Renormings and the Fixed Point Property

Carlos Alberto Hernández Linares
María A. Japón Pineda

Department of Mathematical Analysis
Faculty of Mathematics
University of Seville
Seville, Spain.

April 2010

Outline

Definitions

Definition

Let $T : C \to C$ be a mapping. We say that T has a fixed point if there exists $x \in C$ such that $Tx = x$.

Theorem (Banach contraction)

Let X be a Banach space and C a closed subset of X. If $T : C \to C$ is a contraction, i.e.

$$\|Tx - Ty\| \leq k\|x - y\|, \forall x, y \in C, \text{ with } k < 1,$$

then T has a fixed point.

Definitions

Definition

A mapping $T : C \to C$ is non-expansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$

Banach's theorem does not hold for non-expansive mappings.

Definition

A Banach space X has the fixed point property (FPP) if every non-expansive mapping $T : C \to C$, where C is a closed convex bounded subset of X, has a fixed point.

Fixed Point and Reflexivity

$$\left.\begin{array}{l}\text{Uniformly smooth}(\Rightarrow Reflexivity)\\ \text{Uniformly Convex}(\Rightarrow Reflexivity)\\ \text{Normal Structure + Reflexivity}\\ \text{Uniformly Kadec Klee + Reflexivity}\\ \text{Uniformly Opial Condition + Reflexivity}\\ \vdots\\ \text{etc + Reflexivity}\end{array}\right\} \Rightarrow FPP$$

FPP $\Rightarrow$ Reflexivity ?

ℓ_1 does not have the FPP

Theorem

ℓ_1 does not have the FPP.

Proof: We consider

$$C = \{x = (x_i) \in \ell_1 : \forall i \in \mathbf{N} \ x_i \geq 0, \|x\|_1 = 1\}.$$

The set C is a closed convex bounded subset of ℓ_1. Let $T : C \to C$ be the mapping given by

$$T(x_1, x_2, \dots) = (0, x_1, x_2, \dots.)$$

T is a non-expansive mapping and fixed point free.

The main question.

If X fails to have the FPP, can X be renormed to have the FPP?

In particular, can ℓ_1 be renormed to have the FPP?

Some answers

Theorem (T. Domínguez Benavides, 2009)

Every reflexive Banach space can be renormed to have the FPP.

Theorem (P. N. Dowling, C. J. Lennard and B. Turett, 1997-1998)

$\ell_1(\Gamma)$, $c_0(\Gamma)$ *and* ℓ_∞ *can not be renormed to have the FPP.*

Some answers

Theorem (P.K. Lin, 2008)

The Banach space ℓ_1 can be renormed to have the FPP.

In ℓ_1 consider the norm given by

$$\left|\left|\left|\sum_{n=1}^{\infty} a_n e_n\right|\right|\right| = \sup_k \gamma_k \left\|\sum_{n=k}^{\infty} a_n e_n\right\|_1,$$

where $\{e_n\}_n$ is the canonical basis on ℓ_1 and $\gamma_k = \frac{8^k}{1+8^k}$.
Then $(\ell_1, |||\cdot|||)$ has the FPP.

Some answers

Remark

$(\ell_1, |||\cdot|||)$ is the first known Banach space with the FPP and non-reflexive.

$$\text{FPP} \not\Leftrightarrow \text{Reflexivity}$$

Objective

If X fails to have the FPP, we try to find a renorming, $|||\cdot|||$, so that $(X, |||\cdot|||)$ has the FPP.

Our assumptions

Let $(X, \|\cdot\|)$ be a Banach space. Let $R_k : X \to [0, \infty)$ $(k \geq 1)$ be a **family of seminorms** such that

$$R_1(x) = \|x\|, \quad \text{and } \forall k \geq 2 \quad R_k(x) \leq \|x\|$$

Consider a nondecreasing sequence $\{\gamma_k\} \subset (0, 1)$ so that

$$\lim_k \gamma_k = 1$$

and define

$$|||x||| = \sup_{k \geq 1} \gamma_k R_k(x); \;\; x \in X.$$

Then

$$\gamma_1 \|x\| \leq |||x||| \leq \|x\|.$$

Our assumptions

Consider $(X, \|\cdot\|)$ endowed with a **linear topology** τ. Assume that the **family of seminorms** and the **linear topology** satisfy the following properties:

1. $\lim_k R_k(x) = 0$ for all $x \in X$.
   For all $k \geq 1$ and for every norm-bounded $x_n \to 0$ in τ:
2. $$\limsup_n R_k(x_n) = \limsup_n \|x_n\|.$$
3. For all $x \in X$,
   $$\limsup_n R_k(x_n + x) = \limsup_n R_k(x_n) + R_k(x).$$

Example

Consider $(\ell_1, \|\cdot\|_1)$ with its usual norm. Let $\{R_k(\cdot)\}$ be a family of seminorms given by

$$R_1(x) = \|x\|_1,$$

$$R_k(x) = \left\| \sum_{n=k}^{\infty} x_n e_n \right\|_1 \quad \forall k \geq 2,$$

where $x = \sum_{n=1}^{\infty} x_n e_n \in \ell_1$.

