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COMPOSITION OPERATORS ON THE WEIGHTED BERGMAN-NEVANLINNA CLASSES

SOMDATT D. SHARMA AND AJAY K. SHARMA

ABSTRACT. In this paper we use an α-Carleson measure and a vanishing Carleson measure to characterize bounded and compact composition operators on weighted Bergman-Nevanlinna spaces.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. For each analytic self-map ϕ of $\mathbb{D}$, the composition operator is defined by $C_\phi f = f \circ \phi$ for all f analytic on $\mathbb{D}$. It is well known that C_ϕ is a bounded linear operator on the Hardy spaces H^p of the unit disk, $0 < p < \infty$, as well as on the weighted Bergman spaces A^p_α of the unit disk, $0 < p < \infty$. Compact composition operators are among the most studied composition operators on these spaces. The study of compact composition operators on H^2 of $\mathbb{D}$ was initiated by H. J. Schwartz [S] in his unpublished thesis in the late sixties. He proved that if C_ϕ is compact, then $|\phi^*| < 1$ a.e. on the unit circle. In other words, C_ϕ is not compact whenever the set $\{|\phi^*| = 1\}$ has positive measure. Schwartz also proved that this condition is not sufficient by showing that the composition operator induced by $\phi(z) = \frac{1+z}{2}$ is not compact, even though the range of ϕ touches the unit disk just at one point. The complete characterization of ϕ for which C_ϕ is compact on H^2 have been given by Shapiro [Sh1-2] and McCluer [Mac].

As operators on the Nevanlinna class composition operators were first studied by Masri in his thesis [Mas], where he obtained several necessary conditions

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and sufficient conditions on ϕ for the operator C_ϕ to be compact. The compactness of C_ϕ as an operator on the Nevanlinna classes N and N^p has been studied by Choa and Kim, see [ChK1] and [ChK2]. Our goal in the present work is to characterize those holomorphic self-maps ϕ of $\mathbb{D}$ that induce compact composition operators on the weighted Bergman-Nevanlinna class A^0_α. Our criterion provides a complete characterization of those ϕ for which C_ϕ is compact on weighted Bergman-Nevanlinna classes. We will show that, like in the case of Hardy spaces and weighted Bergman spaces, every analytic self-map ϕ of $\mathbb{D}$ induces a bounded composition operator on the weighted Bergman-Nevanlinna class A^0_α and that C_ϕ is compact on the Bergman-Nevanlinna class A^0_α if and only if it is compact on any of the weighted Bergman space A^p_α, $0 < p < \infty$.

It is known that C_ϕ is compact on the Nevanlinna class N if and only if it is compact on H^2 [ChK1] and that for an arbitrary ϕ the compactness problem for C_ϕ on H^p spaces is quite different from the one on weighted Bergman spaces [MaS]. MacCluer and Shapiro [MaS] gave a nice example of analytic self-map of $\mathbb{D}$, which induces a compact composition operator on weighted Bergman space, but does not induce a compact composition operator on the Hardy spaces. As a matter of fact, they established the existence of an inner function ϕ (holomorphic on $\mathbb{D}$ with modulus ≤ 1 everywhere on $\mathbb{D}$ and radial limits of modulus 1 almost everywhere on $\partial\mathbb{D}$) such that C_ϕ is compact on A^p_α for all $0 < p < \infty$ and $\alpha > -1$. However, it is well known that no inner function can induce a compact composition operator on any of the H^p spaces [MaS] and, therefore, the space N. In fact, they cited the following example.

Example 1.1. Let

$$\phi(z) = \exp \int_{\partial\mathbb{D}} \frac{z+\zeta}{z-\zeta} d\mu(\zeta),$$

where μ is a Borel measure on $\partial\mathbb{D}$ that is singular with respect to linear Lebesgue measure and

$$\int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{|\zeta-\omega|^2} = \infty$$

at every $\omega \in \partial\mathbb{D}$. Then ϕ is a singular inner function which induces compact composition operator on A^p_α but not on the Hardy spaces H^p.

