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ON SPECTRAL STRUCTURE OF BOUNDED LINEAR OPERATORS ON REFLEXIVE BANACH SPACES

STANISLAV A. SHKARIN AND OLEG G. SMOLYANOV

ABSTRACT. A descriptive characterization of point, continuous, and residual spectra of operators acting on a separable Hilbert space is obtained. The possible point spectra of bounded linear operators acting on ℓ_p, $1 < p < \infty$ are characterized.

1. INTRODUCTION

As usual $\mathbb{C}$ is the field of complex numbers, $\mathbb{R}$ is the field of real numbers, $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is the set of positive integers. We also denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. All vector spaces in this paper are assumed to be over the field $\mathbb{C}$ and all topological vector spaces are assumed to be Hausdorff. For a vector space X and a linear operator $T : D_T \longrightarrow X$ defined on a linear subspace D_T of X, the set

$$\sigma_p(T) = \{z \in \mathbb{C} : \dim \{x \in D_T : Tx = zx\} = \nu(z) > 0\}$$

is called the *point spectrum* [2] of T. The number $\nu(z) \in \overline{\mathbb{N}}$ for $z \in \sigma_p(T)$ is called the *multiplicity* of z. For any $n \in \overline{\mathbb{N}}$ we denote

$$\sigma_{p,n}(T) = \{z \in \sigma_p(T) : \nu(z) = n\} \quad \text{and} \quad \sigma_p^n(T) = \{z \in \sigma_p(T) : \nu(z) \geqslant n\}.$$

Note that $\sigma_p(T) = \sigma_p^1(T)$ and $\sigma_p^\infty(T) = \sigma_{p,\infty}(T)$.

For a topological vector space X, an operator $T : D_T \longrightarrow X$ is said to be *closed* [2] if its graph $\Gamma_T = \{(x, Tx) : x \in D_T\}$ is closed in $X \times X$ and T is called

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densely defined [2], if D_T is dense in X. The *spectrum* of $T : D_T \subseteq X \longrightarrow X$ is [2] the set

$$\sigma(T) = \mathbb{C} \setminus \left\{ z \in \mathbb{C} : \begin{array}{c} \text{the operator } T - zI \text{ has continuous} \\ \text{densely defined inverse} \end{array} \right\},$$

where $I : X \longrightarrow X$ is the identity operator. The sets

$$\sigma_c(T) = \{z \in \sigma(T) \setminus \sigma_p(T) : \text{the set } (T - zI)(D_T) \text{ is dense in } X\} \quad \text{and}$$
$$\sigma_r(T) = \{z \in \sigma(T) \setminus \sigma_p(T) : \text{the set } (T - zI)(D_T) \text{ is not dense in } X\}$$

are called *continuous spectrum* and *residual spectrum* [2] of T, respectively. Obviously, $\sigma(T)$ is the disjoint union of three sets $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$.

For any non-empty compact set $K \subset \mathbb{R}$, Kalisch [3] constructed a bounded linear operator T acting on a separable Hilbert space such that $\sigma(T) = \sigma_p(T) = K$. Using a similar construction, Nikolskaia [8] proved that a set $A \subset \mathbb{C}$ is the point spectrum of a linear continuous operators on a separable Hilbert space if and only if A is a bounded F_σ-set. R. Kaufmann [4, 5, 6, 7] proved that a set $A \subset \mathbb{C}$ is a point spectrum of a bounded linear operator on a separable Banach space if and only if A is bounded and is a Souslin set, that is, the continuous image of a complete separable metric space. The result of Kaufmann was strengthened by the authors in the following way [11].

Theorem S. *Let A, B, C be three disjoint subsets of $\mathbb{C}$. Then the following conditions are equivalent.*

(S1) *There exists a bounded linear operator T acting on a separable Banach space X such that $A = \sigma_p(T)$, $B = \sigma_c(T)$ and $C = \sigma_r(T)$.*

(S2) *The set $A \cup B \cup C$ is non-empty and compact, the sets A, $\mathbb{C} \setminus B$ and $\mathbb{C} \setminus C$ are Souslin and there exists an F_σ-set D and an operator T such that $\sigma_p(T) \cup D = \sigma_p(T) \cup \sigma_r(T)$.*

In the present work we provide the following characterization of the sets $\sigma_{p,n}(T)$, $\sigma_p^n(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ for bounded and for closed densely defined linear operators acting on a separable Hilbert space.

Theorem 1. I. *Let T be a closed densely defined linear operator acting on a separable Hilbert space H. Then $\sigma_p^n(T)$ is an F_σ-set for any $n \in \mathbb{N}$ and $\sigma_c(T)$ is a G_δ-set.*

II. *Let $K \subset \mathbb{C}$ be non-empty and compact and $K = A \cup B \cup C$, where $A \cap B = A \cap C = B \cap C = \varnothing$, A is an F_σ-set and B is a G_δ-set. Let also A_n be a decreasing sequence of F_σ-sets such that $A = A_1$. Then there exists a bounded linear operator T acting on a separable infinite dimensional Hilbert space such that $\sigma_p^n(T) = A_n$ for any $n \in \mathbb{N}$, $\sigma_c(T) = B$ and $\sigma_r(T) = C$.*

III. *Let $K \subset \mathbb{C}$ be a closed set and $K = A \cup B \cup C$, where $A \cap B = A \cap C = B \cap C = \varnothing$, A is an F_σ-set and B is a G_δ-set. Let also A_n be a decreasing sequence of F_σ-sets such that $A = A_1$. Then there exists a closed densely defined linear operator T acting on a separable infinite dimensional Hilbert space such that $\sigma_p^n(T) = A_n$ for any $n \in \mathbb{N}$, $\sigma_c(T) = B$ and $\sigma_r(T) = C$.*

Since the spectrum of a bounded linear operator on a Hilbert space is non-empty and compact and the spectrum of a closed densely defined linear operator on a Hilbert space is closed, Theorem 1 provides a complete description of all possible $\sigma_p^n(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ for bounded and for closed densely defined operators acting on ℓ_2. Note also that Theorem 1, even in its point spectrum part, can not be obtained from the constructions used by Nikolskaia or Kalisch, because the latter do not provide the full variety of the parts $\sigma_p^n(T)$ of the point spectrum. Unfortunately the proof of Theorem 1 does not admit any straightforward modification applicable to any single Banach space different from ℓ_2. The question whether Theorem 1 remains true if one replaces the separable Hilbert space by, for instance, ℓ_p for $1 < p < \infty$, $p \neq 2$ remains open. However, using a completely different approach, we prove an analogue of Nikolskaia's theorem for these spaces.