Let $\tau = \sigma(\ell_1, c_0)$. Then the family of seminorms and the topology τ are in the above conditions.

Our result

With the above assumptions on $(X, \|\cdot\|)$ and the family of seminorms $\{R_k(\cdot)\}$ we get the following.

Main Theorem (Hernández and Japón, 2010)

If every bounded sequence in X has a τ-convergent subsequence then $(X, |||\cdot|||)$ has the FPP.

Example

Lin's result can be derived from the Main Theorem defining the seminorms

$$R_k(x) = \left\| \sum_{n=k}^{\infty} x_n e_n \right\|_1$$

and taking τ as the weak-star topology associated to the duality $\sigma(\ell_1, c_0)$.

The condition $\gamma_k = \frac{8^k}{1+8^k}$ can be dropped.

Example

We can obtain other renormings in ℓ_1 that have the FPP. For instance, let $p > 1$ and for $k \geq 2$ define for $x = (a_n) \in \ell_1$

$$R_k(x) = \sum_{n=2k}^{\infty} |a_n| + \left(\sum_{n=k}^{2k-1} |a_n|^p \right)^{\frac{1}{p}},$$

and $R_1(x) = \|x\|_1$.

Then $(\ell_1, |||\cdot|||)$ has the FPP.

Corollary

Let $\{X_n\}$ be a sequence of finite dimensional Banach spaces and consider

$$X = \oplus_1 \sum_n X_n = \left\{ x = (x_n) : x_n \in X_n, \|x\| = \sum_n \|x_n\|_{X_n} < \infty \right\}.$$

Then X can be renormed to have the FPP.

Proof: Define the seminorms

$$R_k(x) = \sum_{n=k}^{\infty} \|x_n\|_{X_n}$$

and let τ be the weak star topology where the predual of X is

$$E = \left\{ x = (x_n) : \quad x_n \in X_n, \ \lim \|x_n\|_{X_n} = 0, \quad \|x\| = \sup_n \|x_n\|_{X_n} \right\}.$$

Example

Let $1 < p < \infty$ be and

$$X = \oplus_1 \sum_n \ell_p^n.$$

X can be renormed to have the FPP. Moreover X is non-reflexive and it is not isomorphic to any subspace of ℓ_1.

If X were isomorphic to ℓ_1 then

$$1 = type(\ell_1) = type(X) = type(\ell_p) = \min\{2, p\}$$

Let G be a locally compact group. $B(G)$ its Fourier-Stieltjes algebra.

Theorem (A.T.-M Lau and M. Leinert, 2008)

$B(G)$ has the FPP $\Leftrightarrow$ G is finite.

Corollary (of the Main Theorem)

If G is a separable compact group, $B(G)$ can be renormed to have the FPP.

Proof:

$$B(G) = \oplus_1 \sum_n \mathfrak{T}(H_n),$$

where H_n is a finite dimensional Hilbert space and $\mathfrak{T}(H_n)$ is the trass class operator on H_n.

Application to subspaces of $L_1(\mu)$

Consider (Σ, Ω, μ) a σ-finite measure space. Let $\Omega = \cup_n \mathcal{A}_n$ with $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ and $\mu(\mathcal{A}_n) < +\infty$ for all $n \in \mathbf{N}$. We define for all $x \in L_1(\mu)$

$$R_1(x) = \|x\|_1 = \int_\Omega |x| d\mu,$$

$$R_k(x) = \sup\left\{\int_{E\cap\mathcal{A}_k} |x| d\mu : \mu(E) < \frac{1}{k}\right\} + \|x\chi_{\mathcal{A}_k^c}\|_1; \text{ for } k \geq 2.$$

$$\begin{aligned} \tau &:= \text{the topology of locally convergence in measure } (lcm) \\ (&\equiv \text{the topology of convergence a.e., up to subsequences.)} \end{aligned}$$

$$d_\tau(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(\mathcal{A}_n)} \int_{\mathcal{A}_n} \frac{|x-y|}{1+|x-y|} d\mu; \; x, y \in L_1(\mu).$$

Application to subspaces of $L_1(\mu)$

For a nondecreasing sequence $\{\gamma_k\}$ in $(0,1)$ such that $\lim_k \gamma_k = 1$ we define a equivalent norm on $L_1(\mu)$ as

$$|||x||| = \sup_k \gamma_k R_k(x).$$

Theorem

The seminorms $R_k(\cdot)$ defined above satisfy the properties of the Main Theorem. Thus the following holds:
If X is a subspace of $L_1(\mu)$ such that B_X is lcm-relatively compact then X can be renormed to have the FPP.