2. Preliminaries

Let $H(\mathbb{D})$ denote the space of all holomorphic functions on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. For each $\alpha \in (-1, \infty)$, we set $d\nu_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$, $z \in \mathbb{D}$. Then $d\nu_\alpha$ is a probability measure on $\mathbb{D}$. For $0 < p < \infty$ the weighted Bergman space A^p_α is defined as

$$A^p_\alpha = \{f \in H(\mathbb{D}) : ||f||_{p,\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p d\nu_\alpha(z)\right)^{1/p} < \infty\}.$$

Note that $||f||_{p,\alpha}$ is a true norm only if $1 \leq p < \infty$. When $0 < p < 1$, A^p_α is an F-space with respect to the translation invariant metric defined by $d_p(f,g) = ||f-g||^p_{p,\alpha}$. The growth restrictions of functions in the Bergman space is essential in our study. To this end, the following sharp estimate will be useful.

Lemma 2.1 [HKZ] *Let $f \in A^p_\alpha$. Then for every z in $\mathbb{D}$, we have*

$$|f(z)| \leq \frac{||f||_{p,\alpha}}{(1-|z|^2)^{(2+\alpha)/2}}$$

with equality if and only if f is a constant multiple of the function

$$k_a(z) = \left(\frac{1-|z|^2}{(1-\overline{a}z)^2}\right)^{\frac{2+\alpha}{p}}.$$

It can be easily shown that

$$||k_a||^p_{p,\alpha} \approx 1$$

with constant depending only on α and p [Sh1, p. 400]. The weighted Bergman-Nevanlinna class A^0_α is defined by

$$A^0_\alpha = \{f \in H(\mathbb{D}) : ||f||_{0,\alpha} = \int_{\mathbb{D}} \log^+ |f(z)| d\nu_\alpha(z) < \infty\},$$

where $\log^+ x = \max(\log x, 0)$. The space A^0_α appears in the limit as $p \to 0$ of the weighted Bergman space A^p_α, in the sense of

$$\lim_{p\to 0} \frac{t^p - 1}{p} = \log^+ t, \qquad 0 < t < \infty.$$

Of course, we are abusing of the term norm since it fails to satisfy the properties of a norm, but in this case $(f,g) \to ||f-g||_{0,\alpha}$ defines a translation invariant metric on A^0_α and this turns A^0_α into a complete metric space. Obviously, the inequality

$$\log^+ x \leq \log(1+x) \leq 1 + \log^+ x, \qquad x \geq 0$$

implies that $f \in A^0_\alpha$ if and only if

$$\int_{\mathbb{D}} \log(1+|f(z)|) d\nu_\alpha(z) < \infty$$

for f holomorphic on $\mathbb{D}$.

α-Carleson measures. For $\zeta \in \partial\mathbb{D}$ and $0 < \delta < 2$, let $S(\delta,\zeta) = \{z \in \mathbb{D} : |z-\zeta| < \delta\}$. A positive Borel measure μ on $\mathbb{D}$ is called *α-Carleson measure* if

$$\sup_{\delta>0} \sup_{\zeta\in\partial\mathbb{D}} \frac{\mu(S(\delta,\zeta))}{\delta^{\alpha+2}} < \infty,$$

and a vanishing Carleson measure if

$$\lim_{\delta\to 0} \sup_{\zeta\in\partial\mathbb{D}} \frac{\mu(S(\delta,\zeta))}{\delta^{\alpha+2}} = 0.$$

We use the sets $S(\delta,\zeta)$ as the Carleson sets along with a more convenient choice of pseudo-hyperbolic disks. We now incorporate a few lines from Axler's paper [Ax] for the sake of a more self-contained exposition.

