

ON THE GROWTH OF HARDY AND BERGMAN NORMS OF FUNCTIONS IN THE DIRICHLET SPACE

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ABSTRACT. We review the Chang-Marshall inequality of Moser-Trudinger type for the Dirichlet space. We then use a weaker version of this result to derive a sharp asymptotic estimate for Hardy and Bergman norms of a Dirichlet function for large exponents.

INTRODUCTION

Denote by H^p and A^p respectively the standard Hardy and Bergman spaces of the unit disk \mathbb{D} , $0 < p < \infty$. The space $A^\infty = H^\infty$ consists of all bounded analytic functions in \mathbb{D} . Let \mathcal{D} denote the Dirichlet space of all analytic functions in \mathbb{D} such that $f' \in A^2$.

It is well known that $\mathcal{D} \subset H^p \subset A^p$ for all $p \in (0, \infty)$. However, $\mathcal{D} \not\subset H^\infty$; that is, there exist unbounded functions in \mathcal{D} . For any such function f we obviously have $\lim_{p \rightarrow \infty} \|f\|_{H^p} = \|f\|_{H^\infty} = \infty$. The *main result* of this note consists in quantifying this in asymptotic form as follows:

We have $\|f\|_{H^p} = o(p^{1/2})$ as $p \rightarrow \infty$ (and likewise for the A^p norm). The exponent one-half cannot be improved.

The proof uses two main tools: an inequality of Chang-Marshall (Moser-Trudinger) type and a theorem on the Taylor coefficients of certain logarithmic functions in the Dirichlet space.

2000 *Mathematics Subject Classification.* 31C25, 30H05.

Revised September 10, 2004.

The author is supported by MCyT grant BFM2003-07294-C02-01, Spain.

1. BACKGROUND

We begin by reviewing the basic concepts and collecting the essential facts that will be needed later.

1.1. Hardy spaces. As is customary, we denote by H^p the standard Hardy space of all functions analytic in the unit disk \mathbb{D} for which

$$\|f\|_{H^p} = \sup_{0 < r < 1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

The functions in any of these spaces have radial limits $f(e^{i\theta})$ almost everywhere on the unit circle \mathbb{T} .

The space H^2 admits the well known formula for norm computation: if $f \in H^2$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is its Taylor series in \mathbb{D} , then

$$(1) \quad \|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

1.2. Bergman spaces. Let dA denote the Lebesgue area measure, normalized so that $A(\mathbb{D}) = 1$. If $0 < p < \infty$, the *Bergman space* A^p is the set of all analytic functions f in the unit disk \mathbb{D} with finite $L^p(\mathbb{D}, dA)$ norm:

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p d\theta r dr < \infty.$$

Note that $\|f\|_{A^p}$ is a true norm if and only if $1 \leq p < \infty$ and, in this case, A^p is a Banach space. When $0 < p < 1$, A^p is still complete with respect to the metric defined by $d_p(f, g) = \|f - g\|_{A^p}^p$.

It follows easily from the formula for A^p norm above and from the fact that the integral means $\left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$ are increasing with r that $\|f\|_{A^p} \leq \|f\|_{H^p}$ and, therefore, $H^p \subset A^p$ for all p .

Formula (1) has its Bergman space analogue: if $f \in A^2$ and (a_n) is the sequence of its Taylor coefficients, then

$$(2) \quad \|f\|_{A^2}^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.$$

1.3. The Dirichlet space and Beurling's estimate. The Dirichlet space \mathcal{D} is the set of all analytic functions f in \mathbb{D} with finite Dirichlet integral. The norm in \mathcal{D} is usually given by

$$(3) \quad \|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) = |a_0|^2 + \sum_{n=1}^{\infty} n|a_n|^2 < \infty.$$

When f is a univalent (one-to-one) map, then the Jacobian of the change of variable $w = f(z)$ is precisely $|f'(z)|^2$, so we get

$$\|f\|_{\mathcal{D}}^2 = \int_{f(\mathbb{D})} dA(w) = A[f(\mathbb{D})] < \infty.$$

In general, $f \in \mathcal{D}$ means that the image Riemann surface $f(\mathbb{D})$ has finite area.

It is immediate from (1) and (3) that $\mathcal{D} \subset H^2$. It is actually a well known fact, although a bit more difficult to prove, that $\mathcal{D} \subset H^p$ for all $0 < p < \infty$ ([D], Chapter 6, Exercise 7). In any event, such inclusions are an easy consequence of the (not so easy) inequalities of Moser-Trudinger type that will be discussed here.

