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COMPOSITION OPERATORS WITH LINEAR FRACTIONAL SYMBOLS AND THEIR ADJOINTS

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ABSTRACT. We characterize all linear fractional maps of the disk into itself in terms of their coefficients. We also prove the formula for the adjoint of a composition operator with a linear fractional symbol acting on the quotient Dirichlet space due to Gallardo and Montes by a method different from theirs.

INTRODUCTION

As usual, we will denote by $\mathbb{D}$ the unit disk in the complex plane: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We will call φ a *self-map* of $\mathbb{D}$ if it is a holomorphic (analytic) map in $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. The *composition operator* with *symbol* φ is defined as $C_\varphi f(z) = f(\varphi(z))$, for any self-map φ. The subject of composition operators has been an active area of research for more than thirty years (*cf.* [8] and [2]).

We will study such operators acting in the quotient Dirichlet space. Denoting by dA the normalized area Lebesgue measure $\pi^{-1}dxdy$, the *Dirichlet space* $\mathcal{D}$ is defined as the Hilbert space of analytic functions in $\mathbb{D}$ whose derivative is of square integrable modulus with respect to dA.

For $f,\ g \in \mathcal{D}$ with Taylor series around $z = 0$ given by $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ respectively, we define their inner product in $\mathcal{D}$ as

$$\langle f, g\rangle_{\mathcal{D}} = a_0\overline{b_0} + \sum_{n=1}^{\infty} n\ a_n\overline{b_n}. \tag{1}$$

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If we want to work in the quotient space $\mathcal{D}_0 = \mathcal{D}/\mathbb{C}$, the Dirichlet space modulo constant functions, the inner product is defined by

$$\langle f, g\rangle_{\mathcal{D}_0} = \sum_{n=1}^{\infty} n\ a_n \overline{b_n}. \tag{2}$$

The problem of the computation of the adjoint of a composition operator with linear fractional symbol was first solved by Cowen [1] in the Hardy space. Later on, Hurst [6] using an analogous argument obtained the solution in the weighted Bergman spaces A^2_α. In 2003, Gallardo and Montes [3] computed the adjoint of a composition operator acting in the Dirichlet space by a different method from those used by Cowen and Hurst.

In this paper we first give a characterization in terms of the coefficients of all linear fractional maps φ that are self-maps of the disk and then review the transformation $\varphi \mapsto \varphi^*$ between linear fractional maps used by Cowen [1], [2] and also by other authors [3], [5].

In the second part of the paper, we use a new reasoning with the orthogonal basis of the quotient Dirichlet space to obtain the formula for the adjoint of a composition operator with linear fractional symbol C^*_φ when C_φ acts in $\mathcal{D}_0$.

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1. Preliminaries

1.1. **Pseudo-hyperbolic disks.** Denote by φ_a the disk automorphism which is an involution and interchanges the points 0 and a:

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z}\,, \qquad a \in \mathbb{D}\,. \tag{3}$$

The *pseudo-hyperbolic disk* of center a and radius r is defined as

$$\Delta(a,r) = \{z \in \mathbb{D} : |\varphi_a(z)| < r\} \qquad (0 < r < 1)\,.$$

Observe that it is a Euclidean disk since

$$\Delta(a,r) = \varphi_a(D(0,r)). \tag{4}$$

Conversely, every Euclidean disk $D(c,R) = \{z \in \mathbb{C}\colon |z-c| < R < 1\}$ is a pseudo-hyperbolic disk whose pseudo-hyperbolic radius r and center a can be computed according to the following formulas:

$$r = \frac{1+R^2-|c|^2-\sqrt{(1+R^2-|c|^2)^2-4R^2}}{2R}\,,$$

$$a = \frac{2|c|}{1-R^2+|c|^2+\sqrt{(1-R^2+|c|^2)^2-4R^2}}\,.$$

See Lemma 4 of [9].

1.2. **Linear fractional self-maps of the disk.** We say that φ is a *linear fractional map* if it has the form $\varphi(z) = \frac{az+b}{cz+d}$ for complex numbers a, b, c, d such that $ad - bc \neq 0$.