Theorem 2. *Let $1 < p < \infty$ and $A \subset \mathbb{C}$. Then there exists a bounded linear operator $T : \ell_p \longrightarrow \ell_p$ for which $\sigma_p(T) = A$ if and only if A is a bounded F_σ-set.*

Note also that there are separable reflexive Banach spaces X, for which the family of the sets $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ for bounded linear operators T acting on X is much poorer than for $X = \ell_2$. For instance, if one takes X being hereditarily indecomposable [1], then $\sigma_r(T) \cup \sigma_c(T) \subseteq \{0\}$ and $\sigma_p(T)$ is countable for any bounded linear operator T acting on X.

It is also worth noting that the sets appearing as spectra of continuous linear operators acting on a separable Fréchet space were characterized by Shkarin [10] (necessary conditions on a set to be such a spectrum were earlier obtained by Slodowski [12]); namely, $A \subset \mathbb{C}$ is the spectrum of some linear continuous operator acting on a separable Fréchet space if and only if A is a $G_{\delta\sigma}$-set.

2. Properties of spectral parts for general separable reflexive Banach spaces

Proposition 1. *Let X be a topological vector space and $T : D_T \longrightarrow X$ be a linear operator whose graph is a union of countably many metrizable compact sets. Then for any $n \in \mathbb{N}$, $\sigma_p^n(T)$ is an F_σ-set.*

Proof. Let $n \in \mathbb{N}$ and A, B be the sets defined by the formulas

$$A = \{((x_1, y_1), \dots, (x_n, y_n), z) \in \Gamma_T^n \times \mathbb{C} : y_j = zx_j \text{ for } 1 \leqslant j \leqslant n\},$$

$$B = \left\{((x_1, y_1), \dots, (x_n, y_n), z) \in A : \begin{matrix}\text{the vectors } x_1, \dots, x_n \\ \text{are linearly independent}\end{matrix}\right\}.$$

Since A is closed in the space $\Gamma_T^n \times \mathbb{C}$, which is a union of countably many metrizable compact sets, the set A is itself a countable union of metrizable compact sets. One can easily verify that B is open in A. Since an open subset of a metrizable compact set is a countable union of metrizable compact sets, we have that B is a countable union of metrizable compact sets. Let now

$$\varphi : B \longrightarrow \mathbb{C}, \quad \varphi((x_1, y_1), \ldots, (x_n, y_n), z) = z.$$

Since φ is continuous and a continuous image of a compact set is again a compact set, we have that the set $\varphi(B) = \sigma_p^n(T)$ is σ-compact and, therefore, is an F_σ-set. □

Corollary 1. *Let a topological vector space X be a countable union of metrizable compact sets and $T : D_T \longrightarrow X$ be a closed linear operator. Then for any $n \in \mathbb{N}$, $\sigma_p^n(T)$ is an F_σ-set.*

For a locally convex topological vector space X, the symbol X' stands for the space of linear continuous functionals on X. As usual for $y \in X'$ and $x \in X$ we write (x, y) instead of $y(x)$. If $T : D_T \longrightarrow X$ is a densely defined linear operator, then the symbol T' stands for the dual operator $T' : D_{T'} \longrightarrow X'$, that is, $D_{T'}$ is the set of $\varphi \in X'$ for which the functional $x \mapsto (Tx, \varphi)$ is continuous on D_T with respect to the topology of X and $T'\varphi \in X'$ is the (unique) continuous extension of this functional. Note that the operator T' is always closed when X' is endowed with the $*$-weak topology $\sigma(X', X)$ (see, for instance, [9]).

Corollary 2. *Let X be a separable metrizable locally convex topological vector space and $T : D_T \subseteq X \longrightarrow X$ be a densely defined linear operator. Then for any $n \in \mathbb{N}$, $\sigma_p^n(T')$ is an F_σ-set.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a base of neighborhoods of zero in X. Then X' is the union of the sets $U_n^\circ = \{y \in X' : |(x, y)| \leqslant 1 \text{ for any } x \in U_n\}$. Alaoglu's theorem [9] implies that U_n° are compact in the $*$-weak topology $\sigma(X', X)$. Since X is separable, the compact spaces $(U_n^\circ, \sigma(X', X))$ are metrizable [9]. Since T' is a closed operator on $(X', \sigma(X', X))$, it remains to apply Corollary 1. □

Proposition 2. *Let X be a locally convex topological vector space and $T : D_T \longrightarrow X$ be a closed densely defined linear operator. Then $\sigma_p(T) \cup \sigma_r(T) = \sigma_p(T) \cup \sigma_p(T')$.*

Proof. Let $z \in \sigma_r(T)$. Then the linear space $(T - zI)(X)$ is not dense in X. By Hahn–Banach theorem [9] there exists $y \in X' \setminus \{0\}$ such that $((T - zI)x, y) = 0$ and therefore $(Tx, y) = (x, zy)$ for any $x \in X$. Hence $y \in D_{T'}$ and $T'y = zy$. Thus $z \in \sigma_p(T')$. Let now $z \in \sigma_p(T')$. Then there exists $y \in D_{T'} \setminus \{0\}$ such that $T'y = zy$. Hence $(Tx, y) = (x, T'y) = (x, zy) = (zx, y)$ and therefore $((T - zI)x, y) = 0$ for any $x \in D_T$. So we have $(T - zI)(D_T) \subseteq \ker y$. Thus the set $(T - zI)(D_T)$ is not dense in X. It follows that $z \in \sigma_r(T) \cup \sigma_p(T)$. □

Proposition 3. *Let X be a separable reflexive Banach space and $T : D_T \longrightarrow X$ be a closed densely defined linear operator. Then for any $n \in \mathbb{N}$, $\sigma_p^n(T)$ is an F_σ-set and $\sigma_c(T)$ is a G_δ-set.*

Proof. Let X_σ be the space X endowed with the weak topology. Since a linear subspace of a Banach space is closed if and only if it is weakly closed and is dense if and only if it is weakly dense, we have that T is a closed densely defined linear operator on the space X_σ. Since closed balls of X are metrizable and compact in the weak topology, we have that X_σ is a countable union of metrizable compact sets. Applying Corollary 1, we obtain that $\sigma_p^n(T)$ is an F_σ-set for any $n \in \mathbb{N}$. Corollary 2 implies that $\sigma_p(T')$ is an F_σ-set. According to Proposition 2, $\sigma_p(T) \cup \sigma_r(T) = \sigma_p(T) \cup \sigma_p(T')$. Hence $\sigma_p(T) \cup \sigma_r(T)$ is an F_σ-set. Since $\sigma(T)$ is closed, we have that $\sigma_c(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_r(T))$ is a G_δ-set. □

Remark 1. Recall that a Banach space X is called *quasireflexive* if $\dim X''/X < +\infty$. In particular, any reflexive Banach space is quasireflexive. Slightly modifying the first part of the proof, one can see that Proposition 3 remains true if reflexivity is replaced by quasireflexivity.

