Universality and lineability: new trends

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Contenidos

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First examples, I

Fekete, 1914

There exists a real power series $\sum_{n=1}^{\infty} a_n x^n$ with the following property: for each continuous function $g : [-1, 1] \to \mathbb{R}$ with $g(0) = 0$, there exists $(n_k) \uparrow\subset \mathbb{N}$ such that $\sum_{n=1}^{n_k} a_n x^n \to g(x)$ $(k \to \infty)$ unif.

- This is surprising, because every power series is the Taylor series of some function in $C^\infty(\mathbb{R})$.
  [Borel, 1895]

Birkhoff, 1929

There exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that the sequence of its translates $\{z \mapsto f(z+n) : n \in \mathbb{N}\}$ is dense in $H(\mathbb{C})$.

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MacLane, 1952

There exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that the sequence of its derivatives $\{f^{(n)} : n \in \mathbb{N}\}$ is dense in $H(\mathbb{C})$.

¿What do these 3 examples share?

They are objects with chaotic behaviour which, after a limit process, approximate each element of a maximal class of objects.

The preceding considerations lead to the following concept.

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Concepts

Definition

Assume that X and Y are TVs and that $T_n : X \to Y$ ($n \geq 1$) is a sequence of continuous mappings. We say that (T_n) is universal provided that there is an element $x_0 \in X$, called universal for (T_n), such that $\overline{\{T_n x_0 : n \in \mathbb{N}\}} = Y$.

Definition

If X is a TVS and $T \in L(X)$, then T is called hypercyclic whenever the sequence of iterates $T^n : X \to X$ ($n \geq 1$) is universal. The corresponding vectors $x_0 \in X$ with dense orbit are called hypercyclic for T.

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Remarks

- The word hypercyclic was coined by Beauzamy in 1980. It reinforces the notion of cyclic operator: an operator $T \in L(X)$ is called cyclic if there is a vector $x_0 \in X$ such that $\overline{\text{span}}\{x, Tx, T^2x, ...\} = X$.
- With the preceding terminology, we get that the sequence $T_n : (a_n) \in \mathbb{R}^{\mathbb{N}} \mapsto \sum_{k=1}^{n} a_k x^k \in (C_0[0,1], \|\cdot\|_\infty)$ $(n \geq 1)$ is universal.
- The traslation op. $f \mapsto f(\cdot + 1)$ and the differentiation op. $f \mapsto f'$ are hypercyclic on $H(\mathbb{C})$.
- (T_n) universal $\implies$ Y is separable.
- If an operator T is hypercyclic, the set $HC(T)$ of HC vectors is dense in X.

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A sufficient condition

- Relation with the invariant subspace problem and the invariant subset problem: Given $T \in L(X)$, each vector of $X \setminus \{0\}$ is cyclic [hypercyclic, resp.] $\Longleftrightarrow$ X lacks closed T-invariant nontrivial subspaces [subsets, resp.]
  Read (1988) found in ℓ_1 an operator for which any nonzero vector is HC.

Birkhoff, 1920

Let $T_n : X \to Y$ $(n \geq 1)$ be a sequence of continuous mappings between two TSs, with X Baire and Y 2nd countable. TFAE:
(a) The subset $U((T_n))$ of universal els. is dense in X.
(b) $U((T_n))$ is residual.
(c) (T_n) is transitive, that is, given nonempty open sets $U \subset X$, $V \subset Y$, there exists $n \in \mathbb{N}$ such that $T_n(U) \cap V \neq \emptyset$.

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Necessary conditions

- Thus, if X is a separable F-space we have: $T \in L(X)$ is HC $\iff$ T is transitive. In such a case, $HC(T)$ es residual.

Rolewicz, 1969

If $T \in L(X)$ is HC then $\dim(X) = \infty$. If in addition X is locally convex, then $\sigma_P(T^*) = \emptyset$.

Kitai, 1982

If X is a complex Banach space and $T \in L(X)$ is HC then T is not compact and $\sigma(T) \cap \mathbb{T} \neq \emptyset$.

