Long-time behavior of a Cahn-Hilliard-Navier-Stokes vesicle-fluid interaction model

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The Model: Interaction of a vesicle membrane with a fluid field in 3D

Evolution of vesicles immersed in an incompressible Newtonian fluid. Volume and area are conserved.

System:

- Navier-Stokes eqs.+stress term membrane.
- Incompressibility eq.
- Phase-field eq. of Cahn-Hilliard type (diffuse interface)

Phase-field equation

$F(\phi) = \frac{1}{4}(\phi^2 - 1)^2$ Ginzburg-Landau potential,

$M, \alpha > 0$ constants, ε interface width,

$A(\phi) = \int_\Omega \left(\frac{\varepsilon}{2} |\nabla \phi|^2 + \frac{1}{\varepsilon} F'(\phi) \right) dx$, area surface.

Bending energy

$$E_b(\phi) = \frac{1}{2\varepsilon} \int_\Omega (-\varepsilon \Delta \phi + \frac{1}{\varepsilon} F'(\phi))^2 \, dx + \frac{1}{2} M (A(\phi) - \alpha)^2$$

Cahn-Hilliard equation

$$\partial_t \phi + \boldsymbol{u} \cdot \nabla \phi = -\Delta \frac{\delta E_b(\phi)}{\delta \phi}$$

Phase-field equation

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Cahn-Hilliard equation

$$\partial_t \phi + \boldsymbol{u} \cdot \nabla \phi = -\Delta \frac{\delta E_b(\phi)}{\delta \phi}$$

Other phase-field equation

$$B(\phi) = \int_\Omega \phi \, dx, \text{ volume.}$$

Bending energy

$$E_b(\phi) = \frac{1}{2\varepsilon}\int_\Omega (-\varepsilon\Delta\phi + \frac{1}{\varepsilon}F'(\phi))^2 \, dx + \frac{1}{2}M_1(A(\phi) - \alpha)^2 + \frac{1}{2}M_2(B(\phi) - \beta)^2$$

Allen-Cahn equation

$$\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi = -\frac{\delta E_b(\phi)}{\delta\phi}$$

Q. Du,M. Li,C. Liu'07

Phase-field equation

$$F(\phi) = \frac{1}{4}(\phi^2 - 1)^2 \text{ Ginzburg-Landau potential,}$$
$$M, \alpha > 0 \text{ constants,} \qquad \varepsilon \text{ interface width,}$$
$$A(\phi) = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla \phi|^2 + \frac{1}{\varepsilon} F'(\phi) \right) dx, \text{ area surface.}$$

Bending energy

$$E_b(\phi) = \frac{1}{2\varepsilon} \int_\Omega (-\varepsilon \Delta \phi + \frac{1}{\varepsilon} F'(\phi))^2 \, dx + \frac{1}{2} M(A(\phi) - \alpha)^2$$

Cahn-Hilliard equation

$$\partial_t \phi + \boldsymbol{u} \cdot \nabla \phi = -\Delta \frac{\delta E_b(\phi)}{\delta \phi}$$

Chemical potential:

$$w := \frac{\delta E_b(\phi)}{\delta \phi} = \varepsilon \Delta^2 \phi + G(\phi)$$

where

$$G(\phi) := -\frac{1}{\varepsilon}\Delta F'(\phi) + \frac{1}{\varepsilon^2}\left(-\varepsilon\Delta\phi + \frac{1}{\varepsilon}F'(\phi)\right) F'(\phi) + M(A(\phi) - \alpha)\left(-\varepsilon\Delta\phi + \frac{1}{\varepsilon}F'(\phi)\right).$$

Cahn-Hilliard equations

$$\partial_t \phi + \boldsymbol{u} \cdot \nabla \phi - \Delta w = 0,$$

$$\varepsilon \Delta^2 \phi + G(\phi) - w = 0.$$

The equations of the model

Changing the variables

$$\psi(x,t) := \phi(x,t) - m_0 \text{ and } z := w - \langle w \rangle \text{ with } m_0 = \langle \phi(x,0) \rangle$$