For $w \in \mathbb{D}$, let τ_w be the function defined by

$$\tau_w(z) = \frac{w - z}{1 - \overline{w}z}$$

for $z \in \mathbb{D}$. The function τ_w is an automorphism of $\mathbb{D}$. For w and z in $\mathbb{D}$, the *pseudo-hyperbolic distance* d between w and z is defined by

$$d(w,z) = |\tau_w(z)|.$$

For $0 < r < 1$ and $w \in \mathbb{D}$, denote by $D(w,r)$ the disk whose pseudo-hyperbolic center is w and whose pseudo-hyperbolic radius is r, that is,

$$D(w,r) = \left\{ z \in \mathbb{D} : \left| \frac{w - z}{1 - \overline{w}z} \right| < r \right\}.$$

Since τ_w is a linear fractional transformation, the pseudo-hyperbolic disk $D(w,r)$ is also a Euclidean disk. Except for the special case $D(0,r) = r\mathbb{D}$, the Euclidean center and Euclidean radius of $D(w,r)$ do not coincide with its pseudo-hyperbolic center and pseudo-hyperbolic radius. The Euclidean centre and Euclidean radius of $D(w,r)$ are

$$\frac{1 - r^2}{1 - r^2|w|^2} w \quad \text{and} \quad \frac{1 - |w|^2}{1 - r^2|w|^2} r,$$

respectively. For $w \in \mathbb{D}$, it is easy to verify that τ_w is its own inverse under composition $(\tau_w \circ \tau_w)(z) = z$ for all $z \in \mathbb{D}$. Another simple calculation shows that τ_w preserves pseudo-hyperbolic distances, that is,

$$d(\lambda, z) = d(\tau_w(\lambda), \tau_w(z))$$

for all $\lambda, z \in D$. Thus τ_w maps a pseudo-hyperbolic disk centered at the point λ to the pseudo-hyperbolic disk centered at $\tau_w(\lambda)$:

$$\tau_w(D(\lambda, r)) = D(\tau_w(\lambda), r)$$

for all $\lambda \in \mathbb{D}$ and $r \in (0,1)$.

By $|D(w,r)|_A$ we denote the area of $D(w,r)$.

Lemma 2.2. [Ax] *If $w \in D$ and $0 < r < 1$, then*

(i) $|D(w,r)|_A = \pi r^2 (1 - |w|^2)^2 (1 - r^2|w|^2)^{-2}$.

(ii) $\inf \left\{ \frac{(1 - |w|^2)^2}{|1 - \overline{w}z|^4} : z \in D(w,r) \right\} = \frac{(1 - r|w|)^4}{(1 - |w|^2)^2}$.

(iii) $\sup \left\{ \frac{(1 - |w|^2)^2}{|1 - \overline{w}z|^4} : z \in D(w,r) \right\} = \frac{(1 + r|w|)^4}{(1 - |w|^2)^2}$.

We also have

$$|D(w,r)|_A \approx (1 - |w|^2)^2 \approx (1 - |z|^2)^2 \approx |D(z,r)|_A,$$

for $z \in D(w,r)$, where $\approx$ means that the two quantities are bounded above and below by constants independent of w. Also for each $D(w,r)$, there is a $\zeta \in \partial\mathbb{D}$ so that $D(w,r) \subset S(\delta,\zeta)$ for $\delta \approx 1-|w|$ and for fixed r, $0<r<1$,

$$(\alpha+1)\int_{D(w,r)}(1-|z|^2)^\alpha dA(z) \approx (1-|w|^2)^{\alpha+2}.$$

Lemma 2.3. [Ax] *For a fixed r, $0<r<1$, there exists a positive constant C depending upon r such that*

$$|f(w)|^p \le \frac{C}{|D(w,r)|_A}\int_{D(w,r)}|f(z)|^p dA(z)$$

for f analytic in $\mathbb{D}$ and $w \in \mathbb{D}$.

Lemma 2.4. [Ax] *Let $0<r<1$. Then there is a sequence $\{a_n\}$ in $\mathbb{D}$ and a positive integer M such that $\cup_{n=1}^{\infty} D(a_n,r) = \mathbb{D}$ and each $z \in \mathbb{D}$ is in, at most, M of the pseudo-hyperbolic disks*

$$D\left(a_1, \frac{1+r}{2}\right), D\left(a_2, \frac{1+r}{2}\right), D\left(a_3, \frac{1+r}{2}\right), \ldots .$$

3. Boundedness of composition operators on A^0_α

In this section we provide a necessary and sufficient condition for boundedness of $C_\phi : A^0_\alpha \to A^0_\alpha$ in terms of a Carleson measure condition satisfied by the pull back measure $\nu_\alpha \circ \phi^{-1}$ on $\mathbb{D}$. We need the following lemma.

