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TOEPLITZ OPERATORS AND DIVISION BY INNER FUNCTIONS

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ABSTRACT. A subspace X of the Hardy space H^1 is said to have the K-property if for any $\psi \in H^\infty$, the Toeplitz operator $T_{\overline{\psi}}$ maps X into itself. This in turn implies that X also has the f-property. This means that $h/I \in X$ whenever $h \in X$ and I is an inner function with $h/I \in H^1$.

In this survey paper we present a list of subspaces of H^1 that have or have not the f- or K-property, showing some of the different techniques and methods used in the subject.

1. INTRODUCTION

Denote by $\mathbb{D}$ the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, and its boundary by $\partial\mathbb{D}$ or $\mathbb{T}$, indistinctly. Denote also by H^p $(0 < p \leq \infty)$ the classical Hardy spaces consisting of those analytic functions f defined on $\mathbb{D}$ such that

$$\|f\|_{H^p} := \sup_{0<r<1} M_p(r, f) < \infty,$$

where $M_p(r, f)$, $0 < r < 1$, are defined as,

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}, \ (0 < p < \infty),$$

$$\text{and} \qquad M_\infty(r,f) = \max_\theta |f(re^{i\theta})|.$$

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A thorough study of these spaces is done in Duren's book [10], from where we review some well known properties in order to motivate the definitions that follow. When $1 \leq p \leq \infty$, H^p is a Banach space with norm given by $\|\cdot\|_{H^p}$, and for $0 < p < 1$ it is an F-space (complete metrizable topological vector space) with distance function given by $d(f,g) = \|f-g\|_{H^p}^p$.

Every function $f \in H^p$ has radial limits almost everywhere on the boundary, and the boundary function of f, defined as,

$$f(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta}), \quad \theta \in \mathbb{R},$$

is in $L^p(\partial\mathbb{D})$ and also $\log|f(e^{i\theta})| \in L^1(\partial\mathbb{D})$. (From now on, the three symbols L^p, $L^p(\partial\mathbb{D})$, and $L^p(\mathbb{T})$, will denote the same space). Besides, the "H^p-norm" (it is only a true norm when $p \geq 1$) of f coincides with the L^p-norm of its boundary function:

$$\|f\|_{H^p} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right)^{1/p}.$$

If $f \in H^p$, $f \not\equiv 0$, then the sequence of its zeros $\{a_n\}$, repeated according to multiplicities, satisfies the so called *Blaschke Condition*:

$$\sum_n (1-|a_n|) < \infty.$$

This condition for a sequence $\{a_n\} \subset \mathbb{D}$ turns out to be also sufficient in order to construct an H^p function whose sequence of zeros (repeated according to multiplicities) is exactly $\{a_n\}$. For that, we just need to consider, for each n, the self-mappings of the unit disk,

$$b_n(z) = z, \qquad \text{if } a_n = 0,$$

$$b_n(z) = \frac{\overline{a_n}}{|a_n|}\frac{a_n - z}{1-\overline{a_n}z}, \qquad \text{if } a_n \neq 0,$$

and realize that the infinite product $\prod_n b_n(z)$ converges absolutely and uniformly on each compact subset of $\mathbb{D}$, defining a function $B \in H^\infty$, called *the Blaschke product associated with* $\{a_n\}$, whose sequence of zeros is exactly $\{a_n\}$, and with the further properties that $|B(z)| < 1$ in $\mathbb{D}$, and $|B(e^{i\theta})| = 1$ a.e..

All this gives rise to a result of F. Riesz that states that any function $f \in H^p$ with $f \not\equiv 0$, can be factored in the form $f(z) = B(z)g(z)$, where $B(z)$ is the Blaschke product associated with the sequence of zeros of f, and g is a zero-free function in H^p with $\|g\|_{H^p} = \|f\|_{H^p}$.

Actually, the previous can be carried further bringing us to a canonical factorization due to Smirnov. Quoting from Duren's book, observe that the function

$$F(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\log|f(e^{it})|dt\right)$$

is analytic in $\mathbb{D}$ and satisfies $|F(e^{i\theta})| = |f(e^{i\theta})|$ a.e. because $\log|f(e^{i\theta})| \in L^1$. Also, since $|f(e^{i\theta})| \in L^p$, an application of the arithmetic-geometric mean inequality shows that $F \in H^p$. Moreover, $|f(e^{i\theta})| = |g(e^{i\theta})|$ a.e. and $|g(z)| \leq |F(z)|$ in $\mathbb{D}$, where g is the zero-free factor associated with f in the Riesz factorization. Hence, the function $S_0(z) = g(z)/F(z)$ is analytic in $\mathbb{D}$ and has the properties

$$0 < |S_0(z)| \leq 1, \qquad \text{and} \quad |S_0(e^{i\theta})| = 1 \text{ a.e.},$$

This shows that $-\log|S_0(z)|$ is a positive harmonic function in $\mathbb{D}$ which vanishes almost everywhere on the boundary. Thus by the Herglotz representation theorem and Fatou's theorem, it can be represented as the Poisson integral of a positive singular measure μ which, by analytic completion, gives $S_0(z) = e^{i\gamma}S(z)$, where γ is a real constant. The function $S(z)$ is called the *singular inner function associated to* μ,

$$S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)\right).$$

Everything together yields the canonical factorization

$$f(z) = e^{i\gamma}\, B(z)\, S(z)\, F(z).$$

The function F above is an *outer function for the class* H^p, i.e., it is a function of the form

$$F(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log\psi(t) dt\right),$$

where $\psi(t) \geq 0$, $\log\psi(t) \in L^1$, and $\psi(t) \in L^p$.