Navier-Stokes-Cahn-Hilliard system

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \Delta \boldsymbol{u} - z\nabla\psi + \nabla\widetilde{q} = 0, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \partial_t \psi + \boldsymbol{u} \cdot \nabla\psi - \Delta z = 0, \\ \varepsilon\Delta^2\psi + \overline{G}(\psi) - z = 0, \end{cases}$$

$$\boldsymbol{u}|_{\partial\Omega} = 0, \quad \partial_n\psi|_{\partial\Omega} = 0, \quad \partial_n\Delta\psi|_{\partial\Omega} = 0, \quad \partial_n z|_{\partial\Omega} = 0,$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \psi(0) = \psi_0 := \phi(x,0) - m_0 \quad \text{in } \Omega.$$

where

$$\overline{G}(\psi) := G(\psi + m_0) - \langle G(\psi + m_0) \rangle,$$

Spaces

$$\boldsymbol{H} = \{\boldsymbol{u} \in \boldsymbol{L}^2;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

$$\boldsymbol{V} = \{\boldsymbol{u} \in \boldsymbol{H}^1;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} = 0 \text{ on } \partial\Omega\}$$

$$L^2_* = \left\{ v \in L^2(\Omega);\ \int_\Omega v = 0 \right\},$$

$$H^k_* = \left\{ v \in H^k(\Omega);\ \int_\Omega v = 0 \right\} \qquad k \geq 1,$$

$$H^2_1 = \{ v \in H^2_*(\Omega);\ \partial_n v|_{\partial\Omega} = 0 \}$$

$$H^k_2 = \{ v \in H^k_*;\ \partial_n v|_{\partial\Omega} = 0,\ \partial_n \Delta v|_{\partial\Omega} = 0 \} \qquad k = 3, 4, 5, 6.$$

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Total energy

$$E_k(\boldsymbol{u}) = \frac{1}{2}\int_\Omega |\boldsymbol{u}|^2, \qquad \overline{E}_b(\psi) = E_b(\psi + m_0)$$

$$\overline{E}(\boldsymbol{u},\psi) = E_k(\boldsymbol{u}) + \overline{E}_b(\psi)$$

Energy Inequality

$$\frac{d}{dt}\overline{E}(\boldsymbol{u}(t),\psi(t)) + |\nabla \boldsymbol{u}(t)|_2^2 + |\nabla z(t)|_2^2 \leq 0.$$

Weak solution

Global weak estimates

$$\boldsymbol{u} \in L^2(0,+\infty;\boldsymbol{V}) \cap L^\infty(0,+\infty;\boldsymbol{H}), \quad z \in L^2(0,+\infty;H^1_*)$$
$$\psi \in L^\infty(0,+\infty;H^2_1).$$

Additional estimate

$$\psi \in L^2_{loc}(0,+\infty;H^5_2)$$

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Strong solution

Strong estimates for velocity for large times

$$\boldsymbol{u} \in L^\infty([T^*_{reg}, +\infty); \boldsymbol{H}^1) \cap L^2([T^*_{reg}, +\infty); \boldsymbol{H}^2),$$

$$\partial_t \boldsymbol{u} \in L^2([T^*_{reg}, +\infty); \boldsymbol{L}^2).$$

Global in time strong estimate for ψ

$$\psi \in L^\infty(0, +\infty; H^3_2).$$

Strong estimates for ψ and z

$$\psi \in L^2_{loc}(0, +\infty; H^6_2), \qquad \partial_t \psi \in L^2_{loc}(0, +\infty; L^2_*)$$

$$z \in L^2_{loc}(0, +\infty; H^2_1).$$

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Convergence at infinite time of strong solution

$$\overline{E}(\boldsymbol{u}(t),\psi(t)) \searrow E_\infty \text{ in } \mathbb{R} \quad \text{as } t\uparrow +\infty.$$