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If $X = c_0$ or ℓ_p $(1 \leq \ell_p < \infty)$, $|\lambda| > 1$, and B denotes the backward shift operator $B : (x_1, x_2, x_3, ...) \in X \mapsto (x_2, x_3, x_4, ...) \in X$, then λB is HC.

Problem. Rolewicz, 1969

Given a separable Banach space X with $\dim(X) = \infty$, does it support a HC operator?

- The main "testing fields" for the search of HC operators are: backward shifts, differentiation operators and composition operators.
  If $\varphi \in H(\Omega, G)$, the composition operator associated to φ is defined as $C_\varphi : f \in H(G) \mapsto f \circ \varphi \in H(\Omega)$.

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Examples of HC operators, II

Seidel y Walsh, 1941

The non-euclidean translation operator $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$, where $\varphi(z) = \frac{z+a}{1+\overline{a}z}$ $[a \neq 0,\ |a| < 1]$ is HC.

Godefroy and Shapiro, 1991

If $\Phi(z) = \sum_{n=1} c_n z^n$ is an entire function of exponential type [$\limsup_{r\to\infty} \log M(r,f)/\log r < \infty$], then the operator $\Phi(D) = \sum_{n=1} c_n D^n : H(\mathbb{C}) \to H(\mathbb{C})$ is HC.

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If $p \in [1,\infty)$ and $\varphi \in \mathrm{Aut}(\mathbb{D})$ is non-elliptic, then the operator $C_\varphi : H^p \longrightarrow H^p$ is HC.

Gallardo and Montes (2004) gave a complete characterization of $\varphi \in LFT(\mathbb{D})$ generating HC C_φ on $S_\nu = \{f(z) = \sum_{n=0} a_n z^n \in H(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 (n+1)^\nu < \infty\}$.

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Examples of HC operators, III. Existence

Montes and LBG, 1995. Grosse-Erdmann and Mortini, 2009

Let $G \subset \mathbb{C}$ be a simply connected or a infinitely connected domain, and $(\varphi_n) \subset \mathrm{Aut}(G)$. Then: $C_{\varphi_n} : H(G) \to H(G)$ $(n \geq 1)$ is universal $\iff$ (φ_n) is runaway, that is, given a compact set $K \subset G$, there is $N = N(K) \in \mathbb{N}$ such that $K \cap \varphi_N(K) = \emptyset$.

Ansari and LBG, 1997; Bonet and Peris, 1998

If X is a separable Fréchet space with $\dim(X) = \infty$ then there exists some HC operator T on X.

- T can be chosen to be onto. If X is Banach, T can be chosen to be bijective and of the form $T = I + K$, with K compact y nilpotent $[\sigma(T) = \{0\}]$.

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Existence and non-existence

Problem

Which [separable, infinite dimensional] TVSs support HC operators?

- [Bonet and Peris (1998)] $\varphi = \oplus_{n\in\mathbb{N}}\mathbb{R}$ does not carry a HC operator.
- [Grosse-Erdmann (1999)] $L^p[0,1]$ $(0 < p < 1)$ carries a HC operator.
- [Shkarin (2010)] $L^p[0,1] \oplus \mathbb{R}$ does not carry a HC operator.
- [Shkarin (2010)] Every normed space with countable dimension carries a HC operator.

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HC semigroups of operators

Definition

Let X be a TVS. A family $\{T_t\}_{t\geq 0} \subset L(X)$ is a strongly continuous semigroup of operators in L(X) if $T_0 = I$, $T_t T_s = T_{t+s}\ \forall t, s \geq 0$, and $\lim_{t\to s} T_t x = T_s x\ \forall s \geq 0, x \in X$. A SCS $\{T_t\}_{t\geq 0}$ is said to be hypercyclic if $\{T_t x :\ t \geq 0\}$ is dense in X for some $x \in X$, called HC for (T_t).

Conejero, Müller and Peris, 2007

Let X be an F-space and $\mathcal{T} = (T_t)_{t\geq 0}$ be a SCS on it. Then: $\mathcal{T}$ is HC $\iff$ each T_u $[u > 0]$ is HC $\iff$ some T_u is HC. In this case, $HC(T_u) = HC(\mathcal{T})\ \forall u > 0$.