Lemma 3.1. *If μ is an α-Carleson measure on $\mathbb{D}$, then there is a constant C such that*

$$\int_{\mathbb{D}} \log(1+|f(w)|)d\mu(w) \le C\int_{\mathbb{D}} \log(1+|f(w)|)d\nu_\alpha(w)$$

for any $f \in A^0_\alpha$.

Proof. Fix $0<r<1$. Pick a sequence $\{a_n\}$ in $\mathbb{D}$ satisfying the conditions of Lemma 2.4. There are constants C', C'' and C''', such that for any $f \in A^0_\alpha$, we have

$$\int_{\mathbb{D}} \log(1+|f(w)|)d\mu(w) \le$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{\infty} \int_{D(a_n,r)} \log(1+|f(w)|)d\mu(w)\\
&\leq \sum_{n=1}^{\infty} \mu(D(a_n,r)) \sup\{\log(1+|f(w)|) : w \in D(a_n,r)\}\\
&\leq C' \sum_{n=1}^{\infty} \frac{\mu(D(a_n,r))}{|D(a_n,r)|_A} \int_{D(a_n,\frac{1+r}{2})} \log(1+|f(w)|)d\nu(w)\\
&\leq C'C'' \sum_{n=1}^{\infty} \frac{\mu(S(1-|a_n|,\zeta))}{(1-|a_n|^2)^2(1-|a_n|^2)^{\alpha}} \int_{D(a_n,\frac{1+r}{2})} \log(1+|f(w)|)d\nu_\alpha(w)\\
&= C'C'' \sum_{n=1}^{\infty} \frac{\mu(S(1-|a_n|,\zeta))}{(1-|a_n|^2)^{\alpha+2}} \int_{D(a_n,\frac{1+r}{2})} \log(1+|f(w)|)d\nu_\alpha(w)\\
&\leq C'C''C''' \sum_{n=1}^{\infty} \int_{D(a_n,\frac{1+r}{2})} \log(1+|f(w)|)d\nu_\alpha(w)\\
&\leq CM \int_{\mathbb{D}} \log(1+|f(w)|)d\nu_\alpha(w), \quad \text{where } C = C'C''C'''.
\end{aligned}
$$

□

Theorem 3.2. *Let ϕ be a holomorphic self-map of $\mathbb{D}$. Then $C_\phi : A^0_\alpha \to A^0_\alpha$ is bounded if and only if the pull back measure $\nu_\alpha \circ \phi^{-1}$ is an α-Carleson measure on $\mathbb{D}$.*

Proof. Suppose C_ϕ is bounded. Assume $0 < \delta < 1$ and $\zeta \in \partial\mathbb{D}$. Take

$$f_a(z) = \exp\left(\frac{(1-|a|^2)^{(\alpha+2)}}{(1-\overline{a}z)^{2(\alpha+2)}}\right),$$

where $a = (1-\delta)\zeta$. Now

$$
\begin{aligned}
||f_a||_{0,\alpha} &= \int_{\mathbb{D}} \log^+ \left|\exp\left(\frac{(1-|a|^2)^{(\alpha+2)}}{(1-\overline{a}z)^{2(\alpha+2)}}\right)\right| d\nu_\alpha(z)\\
&\leq \int_{\mathbb{D}} \frac{(1-|a|^2)^{(\alpha+2)}}{|1-\overline{a}z|^{2(\alpha+2)}} d\nu_\alpha(z)\\
&\leq ||k_a||^2_{2,\alpha}\\
&\approx 1.
\end{aligned}
$$