ω-limit set

Fixed a global weak solution, $(\boldsymbol{u},\psi)$, associated to the initial data, $(\boldsymbol{u}_0,\psi_0)\in \boldsymbol{V}\times H^3_2$,

$$\omega(\boldsymbol{u},\psi)=\{(\boldsymbol{u}_\infty,\psi_\infty)\in \boldsymbol{V}\times H^3_2 : \exists\{t_n\}\uparrow +\infty \text{ s.t.} \\ (\boldsymbol{u}(t_n),\psi(t_n))\to(\boldsymbol{u}_\infty,\psi_\infty) \text{ weakly in } \boldsymbol{H}^1\times H^3_2\}.$$

Set of equilibrium points

$$\mathcal{S}=\{(0,\psi):\psi\in H^4_2(\Omega):\ \varepsilon\Delta^2\psi+\overline{G}(\psi)=0 \text{ a.e in } \Omega\}.$$

Convergence at infinite time of strong solution

Assume that $(\boldsymbol{u}_0, \psi_0) \in \boldsymbol{V} \times H_2^3$, fixed $(\boldsymbol{u}(t), \psi(t), z(t))$ a weak solution in $(0, +\infty)$,

Theorem

$\omega(\boldsymbol{u}, \psi)$ is nonempty and $\omega(\boldsymbol{u}, \psi) \subset S$.
For any $(0, \overline{\psi}) \in \omega(\boldsymbol{u}, \psi) \subset S$, $\overline{E}(\overline{\psi}) = E_\infty$

Theorem

There exists $\overline{\psi} \in H_2^4$, such that $\psi(t) \to \overline{\psi}$ in H_2^3 weakly as $t \uparrow +\infty$, i.e.

$$\omega(\boldsymbol{u}, \psi) = \{(0, \overline{\psi})\}.$$

Tools of proof: BCE, F. Guillén-González'14.

- BCE, F. Guillén-González, M.A. Rodríguez-Bellido'10,
- Lemma of strong continuous dependence with respect to initial data.
- Lojasiewicz-Simon Lemma.

H. Wu, X. Xu'13: Navier-Stokes-Allen-Cahn

Lemma

Let $\Phi, B \in L^1(0, +\infty)$ be two positive functions such that $\Phi \in H^1(0, T)\ \forall T > 0$, which satisfies

$$\Phi'(t) \leq C(\Phi(t)^3 + B(t)).$$

Then, there exists a sufficiently large $T^* \geq 0$ such that $\Phi \in L^\infty(T^*, +\infty)$ and

$$\exists \lim_{t \to +\infty} \Phi(t) = 0.$$

Continuous dependence initial data

If $(\boldsymbol{u}^\varepsilon, \psi^\varepsilon, z^\varepsilon)$, for some $\varepsilon > 0$, and $(\boldsymbol{u}^0, \psi^0, z^0)$ are two regular solutions in $(0, T^*)$ of the problem; associated to the different initial conditions, $(\boldsymbol{u}_0^\varepsilon, \psi_0^\varepsilon) \in \boldsymbol{H}^1 \times H^3_2$ and $(\boldsymbol{u}_0, \psi_0) \in \boldsymbol{H}^1 \times H^3_2$, respectively, then $\boldsymbol{u}^\varepsilon - \boldsymbol{u}^0$, $\psi^\varepsilon - \psi^0$ and $z^\varepsilon - z^0$ depend continuously of the initial values in the following sense: If $\boldsymbol{u}_0^\varepsilon \to \boldsymbol{u}_0$ weakly in $\boldsymbol{H}^1$ (and strongly in $\boldsymbol{L}^2$) and $\psi_0^\varepsilon \to \psi_0$ weakly in H^3_2 (and strongly in H^2_1), then,

$$\begin{aligned} \boldsymbol{u}^\varepsilon - \boldsymbol{u}^0 \to 0 \quad & \text{in } L^\infty(0, T^*; \boldsymbol{L}^2) \cap L^2(0, T^*; \boldsymbol{H}^1), \\ \psi^\varepsilon - \psi^0 \to 0 \quad & \text{in } L^\infty(0, T^*; H^2) \cap L^2(0, T^*; H^5). \end{aligned}$$

Lojasiewicz-Simon inequality

Let $\mathcal{S}$ be the following set of equilibrium points related to the bending energy

$$\mathcal{S} = \{\psi \in H_2^4(\Omega) : \ \varepsilon \Delta^2 \psi + \overline{G}(\psi) = 0 \text{ a.e in } \Omega\}.$$