We now present some properties of linear fractional maps that we will use in Section 3.

The following important lemma was first proved in [1].

LEMMA 1. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional map. Then φ is a self-map of the disk if and only if the linear fractional transformation*

$$\varphi^*(z) = \frac{1}{\overline{\varphi^{-1}(\frac{1}{\overline{z}})}} = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}} \tag{5}$$

is also a self-map of the disk.

Proof. Denote by $\overline{\mathbb{C}}$ the extended complex plane. To prove that φ^* is a self-map of the disk, we just need to observe that φ is a self-map of the disk itself and this implies that $\varphi^{-1}(\overline{\mathbb{C}} \backslash \overline{\mathbb{D}}) \subset \overline{\mathbb{C}} \backslash \overline{\mathbb{D}}$. On the other hand, the map $1/\overline{z}$ is a one-to-one transformation from $\mathbb{D}$ onto $\overline{\mathbb{C}} \backslash \overline{\mathbb{D}}$. These two facts give us the desired conclusion.

The second identity in (5) follows by a direct calculation. □

LEMMA 2. *Let φ and ψ be two linear fractional self-maps of $\mathbb{D}$. Then*

$$(\varphi \circ \psi)^* = \psi^* \circ \varphi^*.$$

Proof. Straightforward. □

Once we know φ is a linear fractional self-map of the disk, Lemma 1 shows us how to build a related linear fractional transformation φ^* from the disk into itself which is useful in the study of composition operators. However, this does not help us in deciding whether φ actually maps $\mathbb{D}$ into itself. We will do this in the next section.

2. A CHARACTERIZATION OF THE LINEAR FRACTIONAL SELF-MAPS OF THE DISK

The following theorem tells us when a linear fractional map is a self-map of the unit disk only in terms of its coefficients a, b, c, d. Due to the lack of a specific reference, we prove it here.

THEOREM 1. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional map. Then $\varphi(\mathbb{D}) \subset \mathbb{D}$ if and only if*

$$|b\overline{d} - a\overline{c}| + |ad - bc| \leq |d|^2 - |c|^2. \tag{6}$$

Proof. First of all, we observe that if φ is a self-map of the disk, then $d \neq 0$ and $|-d/c|$ has to be greater than or equal to 1 whenever $c \neq 0$.

It is known that every univalent self-map of the disk φ such that $\varphi(\mathbb{D})$ is a Euclidean disk of center $A \in \mathbb{D}$ and radius R can be written in the form $\varphi = \lambda R \varphi_\alpha + A$, $|A| + R \leq 1, |\lambda| = 1$ and φ_α as in (3): just observe that $\psi = (\varphi - A)/R$ is a disk automorphism.

We now have to solve the equation

$$\frac{az+b}{cz+d} = \lambda R \varphi_\alpha(z) + A. \tag{7}$$

Recall that $ad - bc \neq 0$. Without loss of generality, we may assume that $d = 1$. So (7) becomes:

$$\frac{az+b}{cz+1} = \frac{-(A\overline{\alpha} + \lambda R)z + (\lambda R \alpha + A)}{-\overline{\alpha}z + 1}.$$

This is only possible when the corresponding coefficients are equal:

$$a = -(A\overline{\alpha} + \lambda R), \quad b = \lambda R \alpha + A, \quad c = -\overline{\alpha}.$$

This yields

$$R = \frac{Ac - a}{\lambda}, \quad A = \frac{b - a\overline{c}}{1 - |c|^2},$$

hence

$$R = \frac{bc - a}{\lambda(1 - |c|^2)} = \frac{|bc - a|}{1 - |c|^2}.$$

Returning to the general case when d is not necessarily 1, we get

$$R = \frac{|bc - ad|}{|d|^2 - |c|^2}. \tag{8}$$

Similarly, in the general case,

$$A = \frac{b\overline{d} - \overline{c}a}{|d|^2 - |c|^2}. \tag{9}$$

So, φ will be a self-map of the disk if and only if

$$|A| + R \leq 1$$

or, equivalently,

$$|b\overline{d} - a\overline{c}| + |ad - bc| \leq |d|^2 - |c|^2.$$

□

We remark that in 1917 Schur gave a criterion in terms of the Taylor coefficients for a general analytic function to be a self-map of the disk. See [4] or [7] for an elegant proof. However, even for linear fractional maps this criterion does not seem easy to use. It is not clear how it would imply our Theorem 1.