Since C_ϕ is bounded, there is a constant K such that

$$||C_\phi f_a||_{0,\alpha} \leq K||f_a||_{0,\alpha} \leq C.$$

That is,

$$\begin{aligned} C &\geq \int_{\mathbb{D}} \log^+ |f_a \circ \phi(z)| d\nu_\alpha(z) \\ &= \int_{\mathbb{D}} \Re\left(\frac{(1-|a|^2)^{(\alpha+2)}}{(1-\overline{a}\phi(z))^{2(\alpha+2)}}\right) d\nu_\alpha(z) \\ &= \int_{\mathbb{D}} \Re\left(\frac{(1-|a|^2)^{(\alpha+2)}}{(1-\overline{a}z)^{2(\alpha+2)}}\right) d\nu_\alpha \circ \phi^{-1}(z). \end{aligned}$$

Now

$$\begin{aligned} \Re\left(\frac{(1-|a|^2)^{(\alpha+2)}}{(1-\overline{a}z)^{2(\alpha+2)}}\right) &= \frac{(1-|a|^2)^{(\alpha+2)}}{(1-|a|)^{2(\alpha+2)}} \Re\left(\frac{1-|a|}{1-\overline{a}z}\right)^{2(\alpha+2)} \\ &= \frac{(1-|a|^2)^{(\alpha+2)}}{(1-|a|)^{2(\alpha+2)}} \Re\left(1+\frac{|a|(1-z\overline{\zeta})}{(1-|a|)}\right)^{-2(\alpha+2)}, (\zeta = \frac{a}{|a|}) \\ &> \frac{(1-|a|^2)^{(\alpha+2)}}{(1-|a|)^{2(\alpha+2)}} \frac{1}{2^{\alpha+2}} \\ &\geq \frac{1}{(2\delta)^{\alpha+2}} \end{aligned}$$

if $\frac{|1-z\overline{\zeta}|}{1-|a|} < \gamma_0$ for some fixed $\gamma_0 > 0$, that is, if $z \in S(\gamma_0\delta, \zeta)$. Thus for $\zeta \in \partial\mathbb{D}$ and $0 < \delta < 1$, we have

$$C \geq \frac{1}{2^{\alpha+2}} \int_{S(\gamma_0\delta,\zeta)} \frac{1}{\delta^{\alpha+2}} d\nu_\alpha \circ \phi^{-1}(z) = \frac{1}{2^{\alpha+2}} \frac{1}{\delta^{\alpha+2}} \nu_\alpha \circ \phi^{-1}(S(\gamma_0\delta,\zeta)),$$

that is, $\nu_\alpha \circ \phi^{-1}(S(\gamma_0\delta,\zeta)) \leq C\delta^{\alpha+2}$ and so $\nu_\alpha \circ \phi^{-1}$ is an α-Carleson measure on $\mathbb{D}$.

Conversely, suppose $\nu_\alpha \circ \phi^{-1}$ is an α-Carleson measure. Then, by Lemma 3.1, we have, for each $f \in A^0_\alpha$,

$$\begin{aligned} ||C_\phi f||_{0,\alpha} &= \int_{\mathbb{D}} \log(1+|(f\circ\phi)(w)|) d\nu_\alpha(w) \\ &= \int_{\mathbb{D}} \log(1+|f(w)|) d\nu_\alpha \circ \phi^{-1}(w) \\ &\leq C \int_{\mathbb{D}} \log(1+|f(w)|) d\nu_\alpha(w) \\ &= C||f||_{0,\alpha}. \end{aligned}$$

This completes the proof. □

Remark. Theorem 3.2 above and Theorem 4.3 of MacCluer and Shapiro [MaS] assert that C_ϕ is bounded on A^p_α as well as on A^0_α if and only if $\nu_\alpha \circ \phi^{-1}$ is an α-Carleson measure on $\mathbb{D}$. But in view of Theorem 3.4 of MacCluer and Shapiro [MaS], every analytic self-map ϕ of $\mathbb{D}$ induces a bounded composition operator on A^p_α for all $0 < p < \infty$ and $\alpha > -1$. Hence we conclude that every analytic self-map ϕ of $\mathbb{D}$ induces a bounded composition operator on A^0_α.