On the other hand, an *inner function* is a function $I(z)$ analytic in $\mathbb{D}$ with $|I(z)| \leq 1$ and $|I(e^{i\theta})| = 1$ a.e.. The above tells us that any inner function I is uniquely factored in the form $I(z) = e^{i\gamma}\, B(z)\, S(z)$, where γ is a real number, B is a Blaschke product and S is a singular inner function.

The canonical factorization brings us to the following important observation.

Theorem 1. *If $f \in H^p$, $1 \leq p \leq \infty$, and I is an inner function such that $f/I \in H^1$, then $f/I \in H^p$ and $\|f/I\|_{H^p} = \|f\|_{H^p}$.*

The analysis of this property in different subspaces of H^1 is the central topic of this survey.

Definition 1. A subspace X of H^1 is said to have the *f-property* (also called the *property of division by inner functions*) if $h/I \in X$ whenever $h \in X$ and I is an inner function with $h/I \in H^1$.

Rephrasing it, $X \subseteq H^1$ has the f-property if, whenever $F \in H^1$, I is inner and $FI \in X$, one also has that $F \in X$.

This notion, of studying whether the outer part of functions in a given subspace of H^1 remains in the same space, seems to have appeared in the early seventies. We mention especially the works of Havin [27], Korenblum and Korolevič [36] and Korenblum [33, 34] on the subject where a number of subspaces of H^1 were shown to enjoy the f-property.

Let us say here that Havin proved in [27] that the following spaces do satisfy the f-property:

- $\Lambda_{\alpha,n}$ ($n = 0, 1, \dots, 0 < \alpha < 1$), the space of functions $f \in H^1$ such that $f^{(n)} \in \mathrm{Lip}_\alpha(\mathbb{D})$.
- $\Lambda_{0,n}$ ($n = 0, 1 \dots$), the space of functions $f \in H^1$ such that $f^{(n)}$ is continuous in $\mathbb{D}$ and $f^{(n)}(e^{i\theta})$ is smooth in the sense of Zygmund.
- $A^{p,1} = \{f \in H^1 : f' \in L^p(\mathbb{D})\}$, $1 < p < \infty$.
- $H^{p,1} = \{f \in H^1 : f' \in H^p\}$, $1 < p < \infty$.

In view of this, one is driven to think that spaces not having the f-property must be rare. However, the first such example appeared published in 1972, and was given by Gurarii [25] who proved that the space W^+ of analytic functions in $\mathbb{D}$ with absolutely convergent power series does not possess the f-property.

Before getting deeper into the study of our next question, let us introduce the other type of property we are interested in, and that implies the f-property. That will be called "the K-property".

Notice that any function $f \in H^1$ can be recovered from its boundary function by the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{1 - e^{-it} z} dt, \qquad z \in \mathbb{D}.$$

This is nothing else but the Szegö projection of the boundary function $f(e^{it})$. In general, the *Analytic Szegö Projection* of an $L^1(\partial\mathbb{D})$ function $\psi(e^{it}) \sim \sum_{-\infty}^{\infty} \hat{\psi}(n) e^{int}$ is defined as,

$$\begin{aligned} P\psi(z) =& \frac{1}{2\pi} \int_0^{2\pi} \frac{\psi(e^{it})}{1 - e^{-it} z} dt \\ =& \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^{\infty} \psi(e^{it}) e^{-int}\, z^n\, dt \\ =& \sum_{n=0}^{\infty} \hat{\psi}(n) z^n, \end{aligned}$$

which, in fact, is analytic in $\mathbb{D}$. Notice that, according to the M. Riesz theorem on conjugate functions, the analytic Szegö projection operator maps $L^p(\partial\mathbb{D})$ boundedly onto H^p, if $1 < p < \infty$. However, this is not true either for $p = 1$ or for $p = \infty$.

Returning to our main stream, observe that if h is analytic in $\mathbb{D}$, I is inner, and $h/I \in H^1$, then, using that $|I(e^{i\theta})| = 1$ a.e.,

$$\text{(1)} \qquad \tfrac{h}{I}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\frac{h}{I}(e^{it})}{1 - e^{-it} z} dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{h(e^{it})\, \overline{I(e^{it})}}{1 - e^{-it} z} dt = P(h\overline{I})(z).$$

Definition 2. Given $v \in L^\infty(\partial\mathbb{D})$, the *Toeplitz operator* T_v *associated with the symbol* v is defined as

$$T_v f\,(z) = P(vf)(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{v(\zeta)f(\zeta)}{\zeta - z}d\zeta, \qquad f \in H^1,\ z \in \mathbb{D}.$$

Many basic facts about Toeplitz operators can be found in Chapter 7 of [9] and in Appendix 4 of [38], for example.