3. Proof of Theorem 1

We need some additional notation and auxiliary lemmas.

Let $S = S(\mathbb{R}^2)$ be the Schwarz space of rapidly decreasing infinitely differentiable functions on the plane; let S' be the dual space of the Fréchet space S (it is usually called the space of Schwarz distributions [9]) and let $\Phi : S' \longrightarrow S'$ be the Fourier transform. Let also $\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ be given by the formula $\alpha(x, y) = (1 + x^2)^{-1}(1 + y^2)^{-1}$.

Consider the space $E = \{f \in S' : \alpha \cdot \Phi f \in L_2(\mathbb{R}^2)\}$ endowed with the inner product

$$(f, g)_E = (\alpha \cdot \Phi f, \alpha \cdot \Phi g)_{L_2(\mathbb{R}^2)} = \iint\limits_{\mathbb{R}^2} \alpha^2(x, y)\Phi f(x, y)\overline{\Phi g(x, y)}\, dx\, dy.$$

Since the map $f \mapsto \alpha \cdot \Phi f$ is a linear homeomorphism of S' onto S', we have that E is a Hilbert space and the topology of E defined by the inner product $(\cdot, \cdot)_E$ is stronger than the topology of S'. Note also that $L_2(\mathbb{R}^2) \subset E$ and the topology of E is weaker than the natural Hilbert space topology of $L_2(\mathbb{R}^2)$. We shall also use the following notation. For two functions A and B defined on the same set we write $A \ll B$ if there exists $c > 0$ such that $|A| \leqslant c|B|$. For $\varphi \in S$ denote

$$p(\varphi) = \iint\limits_{\mathbb{R}^2} \left| \left(1 + \frac{\partial^4}{\partial x^4}\right) \left(1 + \frac{\partial^4}{\partial y^4}\right) \varphi(x, y) \right| dx\, dy.$$

Clearly p is a continuous norm on the locally convex topological vector space S.

Lemma 1. *$\|\varphi \cdot f\|_E \ll p(\varphi)\|f\|_E$ for $f \in E$ and $\varphi \in S$.*

Proof. By definition $\|\varphi \cdot f\|_E = \|\alpha \cdot \Phi(\varphi \cdot f)\|_{L_2(\mathbb{R}^2)} \ll \|\alpha \cdot (\Phi\varphi * \Phi f)\|_{L_2(\mathbb{R}^2)}$, where $*$ denotes the convolution of functions. Using the definition of p and the well-known properties of the Fourier transform, we obtain $\Phi\varphi \ll p(\varphi) \cdot \beta$, where $\beta(x,y)=(1+x^4)^{-1}(1+y^4)^{-1}$. Hence

$$\|\varphi\cdot f\|_E^2 \ll p^2(\varphi)\|\alpha\cdot(\beta*\Phi f)\|^2_{L_2(\mathbb{R}^2)} = p^2(\varphi)\iint\limits_{\mathbb{R}^2}\alpha^2(x,y)\times$$

$$\times\iint\limits_{\mathbb{R}^2}|\Phi f(u,v)|^2\beta(x-u,y-v)\,du\,dv\iint\limits_{\mathbb{R}^2}|\Phi f(s,t)|^2\beta(x-s,y-t)\,ds\,dt\;dx\,dy=$$

$$=p^2(\varphi)\iiiint\limits_{\mathbb{R}^4}|\Phi f(u,v)\Phi f(s,t)|\int\limits_{\mathbb{R}}\frac{\beta(x-u,x-s)dx}{(1+x^2)^2}\int\limits_{\mathbb{R}}\frac{\beta(y-v,y-t)dy}{(1+y^2)^2}du\,dv\,ds\,dt.$$

One can easily verify that

$$\int\limits_{\mathbb{R}}\frac{dx}{(1+x^2)^2(1+(x-u)^4)(1+(x-s)^4)}\ll\frac{1}{1+(u-s)^4}\left[\frac{1}{(1+u^2)^2}+\frac{1}{(1+s^2)^2}\right].$$

Thus

$$\|\varphi\cdot f\|_E^2\ll p^2(\varphi)\iiiint\limits_{\mathbb{R}^4}\frac{|\Phi f(u,v)\Phi f(s,t)|}{1+(u-s)^4}\left[\frac{1}{(1+u^2)^2}+\frac{1}{(1+s^2)^2}\right]\times$$

$$\times\frac{1}{1+(v-t)^4}\left[\frac{1}{(1+v^2)^2}+\frac{1}{(1+t^2)^2}\right]du\,dv\,ds\,dt.$$

Performing the change of variables $a=u-s$, $b=v-t$ (we pass from variables u,v,t,s to variables u,v,a,b) in the last integral and denoting $g=\alpha\cdot\Phi f$, we have

$$\|\varphi\cdot f\|_E^2\ll p^2(\varphi)\iint\limits_{\mathbb{R}^2}\left[\iint\limits_{\mathbb{R}^2}\frac{|g(u,v)g(u-a,v-b)|}{1+a^4}\left[\frac{1+(u-a)^2}{1+u^2}+\frac{1+u^2}{1+(u-a)^2}\right]\times\right.$$

$$\left.\times\frac{1}{1+b^4}\left[\frac{1+(v-b)^2}{1+v^2}+\frac{1+v^2}{1+(v-b)^2}\right]du\,dv\right]da\,db.$$

Since $g\in L_2(\mathbb{R}^2)$, we obtain

$$\iint\limits_{\mathbb{R}^2}|g(u,v)g(u-a,v-b)|\,du\,dv\leqslant\|g\|^2_{L_2(\mathbb{R}^2)}=\|f\|_E^2.$$

Hence,

$$\|\varphi\cdot f\|_E^2\ll p^2(\varphi)\|f\|_E^2\iint\limits_{\mathbb{R}^2}\frac{1}{1+a^4}\sup_{u\in\mathbb{R}}\left[\frac{1+(u-a)^2}{1+u^2}+\frac{1+u^2}{1+(u-a)^2}\right]\frac{1}{1+b^4}\times$$

$$\times\sup_{v\in\mathbb{R}}\left[\frac{1+(v-b)^2}{1+v^2}+\frac{1+v^2}{1+(v-b)^2}\right]da\,db\ll p^2(\varphi)\|f\|_E^2\iint\limits_{\mathbb{R}^2}\frac{1}{1+a^2}\frac{1}{1+b^2}\,da\,db$$