Obviously, the space \mathcal{D} is not contained in H^∞ (for example, there are unbounded conformal maps of \mathbb{D} onto domains of finite area).

It is often convenient to consider the closed subspace

$$\mathcal{D}_0 = \{f \in \mathcal{D} : f(0) = 0\}.$$

Since $\mathcal{D} \subset H^2$, each function f in \mathcal{D} has radial limits $f(e^{i\theta})$ almost everywhere. Let $E_\lambda = \{\theta \in [0, 2\pi] : |f(e^{i\theta})| > \lambda\}$ and let $|E_\lambda|$ be the normalized arc measure of this set on the unit circle \mathbb{T} , *i.e.*, the boundary *distribution function* of f . In his famous doctoral thesis in 1933, Beurling [Be] obtained the following estimate on this distribution function for the functions in the unit ball of \mathcal{D}_0 .

THEOREM A. *If $f \in \mathcal{D}$, $f(0) = 0$, and $\|f\|_{\mathcal{D}} \leq 1$ then $|E_\lambda| \leq e^{-\lambda^2+1}$.*

He also showed that this deep result is sharp by using a family of logarithmic functions. It seems that it was observed only much later that Beurling's estimate implies another important inequality of Moser-Trudinger type.

1.4. The Chang-Marshall inequality. The integrability of exponential expressions of the functions whose derivative has certain integrability properties (in relation to the critical Sobolev index) has been a subject of study for several decades.

Take as an example the following variant of the Sobolev imbedding theorem, due to Hardy and Littlewood in the case of analytic functions: whenever $0 < p < 2$ and $f' \in A^p$, we have $f \in A^{\frac{2p}{2-p}}$. But what can we say about the integrability of f in the critical case $f \in \mathcal{D}$? The answer clearly cannot be that $f \in H^\infty$, as we observed in Subsection 1.3, so it should ideally again be expressed by some integrability condition on f . It turns out that if $f \in \mathcal{D}$, then it has the following property:

$$\int_{\mathbb{D}} e^{|f(z)|^2} dA(z) < \infty.$$

We can actually get a little more, but not much more!

Important results in this respect (in the more general context of real variables) are due to N. Trudinger in the late 1960's and J. Moser in the early 1970's, which is why results with this flavor are usually referred to as the

Moser-Trudinger inequalities. For a detailed bibliography, see Lecture 3 of [Ch], for example.

By integrating in polar coordinates, keeping in mind that the integral means $\left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}\right)^{1/p}$ are increasing with r , it is easy to see that

$$\int_{\mathbb{D}} e^{\alpha|f(z)|^2} dA(z) \leq \frac{1}{2\pi} \int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} d\theta \quad (\alpha > 0).$$

It may come as a surprise that even these larger integrals over the unit circle will still be finite when $f \in \mathcal{D}$. One way of proving this is, as indicated in [CM], by using Beurling's Theorem A and a nice trick due to Garnett. An alternative and simpler proof via Green's formula is given in the forthcoming paper [PV].

THEOREM B. *For every fixed f in \mathcal{D} and for all $\alpha > 0$ we still have*

$$\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} d\theta < \infty.$$

Proof. We first prove the statement in the easier case $\alpha < 1$. Applying Fubini's theorem to a function g , increasing on $[0, \infty)$ and absolutely continuous function on every closed interval of this semi-axis (as in [R], Theorem 8.16), we get

$$\begin{aligned} \int_0^{2\pi} g(|f(e^{i\theta})|) d\theta - 2\pi g(0) &= \int_0^{2\pi} \left(\int_0^{|f(e^{i\theta})|} g'(\lambda) d\lambda \right) d\theta \\ &= 2\pi \int_0^\infty |E_\lambda| g'(\lambda) d\lambda. \end{aligned}$$

By choosing $g(\lambda) = e^{\alpha\lambda^2}$ and taking into account Beurling's Theorem A, we get

$$(4) \quad \int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} \frac{d\theta}{2\pi} = 1 + 2\alpha \int_0^\infty \lambda e^{\alpha\lambda^2} |E_\lambda| d\lambda < \infty$$

for any $\alpha < 1$.