Notice that (1), can be written as $\frac{h}{I} = T_{\bar{I}}(h)$. Here, of course, the symbol $\overline{I}$ is understood as the boundary function associated with $\overline{I}$.

Definition 3. A subspace X of H^1 is said to have the K-property if $T_{\overline{\psi}}(X) \subseteq X$ for any $\psi \in H^\infty$.

From the above discussion we obtain that the K-property implies the f-property. Also, the fact that the Szegö projection maps $L^p(\partial\mathbb{D})$ boundedly onto H^p whenever $1 < p < \infty$ implies easily the following.

Theorem 2. *If* $1 < p < \infty$, *then* H^p *has the* K*-property.* ☐

As we mentioned above, the purpose of this survey is to build a (non-exhaustive) list of subspaces of H^1 that have or have not the f- or K-property. Actually, we are mainly interested in showing some of the different techniques and methods used in the subject.

2. Representation formulae

The Dirichlet space $\mathcal{D}$ consists of those functions $f(z) = \sum_0^\infty a_n z^n$ analytic in $\mathbb{D}$ with Dirichlet integral $\mathcal{D}(f)$ finite, i.e.,

$$\mathcal{D}(f) := \Big(\frac{1}{\pi}\iint_{\mathbb{D}} |f'(z)|^2 dA(z)\Big)^{1/2} \equiv \Big(\sum_{n=1}^{\infty} n|a_n|^2\Big)^{1/2} < \infty.$$

Theorem 3. *The Dirichlet space has the* f*-property.*

Proof. Carleson [8] obtained a simple explicit formula for the Dirichlet integral of a given function f involving the zeros of f and its boundary values. Namely, he proved that if $F \in H^2$ and $I = BS$, where B is the Blaschke product with sequence of zeros $\{a_n\}$ and S is the singular inner function generated by the measure μ then

$$\begin{aligned}\mathcal{D}(FI)^2 =& D(F)^2 \\ &+\frac{1}{2\pi}\int_0^{2\pi} |F(e^{it})|^2 \left(\sum_n \frac{1-|a_n|^2}{|e^{it}-a_n|^2} + 2\int_0^{2\pi}\frac{d\mu(\xi)}{|e^{it}-\xi|^2}\right) dt.\end{aligned}$$

By means of this representation formula, one obtains easily that, whenever $F \in H^2$ and I is inner,

$$\mathcal{D}(F) \le \mathcal{D}(FI),$$

from where we deduce that if $FI \in \mathcal{D}$, then so is F, concluding the result. ☐

Given a nondecreasing weight sequence $w = \{w_n\}_{n=1}^{\infty}$, $w_n \geq 0$, we define the Dirichlet-type space D_w as the set of functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$ for which

$$\mathcal{D}_w^2(f) := \sum_{n=1}^{\infty} w_n |a_n|^2 < \infty.$$

Dyakonov [11] proved an analogue of Carleson formula for these spaces which gave rise to "orthogonality relations" of the form

$$\mathcal{D}_w^2(FI) = \mathcal{D}_w^2(F) + R_w(F, I), \quad F \in H^2, \quad I \text{ inner},$$

with $R_w(F, I) > 0$. This immediately yields that the spaces D_w have the f-property.

Let us remark here that Rabindranathan [40] and Korenblum and Faĭvyševskiĭ[35] independently had proved that these spaces in fact have the stronger K-property.

Another instance in which a representation formula plays an important role in deciding whether the given subspace of H^1 satisfies the f-property is in the case of $BMOA$, the space of H^1 functions whose boundary values have bounded mean oscillation; and in the case of $VMOA$, consisting of those $BMOA$ functions with vanishing mean oscillation, also characterized as the closure of the polynomials in $BMOA$. A good account on $BMOA$ may be found in the survey paper [20].

Theorem 4. *$BMOA$ and $VMOA$ have the f-property.*

Proof. Among the many characterizations of $BMOA$ and $VMOA$ we rest upon the one using the Garsia norm:

$$f \in BMOA \iff \sup_{a \in \mathbb{D}} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \frac{1 - |a|^2}{|1 - \bar{a}e^{i\theta}|^2} d\theta \; - \; |f(a)|^2 \right) < \infty,$$

$$f \in VMOA \iff \lim_{|a| \to 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \frac{1 - |a|^2}{|1 - \bar{a}e^{i\theta}|^2} d\theta \; - \; |f(a)|^2 \right) = 0.$$