Observe that using both Theorem 1 and Lemma 1, we have the following curious fact about complex numbers.

Corollary 1. *Let a, b, c, d be complex numbers such that $ad-cb\neq 0$. Then*

$$|b\overline{d}-a\overline{c}|+|ad-bc|\leq |d|^2-|c|^2$$

if and only if

$$|\overline{c}d-\overline{a}b|+|ad-bc|\leq |d|^2-|b|^2.$$

3. Adjoints of composition operators with linear fractional self-maps of $\mathbb{D}$ as symbols

One of the major problems in the study of composition operators is the lack of a reasonable representation for the adjoint C_φ^*. It is known that if we denote by K_ω the reproducing kernel function, the formula $C_\varphi^*(K_\omega)=K_{\varphi(\omega)}$ holds in any of the spaces H^2, A^2, or $\mathcal{D}$. Beyond this fact, not much is known about the adjoints of composition operators. We review some important results.

Using the reproducing kernel functions, Cowen [1] obtained an expression for the adjoint of a composition operator with linear fractional self-map of the disk as symbol when it acts in the Hardy space H^2. His result was later extended using the same method to the weighted Bergman spaces A^2_α by Hurst [6].

The argument used by Cowen and Hurst is related with the particular form of the reproducing kernel functions of the Hardy and Bergman spaces and hence cannot be adapted to the Dirichlet space. However, the computation of C_φ^* for linear fractional symbol φ when C_φ acts in the Dirichlet space has been obtained recently by Gallardo and Montes using a reasoning related to fixed points and similarity to unitary operators in certain cases [3].

In this section we present a different method in terms of the orthogonal bases of the spaces mentioned above which is available for all of them and we also prove that only a few composition operators have another composition operator as adjoint.

Theorem 2. *Let C_φ be a composition operator acting in the Dirichlet space modulo constant functions, the following statements are equivalent:*

(i) *There exists a self-map of the disk ψ such that $C_\varphi^*=C_\psi$.*

(ii) *$\varphi(z)=\frac{az+b}{cz+d}$ is a linear fractional self-map of the disk.*

Moreover, if the conditions above are satisfied, ψ equals the map φ^* given by (5).

Proof. We first prove that (i) $\Longrightarrow$(ii).

Let $C_\varphi^*=C_\psi$ for some self-maps φ, ψ of the disk, where $\varphi(z)=\sum_{n=0}^\infty a_nz^n$, $\psi(z)=\sum_{n=0}^\infty b_nz^n$.

For all $n,m\in\mathbb{N}$, $\langle C_\psi z^n,z^m\rangle_{\mathcal{D}_0}=\langle z^n,C_\varphi z^m\rangle_{\mathcal{D}_0}$. Taking $n=1$ we get $\langle\psi,z^m\rangle_{\mathcal{D}_0}=\langle z,\varphi^m\rangle_{\mathcal{D}_0}$.

Keeping in mind that $\langle Az^n, Bz^m\rangle_{\mathcal{D}_0} = nA\overline{B}$ when $n = m$ and zero otherwise, we have for all $m \geq 1$

$$mb_m = \overline{(\varphi^m)'(0)} \iff b_m = \overline{\varphi^{m-1}(0)\varphi'(0)} \iff b_m = \overline{a_0^{m-1}a_1}.$$

Using the last equality,

$$\begin{aligned}\psi(z) = \sum_{n=0}^{\infty} b_n z^n &= b_0 + \sum_{n=1}^{\infty} \overline{a_0^{n-1}a_1} z^n \\ &= b_0 + \frac{\overline{a_1}z}{1-\overline{a_0}z} \\ &= \frac{(\overline{a_1}-\overline{a_0}b_0)z + b_0}{1-\overline{a_0}z}.\end{aligned}$$

Thus, ψ is a linear fractional map.