Application to subspaces of $L_1(\mu)$

Remark 1

If μ is finite. Consider $\mathcal{A}_k = \Omega$, then

$$R_k(x) = \sup\left\{\int_E |x| d\mu : \mu(E) < \frac{1}{k}\right\}; \text{ for } k \geq 2.$$

Application to subspaces of $L_1(\mu)$

Remark 2

Assume now that $\Omega = \mathbf{N}$ and μ is the counting measure defined on the subsets of $\mathbf{N}$. Then the space $L_1(\mu)$ becomes the sequence space ℓ_1. Taken $A_1 = \emptyset$ and $A_n = \{1, \ldots, n-1\}$ for $n \geq 2$ so

$$R_k(x) = \|x\chi_{A_k^c}\|_1 = \sum_{n=k}^{\infty} |x(n)|; \text{ for all } k \in \mathbf{N}$$

$$lcm = \sigma(\ell_1, c_0) \text{ in norm-bounded subsets.}$$

In this case we recover the Lin's renorming taken $\gamma_k = \frac{8^k}{1+8^k}$.

Other results

Corollary

Let X be a closed subspace for $L_1(\mu)$. If X is a dual space such that the lcm-topology coincides with the w^*-topology on B_X, then $(X, |||\cdot|||)$ has the FPP.

Application: The Bergman Space

$L_a(\mathbb{D}) := \{f \in L_1(\mathbb{D}) : f \text{ is an analytic function on } \mathbb{D}\}$.
$L_a(\mathbb{D})$ is a dual space and τ = topology convergence in measure = weak*-topology.
Then $(L_a(\mathbb{D}), |||\cdot|||)$ has the FPP.

Other results

Example (Godefroy, N.J. Kalton, D. Li, 1995)

There exists a subspace X of $L_1[0,1]$ such that the unit ball B_X is compact for the topology of convergence in measure (but it is not locally convex for this topology). Then X can be renormed to have the FPP.

Remark

The topology of convergence in measure does not coincide with any dual topology.

Other results

Example (J. Bourgain, H.P. Rosenthal, 1980)

There exists a subspace X of $L_1[0,1]$ such that X fails to have the Radon-Nikodym property and every bounded sequence has a subsequence converging in measure. Therefore, X can be renormed to have the FPP.

Remark

X is not isomorphic to a subspace of ℓ_1 because X fails the Radon-Nikodym property.

C.A. Hernández-Linares and M.A. Japón. *A renorming in some Banach spaces with applications to fixed point theory.* J. Funct. Anal. 258 (2010), 3452-3468.

Non-commutative L_1-spaces

Let $\mathcal{M}$ be a finite von Neumann algebra.
Let $L_1(\mathcal{M})$ be the non-commutative L_1-space corresponding to $\mathcal{M}$, i.e. $L_1(\mathcal{M})$ is the predual of $\mathcal{M}$ ($\mathcal{M}_*$).

$$\mathcal{M} \text{ commutative } \Rightarrow L_1(\mathcal{M}) = L_1(\mu).$$

We can generalize our renorming techniques to non-commutative $L_1(\mathcal{M})$-spaces.

$L_1(\mathcal{M})$ does not have the FPP.

Can $L_1(\mathcal{M})$ be renormed to have the FPP?

A little bit of background

Definition

A *von Neumann algebra* is a subalgebra $\mathcal{M}$ of $B(H)$ which is self-adjoint (if $x \in \mathcal{M}$ implies $x^* \in \mathcal{M}$), contains $\mathbf{1}$ (the identity operator) and it is closed in the weak operator topology (WOT).

Remark

If H is a separable infinite dimensional Hilbert space, every $T \in B(H)$ has a matrix representation in the form

$$T = ((Te_i, e_j))_{i \geq 1; j \geq 1},$$

so a von Neumann algebra is a unital sub-algebra of $B(H)$ which is closed in the topology of coordinatewise convergence (WOT).

A little bit of background

Assume H is a separable Hilbert space.