$$\ll p^2(\varphi)\|f\|_E^2.$$

Thus $\|\varphi\cdot f\|_E\ll p(\varphi)\|f\|_E$. □

Let also $\gamma : \mathbb{R}^2 \longrightarrow \mathbb{C}$, $\gamma(x,y) = x + iy$ and $A \subset \mathbb{D} = \{z \in \mathbb{C} : |z| \leqslant 1\}$ be an infinite locally compact set. Then there exists a sequence of compact sets $K_n \subset \mathbb{C}$ such that K_n is contained in the interior of K_{n+1} in A for any $n \in \mathbb{N}$ and $A = \bigcup_{n=1}^{\infty} K_n$. Since A is locally compact, we also have that the set $\overline{A} \setminus A$ is compact. Hence for any $n \in \mathbb{N}$ there exists an infinitely differentiable function $\varphi_n : \mathbb{R}^2 \longrightarrow [0,1]$ with bounded support such that $\varphi_n\big|_{\gamma^{-1}(K_n)} \equiv 0$ and $\varphi_n\big|_{\gamma^{-1}(\overline{A}\setminus A)} \equiv 1$. Consider the space

$$E_A = \{f \in E : \operatorname{supp} f \subseteq \gamma^{-1}(\overline{A}) \text{ and } \|f\|_A < +\infty\},$$

where $\|f\|_A^2 = \|f\|_E^2 + \sum_{n=1}^{\infty} \|f \cdot \varphi_n\|_E^2$ and $\operatorname{supp} f$ is the support of the generalized function f. It is straightforward to verify that $(E_A, \|\cdot\|_A)$ is a Hilbert space.

For any $(x,y) \in \mathbb{R}^2$, the symbol $\delta_{x,y}$ stands for the Dirac's δ-function concentrated in the point (x,y). Since the function $\Phi\delta_{x,y}$ is bounded, we have that $\delta_{x,y} \in E$. If $(x,y) \in \mathbb{R}^2$ is such that $x + iy \in A$, then there exists $n \in \mathbb{N}$ for which $x + iy \in K_n$. Hence $\delta_{x,y} \cdot \varphi_m = 0$ for $m \geqslant n$. Therefore

$$\|\delta_{x,y}\|_A^2 = \|\delta_{x,y}\|_E^2 + \left(1 + \sum_{k=1}^{n-1} \varphi_n^2(x,y)\right) < +\infty.$$

It follows that $\delta_{x,y} \in E_A$ for $x+iy \in A$. Let us verify that the map $x+iy \mapsto \delta_{x,y}$ is continuous from A to the Hilbert space E_A. Let $x_n + iy_n \in A$, $(x_n, y_n \in \mathbb{R})$, $x_n \to x$, $y_n \to y$ and $x+iy \in A$. Then there exists $n \in \mathbb{N}$ for which $x+iy \in K_n$. Since K_n is contained in the interior of K_{n+1} in A, we have $x_m + iy_m \in K_{n+1}$ for sufficiently large m. For such m,

$$\|\delta_{x_m,y_m} - \delta_{x,y}\|_A^2 = \|\delta_{x_m,y_m} - \delta_{x,y}\|_E^2 + \sum_{k=1}^{n} \|\varphi_k(x_m,y_m)\delta_{x_m,y_m} - \varphi_k(x,y)\delta_{x,y}\|_E^2.$$

The definition of $\|\cdot\|_E$ and the Lebesgue theorem imply that $\|\delta_{x_m,y_m} - \delta_{x,y}\|_A \to 0$ as $m \to +\infty$. The continuity of the map $x+iy \mapsto \delta_{x,y}$ is verified. Let now H_A be the closure in E_A of the linear span of the set $\{\delta_{x,y} : x+iy \in A\}$. Since the map $x+iy \mapsto \delta_{x,y}$ is continuous, H_A is separable as a closed linear span of a separable set. Thus $(H_A, \|\cdot\|_A)$ is a separable Hilbert space. Consider now the operator $T : S' \longrightarrow S'$ defined by the formula

$$Tf = \gamma \cdot f.$$

Lemma 2. *$T(H_A) \subseteq H_A$ and the restriction T_A of T to H_A considered as an operator on the Hilbert space H_A is bounded. Moreover, $\sigma(T_A) = \overline{A}$, $\sigma_p(T_A) = \sigma_{p,1}(T_A) = A$ and the operator $T_A - zI$ has dense range in H_A for any $z \in \mathbb{C}$ which is not an isolated point of A. Moreover, there exist a constant $c \geqslant 1$ and a decreasing continuous function $a : (0,+\infty) \longrightarrow (0,+\infty)$, which do not depend on A such that $\|T_A\| \leqslant c$ and $\|(T_A - zI)^{-1}\| \leqslant a(\operatorname{dist}(z,A))$ for $z \in \mathbb{C} \setminus \overline{A}$.*

Proof. Let $\eta \in S$. According to Lemma 1 for any $f \in H_A$,

$$\begin{aligned}\|\eta \cdot f\|_A^2 &= \|\eta \cdot f\|_E^2 + \sum_{n=1}^{\infty} \|\eta \cdot f \cdot \varphi_n\|_E^2 \\ &\leqslant p^2(\eta)\left(\|f\|_E^2 + \sum_{n=1}^{\infty} \|f \cdot \varphi_n\|_E^2\right) \\ &= p^2(\eta)\|f\|_A^2.\end{aligned}$$

Choose an infinitely differentiable function γ_0 with compact support such that $\gamma_0(x,y) = \gamma(x,y)$ for $x^2+y^2 \leqslant 1$. Since the support of any $f \in E_A$ is contained in the set $\gamma^{-1}(\overline{A}) \subseteq \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leqslant 1\}$, we have $Tf = \gamma_0 \cdot f$ and $\operatorname{supp} Tf \subseteq \operatorname{supp} f \subseteq \gamma^{-1}(\overline{A})$ for any $f \in E_A$. From the fact that $Tf = \gamma \cdot f$ and Lemma 1, it follows that $\|Tf\|_A \leqslant p(\gamma_0)\|f\|_A$ for any $f \in E_A$, where p is the norm defined by at the beginning of Section 3. Hence there exists a constant $c \geqslant 1$ such that $\|Tf\|_A \leqslant c\|f\|_A$ for any $f \in E_A$. Thus the operator $T\big|_{E_A}$ acts boundedly on the Hilbert space E_A and $\|T\big|_{E_A}\| \leqslant c$. Let $x, y \in \mathbb{R}$ be such that $x+iy \in A$. Then $T\delta_{x,y} = \gamma(x,y)\delta_{x,y} \in H_A$. Hence T maps the dense in H_A linear span of the set $\{\delta_{x,y} : (x,y) \in \gamma^{-1}(A)\}$ into H_A. Since the operator $T\big|_{E_A}$ is bounded with respect to the norm $\|\cdot\|_A$ and H_A is complete, we obtain that $T(H_A) \subseteq H_A$ and $\|T_A\| \leqslant c$.

