Mathematica Balkanica

New Series Vol. 14, 2000, Fasc. 1-2

Compact Mappings and Proper Mappings Between Banach Spaces that Share a Value

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Sufficient conditions are given to assert that differentiable compact mappings and differentiable proper mappings between Banach spaces share a value. The conditions involve Fredholm operators. The proof of the result is constructive and is based upon continuation methods.

AMS Subj. Classification: Primary 58C30, Secondary 65H20, 47H17

Key Words: zero point, continuation methods, C^1-homotopy, surjective implicit function theorem, proper mapping, compact mapping, Fredholm mapping

1. Preliminaries

In the whole this paper we assume that X and Y are two real Banach spaces. If $F : X \to Y$ is a continuous mapping, then one way of solving the equation

$$F(x) = 0 \tag{1}$$

is to embed (1) in a continuum of problems

$$H(t, x) = 0 \quad (0 \leq t \leq 1), \tag{2}$$

where we can solve easily the problem (2) when $t = 0$, and when $t = 1$ it becomes (1). If it is possible to continue the solution for $t = 0$ through $t = 1$ then (1) is solved. This method is called continuation respect to a parameter [1–6, 13–16].

In this paper sufficient conditions in order to prove that a differentiable mapping F has at least one zero are given. Other sufficient conditions to guarantee the existence of zero points in a finite dimensional setting have been given by the author in other several papers [8–12]. We use here continuation methods. The proof supplies the existence of implicitly defined curves reaching zero points

[5–7]. The key is the use of the Surjective Implicit Function Theorem in Banach spaces and the properties of the Fredholm C^1-mappings (see [7] and [13]). If F is a C^1-mapping, then $F'(x)$ is a continuous linear mapping from X into Y for every $x \in X$ and, in addition the function $x \in X \to \|F'(x)\| \in [0, \infty)$ is continuous.

We briefly recall some concepts about Banach-valued Mappings. Assume that $F : A \to Y$ is a continuous mapping, where $A \subset X$, F is said to be compact whenever the image $F(B)$ is relatively compact (i.e. its closure $\overline{F(B)}$ is compact in Y) for every bounded subset $B \subset A$. F is said to be proper whenever the preimage $F^{-1}(K)$ of every compact subset $K \subset Y$ is again a compact subset of A. If $a \in A$, A is open, F is compact and the derivative $F'(a)$ exists, then $F'(a) : X \to Y$ is a linear compact application (see [13, p.296]). If again A is open, then F is said to be Fredholm mapping if and only if F is a C^1-mapping, and $F'(x) : X \to Y$ is a linear Fredholm operator for all $x \in A$. That $L : X \to Y$ is a linear Fredholm operator means that L is linear and continuous and both the numbers dim(ker(L)) and codim($L(X)$) are finite. The number

$$\text{ind}(L) = \dim(\ker(L)) - \text{codim}(L(X))$$

is called the index of L. From the continuity of $x \to \|F'(x)\|$ is derived that ind($F'(x)$) is constant on A if A is connected. Fredholm mappings are important in bifurcations theory and historically they arose from the consideration of a linear equation $Lx = y$ $(x \in X)$ for given $y \in Y$ and its dual equation $L^*x^* = y^*$ $(x^* \in Y^*)$ for given $y^* \in X^*$ and from the desire of finding a class of linear operators for which the "good" properties of classical linear systems of equations in finite dimensional spaces are preserved as much as possible.

Finally we state for future references the Surjective Implicit Function Theorem (see [13, p.176]).

Theorem 1. *Assume that X, Y and Z are Banach spaces and that U is an open subset of $X \times Y$. Suppose that $(x_0, y_0) \in U$, and $F : U \to Z$ is a C^1-mapping, and $F(x_0, y_0) = 0$. Also suppose that the partial derivative $D_2F(x_0, y_0) : Y \to Z$ is surjective and that the nullspace $Y_1 = Ker(D_2F(x_0, y_0))$ is a complementary subspace of Y, that is, there exists a Banach subspace $Y_2 \subset Y$ such that $Y = Y_1 \oplus Y_2$ (topological direct sum). Then there exists an open subset $U_1 \subset X \times Y_1$ with $(x_0, 0) \in U_1$ and a C^1-mapping*

$$v : U_1 \to Y_2$$

satisfying:

a) $v(x_0, 0) = 0$,

b) $F(x, y_0 + y_1 + v(x, y_1)) = 0$ *for every* $(x, y_1) \in U_1$.

2. Mappings sharing a value

Clearly, if we get a decomposition $F = f - g$, then F has a zero if and only if f and g share a value, that is, there is $x \in X$ with $f(x) = g(x)$. We establish our result in terms of f, g.