... Hence, at least theoretically and in the setting of F-spaces, the problems that could be posed for hypercyclicity of semigroups come down to problems for single operators.

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Let X be an F-space and $\mathcal{T} = (T_t)_{t\geq 0}$ be a SCS on it. Then:
$\mathcal{T}$ is HC $\iff$ each T_u $[u > 0]$ is HC $\iff$ some T_u is HC.
In this case, $HC(T_u) = HC(\mathcal{T})\ \forall u > 0$.

... Hence, at least theoretically and in the setting of F-spaces, the problems that could be posed for hypercyclicity of semigroups come down to problems for single operators.

HC semigroups of operators

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Holomorphic monsters, I

Luh, 1985

If $G \subset \mathbb{C}$ is a s.c. domain, a holomorphic monster on G is a function $f \in H(G)$ satisfying: given $g \in H(\mathbb{D})$, $\xi \in \partial G$ and any derivative or antiderivative F of f of any order, there are sequences $a_n \to 0$ and $b_n \to \xi$ such that $a_n z + b_n \in G$ $(n \geq 1,\ z \in \mathbb{D})$ and

$$F(a_n z + b_n) \to g(z) \text{ in } H(\mathbb{D}).$$

Luh, 1985. Grosse-Erdmann, 1987

There are holomorphic monsters, and in fact they form a residual set in $H(G)$.

- M.C. Calderón and LBG (2000) conceived the notion of holomorphic T-monster, where $T \in L(H(G))$: simply replace F above by Tf.

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Holomorphic monsters, II

- By considering countable families $(T_n) \subset L(H(G))$ and the theory of universality, it is possible to extend the theory of holomorphic monsters.

Theorem

(a) [Calderón and LBG, 2000] If $G \subset \mathbb{C}$ is a domain, $\Phi \neq 0$ is an entire function of exponential type and $\lambda \in \mathbb{C}$ then there are T-monsters in $H(G)$ for the operators $T = \Phi(D)$ and $(Tf)(z) = \lambda f(z) + \int_a^z \Phi(z-t) f(t)\, dt$ [here if G is s. connected].
(b) [Calderón and LBG, 2001] There are no Luh-monsters in H^p $(1 \leq p < \infty)$. For any polynomial $P \neq 0$, there are $P(D)$-monsters in H^p.
(c) [Calderón, Grosse-E. and LBG, 2002] If $\varphi \in H(G, G)$ then there are C_φ-monsters in $H(G) \iff$ for every $V \in O(\partial G)$ the set $\varphi(V \cap G)$ is not relatively compact in G.

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Universal Taylor series, I

- In 1905 Porter discovered the phenomenon of overconvergence: some power series possess subsequences for their partial sums being convergent beyond the circle of convergence.

Nestoridis, 1996

There are universal Taylor series (UTS) in $H(\mathbb{D})$, that is, functions $f(z) = \sum_{n=0}^{\infty} f_n z^n \in H(\mathbb{D})$ satisfying that, for every compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with $\mathbb{C} \setminus K$ connected and every $h \in A(K) := C(K) \cap H(K^0)$, $\exists\, (\lambda_n) \uparrow \subset \mathbb{N}_0$ such that

$$S(\lambda_n, f, z) := \sum_{k=0}^{\lambda_n} f_k z^k \longrightarrow h \quad \text{unif. on } K.$$

- Luh (1970) and Chui and Parnes (1971) had proved a similar property but with $K \subset \mathbb{C} \setminus \overline{\mathbb{D}}$.
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Universal Taylor series, II

- The last result can be extended by using summability methods.