Thus, if $FI \in BMOA$ ($VMOA$) with $F \in H^1$ and I inner, then

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} |FI(e^{i\theta})|^2 \frac{1 - |a|^2}{|1 - \bar{a}e^{i\theta}|^2} d\theta \; - \; |FI(a)|^2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^2 \frac{1 - |a|^2}{|1 - \bar{a}e^{i\theta}|^2} d\theta - |F(a)|^2 \; + \; \overbrace{|F(a)|^2 \big(1 - |I(a)|^2\big)}^{\geq 0} \\ &\geq \frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^2 \frac{1 - |a|^2}{|1 - \bar{a}e^{i\theta}|^2} d\theta - |F(a)|^2. \end{aligned}$$

From here it is obvious that F itself is in $BMOA$ ($VMOA$). ☐

Dyakonov [13] has introduced recently the notion of a Garsia-type norm (GTN) on a Banach space of analytic functions in $\mathbb{D}$: Let X be such an space and assume that $X \subset H^p$ for some $p \in (0, \infty)$. Write $|H^p|$ for the set of nonnegative functions $g \in L^p(\partial\mathbb{D})$ satisfying either $\log g \in L^1(\partial\mathbb{D})$ or $g = 0$ a.e.; these are precisely the boundary values of the moduli of H^p-functions. Suppose that there exists a mapping $\Psi : |H^p| \times \mathbb{D} \to [0, \infty]$ and a function $k : \mathbb{D} \to (0, \infty)$ with the following properties:

(a) $\Psi(cg, z) = c^p \Psi(g, z)$, whenever $g \in |H^p|$ and $z \in \mathbb{D}$.

(b) $\Psi(|f|, z) \geq |f(z)|^p$, for all $f \in H^p$ and $z \in \mathbb{D}$.

(c) The quantity

$$N_{p,\Psi,k}(f) := \sup_{z\in\mathbb{D}} \frac{\{\Psi(|f|, z) - |f(z)|^p\}^{1/p}}{k(z)}, \quad f \in H^p, \tag{2}$$

is comparable to $\|f\|_X$ with constants not dependidng on f (it is understood that $\|f\|_X = \infty$ if $f \in H^p \setminus X$).

Then we say that $N(\cdot)$ is a GTN on X.

Of course, the first example of a Garsia-type norm is the classical Garsia norm in $BMOA$ and, just as it happens in this space, if X has a Garsia-type norm then it satisfies the f-property.

Dyakonov proved in [12] that, for $0 < \alpha < 1$, the Lipschitz spaces $\Lambda_\alpha = \Lambda_{\alpha,0}$ have a GTN and, hence, the f-property. As mentioned above this was first proved by Havin [27].

Dealing with the perhaps more natural Lipschitz space $\Lambda_1 = \Lambda_{1,0}$ is harder. N. A. Širokov [48] proved that Λ_1 has in fact the f-property. His proof was very involved. Dyakonov [13] has proved that Λ_1 has a GTN obtaining in this way an alternative proof of Širokov's result simpler than the original one.

3. Duality

The idea of Hedenmalm of using a duality argument to study the K-property in different subspaces of H^1 has been very fruitful. Most of the results in this section are from his paper [28].

Recall that $BMOA$ is the dual space of H^1 under the pairing

$$\langle f, g\rangle = \lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\, \overline{g(e^{i\theta})}\, d\theta, \qquad f \in H^1,\ g \in BMOA.$$

Theorem 5 (Hedenmalm [28]). *$BMOA$ has the K-property.*

Proof. For a given $\psi \in H^\infty$, we claim that the Toeplitz operator $T_{\overline{\psi}}$ defined in $BMOA$ is in fact the adjoint operator of the operator M_ψ, multiplication by ψ, which of course, is continuous from H^1 to H^1. Under the validity of this claim, $T_{\overline{\psi}}$ is also continuous from $(H^1)^* = BMOA$ to $(H^1)^* = BMOA$, concluding the result.

To prove the claim, take $f \in H^2$, which is dense in H^1, and $g \in BMOA$ ($\psi g \in H^2$). Then, recalling that P denotes the analytic Szegö projection, we have

$$\begin{aligned}\langle f,(M_\psi)^*(g)\rangle &= \langle M_\psi(f),g\rangle\\ &= \lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}(\psi f)(re^{i\theta})\overline{g(e^{i\theta})}d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}(\psi f)(e^{i\theta})\overline{g(e^{i\theta})}d\theta \qquad [(\psi f),g\in H^2]\\ &= \frac{1}{2\pi}\int_0^{2\pi}f(e^{i\theta})\overline{(\overline{\psi}g)(e^{i\theta})}d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}f(e^{i\theta})\overline{P(\overline{\psi}g)(e^{i\theta})}d\theta \qquad \begin{bmatrix}\text{the non-analytic}\\ \text{part of } \overline{\psi}g \text{ is}\\ \text{annihilated by}\\ f\in H^2\end{bmatrix},\end{aligned}$$

from where we deduce that $(M_\psi)^* = T_{\overline{\psi}}$, and the claim is proved. □

A similar argument can be applied to H^1 obtaining the following result.