Theorem 2. *Suppose that* $f, g : X \to Y$ *are two* C^1*-mappings satisfying the following conditions:*

i) f *has at least one zero.*

ii) f *is proper and Fredholm.*

iii) g *is compact.*

(iv) Given $t \in [0, 1]$ *and* $x \in X$, *then either* $f(x) \neq g(x)$ *or the linear mapping* $f'(x) - tg'(x)$ *is surjective.*

Then f *and* g *share a value.*

Proof. *First step.* By hypothesis (i), there is $x_0 \in X$ such that $f(x_0) = 0$. Choose any open ball $D_1 \subset X$ containing x_0. Let us define the set

$$V = g(A),$$

where

$$A = \{x \in D_1 : \text{ there is } t = t(x) \in [0, 1] \text{ such that } f(x) = tg(x)\}.$$

Observe that A is not emtpy, because $f(x_0) = 0 = 0.g(x_0)$, and so $x_0 \in A$. Note also that $V \subset g(D_1)$, and D_1 is bounded, therefore $g(D_1)$ is relatively compact (by (iii)) and so V is relatively compact too. We also have that the subset $[0, 1] \times \bar{V}$ is compact in the topological product space$\mathbf{R} \times Y$. We now construct the "cone"

$$V' = \{ty : t \in [0, 1], y \in \bar{V}\}.$$

This set is compact in Y, because it is image of $[0, 1] \times \bar{V}$ under the continuous mapping

$$(t, y) \in [0, 1] \times Y \to ty \in Y.$$

Next, we consider the subset of X given by

$$V'' = f^{-1}(V').$$

Again V'' is a compact set, because f is proper (by (ii)) and V' is compact.

Second step. If we fix a point $x \in X$ then $f'(x)$ is a Fredholm linear operator by (ii) and $tg'(x)$ is a compact operator for given scalar t. Finally, from [13, pp. 366–367], we have that if we fix $x \in X$ and $t \in [0,1]$, then $f'(x) - tg'(x)$ is a linear Fredholm operator and

$$\operatorname{ind}(f'(x) - tg'(x)) = \operatorname{ind}(f'(x))$$

for every $t \in [0,1]$.

On the other hand, as noted in Section 1, the index $\operatorname{ind}(f'(x))$ is constant when x runs over X. Hence

$$\operatorname{ind}(f'(x) - tg'(x)) = \text{a constant}.$$

for all $t \in [0,1]$ and all $x \in X$, and in particular for all $x \in V''$.

Third step. In this paragraph we will use Theorem 1 together with the continuation method to prove that f and g share a value. To this end let us construct the following C^1-homotopy:

$$H : \mathbf{R} \times X \to Y$$

given by

$$H(t,x) = f(x) - tg(x),$$

where $\mathbf{R}$ is the real line. Clearly $H(0,x_0) = 0$. In addition, the partial derivative

$$D_2H(0,x_0) = f'(x_0)$$

is surjective (as a linear mapping from X into Y) and the subspace

$$X_1 = \operatorname{Ker}(D_2H(0,x_0))$$

is complementary in X. This is true because $f'(x) - tg'(x)$ is Fredholm for all $t \in [0,1]$ and all $x \in V''$ (so $\operatorname{Ker}(f'(x) - tg'(x))$ is always finite dimensional) and the surjectivity condition is satisfied from hypothesis (iv) because $f(x_0) = 0$.

We will prove that every curves exist linking the zero x_0 of $H(0,x)$ with a zero of $H(1,x)$.

Let, more in general, $(t_0, a) \in \mathbf{R} \times X$ be a point such that

$$H(t_0, a) = 0.$$

Then Theorem 1 may be applied and we conclude that there exist positive numbers r_0 and r and a C^1-mapping

$$v : (t_0 - r_0, t_0 + r_0) \times B(0, r_0) \subset \mathbf{R} \times X_1 \to X_2$$

that satisfies

$$v(t_0, 0) = 0$$

and

$$H(t, a + x_1 + v(t, x_1)) = 0$$

for any fixed $t \in (t_0 - r_0, t_0 + r_0)$, for all $(t, x_1) \in (t_0 - r_0, t_0 + r_0) \times B(0, r_0)$ and $\|v(t, x_1) + x_1\| \leq r$. We have denoted here $\|.\|$ = the norm on X, $X_1 = \text{Ker}(D_2H(t_0, a)) = \text{Ker}(f'(a) - tg'(a))$, $B(0, r_0)$ = the open ball with center zero and radius r_0 in X_1 and X_2 = the complement of X_1 in X as closed subspace. So, each point $x = a + x_1 + v(t, x_1)$ is a solution for $f(x) - tg(x) = 0$.

From the proof of Implicit Function Theorem [13, pp.149–155] obtained trough the Banach Fixed-Point Theorem together with the compactness of $[0, 1] \times V''$ and the continuity of $H(t, x)$ and $D_2H(t, x)$, the constants r_0 and r can be taken independently on the particular point $(t_0, a) \in [0, 1] \times V''$, since the proof of the Sobrejective Implicit Function Theorem ([13], p.177) really uses a restriction of $D_2H(x, y)$ and $H(x, y)$ in a convenient shape.

Then from the compacteness of $[0, 1]$ we can reach the level $t = 1$ from the level $t = 0$ in a finite number of steps. X being connected, there is a solution curve $C \subset [0, 1] \times X$ with a point $(1, x_0^*) \in [0, 1] \times X$ with $H(t, x) = 0$ for all $(t, x) \in C$, and so

$$H(1, x_0^*) = 0,$$

that is,

$$f(x_0^*) = g(x_0^*),$$

as we wanted. The proof is finished. ∎

Acknowledgement. This work is partially supported by D.G.E.S. PB 96-1338-CO2-01 and the Junta de Andalucia.

I want to thank Prof. L. Bernal for helpful comments and suggestions.

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Received: 06.11.1998