To prove the statement for arbitrary $0 < \alpha < \infty$, we follow the observation due to Garnett from p. 1016 of [CM]. If $f(z) = \sum_{n=0}^\infty a_n z^n$, there is obviously a polynomial P and $g \in \mathcal{D}$ such that $f = P + g$, $g(0) = 0$, and $\|\sqrt{3\alpha}g\|_{\mathcal{D}} \leq 1$, whence by (4) we have

$$\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} \frac{d\theta}{2\pi} \leq \int_0^{2\pi} e^{2\alpha(|P|^2 + |g|^2)} \frac{d\theta}{2\pi} \leq e^{2\alpha\|P\|_\infty^2} \int_0^{2\pi} e^{2\alpha|g(e^{i\theta})|^2} d\theta < \infty,$$

which proves the statement. \square

Even though the integrals considered above are finite for all positive α , they need not be uniformly bounded for all α ; in fact, whenever $\alpha > 1$ they are not (even if we assume that $f \in \mathcal{D}_0$)! This is shown by the same extremal logarithmic functions used by Beurling (see [CM]). In their celebrated paper

[CM], Chang and Marshall proved the following impressive result, now usually referred to as the *Chang-Marshall inequality*:

$$\sup \left\{ \int_0^{2\pi} e^{|f(e^{i\theta})|^2} d\theta : \|f\|_{\mathcal{D}} \leq 1, f(0) = 0 \right\} < \infty,$$

thus answering the important open question at that time about the uniform estimate when $\alpha = 1$. Later on, Marshall [M] simplified the initial (very difficult) proof of this statement. Mathematicians such as Essén and Carleson (and many others) have also been working on related problems.

We mention the uniform Chang-Marshall inequality with $\alpha = 1$ primarily as an important historical development but we will not need the full strength of the result. For our purpose, Theorem B (also from [CM]) will suffice. It should also be pointed out that Beurling's Theorem A alone will not be enough to deduce our main result.

2. ASYMPTOTIC FORMULAS FOR HARDY AND BERGMAN SPACE NORMS OF FUNCTIONS IN THE DIRICHLET SPACE

The notation $a_n \asymp b_n$ for two positive sequences will mean that the finite (nonzero) limit $\lim_{n \rightarrow \infty} a_n/b_n$ exists, while $a_n \lesssim b_n$ will mean that $a_n \leq C b_n$ for some fixed positive constant C and all n large enough. Similar notation will be used below for positive functions $u(p)$ of a positive real variable p instead of sequences.

It is a standard exercise to check that H^p norms increase as p increases and that $\lim_{p \rightarrow \infty} \|f\|_{H^p} = \|f\|_{H^\infty}$. In particular, if f is an unbounded function in \mathcal{D} , we have $\lim_{p \rightarrow \infty} \|f\|_{H^p} = \infty$. This can be quantified as a precise asymptotic relation for the Hardy norms as $p \rightarrow \infty$.

Observe that, if $\|f\|_{\mathcal{D}} \leq 1$ and $f(0) = 0$, then the formula for the distribution function used earlier, Beurling's Theorem A, the change of variable $t = \lambda^2$, and Stirling's formula imply

$$\begin{aligned} \int_0^{2\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi} &= p \int_0^\infty \lambda^{p-1} |E_\lambda| d\lambda \\ &\leq pe \int_0^\infty \lambda^{p-1} e^{-\lambda^2} d\lambda \\ &= \frac{pe}{2} \Gamma\left(\frac{p}{2}\right) \\ &\asymp \left(\frac{p}{2e}\right)^{\frac{p+1}{2}}, \end{aligned}$$

hence $\|f\|_{H^p} \lesssim \sqrt{p}$ as $p \rightarrow \infty$ (and, in particular, $f \in H^p$ for all p). However, this can be improved to a "little-oh" estimate, as will be shown below. The following auxiliary result will be useful.

THEOREM C. For every real β , the Taylor coefficients a_n of the function

$$(5) \quad F(z) = \left(\log \frac{2}{1-z} \right)^\beta$$

have the property that $a_n \asymp n^{-1}(\log n)^{\beta-1}$ as $n \rightarrow \infty$.

Theorem C is stated as Theorem 2.31 and proved on p. 192 of the classical monograph [Z], hence we omit its proof. We are now ready to prove our main result.

THEOREM 1. (a) If $f \in \mathcal{D}$, then its H^p norm enjoys the following asymptotic estimate:

$$(6) \quad \|f\|_{H^p} = o(p^{1/2}) \quad \text{as } p \rightarrow \infty.$$

The exponent $1/2$ is best possible; that is, for every $\varepsilon > 0$, there exists a function $F_\varepsilon \in \mathcal{D}$ such that $p^{-(1/2-\varepsilon)}\|F_\varepsilon\|_{H^p} \rightarrow \infty$ as $p \rightarrow \infty$.

(b) If $f \in \mathcal{D}$, then its A^p norm also enjoys the estimate:

$$(7) \quad \|f\|_{A^p} = o(p^{1/2}) \quad \text{as } p \rightarrow \infty,$$

and the exponent $1/2$ is best possible in the same sense as in (a).