4. Compactness of composition operators on A^0_α

Before proving the main result of this section, we recall that C_ϕ is compact on A^0_α if and only if for every sequence $\{f_n\}$ which is bounded in A^0_α and converges to 0 uniformly on compact subsets of $\mathbb{D}$, we have $||C_\phi f_n||_{0,\alpha} \to 0$.

We now characterize compact composition operators on A^0_α.

Theorem 4.1. *Let ϕ be a holomorphic self-map of $\mathbb{D}$. Then $C_\phi : A^0_\alpha \to A^0_\alpha$ is compact if and only if the measure $\nu_\alpha \circ \phi^{-1}$ is a vanishing Carleson measure on $\mathbb{D}$.*

Proof. Suppose C_ϕ is compact. Let $\zeta \in \partial\mathbb{D}$ and $0 < \delta < \frac{1}{2}$. Consider the family of functions

$$f_a(z) = (1-|a|)^{\alpha+2} \exp\left(\frac{(1-|a|^2)^{(\alpha+2)}}{(1-\overline{a}z)^{2(\alpha+2)}}\right),$$

where $a = (1-\delta)\zeta$ for some $\zeta \in \partial\mathbb{D}$. Clearly $f_a \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. As in the proof of Theorem 3.2, there exists a positive constant C such that

$$||f_a||_{0,\alpha} \leq C.$$

Again as in the proof of Theorem 3.2, there exist $\gamma_0 > 0$ such that if $z \in S(\gamma_0\delta, \zeta)$, then

$$\Re\left(\frac{(1-|a|^2)^{(\alpha+2)}}{(1-\overline{a}z)^{2(\alpha+2)}}\right) \geq \frac{1}{2^{\alpha+2}\delta^{\alpha+2}}$$

and so

$$\begin{aligned}\log^+|f_a(z)| &\geq \log^+\left((1-|a|)^{\alpha+2}\exp\left(\Re\frac{(1-|a|^{\alpha+2})}{(1-\overline{a}z)^{2(\alpha+2)}}\right)\right)\\ &\geq \log^+\left(\delta^{\alpha+2}\exp\frac{1}{2^{\alpha+2}\delta^{\alpha+2}}\right).\end{aligned}$$

Therefore, we have for any $\zeta \in \partial\mathbb{D}$ and $0 < \delta < 1$

$$\begin{aligned}\log^+\left(\delta^{\alpha+2}\exp\frac{1}{2^{\alpha+2}\delta^{\alpha+2}}\right)\nu_\alpha\circ\phi^{-1}(S(\gamma_0\delta,\zeta)) &\leq \int_{S(\gamma_0\delta,\zeta)} \log^+|f_a(z)|\, d\nu_\alpha\circ\phi^{-1}(z)\\ &\leq \int_{\mathbb{D}} \log^+|f_a\circ\phi(z)|\, d\nu_\alpha(z)\\ &= ||C_\phi f_a||_{0,\alpha}.\end{aligned}$$

But compactness of C_ϕ forces $||C_\phi f_a||_{0,\alpha}$ to tend to zero as $\delta \to 0$, which implies that

$$\lim_{\delta\to 0}\left(\log^+\left(\delta^{\alpha+2}\exp\frac{1}{2^{\alpha+2}\delta^{\alpha+2}}\right)\nu_\alpha\circ\phi^{-1}(S(\gamma_0\delta,\zeta))\right) = 0$$

uniformly on $\zeta \in \partial\mathbb{D}$. Now, since

$$\lim_{\delta\to 0} \delta^{\alpha+2}\Big(\log^+ \delta^{\alpha+2} \exp\frac{1}{2^{\alpha+2}\delta^{\alpha+2}}\Big) = \lim_{t\to\infty} \frac{1}{t^{\alpha+2}}\Big(\frac{t^{\alpha+2}}{2^{\alpha+2}} - (\alpha+2)\log t\Big)$$
$$= \frac{1}{2^{\alpha+2}} > 0,$$

it follows that

$$\lim_{\delta\to 0} \frac{\nu_\alpha \circ \phi^{-1}(S(\gamma_0\delta,\zeta))}{\delta^{\alpha+2}} = 0 \text{ uniformly on } \zeta.$$

Thus $\nu_\alpha \circ \phi^{-1}$ is a vanishing Carleson measure on $\mathbb{D}$.