Definition

A von Neumann algebra $\mathcal{M}$ is finite when

$$T \in \mathcal{M} \text{ and } TT^* = \mathbf{1} \Rightarrow T^*T = \mathbf{1}.$$

Let $\mathcal{M}_+$ be the cone of all positive elements of $\mathcal{M}$, that is,

$$\mathcal{M}_+ = \{x \in \mathcal{M} : \langle xh|h\rangle \geq 0, \text{ for all } h \in H\}.$$

$$P(\mathcal{M}) := \{p \in \mathcal{M} : p \text{ is a projection}\}$$

A little bit of background

Definition

A trace on a von Neumann algebra $\mathcal{M}$ is a map $\tau : \mathcal{M}_+ \to [0, \infty]$ satisfying:

1) $\tau(x + y) = \tau(x) + \tau(y)$, for all $x, y \in \mathcal{M}_+$.
2) $\tau(\lambda x) = \lambda\tau(x)$; $x \in \mathcal{M}_+$, $\lambda \in [0, +\infty]$.
3) $\tau(xx^*) = \tau(x^*x)$ for all $x \in \mathcal{M}$.

The trace τ is said to be

4) normal: if for each $x_\alpha \uparrow x$ in $\mathcal{M}_+$ we have $\tau(x_\alpha) \uparrow \tau(x)$.
5) faithful: if $\tau(x) = 0$ implies that $x = 0$ for all $x \in \mathcal{M}_+$.
6) finite: if $\tau(\mathbf{1}) < +\infty$.

A little bit of background

A little bit of background

In a finite von Neumann algebra there always exists a normal faithful finite trace.

Example

$\mathcal{M} = L_\infty(\mu)$, $H = L_2(\mu)$. For $f \in L_\infty(\mu)$

$$f: \begin{array}{ccc} L_2(\mu) & \to & L_2(\mu) \\ g & \mapsto & fg \end{array}, \quad \tau(f) = \int f d\mu$$

and $\mathcal{M}_* = L_1(\mu)$;

A little bit of background

Define for all $x \in L_1(\mathcal{M})$

$$R_1(x) := \|x\|_1 = \tau(|x|)$$

$$R_k(x) := \sup\{\|xp\|_1 : p \in \mathcal{P}(\mathcal{M}), \tau(p) < 1/k\}, k \geq 2.$$

The linear topology

Assume that $\mathcal{M}$ is a finite von Neumann algebra ($\tau(1) < +\infty$). Consider the measure topology defined by the neighborhoods of zero

$$N(\epsilon, \delta) = \{x \in \mathcal{M} : \exists\, p \in \mathcal{P}(\mathcal{M}) \text{ such that } \|xp\|_\infty \leq \varepsilon \text{ and } \tau(1-p) \leq \delta\}$$

for $\epsilon, \delta > 0$. (E. Nelson, 1974)

The theorem

Theorem (Hernández and Japón, 2010)

Let $\mathcal{M}$ be a finite von Neumann algebra. If the unit ball is compact for the measure topology, then $L_1(\mathcal{M})$ can be renormed to have the FPP.

Applications

Example (The hyperfinite II_1 factor)

Let $(R, \tau) = \bigotimes_{n \geq 1}(M_2, \sigma_2)$ be the von Neumann algebra tensor product, M_2 denotes the complex 2×2 matrices and
$\sigma_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}(a + d)$.

Definition

A von Neumann algebra is:

1. a factor if $x \in \mathcal{M}$ and $xy = yx$ for all $y \in \mathcal{M}$ implies $x = \lambda \mathbf{1}$ for some $\lambda > 0$.
2. of type II_1 if it is finite and it does not contain any nonzero abelian projection.
3. hyperfinite if there exists a sequence $\mathcal{M}_n \subset \mathcal{M}_{n+1}$ of finite-dimensional von Neumann algebras such that $\mathcal{M}$ is the closure of $\cup_n \mathcal{M}_n$ with respect to the WOT.

Theorem (F.J. Murray and J. von Neumann, 1943)

(R, τ) is the unique, up to isomorphism, hyperfinite II_1 factor.

Applications

Theorem

$L_1(R)$ can be renormed to have the FPP.

Theorem (U. Haagerup, H. P. Rosenthal & F. A. Sukochev, 2000)

If $\mathcal{M}$ is an arbitrary hyperfinite von Neumann algebra of type II_1, then $L_1(\mathcal{M})$ is isomorphic to $L_1(R)$.

Corollary

If $\mathcal{M}$ is any hyperfinite von Neumann algebra of type II_1. Then $L_1(\mathcal{M})$ can be renormed to have the FPP

Applications

Carlos A. Hernández-Linares and Maria A. Japón Pineda, *Renormings and Fixed Point Property in non-commutative L_1-spaces. Preprint.*