Repeating the process above and interchanging ψ and φ, we have:

$$\varphi(z) = \frac{(\overline{b_1}-\overline{b_0}a_0)z + a_0}{1-\overline{b_0}z} = \frac{(a_1-\overline{b_0}a_0)z + a_0}{1-\overline{b_0}z}.$$

That is, φ is a linear fractional self-map of the disk. Comparing with equation (5), we see that $\psi = \varphi^*$.

We now prove (ii) $\Longrightarrow$ (i). Recall that every self-map φ of the unit disk such that $\varphi(\mathbb{D})$ is a Euclidean disk of center A and radius R can be written in the form $\varphi(z) = \lambda R\varphi_\alpha(z) + A$, where $|\lambda| = 1$ and $\alpha = \varphi^{-1}(A)$. Using this fact, we will obtain the adjoint of the operator C_φ by computing the adjoints of composition operators with more elementary symbols.

We first compute the adjoint of the composition operator C_ℓ where $\ell(z) = \mathcal{A}z + \mathcal{B}$, $|\mathcal{A}| + |\mathcal{B}| \leq 1$. Note that, even though ℓ is linear, ℓ^* is not: $\ell^*(z) = \overline{\mathcal{A}}z/(1-\overline{\mathcal{B}}z)$. We will see that $\langle z^n, C_\ell z^m\rangle_{\mathcal{D}_0} = \langle C_\psi z^n, z^m\rangle_{\mathcal{D}_0}$ holds for $\psi = \ell^*$ and for all n, $m \in \mathbb{N}$ and this will prove that $C_\ell^* = C_{\ell^*}$.

Using the inner product given by (2), we have

$$\begin{aligned}\langle z^n, C_\ell z^m\rangle_{\mathcal{D}_0} &= \langle z^n, (\mathcal{A}z+\mathcal{B})^m\rangle_{\mathcal{D}_0} \\ &= \sum_{j=0}^{m} \frac{m!}{(m-j)!j!}\overline{\mathcal{B}}^{m-j}\overline{\mathcal{A}}^j \langle z^n, z^j\rangle_{\mathcal{D}_0} \\ &= \begin{cases} \dfrac{m!}{(m-n)!(n-1)!}\overline{\mathcal{B}}^{m-n}\overline{\mathcal{A}}^n, & \text{if } m \leq n; \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle C_{\ell^*} z^n, z^m\rangle_{\mathcal{D}_0} &= \left\langle \left(\frac{\overline{\mathcal{A}}z}{1-\overline{\mathcal{B}}z}\right)^n, z^m \right\rangle_{\mathcal{D}_0} \\
&= \left\langle \overline{\mathcal{A}}^n z^n \sum_{j=0}^{\infty} \frac{(j+n-1)!}{j!(n-1)!} \overline{\mathcal{B}}^j z^j, z^m \right\rangle_{\mathcal{D}_0} \qquad (10)\\
&= \sum_{k=n}^{\infty} \frac{(k-1)!}{(k-n)!(n-1)!} \overline{\mathcal{B}}^{k-n} \overline{\mathcal{A}}^n \langle z^k, z^m\rangle_{\mathcal{D}_0} \\
&= \begin{cases} \dfrac{m!}{(m-n)!(n-1)!} \overline{\mathcal{B}}^{m-n} \overline{\mathcal{A}}^n, & \text{if } m \le n; \\ 0, & \text{otherwise.} \end{cases} \\
&= \langle z^n, C_\ell z^m\rangle_{\mathcal{D}_0}.
\end{aligned}$$