Let $\rho \in C^\infty[0,\infty)$ be such that $\rho\big|_{[0,1/2)\cup[9c,+\infty)} \equiv 0$, $\rho\big|_{[1,8c]} \equiv 1$ and $x_0, y_0 \in \mathbb{R}$ be such that $z = x_0 + iy_0 \in \mathbb{C} \setminus \overline{A}$ and $|z| \leqslant 2c$. Denote

$$\eta(x,y) = \frac{1}{x+iy-z}\rho\Big(\frac{|x+iy-z|}{\operatorname{dist}(z,\overline{A})}\Big).$$

Clearly η is an infinitely differentiable function with bounded support and $\eta\big|_{\gamma^{-1}(\overline{A})} = \frac{1}{\gamma-z}\big|_{\gamma^{-1}(\overline{A})}$. Since the support of any $f \in H_A$ is contained in the set $\gamma^{-1}(\overline{A})$, we have that $\eta \cdot (T_A - zI)f = (T - zI)(\eta \cdot f) = f$ for any $f \in H_A$. Since $Tf = \gamma \cdot f$, the operator $T - zI$ is invertible and $\|(T_A - zI)^{-1}\| \leqslant p(\eta)$. It is straightforward to verify that $p(\eta) \leqslant (\operatorname{dist}(z,A))^{-6}$. Hence there exists a constant $c_2 > 0$ for which $\|(T_A - zI)^{-1}\| \leqslant c_2 \operatorname{dist}(z,A)^{-6}$ for $z \in \mathbb{C}$, $|z| \leqslant 2c$ and $z \notin \overline{A}$. If $|z| > 2c$ then the estimate $\|T_A\| \leqslant c$ implies that $T_A - zI$ is invertible and $\|(T_A - zI)^{-1}\| \leqslant 2|z|^{-1} \leqslant 4\operatorname{dist}(z,A)^{-1}$. Hence $\sigma(T_A) \subseteq \overline{A}$ and $\|(T_A - zI)^{-1}\| \leqslant a(\operatorname{dist}(z,A))$, where $a(t) = \max\{c_2 t^{-6}, 4t^{-1}\}$. Obviously, $a : (0,+\infty) \longrightarrow (0,+\infty)$ is continuous and decreasing.

Note that the spectrum of the operator T is the entire complex plane $\mathbb{C}$, is purely point spectrum of multiplicity 1 and for any $z = x+iy \in \mathbb{C}$ $(x,y \in \mathbb{R})$ the one-dimensional space $\ker(T - zI)$ is spanned by $\delta_{x,y}$. Hence

$$\sigma_p(T_A) = \sigma_{p,1}(T_A) = \{x+iy : x,y \in \mathbb{R} \text{ and } \delta_{x,y} \in H_A\}.$$

For $x+iy \in A$, $\delta_{x,y} \in H_A$ by definition of H_A. If $x+iy \notin \overline{A}$, then $\operatorname{supp}\delta_{x,y} \not\subseteq \overline{A}$ and therefore $\delta_{x,y} \notin E_A$. Hence $\delta_{x,y} \notin H_A$. If $x+iy \in \overline{A} \setminus A$, then $\varphi_n(x,y) = 1$ for any $n \in \mathbb{N}$. Hence $\delta_{x,y} \cdot \varphi_n = \delta_{x,y}$ for any $n \in \mathbb{N}$ and the terms of the series from the definition of the norm $\|\delta_{x,y}\|_A$ are all equal to the same positive number and therefore the series diverges. Hence $\delta_{x,y} \notin E_A$ and therefore $\delta_{x,y} \notin H_A$. From the last display we have $\sigma_p(T_A) = \sigma_{p,1}(T_A) = A$. This equality together with the already proven inclusion $\sigma(T_A) \subseteq \overline{A}$ imply that $\sigma(T_A) = \overline{A}$.

It remains to verify the density in H_A of the range of the operator $T_A - zI$ when $z = x_0+iy_0 \in \mathbb{C}$ is not an isolated point of A. By definitions of T_A and H_A we have that $\delta_{x,y} \in (T_A - zI)(H_A)$ for any $(x, y) \in \gamma^{-1}(A)\backslash\{(x_0, y_0)\}$. If $z \notin A$ we have that $\{\delta_{x,y} : (x, y) \in \gamma^{-1}(A)\} \subset (T_A - zI)(H_A)$. Let $z \in A$. Since z is not an isolated point of A and the map $x+iy \mapsto \delta_{x,y}$ from A to H_A is continuous, we have that δ_{x_0,y_0} is a limit point of the set $\big\{\delta_{x,y} : (x, y) \in \gamma^{-1}(A)\backslash\{(x_0, y_0)\}\big\}$ in H_A. Thus, in any case the set $\{\delta_{x,y} : (x, y) \in \gamma^{-1}(A)\}$ is contained in the closure of $(T_A - zI)(H_A)$. Hence $(T_A - zI)(H_A)$ is dense in H_A since the set $\{\delta_{x,y} : (x, y) \in \gamma^{-1}(A)\}$ has dense linear span H_A. □

Lemma 3. *Let $K \subset \mathbb{C}$ be a nonempty compact set. Then there exists a bounded linear operator C_K acting on a separable infinite dimensional Hilbert space such that $\sigma(C_K) = \sigma_c(C_K) = K$.*

Proof. Let $H_0 = L_2[0, 1]$ and $T_0 : H_0 \longrightarrow H_0$ be the classical Volterra operator: $T_0 f(t) = \int\limits_0^t f(s)\, ds$. It is well-known that T_0 is a bounded linear operator and $\sigma(T_0) = \sigma_c(T_0) = \{0\}$. Let $\{z_n\}$ be a sequence dense in K, $H_n = H_0$ for any $n \in \mathbb{N}$ and $H = \bigoplus\limits_{n=1}^{\infty} H_n$ be the Hilbert direct sum of the Hilbert spaces H_n. Then H is a separable Hilbert space. Define the operator $C_K : H \longrightarrow H$ by the formula $(C_K x)_n = (T_0 - z_n I)x_n$. It is straightforward to verify that this operator satisfies the desired properties. □

Lemma 4. *There exists a bounded linear operator T_1 acting on ℓ_2 such that $\|T_1\| \leqslant 1$, $\sigma_p(T_1) = \sigma(T_1) = \sigma_{p,1}(T_1) = \{0\}$ and the range of T_1 is dense.*

Proof. One can easily verify that the weighted backward shift $(T_1 x)_n = x_{n+1}/(n + 1)$ satisfies the desired conditions. □

Lemma 5. *There exist a constant $c_1 > 0$ and a decreasing continuous function $a_1 : (0, +\infty) \longrightarrow (0, +\infty)$ such that for any non-empty σ-compact set $A \subset \mathbb{D}$ there exists a bounded linear operator Q_A acting on a separable infinite dimensional Hilbert space such that $\sigma(Q_A) = \overline{A}$, $\sigma_p(Q_A) = \sigma_{p,1}(Q_A) = A$, the range of the operator $(Q_A - zI)$ is dense for any $z \in \mathbb{C}$, $\|Q_A\| \leqslant c_1$ and $\|(Q_A - zI)^{-1}\| \leqslant a_1(\operatorname{dist}(z, A))$ for $z \in \mathbb{C} \setminus \overline{A}$.*