Theorem 6. *H^1 does not have the K-property.*

Proof. Assume by contradiction that $T_{\overline{\psi}}(H^1)\subseteq H^1$ for all $\psi\in H^\infty$. Then by the Closed Graph Theorem all these $T_{\overline{\psi}}$ are continuous from H^1 to H^1. So their adjoints $(T_{\overline{\psi}})^*$ are continuous from $BMOA$ to $BMOA$.

Let us see the aspect of these adjoint operators. For $\psi\in H^\infty$, $f\in H^2$ (dense in H^1), and $g\in BMOA\subset H^2$,

$$\begin{aligned}\langle f,(T_{\overline{\psi}})^*g\rangle &= \langle T_{\overline{\psi}}f,g\rangle\\ &= \lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}P(\overline{\psi}f)(re^{it})\overline{g(e^{it})}dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}P(\overline{\psi}f)(e^{it})\overline{g(e^{it})}dt \qquad [P(\overline{\psi}f),\ g\in H^2]\\ &= \frac{1}{2\pi}\int_0^{2\pi}f(e^{it})\overline{(\psi g)(e^{it})}dt \qquad \begin{bmatrix}\text{the non-analytic}\\ \text{part of } \overline{\psi}f \text{ is}\\ \text{annihilated by}\\ g\in H^2\end{bmatrix},\end{aligned}$$

so $(T_{\overline{\psi}})^*$ is the operator M_ψ, multiplication by ψ. The continuity of such operator from $BMOA$ to $BMOA$ implies that H^∞ is a subspace of the space of multipliers of $BMOA$, but we know that this is false: there exist Blaschke products that are not multipliers of $BMOA$, not even of the Bloch space (to be defined later). See for instance [37, 19]. □

Denote by $\mathcal{A}$ the disk algebra. This is the space of analytic functions in $\mathbb{D}$ that admit a continuous extension to $\overline{\mathbb{D}}$. We can see that $\mathcal{A}$ has the f-property arguing as in the proof of Theorem 6.3 in p. 78 of [18]:

Suppose $f \not\equiv 0$, $f = FI \in \mathcal{A}$ with I inner and $F \in H^1$ (hence, $F \in H^\infty$). Let $K = \{\xi \in \mathbb{T} : f(\xi) = 0\}$. Then K is a closed set of (one-dimensional) measure zero. Now, if $\xi \in \mathbb{T} \setminus K$ then $|f(\xi)| > 0$ and, hence, $I(z_n)$ cannot tend to zero for any sequence of point $\{z_n\} \subset \mathbb{D}$ tending to ξ. Then, using Theorem 6.1 and Theorem 6.2 in Chapter II of [18], we deduce that I is analytic across $\mathbb{T} \setminus K$ and then it follows that F is continuous at each point of $\mathbb{T} \setminus K$. Notice that we certainly have $|F| = |f|$ on $\mathbb{T} \setminus K$. Set now $F(\xi) = 0$ for all $\xi \in K$ (since K has measure zero, defining F in this way at the points of K causes no problem at all). Then $|F| = |f|$ on $\mathbb{T}$ and $|F| = 0$ on K. Since f is continuous on $\mathbb{T}$, it follows that F is continuous at each point of K. Thus we have that F is continuous on $\mathbb{T}$. The Poisson representation now implies that $F \in \mathcal{A}$.

On the other hand, using duality arguments, we can prove that $\mathcal{A}$ does not possess the K-property. Hruscev and Vinogradov [29] proved that the dual space of $\mathcal{A}$ is identified with $\mathcal{K}$, the space of Cauchy integrals of complex Borel measures on $\partial\mathbb{D}$. In the same paper, they also characterized the inner functions that are multipliers of $\mathcal{K}$, which for our purposes it suffices to know that they are not all of them.

Theorem 7 (Hedenmalm [28]). *$\mathcal{A}$ does not have the K-property.*

Proof. The same duality argument as before would show that if $\mathcal{A}$ has the K-property, then H^∞ would become subspace of the space of multipliers of $\mathcal{K}$, and, as we have just said, this is not true. □

4. Polynomial approximation

Our next aim is studying the K-property in $VMOA$. We recall that $VMOA$ is the closure of the polynomials in $BMOA$ (see e.g. Theorem 5.5 of [20]). Thus, it becomes interesting to see the behavior of Toeplitz operators on polynomials.

Lemma 8. *If $\psi \in H^\infty$ and p is a polynomial, then $T_{\overline{\psi}}p$ is again a polynomial.*

Proof. Write $\psi(z) = \sum_{n=0}^{\infty} a_n z^n$ and $p(z) = \sum_{n=0}^{N} b_n z^n$, then, for almost every θ,

$$\overline{\psi(e^{i\theta})}\, p(e^{i\theta}) = \Big(\sum_{n=0}^{\infty} \overline{a_n} e^{-in\theta}\Big)\Big(\sum_{n=0}^{N} b_n e^{in\theta}\Big) = \sum_{n=-\infty}^{N} \underbrace{\Big(\sum_{k=\max\{n,0\}}^{N} \overline{a_{|n-k|}} b_k\Big)}_{=c_n} e^{in\theta},$$

so that $T_{\overline{\psi}}p\,(z) = P(\overline{\psi}p)(z) = \sum_{n=0}^{N} c_n z^n$ is indeed a polynomial. □

With this, we are able to prove the following.