Definition

Let $\mathcal{A} = [\alpha_{n\nu}]_{n,\nu=0}^{\infty}$ be an infinite matrix in $\mathbb{C}$. We say that $\mathcal{A}$ is a C-matrix if:

- $\forall n \in \mathbb{N}_0$, $\lim_{\nu\to\infty} |\alpha_{n\nu}|^{1/\nu} = 0$.
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If $\mathcal{A}$ is a C-matrix, a function $f \in H(\mathbb{D})$ is called a $\mathcal{A}$-universal Taylor series if it satisfies the same property as a UTS but replacing $S(n, f, z)$ by $S_{\mathcal{A}}(n, f, z) := \sum_{\nu=0}^{\infty} \alpha_{n\nu} S(\nu, f, z)$.

Melas and Nestoridis, 2001; Calderón, Luh and LBG, 2006

Given $\mathcal{A}$ as before, there is a residual subset in $H(\mathbb{D})$ consisting of $\mathcal{A}$-UTSs.

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Frequent hypercyclicity, I

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Let X be a TVS. Then an operator $T \in L(X)$ is said to be frequent hypercyclic if $\exists x \in X$ s.t., for every nonempty open set $U \subset X$, $\liminf_{n\to\infty} \frac{\operatorname{card}\{k \in \{1,\dots,n\} : T^k x \in U\}}{n} > 0.$

- Replacing T^n by $T_n \in L(X,Y)$ one reaches the notion of frequent universal sequence (FU) of mappings.
- Connection with Ergodic Theory: X separable F-space, $T \in L(X)$ and $\exists \mu$ Borel probability measure with $\operatorname{supp}(\mu) = X$ s.t. T is μ-ergodic $\Longrightarrow$ T is FHC.

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The following ops. are FHC: any translation $\tau_a f = f(\cdot + a)$ on $H(\mathbb{C})$, any C_φ on $H(\mathbb{D})$ with non-elliptic $\varphi \in \operatorname{Aut}(\mathbb{D})$, and any multiple λB $(|\lambda| > 1)$ of the b.w.s. on c_0 or ℓ_p $(1 \le p < \infty)$.

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Assume that Φ is a nonconstant entire function of exponential type. Then $\Phi(D)$ is FHC.

- There is not residuality in these examples: $FHC(\tau_a)$, $FHC(C_\varphi)$, $FHC(\lambda B)$ and $FHC(\Phi(D))$ are of first category.

Theorem

(a) [Shkarin, 2009] There are Banach spaces which do not support FHC operators.
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Suppose that $\varphi \in LFT(\mathbb{D})$ is not a parabolic automorphism. We have: C_φ is FHC on $S_\nu \iff C_\varphi$ is HC.

LBG, 2012

If $(a_n) \subset \mathbb{C}$ is a sequence such that $\lim_{k\to\infty} \inf_{n\in\mathbb{N}} |a_{n+k} - a_n| = +\infty$ then the sequence of translations (τ_{a_n}) is frequently universal on $H(\mathbb{C})$.

Problems

- What sequences $(\varphi_n(z) = a_n z + b_n) \subset \text{Aut}(\mathbb{C})$ satisfy that (C_{φ_n}) is FU on $H(\mathbb{C})$? Recall [Montes and LBG, 1995] that (C_{φ_n}) is universal $\iff \{\min\{|b_n|, |b_n/a_n|\}\}_{n\geq 1}$ is unbounded. Also, complete the parabolic case in S_ν.

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Some problems and size of sets

Problems

- Characterize the class of TVSs supporting FHC operators.
- Are there FHC operators such that $FHC(T)$ is residual or at least of 2nd category?

Recall that if X is an F-space and $T \in L(X)$ is HC then $HC(T)$ is residual, that is, topologically large.
Might it be, in some sense, algebraically large?
A handicap: $HC(T)$ is not a vector space and $0 \notin HC(T)$.
But ... is it possible to find "large" vector spaces contained, except for 0, in $HC(T)$?
This question can be put into a more general setting ...