Proof. Pick an increasing sequence K_n $(n \in \mathbb{N})$ of compact sets such that $A = \bigcup_{n=1}^{\infty} K_n$. Let $K_0 = \varnothing$ and $A_n = K_n \setminus K_{n-1}$ for $n \in \mathbb{N}$. Then the sets A_n are locally compact as open subsets of compact spaces. The set A_n (as for any subset of a separable metrizable set) can be decomposed as $A_n = A_n^c \cup A_n^u$, where the set A_n^c is finite or countable, A_n^u is closed in A_n and does not have isolated points and $A_n^c \cap A_n^u = \varnothing$. Let $A^c = \bigcup_{n=1}^{\infty} A_n^c$. Clearly A^c is finite or countable. Let $\mathcal{N} = \{n \in \mathbb{N} : A_n^u \neq \varnothing\}$.

If $n \in \mathcal{N}$, then let $H_n = H_{A_n^u}$ be the space from Lemma 2 and $L_n = T_{A_n^u}$. By Lemma 2

$$(1) \qquad \sigma(L_n) = \overline{A_n^u} \quad \text{and} \quad \sigma_p(L_n) = \sigma_{p,1}(L_n) = A_n^u;$$

$$(2) \qquad \|L_n\| \leqslant c, \quad \text{and} \quad \|(L_n - zI)^{-1}\| \leqslant a(\operatorname{dist}(z, A_n^u)).$$

If $w \in A^c$, let $G_w = \ell_2$ and consider the linear operator M_w acting on G_w defined by the formula $M_w = T_1 - wI$, where T_1 is the operator furnished by Lemma 4. Taking into account the inclusion $A^c \subseteq \{z \in \mathbb{C} : |z| \leqslant 1\}$ and Lemma 4 we obtain that for any $w \in A^c$,

$$(3) \qquad \sigma(M_w) = \{w\} \quad \text{and} \quad \sigma_p(M_w) = \sigma_{p,1}(M_w) = \{w\};$$

$$(4) \qquad \|M_w\| \leqslant 2 \quad \text{and} \quad \|(L_n - wI)^{-1}\| \leqslant \widetilde{a}(|z - w|),$$

where $\widetilde{a}(t) = \sup_{z \in \mathbb{C},\ |z| \geqslant t} \|(T_1 - z)^{-1}\|$. Clearly the function $\widetilde{a} : (0, +\infty) \longrightarrow (0, +\infty)$ is continuous and decreasing.

Let now $H = \bigoplus_{n \in \mathcal{N}} H_n$, $G = \bigoplus_{w \in A^c} G_z$ and the operators $L : H \longrightarrow H$, $M : G \longrightarrow G$ are defined by the formulas $(Lx)_n = L_n x_n$ and $(Mx)_w = M_w x_w$. Finally let $X = H \oplus G$ and $Q_A : X \longrightarrow X$, $Q_A(h + g) = Lh + Mg$. The first inequalities from (2) and (4) imply that Q_A is a well-defined bounded linear operator acting on the separable infinite dimensional Hilbert space X and $\|Q_A\| \leqslant c_1 = \max\{c, 2\}$. The second inequalities from (2) and (4) imply that $\sigma(Q_A) \subseteq \overline{A}$ and $\|(Q_A - zI)^{-1}\| \leqslant a_1(\operatorname{dist}(z, A))$ for $z \in \mathbb{C} \setminus \overline{A}$, where $a_1 = \max\{a, \widetilde{a}\}$. Moreover, the second relations from (1) and (3) and disjointness of A_n^u and A^c imply that

$$\sigma_p(Q_A) = \sigma_{p,1}(Q_A) = \bigcup_{n \in \mathcal{N}} A_n^u \cup A^c = A.$$

This formula, the inclusion $\sigma(Q_A) \subseteq \overline{A}$ and closeness of $\sigma(Q_A)$ imply that $\sigma(Q_A) = \overline{A}$. Density of the range of $Q_A - zI$ for any $z \in \mathbb{C}$ follows from the fact that the operator $Q_A - zI$ is a direct sum of the operators $L_n - zI$ and $M_w - zI$, which have dense ranges according to Lemmas 2 and 4. □

Lemma 6. *Let $K \subset \mathbb{C}$ be non-empty and compact and $A \subseteq K$ be a σ-compact set. Then there exists a bounded linear operator $R_{K,A}$ on a separable infinite dimensional Hilbert space such that $\sigma(R_{K,A}) = K$, $\sigma_r(R_{K,A}) = A$ and $\sigma_c(R_{K,A}) = K \setminus A$.*

Proof. Without loss of generality we may assume that $K \subseteq \mathbb{D}$. If $A = \varnothing$, then we can take $R_{K,A} = C_K$, where C_K is the operator furnished by Lemma 3. If $A \neq \varnothing$ consider the set $\widetilde{A} = \{\overline{z} : z \in A\}$. Let H_1 and H_2 be separable infinite dimensional Hilbert spaces. By Lemma 3, there exists a bounded linear operator $C_K : H_1 \longrightarrow H_1$ such that $\sigma(C_K) = \sigma_c(C_K) = K$. By Lemma 5, there exists a bounded linear operator $Q_{\widetilde{A}} : H_2 \longrightarrow H_2$ such that $\sigma(Q_{\widetilde{A}}) =$

$\overline{\widetilde{A}}$, $\sigma_p(Q_{\widetilde{A}}) = \widetilde{A}$ and the operator $Q_{\widetilde{A}} - zI$ has dense range for any $z \in \mathbb{C}$. Let $H = H_1 \oplus H_2$ and $R_{K,A} : H \longrightarrow H$ be the direct sum of $Q^*_{\widetilde{A}}$ and C_K: $R_{K,A}(h_1 + h_2) = C_K h_1 + Q^*_{\widetilde{A}} h_2$, where $Q^*_{\widetilde{A}}$ is the adjoint of $Q_{\widetilde{A}}$.