Theorem 9 (Hedenmalm [28]). *$VMOA$ has the K-property.*

Proof. Take $\psi \in H^\infty$. Since $BMOA$ has the K-property $T_{\overline{\psi}}(BMOA) \subseteq BMOA$, so by the Closed Graph Theorem $T_{\overline{\psi}}$ is continuous from $BMOA$ to $BMOA$. Now, if $g \in VMOA$, then it can be approximated in $BMOA$ by a sequence of polynomials $\{q_n\}$. Hence, by the continuity of $T_{\overline{\psi}}$ and Lemma 8, $T_{\overline{\psi}}(q_n)$ are polynomials that approach $T_{\overline{\psi}}(g)$ in the $BMOA$ norm, so $T_{\overline{\psi}}(g) \in VMOA$. □

Now, using that disk algebra $\mathcal{A}$ is the closure of the polynomials in H^∞ and that it does not have the K-property, Hedenmalm [28] obtained the following result.

Theorem 10 (Hedenmalm [28])**.** *If X is a subspace of H^∞ containing $\mathcal{A}$, then it fails to have the K-property.*

Proof. Assume by contradiction that X has the K-property. Then for any $\psi \in H^\infty$,

$$T_{\overline{\psi}}(\mathcal{A}) \subseteq T_{\overline{\psi}}(X) \subseteq X \subseteq H^\infty. \tag{3}$$

By the Closed Graph Theorem $T_{\overline{\psi}}$ is then continuous from X to X, for any $\psi \in H^\infty$.

Since $\mathcal{A}$ does not have the K-property, we take $\varphi \in H^\infty$ such that $T_{\overline{\varphi}}(\mathcal{A}) \not\subseteq \mathcal{A}$, and however, by (3), $T_{\overline{\varphi}}$ is still continuous in $\mathcal{A}$.

Now, if $g \in \mathcal{A}$, then it can be approximated in H^∞ by a sequence of polynomials $\{q_n\}$. Hence, by the continuity of $T_{\overline{\varphi}}$ and Lemma 8, $T_{\overline{\varphi}}(q_n)$ are polynomials that approach $T_{\overline{\varphi}}(g)$ in the H^∞ norm. So $T_{\overline{\varphi}}(g)$ belongs to A, and this would prove that $T_{\overline{\varphi}}(\mathcal{A}) \subseteq \mathcal{A}$, which is a contradiction. □

As immediate consequences of the previous result we have the following.

Corollary 11. *H^∞ and $VMOA \cap H^\infty$ lack the K-property, although they have the f-property.* □

5. Pseudoanalytic extension

Taking as a general basis the works of Dynkin [15, 16, 17], given a function $f \in H^1$, a *pseudoanalytic extension of f to $\mathbb{C}$* is a function F defined in $\mathbb{C} \setminus \partial\mathbb{D}$ which is an extension of f (meaning $F_{|\mathbb{D}} = f$) such that F is of class $\mathcal{C}^1$ in $\mathbb{D}_- \overset{\text{def}}{=} \{z \in \mathbb{C} \ : \ |z| > 1\}$ and, $F(Re^{it}) \to f(e^{it})$, as $R \to 1^+$, both a.e. and in $L^1(\partial\mathbb{D})$. Observe that one such extension is given by just reflecting with respect to $\partial\mathbb{D}$ ($F(z) = f(1/\overline{z})$, $z \in \mathbb{D}_-$), although there are some other ways of constructing pseudoanalytic extensions, like for instance, by global polynomial approximates [15, 16], or by local polynomial approximates [17]. Pseudoanalytic continuation is just one generalization of the classical Weierstrassian notion of analytic continuation. A good number of authors (including H. Poincaré, É. Borel, A. Beurling and H. S Shapiro) have obtained other generalizations by various methods and for a variety of reasons. We mention the recently published monograph [41] as an excelent book where distinct methods of "generalized analytic continuation" are studied in a unifying context.

The relevance of the pseudoanalytic continuation in our work comes from the fact that having a characterization of a given subspace X of H^1 in terms of pseudoanalytic extension, in many cases helps to determine whether X has the f- property and/or the K-property. To illustrate this, let us work with the Lipschitz spaces Λ_ω.

We say that a continuous function $\omega\colon [0,\infty) \to \mathbb{R}$ with $\omega(0) = 0$ is a *majorant* if $\omega(t)$ is increasing and $\omega(t)/t$ is non-increasing for $t > 0$. If, in addition, there is a constant $C = C(\omega) > 0$ such that

$$\int_0^\delta \frac{\omega(t)}{t} dt + \delta \int_\delta^\infty \frac{\omega(t)}{t^2} dt \le C \cdot \omega(\delta), \qquad 0 < \delta < 1,$$

then we say that ω is a *regular majorant*. (See [12] for the terminology, these functions are also called Dini weights satisfying the b_1 condition, cf. [6].)

