the divergence arising in the Lab frame, it introduces the difficulty of having to perform a different boost inside the integral for each pair of holes $(\mathbf{h}_1, \mathbf{h}_2)$.

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