Proof. (a) Let $f \in \mathcal{D}$. It suffices to prove (6) for $p = 2n$: the norms $\|f\|_{H^p}$ increase with p , so the general statement will follow from the inequality

$$\frac{\|f\|_{H^{2n}}}{(2n+2)^{1/2}} \leq \frac{\|f\|_{H^p}}{p^{1/2}} \leq \frac{\|f\|_{H^{2n+2}}}{(2n)^{1/2}},$$

where $2n \leq p < 2n+2$. Now by part (b) of Theorem B, for arbitrary positive α we have

$$(8) \quad \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int_0^{2\pi} |f(e^{i\theta})|^{2n} \frac{d\theta}{2\pi} = \int_0^{2\pi} e^{\alpha |f(e^{i\theta})|^2} \frac{d\theta}{2\pi} < \infty.$$

The general term of the series above must eventually be smaller than one, hence

$$\frac{\|f\|_{H^{2n}}}{(n!)^{1/(2n)}} < \frac{1}{\sqrt{\alpha}}, \quad \text{for all } n \geq N_\alpha.$$

It follows from here by Stirling's formula that

$$\limsup_{n \rightarrow \infty} \frac{\|f\|_{H^{2n}}}{n^{1/2}} \leq \frac{C}{\sqrt{\alpha}}.$$

Since this is true for all positive α , we conclude that

$$\lim_{n \rightarrow \infty} \frac{\|f\|_{H^{2n}}}{n^{1/2}} = 0,$$

so (6) follows.

To see that the exponent one-half is best possible, let $\varepsilon > 0$ be arbitrary and choose $\beta = (1 - \varepsilon)/2$. Consider the function $F = F_\varepsilon$ given by (5) with ε and β as above. Since $\beta < 1/2$, by (3) and Theorem C it follows that

$$\|F_\varepsilon\|_{\mathcal{D}}^2 = \sum_{n=1}^{\infty} n|a_n|^2 \asymp \sum_{n=1}^{\infty} n^{-1}(\log n)^{2\beta-2} < \infty,$$

and so $F_\varepsilon \in \mathcal{D}$. Again by Theorem C, the Taylor coefficients $a_{n,p}$ of the function

$$F_\varepsilon(z)^{p/2} = \left(\log \frac{2}{1-z}\right)^{p\beta/2}$$

behave asymptotically like $n^{-1}(\log n)^{(p\beta)/2-1}$. We are allowed to choose β so that $p\beta > 2$. By (1) we have

$$(9) \quad \|F_\varepsilon\|_{H^p}^p = \|F_\varepsilon^{p/2}\|_{H^2}^2 = \sum_{n=1}^{\infty} |a_{n,p}|^2 \asymp \sum_{n=1}^{\infty} n^{-2}(\log n)^{p\beta-2}.$$

The latter series is equiconvergent with the integral

$$(10) \quad \int_1^\infty \frac{1}{x^2}(\log x)^{p\beta-2} dx = \int_0^\infty e^{-t} t^{p\beta-2} dt = \Gamma(p\beta - 1),$$

which, by Stirling's formula and for large p , is asymptotically equivalent to

$$\frac{(p\beta - 1)^{p\beta-3/2}}{e^{p\beta-1}} \asymp \left(\frac{\beta}{e}\right)^{p\beta} p^{p\beta-3/2} = a^p p^{(p\varepsilon-3)/2}.$$

When divided by $p^{p/2-p\varepsilon}$, this behaves like $a^p p^{(p\varepsilon-3)/2}$ and hence tends to infinity as $p \rightarrow \infty$.

(b) We only have to worry about proving the sharpness, but this is quite similar for the A^p spaces too: instead of (9), using (2) one obtains

$$\|F_\varepsilon\|_{A^p}^p \asymp \sum_{n=1}^{\infty} n^{-3}(\log n)^{p\beta-2},$$

and instead of (10):

$$\int_0^\infty e^{-2t} t^{p\beta-2} dt = 2^{-(p\beta-1)} \Gamma(p\beta - 1).$$

The rest is completely analogous to the end. □

The exponent obtained from the apparently crude estimate (on the n -th term of a convergent series) turned out to be the best one. The heuristics behind this is that the remainder of a series of exponential type behaves asymptotically like its general term (and H^p , A^p norms increase with p).

ACKNOWLEDGMENTS

The author would like to thank Miroslav Pavlović for pointing out an inaccuracy in an earlier draft and the referee for discovering several misprints.

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