Let ω be a regular majorant. The weighted Lipschitz space Λ_ω is defined as,

$$\Lambda_\omega = \{ f \in \mathcal{A} \ : \ \|f\|_{\Lambda_\omega} \overset{\text{def}}{=} \sup_{\substack{z_1, z_2 \in \mathbb{D} \\ z_1 \ne z_2}} \frac{|f(z_1) - f(z_2)|}{\omega(|z_1 - z_2|)} < \infty \}.$$

Two main reasons explain why the weight is restricted to be a regular majorant. In the first place, Λ_ω admits this other equivalent definition, as proved in [12],

$$\Lambda_\omega = \{ f \in \mathcal{A} \ : \ \sup_{\substack{\zeta_1, \zeta_2 \in \partial\mathbb{D} \\ \zeta_1 \ne \zeta_2}} \frac{|f(\zeta_1) - f(\zeta_2)|}{\omega(|\zeta_1 - \zeta_2|)} < \infty \},$$

and, in the second place, the following characterizations hold.

Theorem 12. *Let ω be a regular majorant and $f \in H^1$. The following conditions are equivalent.*

(a) $f \in \Lambda_\omega$,

(b) $M_\infty(r, f') = \mathrm{O}\Big(\dfrac{\omega(1-r)}{1-r}\Big)$, *as* $r \to 1^-$,

(c) *f has a pseudoanalytic extension to $\mathbb{C}$, F, with $F(z) = \mathrm{O}(1)$, as $|z| \to \infty$, such that*

$$|\overline{\partial} F(z)| \le C \frac{\omega(|z| - 1)}{|z| - 1}, \quad z \in \mathbb{D}_-. \tag{4}$$

Recall that the Cauchy-Riemann operator $\overline{\partial}$ is defined as

$$\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{x} + i\frac{\partial}{y}\Big), \quad z = x + iy.$$

Some remarks are in order here. When $\omega(t) = t^\alpha$, $0 < \alpha < 1$, then (a) $\Longleftrightarrow$ (b) is a classical result of Hardy and Littlewood [26], while for an arbitrary regular majorant, (a) $\Longleftrightarrow$ (b) appears in [6]. Also, (a) $\Longleftrightarrow$ (c) is a refined version of Lemma 7 in [12]. This lemma is the key ingredient used by Dyakonov to obtain some characterizations of the functions in Λ_ω in terms of their modulii. We remark that M. Pavlović has obtained in [39] simpler proofs of these results. The proof of (a) $\Longleftrightarrow$ (c) in our theorem simply needs a little adaptation of what is done by Dyakonov in the proof of his lemma.

Proof of (a) $\Longleftrightarrow$ (c). Assume that $f \in \Lambda_\omega$. Consider the pseudoanalytic extension of f given by reflection with respect to $\partial\mathbb{D}$, i.e., $F(z) = f(1/\overline{z})$, $z \in \mathbb{D}_-$. Clearly, F is bounded. It only remains to prove (4). Since (a) and (b) are equivalent, and ω is increasing, one gets, for $z \in \mathbb{D}_-$,

$$|\overline{\partial}F(z)| = \left|f'\Big(\frac{1}{\overline{z}}\Big)\right|\left|\frac{1}{\overline{z}}\right|^2 \le C\frac{\omega\big(1-\frac{1}{|z|}\big)}{1-\frac{1}{|z|}}\frac{1}{|z|^2} \le C\frac{\omega(|z|-1)}{|z|-1}.$$

Conversely, assume that $f \in H^1$ and F is a pseudoanalytic extension of f with $F(z) = \mathrm{O}(1)$, as $|z| \to \infty$, satisfying (4). Then for $z \in \mathbb{D}$ and $\rho > 1$, the Cauchy-Green integral formula applies and gives

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=\rho}\frac{F(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\iint_{\{1<|\zeta|<\rho\}}\frac{\overline{\partial}F(\zeta)}{\zeta - z}dA(\zeta).$$

Differentiate with respect to $z \in \mathbb{D}$, and obtain

$$f'(z) = \frac{1}{2\pi i}\int_{|\zeta|=\rho}\frac{F(\zeta)}{(\zeta - z)^2}d\zeta - \frac{1}{\pi}\iint_{\{1<|\zeta|<\rho\}}\frac{\overline{\partial}F(\zeta)}{(\zeta - z)^2}dA(\zeta), \qquad \rho > 1.$$