Since for any bounded linear operator Q acting on a Hilbert space, $\sigma(Q^*) = \{\overline{z} : z \in \sigma(Q)\}$, we have that $\sigma(Q^*_{\widetilde{A}}) = \overline{A} \subseteq K$. Let us verify that $\sigma_p(Q^*_{\widetilde{A}}) = \varnothing$. Indeed, let $x \in H_2$ and $Q^*_{\widetilde{A}} x = zx$ for some $z \in \mathbb{C}$. Then $(Q^*_{\widetilde{A}} x, y) = (zx, y)$ for any $y \in H_2$. Hence $(x, Q_{\widetilde{A}} y - \overline{z} y) = 0$ for any $y \in H_2$. Density of the range of $Q_{\widetilde{A}} - zI$ implies that $x = 0$. Hence $\sigma_p(Q^*_{\widetilde{A}}) = \varnothing$. Taking into account the mentioned properties of $Q_{\widetilde{A}}$ and C_K, we obtain that $\sigma(R_{K,A}) = K$, $\sigma_p(R_{K,A}) = \varnothing$ and $\sigma_r(R_{K,A}) = \sigma_r(Q^*_{\widetilde{A}})$. It remains to verify that $\sigma_r(Q^*_{\widetilde{A}}) = A$. Let $z \in \mathbb{C}$. Then $\overline{(Q^*_{\widetilde{A}} - zI)(H_2)} = \ker(Q_{\widetilde{A}} - \overline{z}I)^{\perp}$. Since $\sigma_p(Q^*_{\widetilde{A}}) = \varnothing$, we have that $z \in \sigma_p(Q^*_{\widetilde{A}})$ if and only if $\overline{z} \in \sigma_p(Q_{\widetilde{A}}) = \widetilde{A}$, which implies the equality $\sigma_r(Q^*_{\widetilde{A}}) = A$. □

Now we can prove Theorem 1. Part I follows from Lemma 5.

Part II. Without loss of generality we may assume that K is contained in $\mathbb{D}$. By Lemma 6, there exists a bounded linear operator $L_0 = R_{K,A\cup C}$ acting on a separable infinite dimensional Hilbert space H_0 such that $\sigma_r(L_0) = A \cup C$, $\sigma(L_0) = K$ and $\sigma_c(L_0) = B$. Since A_n is σ-compact, Lemma 5 implies the existence of a bounded linear operator $L_n = Q_{A_n}$ acting on a separable infinite dimensional Hilbert space H_n such that $\sigma(L_n) = \overline{A_n}$, $\sigma_p(L_n) = \sigma_{p,1}(L_n) = A_n$, $(L_n - z)(H_n)$ is dense in H_n for any $z \in \mathbb{C}$, $\|L_n\| \leqslant c_1$ and $\|(L_n - zI)^{-1}\| \leqslant a_1(\operatorname{dist}(z, A_n))$ for $z \in \mathbb{C} \setminus \overline{A_n}$, where the constant c_1 and the function a_1 are furnished by Lemma 5.

Let now H be the Hilbert direct sum of H_n: $H = \bigoplus_{n=0}^{\infty} H_n$ and $L : H \longrightarrow H$ be the linear operator defined by the formula $(Lx)_n = L_n x_n$. The estimates $\|L_n\| \leqslant c_1$ imply that L is well-defined and bounded. From the inequalities $\|(L_n - zI)^{-1}\| \leqslant a_1(\operatorname{dist}(z, A_n))$ it follows that

$$\sigma(L) = \overline{\bigcup_{n=0}^{\infty} \sigma(L_n)} = K \quad \text{and} \qquad \sigma_p(L) = \bigcup_{n=0}^{\infty} \sigma_p(L_n) = A.$$

For any $z \in A = A_1$, the multiplicity of the element z of the point spectrum of L is equal to the number of n's for which z belongs to the point spectrum of L_n. Since $\sigma_p(L_n) = A_n$, for any $n \in \mathbb{N}$, and $\sigma_p(L_0) = \varnothing$, we have that $\sigma_p^n(L) = A_n$ for any $n \in \mathbb{N}$. Furthermore,

$$\sigma_r(L) = \left(\bigcup_{n=0}^{\infty} \sigma_r(L_n) \right) \setminus \sigma_p(L) = \sigma_r(L_0) \setminus \sigma_p(L) = (A \cup C) \setminus A = C.$$

Hence the operator $T = L$ satisfies all desired conditions.

Part III. Let $K' = \{z \in K : \operatorname{Re} z \geqslant 0\}$, $K'' = \{z \in K : \operatorname{Re} z \leqslant 0\}$, $A'_n = \{z \in A_n : \operatorname{Re} z \geqslant 0,\ z \neq 0\}$, $C' = \{z \in C : \operatorname{Re} z \geqslant 0, z \neq 0\}$, $A''_n = \{z \in A_n : \operatorname{Re} z < 0, z \neq 0\}$ and $C'' = \{z \in B : \operatorname{Re} z < 0, z \neq 0\}$. Consider the functions $f_1(z) = \frac{z-i}{z+i}$ and $f_2(z) = \frac{z+i}{z-i}$. Let $K^1 = f_1(K') \cup \{1\}$, $K^2 = f_2(K'') \cup \{1\}$, $A^1_n = f_1(A'_n)$, $A^2_n = f_2(A''_n)$, $C^1 = f_1(C')$, $C^2 = f_2(C'')$, $B^1 = K^1 \setminus (A^1_1 \cup C^1)$ and $B^2 = K^2 \setminus (A^2_1 \cup C^2)$. Since f_1 and f_2 are automorphisms of the Riemann sphere, f_1 maps the upper half-plane onto $\mathbb{D}$ and f_2 maps the lower half-plane onto $\mathbb{D}$, we obtain that K^1 and K^2 are non-empty closed subsets of $\mathbb{D}$, A^1_n and A^2_n are decreasing sequences of F_σ-sets, B^1 and B^2 are G_δ-sets and $K^1 = A^1_1 \cup B^1 \cup C^1$, $K^2 = A^2_1 \cup B^2 \cup C^2$, $A^1_1 \cap B^1 = A^1_1 \cap C^1 = B^1 \cap C^1 = \varnothing$, $A^2_1 \cap B^2 = A^2_1 \cap C^2 = B^2 \cap C^2 = \varnothing$. The already proven Part II of Theorem 1 implies the existence of bounded linear operators T_1 and T_2 acting on separable infinite dimensional Hilbert spaces H_1 and H_2 respectively, such that $\sigma(T_j) = K^j$, $\sigma^n_p(T_j) = A^j_n$, $\sigma_r(T_j) = C^j$ and $\sigma_c(T_j) = B^j$ for $j \in \{1, 2\}$. Let $S_1 = i(1 + T_1)(1 - T_1)^{-1}$ and $S_2 = -i(1 + T_2)(1 - T_2)^{-1}$. Since $0 \notin A'_1 \cup C' \cup A''_1 \cup C''$ and $\infty \notin A'_1 \cup C' \cup A''_1 \cup C''$ we have that $-1 \notin \sigma_p(T_j) \cup \sigma_c(T_j)$ and $1 \notin \sigma_p(T_j) \cup \sigma_c(T_j)$ for $j \in \{1, 2\}$. Hence the operators S_j are well-defined closed densely defined linear operators. Moreover, one can easily verify that $\sigma(S_j) = f_j^{-1}(K^j)$, $\sigma^n_p(S_j) = f_j^{-1}(A^j_n)$ and $\sigma_c(S_j) = f_j^{-1}(C^j)$ for $j \in \{1, 2\}$. Now we see that if $0 \notin K$, then the direct sum of S_1 and S_2 satisfies the desired conditions. If $0 \in K$, then Part II of Theorem 1 implies the existence of a bounded linear operator S_3 on a separable Hilbert space H_3, whose spectrum is the one-point set $\{0\}$ and is point of multiplicity n if $0 \in A_n \setminus A_{n+1}$, is point of infinite multiplicity if $0 \in \bigcap_{n=1}^{\infty} A_n$, is residual if $0 \in C$ and is continuous if $0 \in B$. The direct sum of S_1, S_2 and S_3 then satisfies all desired conditions. □