Conversely, suppose that $\nu_\alpha \circ \phi^{-1}$ is a vanishing Carleson measure on $\mathbb{D}$. Then

$$\frac{\nu_\alpha \circ \phi^{-1}(S(\gamma_0\delta,\zeta))}{\delta^{\alpha+2}} \to 0 \text{ uniformly in } \zeta \text{ as } \delta \to 0,$$

that is,

$$\frac{\nu_\alpha \circ \phi^{-1}(D(w,r))}{(1-|w|)^{\alpha+2}} \to 0 \text{ uniformly as } |w| \to 1.$$

Thus, for every $\varepsilon > 0$, we can choose $r_0 > 0$ such that

$$\phi^{-1}(D(w,r)) < \varepsilon(1-|w|)^{\alpha+2} \text{ for all } w \in D \text{ for } |w| > r_0.$$

Suppose $\{f_m\}$ converges to zero weakly in A^0_α. Let $\{w_n\}$ be a sequence as in Lemma 2.4 such that $|w_1| < |w_2| < \cdots$. Then

$$\phi^{-1}(D(w_n,r)) < \varepsilon(1-|w_n|)^{\alpha+2} \text{ for all } w_n \in \mathbb{D} \text{ such that } |w_n| > r_0.$$

Thus

$$\begin{aligned} ||C_\phi f_m||_{0,\alpha} &= \int_{\mathbb{D}} \log(1+|(f_m\circ\phi)(z)|)d\nu_\alpha(z) \\ &= \int_{\mathbb{D}} \log(1+|f_m(z)|)d\nu_\alpha\circ\phi^{-1}(z) \\ &= \int_{|z|\le r_0} \log(1+|f_m(z)|)d\nu_\alpha\circ\phi^{-1}(z) \\ &\qquad + \int_{|z|>r_0} \log(1+|f_m(z)|)d\nu_\alpha\circ\phi^{-1}(z). \end{aligned}$$

Since $\{f_m\}$ converges to zero on each compact subset of $\mathbb{D}$,

$$\lim_{m\to\infty} \int_{|z|\le r_0} \log(1+|f_m(z)|)d\nu_\alpha\circ\phi^{-1}(z) = 0,$$

whereas the second term in the above expression is bounded by

$$\sum_{n=k+1}^{\infty} \int_{\mathbb{D}(w_n,r)} \log(1+|f_m(z)|)d\nu_\alpha\circ\phi^{-1}(z) \le$$

$$\leq \sum_{n=k+1}^{\infty} \nu_\alpha \phi^{-1}(D(w_n,r)) \sup\{\log(1+|f_m(z)|) : z \in D(w_n,r)\}$$
$$\leq C \sum_{n=k+1}^{\infty} \frac{\nu_\alpha \phi^{-1}(D(w_n,r))}{(1-|w|)^{\alpha+2}} \int_{D(w_n,\frac{1+r}{2})} \log(1+|f_m(z)|)d\nu_\alpha(z)$$
$$\leq \varepsilon CM \int_{\mathbb{D}} \log(1+|(f_m(z)|)d\nu_\alpha(z)$$
$$= \varepsilon CM||f_m||_{0,\alpha}.$$

Since $\varepsilon > 0$ was arbitrary , we see that $||C_\phi f_m||_{0,\alpha} \to 0$ strongly. Hence C_ϕ is compact. □

Remark. It can be easily checked that the singular inner function mentioned in Example 1.1 induces a compact composition operator on A^0_α but not on the Nevanlinna class N.

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Department of Mathematics, University of Jammu, Jammu-180 006, (India)
E-mail address: somdatt_sharma@yahoo.co.in