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Lineability: definitions

Aron, Bayart, Gurariy, PérezGª, Quarta, Seoane, LBG. 2004-10

Assume that X is a TVS and μ is a cardinal number. A subset $A \subset X$ is called:

- μ-lineable if $A \cup \{0\}$ contains a vector space M with $\dim(M) = \mu$,
- dense-lineable whenever $A \cup \{0\}$ contains a dense vector subspace of X,
- maximal dense-lineable if $A \cup \{0\}$ contains a dense vector subspace M of X with $\dim(M) = \dim(X)$ [$\Longleftrightarrow \dim(M) = c$, if X a sep. inf-dim. F-space],
- spaceable whenever $A \cup \{0\}$ contains a closed infinite dimensional vector subspace of X, and
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Lineability: definitions

Aron, Bayart, Gurariy, PérezGª, Quarta, Seoane, LBG. 2004-10

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Lineability and HC vectors, I

- With this terminology, let's go back to hypercyclicity.

Herrero-Bourdon-Bès-Wengenroth, 1991-1993-1999-2003

X TVS and $T \in L(X)$ HC $\Longrightarrow$ $HC(T)$ is dense-lineable.

- Their construction gives a subspace M with $\dim(M) = \omega$.

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- Extendable to Fréchet spaces?

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Let X be a separable Fréchet space with a continuous norm, and $T \in L(X)$. Suppose that there are X_0, Y_0 dense in X, $(n_k) \uparrow \subset \mathbb{N}$ and an inf-dim closed subspace $M_0 \subset X$ satisfying:

(a) $T^{n_k}x \to 0 \ \forall x \in X_0$,

(b) for each $y \in Y_0$ there is $(x_k) \subset X_0$ with $x_k \to 0$ and $T^{n_k}x_k \to y$,

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Then $HC(T)$ is spaceable.

• [Montes, 1996] If B is the b.w.s. on c_0 then $HC(2B)$ is not spaceable.

• Sufficient conditions for $FHC(T)$ to be spaceable have been found by Bonilla and Grosse-Erdmann (2011). They apply to τ_a and $\Phi(D)$ with Φ entire transcendental. But, is $FHC(D)$ spaceable in $H(\mathbb{C})$? [$HC(D)$ is spaceable (Shkarin, 2010)].

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Lineability and universality

CalderónM-LBG, 1999/2002.

Let X and Y be TVSs and $(T_n) \subset L(X, Y)$.
(a) If Y is metrizable and (T_{n_k}) is universal for each $(n_k) \uparrow\subset \mathbb{N}$ then $U((T_n))$ is lineable.
(b) If X, Y are metrizable and X is separable and $U((T_{n_k}))$ is dense for each $(n_k) \uparrow\subset \mathbb{N}$ then $U((T_n))$ is dense-lineable.
(c) If X, Y are metrizable, X is Baire and separable and, for each $\nu \in \mathbb{N}$, $(T_{n,\nu})_{n\geq 1} \subset L(X, Y)$ and $U((T_{n_k,\nu}))$ is dense for each $(n_k) \uparrow\subset \mathbb{N}$ then $\bigcap_{\nu\geq 1} U((T_{n,\nu}))$ is dense-lineable.

Consequences

(a) [Calderón and LBG, 2002] The family of Luh-monsters is dense-lineable.
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- Trivially, the set of differentiable functions on $[0,1]$ is dense-lineable in $C[0,1]$, but it is not spaceable [Gurariy, 1966].

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The set of nowhere differentiable functions is spaceable in $C[0,1]$. In fact, any separable inf-dim Banach space is isometrically isomorphic to a space of nowhere differenciable functions $\cup \{0\}$.

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Assume that $G \subset \mathbb{C}^N$ is a domain of holomorphy. Then the set of functions which cannot be holomorphically continued beyond any point of ∂G is dense-lineable and spaceable in $H(G)$.

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Lineability in function spaces, II

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Assume that $G \subset \mathbb{C}$ is a Jordan domain. Then the set of $MCS(G)$ of $f \in H(G)$ having maximal cluster set at any $\xi \in \partial G$ along any curve $\Gamma \subset G$ tending to ∂G with $\partial G \setminus \overline{\Gamma} \neq \emptyset$ is dense-lineable.