If $\overline{\psi} \in \mathcal{S}$, there are three positive constants C, α, and $\theta \in (0, 1/2)$ which depend on $\overline{\psi}$, such that for all $\psi \in H_2^4$ and $\|\psi - \overline{\psi}\|_2 \leq \beta$, then

$$|\overline{E}_b(\psi) - \overline{E}_b(\overline{\psi})|^{1-\theta} \leq C\,|z|_2$$

where $z = z(\psi) := \varepsilon \Delta^2 \psi + \overline{G}(\psi)$.

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Convergence at infinite time of weak solution

ω-limit set

Fixed a global weak solution, $(\boldsymbol{u}, \psi)$, associated to the initial data, $(\boldsymbol{u}_0, \psi_0) \in \boldsymbol{H} \times H^2_1$,

$$\omega(\boldsymbol{u}, \psi) = \{(\boldsymbol{u}_\infty, \psi_\infty) \in \boldsymbol{H} \times H^2_1 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (\boldsymbol{u}(t_n), \psi(t_n)) \to (\boldsymbol{u}_\infty, \psi_\infty) \text{ weakly in } \boldsymbol{H} \times H^2_1\}.$$

Convergence at infinite time of weak solution

Assume that $(\boldsymbol{u}_0, \psi_0) \in \boldsymbol{H} \times H^2_1$, fixed $(\boldsymbol{u}(t), \psi(t), z(t))$ a weak solution in $(0, +\infty)$,

Theorem

$\omega(\boldsymbol{u}, \psi)$ is nonempty and $\omega(\boldsymbol{u}, \psi) \subset S$.
For any $(0, \overline{\psi}) \in \omega(\boldsymbol{u}, \psi) \subset S$ then $\overline{E}(\overline{\psi}) = E_\infty$

Theorem

There exists $\overline{\psi} \in H^4_2$, such that $\psi(t) \to \overline{\psi}$ in H^2_1 weakly as $t \uparrow +\infty$, i.e.

$$\omega(\boldsymbol{u}, \psi) = \{(0, \overline{\psi})\}.$$

Convergence at infinite time of weak solution

Tools of proof:

- H. Petzeltova, E. Rocca, G. Schimperna'13.
- Lojasiewicz-Simon Lemma.

Consider the initial and boundary value problem associated to the problem on the time interval $[t_n, t_n + 1]$ with initial values $\boldsymbol{u}(t_n)$ and $\psi(t_n)$. Define

$$\boldsymbol{u}_n(t) := \boldsymbol{u}(t + t_n)$$

$$\psi_n(t) := \psi(t + t_n)$$

$$z_n(t) := z(\psi(t + t_n))$$

for a.e.

$$t \in [0, 1],$$

$(\boldsymbol{u}_n, \psi_n)$ is a weak solution on the time interval $[0, 1]$.

Lojasiewicz-Simon inequality

Let $\mathcal{S}$ be the following set of equilibrium points related to the bending energy $\overline{E}_b(\psi)$

$$\mathcal{S} = \{\psi \in H_2^4(\Omega) : \varepsilon\Delta^2\psi + \overline{G}(\psi) = 0 \text{ a.e in } \Omega\}.$$

Let $\overline{\psi} \in \mathcal{S}$ and $K > 0$ fixed. Then, for any two sufficiently small constants β and δ, there exists $C > 0$ and $\theta \in (0, 1/2)$ (depending on $\overline{\psi}$, β and δ), such that for all $\psi \in H_2^4$ with $\|\psi\|_3 \leq K$, $\|\psi - \overline{\psi}\|_1 \leq \beta$ and $|\overline{E}_b(\psi) - \overline{E}_b(\overline{\psi})| \leq \delta$, it holds

$$|\overline{E}_b(\psi) - \overline{E}_b(\overline{\psi})|^{1-\theta} \leq C\,|z|_2$$

where $z = z(\psi) := \varepsilon\Delta^2\psi + \overline{G}(\psi)$.