Remark 2. Let $A \subset \mathbb{C}$ be a bounded set. By Theorem 1 there exists a bounded linear operator T on a separable Hilbert space H_0 such that $\sigma_p(T) = \overline{A}$. For any $z \in A$ pick $x_z \in H_0 \setminus \{0\}$ such that $Tx_z = zx_z$, let L be the linear span of the set $\{x_z : z \in A\}$ and $H = \overline{L}$. Then $T\big|_L : L \longrightarrow H$ is a densely defined linear operator on the separable Hilbert space H with point spectrum A. Thus the closeness condition is essential in Theorem 1.

4. Proof of Theorem 2

Lemma 7. *Let $p \in [1, \infty)$ and $K \subset \mathbb{D}$ be a compact set such that $0 \notin K$. Then there exists a bounded linear operator T acting on ℓ_p such that $\|T\| \leqslant 5$ and $\sigma_p(T) = \sigma_{p,1}(T) = K$.*

Proof. We would like to interpret ℓ_p as the space $E = \ell_p(\mathbb{Z}_+ \times \mathbb{Z}_+)$ of sequences $x = \{x_{n,k}\}_{n,k \in \mathbb{Z}_+}$ for which

$$\|x\|^p = \sum_{n,k=0}^{\infty} |x_{n,k}|^p < \infty.$$

Let $U = \{z \in \mathbb{C} : |z| < 2 \text{ and } z \notin K\}$. For any $z \in U$ let $d(z)$ be the distance from the point z to the compact set K and let $U(z) = \{w \in \mathbb{C} : |w - z| < d(z)/2\}$. Clearly U is contained in $\bigcup_{z \in U} U(z)$. Since any family of open subsets of a separable metric space has a countable subfamily with the same union, we have that there exists a sequence z_k of elements of U such that $U \subset \bigcup_{k=1}^{\infty} U(z_k)$. Denote $r_k = d(z_k)/2$.

Consider the operator $T : E \longrightarrow E$ defined by the formula

$$(Tx)_{n,k} = \begin{cases} 2x_{0,k+1} & \text{if } n = 0, \\ 2^{-n}x_{0,0} & \text{if } k = 0 \text{ and } n \geqslant 1, \\ z_n x_{n,k} + r_n x_{n,k-1} & \text{if } k \geqslant 1 \text{ and } n \geqslant 1. \end{cases} \tag{5}$$

One can easily verify that T is bounded, $\|T\| \leqslant 5$ and $0 \notin \sigma_p(T)$. We are going to compute the point spectrum of T. Let $x \in E \setminus \{0\}$ and $z \in \mathbb{C} \setminus \{0\}$ be such that $Tx = zx$. Using (5) we obtain that the equation $Tx = zx$ is equivalent to the following system

$$x_{n,k} = \begin{cases} (z/2)^k x_{0,0} & \text{if } n = 0, \\ \frac{r_n^k 2^{-n}}{z(z-z_n)^k} x_{0,0} & \text{if } n \geqslant 1. \end{cases} \tag{6}$$

The last formula implies that $x = 0$ if $x_{0,0} = 0$. Hence $x_{0,0} = c \neq 0$. From (6) we have $x_{0,k} = c(z/2)^k$. Since $x \in E$, we have $|z| < 2$. Applying (6) once again, we obtain that for any $n \in \mathbb{N}$, $x_{n,k} = \frac{cr_n^k 2^{-n}}{z(z-z_n)^k}$. Since $x \in E$, we have that $|z - z_k| > r_k$, that is $z \notin U(z_k)$. Since the union of $U(z_k)$ contains U, we obtain that $|z| < 2$ and $z \notin U$. Hence $z \in K$. The inclusion $\sigma_p(T) \subset K$ is proved. Let now $z \in K$ and x be the sequence defined by the formula (6) with $x_{0,0} = 1$. Since $|z| \leqslant 1$ and $|z - z_n| \geqslant 2r_n$ for any $n \in \mathbb{N}$, we obtain that $|x_{n,k}| \leqslant \frac{2^{-n-k}}{|z|}$. Hence $x \in E$. Since (6) is equivalent to the equation $Tx = zx$, we have that x is an eigenvector of T corresponding to the eigenvalue z. Hence $\sigma_p(T) \supseteq K$. Thus, $\sigma_p(T) = K$. Since, as we have already mentioned, a vector x satisfying (6) is uniquely determined by its coordinate $x_{0,0}$, we have that all eigenvalues of T are of multiplicity 1. Hence $\sigma_p(T) = \sigma_{p,1}(T) = K$. □

Now we can prove Theorem 2. If $A = \sigma_p(T)$ for some bounded linear operator acting on ℓ_p with $1 < p < \infty$, then according to Proposition 3, A is a bounded F_σ-set. Let A be a bounded F_σ-set. We have to construct a bounded linear operator T on ℓ_p such that $\sigma_p(T) = A$. Without loss of generality, we may assume that $0 \notin A$ and $A \subseteq \mathbb{D}$. Pick a sequence K_n of compact sets for which $A = \bigcup_{n=0}^{\infty} K_n$. According to Lemma 7, for any $n \in \mathbb{Z}_+$, there exists a bounded linear operator T_n on ℓ_p such that $\|T_n\| \leqslant 5$ and $\sigma_p(T_n) = K_n$. Now the direct ℓ_p-sum T of the operators T_n satisfies $\sigma_p(T) = A$ and we are done.

Finally we would like to mention a few open problems.

Question 1. Is Theorem 1 true for operators acting on ℓ_p, $1 < p < \infty$? The same question can be asked about other natural reflexive Banach spaces.

Question 2 (E. Gorin). Does there exist a non-normed separable Fréchet space X such that the spectrum of any linear continuous operator acting on X is bounded?

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E-mail address: `Stanislav.Shkarin@math.uni-wuppertal.de`

Faculty of Mathematics and Mechanics, Moscow State University, 119899 Moscow, Vorobjovy gory, Russia
E-mail address: `Smolyanov@yandex.ru`