- Combinations: [Bonilla-CalderónM-PradoB-LBG, 2009/12] $MCS(\mathbb{D}) \cap UTS(\mathbb{D})$ and $MCS(G) \cap U((C_{\varphi_n}))$ [G Jordan domain, $(\varphi_n) \subset \mathrm{Aut}(G)$ runaway] are spaceable and maximal dense-lineable.
- Aron, Conejero, Peris and Seoane (2007) proved that $f^2 \notin HC(\tau_1)$ for any f entire and τ_1-HC, but there are D-HC functions f with $f^k \in HC(D)$ for all $k \in \mathbb{N}$.

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Given $E \subset \mathbb{T}$ of measure 0, the set $\{f \in C(\mathbb{T}) :$ the Fourier series associated to f diverges at each $t \in E\}$ is algebrable. The algebra can be obtained dense in $C(\mathbb{T})$.

- Aron, Conejero, Peris and Seoane (2010) have proved that the family of everywhere surjective functions $\mathbb{C} \to \mathbb{C}$ contains, except for 0, an uncountable generated algebra.

Bartoszewicz, Glab, Pellegrino and Seoane, 2011

The set $\{f : \mathbb{C} \to \mathbb{C} : \forall$ perfect set $P \subset \mathbb{C}$ and $\forall r \in \mathbb{C}$, $\mathrm{card}\{z \in P : f(z) = r\} = c\}$ is 2^c-algebrable.

Dense-lin. criterium. Aron-GarcíaPacheco-PérezGª-Seoane

If $A, B \subset X$, with X a separable F-space, A lineable, B dense-lineable and $A \supset A + B$ then A is dense-lineable.

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Lineability criteria

- Example: $L^p[0,1] \setminus \bigcup_{q>p} L^q[0,1]$ is dense-lineable.

Spaceability criteria

(a) [Kalton and Wilansky, 1975] If X is a Fréchet space and $Y \subset X$ is a closed linear subspace, with infinite codimension then $X \setminus Y$ is spaceable.
(b) [Ordóñez and LBG, 2012] Assume that $(E, \|\cdot\|)$ is a Banach space of fs $X \to \mathbb{K}$ and that A is a cone in E satisfying:
- Convergence in E implies pointwise convergence of a subsequence.
- $\exists C \in (0, +\infty)$ s.t. $\|f+g\| \geq C\|f\|$ $\forall f, g \in E$ with $\text{supp}(f) \cap \text{supp}(g) = \emptyset$.
- If $f, g \in E$ are such that $f + g \in A$ and $\text{supp}(f) \cap \text{supp}(g) = \emptyset$ then $f, g \in A$.
- $\exists (f_n) \subset E \setminus A$ with pairwise disjoint supports.

Then $E \setminus A$ is spaceable.

- Example: $L^p[0,1] \setminus \bigcup_{q>p} L^q[0,1]$ is spaceable [Botelho-Fávaro-Pellegrino-Seoane-Ordóñez-LBG, 2012].
- It would be interesting to dispose of more dense-lineability, spaceability criteria, and al least one algebrability criterium.

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(a) [Kalton and Wilansky, 1975] If X is a Fréchet space and $Y \subset X$ is a closed linear subspace, with infinite codimension then $X \setminus Y$ is spaceable.
(b) [Ordóñez and LBG, 2012] Assume that $(E, \|\cdot\|)$ is a Banach space of fs $X \to \mathbb{K}$ and that A is a cone in E satisfying:
- Convergence in E implies pointwise convergence of a subsequence.
- $\exists C \in (0,+\infty)$ s.t. $\|f+g\| \ge C\|f\|$ $\forall f, g \in E$ with supp$(f)\cap$ supp$(g) = \emptyset$.
- If $f, g \in E$ are such that $f + g \in A$ and supp$(f)\cap$ supp$(g) = \emptyset$ then $f, g \in A$.
- $\exists (f_n) \subset E \setminus A$ with pairwise disjoint supports.

Then $E \setminus A$ is spaceable.

- Example: $L^p[0,1] \setminus \bigcup_{q>p} L^q[0,1]$ is spaceable [Botelho-Fávaro-Pellegrino-Seoane-Ordóñez-LBG, 2012].
- It would be interesting to dispose of more dense-lineability, spaceability criteria, and al least one algebrability criterium.

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