In (10) we have used the obvious identity

$$\frac{1}{(1-\overline{\mathcal{B}}z)^n} = \frac{1}{\overline{\mathcal{B}}^n\,(n-1)!} \left(\frac{1}{1-\overline{\mathcal{B}}z}\right)^{(n-1)}, \quad z \in \mathbb{D}.$$

It is now left to compute $C^*_{\varphi_\alpha}$ where φ_α is an involutive automorphism as in (3). We apply the change of variable $w = \varphi_\alpha(z)$ whose Jacobian is $|\varphi'_\alpha(z)|^2$. Taking into account that $\varphi_\alpha(\varphi_\alpha(w)) = w$, we have for every $f, g \in \mathcal{D}_0$,

$$\begin{aligned}
\langle f \circ \varphi_\alpha, g\rangle_{\mathcal{D}_0} &= \int_{\mathbb{D}} f'(\varphi_\alpha(z)) \cdot \varphi'_\alpha(z) \cdot \overline{g'(z)}\, dA(z) \\
&= \int_{\mathbb{D}} f'(w) \cdot \overline{g'(\varphi_\alpha(w))} \cdot \overline{\varphi'_\alpha(w)}\, dA(w) \\
&= \langle f, g \circ \varphi_\alpha\rangle_{\mathcal{D}_0}.
\end{aligned}$$

It follows that $C^*_{\varphi_\alpha} = C_{\varphi_\alpha}$.

Finally, recalling that $\varphi = \ell \circ \varphi_\alpha$ and that $\varphi^*_\alpha = \varphi_\alpha$, we obtain

$$C^*_\varphi = C^*_{\ell\circ\varphi_\alpha} = (C_{\varphi_\alpha} C_\ell)^* = C_{\ell^*} C_{\varphi_\alpha} = C_{\varphi_\alpha \circ \ell^*} = C_{\varphi^*} = C_\psi,$$

using also Lemma 2. □

By applying the same method, we obtain the following result known to the experts, but not easy to find explicitly stated in the literature. Note that the statement now refers to the true Dirichlet space instead of $\mathcal{D}_0$.

Proposition 1. *Let C_φ be a composition operator acting in the true Dirichlet space $\mathcal{D}$. Then $C^*_\varphi = C_\psi$ for some self-map of the disk ψ if and only if $\varphi(z) = az$, $a \in \mathbb{D}$. In this case, $\psi(z) = \varphi^*(z) = \overline{a}z$.*

Proof. The sufficiency of the condition $\varphi(z) = az$ follows easily from the equality

$$\langle a^n z^n, z^m\rangle_{\mathcal{D}} = \langle z^n, \overline{a}^n z^m\rangle_{\mathcal{D}}, \quad n, m \in \mathbb{N} \cup \{0\}.$$

To prove the necessity, it is enough to repeat the process in the proof of (i)$\Longrightarrow$(ii) of Theorem 2 and to observe that for every $n,\ m \in \mathbb{N}$,

$$\langle C_\psi z^n, z^m\rangle_{\mathcal{D}} = C_\psi z^n(0)\cdot 0^m + \langle C_\psi z^n, z^m\rangle_{\mathcal{D}_0}.$$

So, whenever $m \neq 0$, $\langle C_\psi z^n, z^m\rangle_{\mathcal{D}} = \langle C_\psi z^n, z^m\rangle_{\mathcal{D}_0}$.

On the other hand, for $m = 0$, taking $n = 1$, by the definition of the inner product in $\mathcal{D}$ we obtain

$$\langle C_\psi z, 1\rangle_{\mathcal{D}} = \langle z, C_\varphi 1\rangle_{\mathcal{D}} = \langle z, 1\rangle_{\mathcal{D}} \Longrightarrow \psi(0) = b_0 = 0,$$
$$\langle C_\varphi z, 1\rangle_{\mathcal{D}} = \langle z, C_\psi 1\rangle_{\mathcal{D}} = \langle z, 1\rangle_{\mathcal{D}} \Longrightarrow \varphi(0) = a_0 = 0.$$

Thus, we have $\varphi(z) = a_1 z$ and $\psi(z) = \overline{a_1} z$. $\square$

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