Observe that, since $F(z) = \mathrm{O}(1)$, as $|z| \to \infty$, the contour integral is $\mathrm{O}(1/\rho)$, as $\rho \to \infty$. Hence, letting $\rho \to \infty$, using (4), and the properties of ω as regular majorant, we obtain,

$$\begin{aligned}
|f'(z)| &= \left|\frac{1}{\pi}\iint_{\{1<|\zeta|\}}\frac{\overline{\partial}F(\zeta)}{(\zeta - z)^2}dA(\zeta)\right| \\
&= \left|\frac{1}{\pi}\int_1^\infty\int_0^{2\pi}\frac{\overline{\partial}F(Re^{i\theta})}{(Re^{i\theta} - z)^2}R\,dR\,d\theta\right| \\
&\le C\int_1^\infty\frac{\omega(R-1)}{R-1}R\underbrace{\frac{1}{\pi}\int_0^{2\pi}\frac{d\theta}{|Re^{i\theta} - z|^2}}_{=2(R^2-|z|^2)^{-1}}\,dR \\
&\le C\int_1^\infty\frac{\omega(R-1)}{(R-1)(R-|z|)}dR \\
&= C\int_0^\infty\frac{\omega(t)}{t(t+1-|z|)}dt \\
&= C\Big(\int_0^{1-|z|} + \int_{1-|z|}^\infty\Big)\frac{\omega(t)}{t(t+1-|z|)}dt \\
\dots &\le C\frac{\omega(1-|z|)}{1-|z|},
\end{aligned}$$

which proves that $f \in \Lambda_\omega$. □

As a consequence of this characterization we have the following result.

Theorem 13. *Let ω be a regular majorant. Then Λ_ω has the K-property.*

Proof. Let $f \in \Lambda_\omega$ and $\psi \in H^\infty$. We have to show that $g \stackrel{\text{def}}{=} T_{\overline{\psi}} f \in \Lambda_\omega$.

Since the boundary function of $(\overline{\psi} f)$ is in L^∞ then, by the M. Riesz theorem on conjugate functions, we have that $g \in H^p$ for all $p < \infty$. Moreover, since $g = P(\overline{\psi} f)$ is the orthogonal projection of $(\overline{\psi} f)$ in H^2, then it is $\overline{\psi} f = g + \overline{h}$ for some $h \in H^2 \subset H^1$. From here we get the expression, valid in $\mathbb{D}$,

$$g = \overline{\psi} f - \overline{h},$$

and on $\partial\mathbb{D}$, it is valid at least in the both desired senses, radially-a.e. and in $L^1(\partial\mathbb{D})$.

By Theorem 12, f has a bounded pseudoanalytic extension F to $\mathbb{C}$, satisfying (4). Consider also the following extensions of $\overline{\psi}$ and $\overline{h}$. For $|z| > 1$, set $z^* = 1/\overline{z} \in \mathbb{D}$, and set

$$\Psi(z) = \overline{\psi(z^*)}, \qquad\qquad H(z) = \overline{h(z^*)},.$$

Finally, consider an extension G of g, given by $G = g$ on $\overline{\mathbb{D}}$ (on the boundary it is understood as radial convergence a.e. and in the $L^1(\partial\mathbb{D})$ sense), and in $\mathbb{D}_-$, set

$$G(z) = F(z)\Psi(z) - H(z),$$

Observe that G is of class $\mathcal{C}^1$ in $\mathbb{D}_-$, because so is F, and because Ψ are H are both analytic in $\mathbb{D}_-$. Also, it is clear that, as $R \to 1^+$, $G(Re^{i\theta}) \to g(e^{i\theta})$ both a.e. and in $L^1(\partial\mathbb{D})$. So G is a pseudoanalytic extension of g, which obviously satisfies that $G(z) = \mathrm{O}(1)$, as $|z| \to \infty$. Let us now check the property (4). Since Ψ and H are analytic in $\mathbb{D}_-$, then $\overline{\partial} G(z) = \Psi(z) \cdot \overline{\partial} F(z)$, $z \in \mathbb{D}_-$. So, using that F satisfies the property (4), there exists a positive constant C such that, for $z \in \mathbb{D}_-$,

$$|\overline{\partial} G(z)| = |\overline{\psi(z^*)}|\, |\overline{\partial} F(z)| \le C \|\psi\|_{H^\infty} \frac{\omega(|z|-1)}{|z|-1}.$$

All this proves that G is a pseudoanalytic extension of g satisfying (4) and $G(z) = \mathrm{O}(1)$, as $|z| \to \infty$. So, by Theorem 12 again, we have that $g \in \Lambda_\omega$ as desired. □

Examples of other spaces that have been characterized in terms of pseudoanalytic extension and, as a result, they have been proved to have the K-property, are the following.

(a) The Besov spaces B^p_s, $s > 0$, $1 \le p < \infty$, consisting of those H^p-functions such that

$$\int_{\mathbb{D}} |f'(z)|^p \big(1 - |z|\big)^{p(1-s)} dA(z) < \infty.$$

See [17].

(b) The Q_p spaces, $0 < p < 1$, consisting of those analytic functions on the unit disk such that

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} |f'(z)|^2 \log^p \left|\frac{1-\overline{a}z}{z-a}\right| dA(z) < \infty.$$

See [14]. Here we should mention that Q_p is defined for all $p \geq 0$, and that they arose in connection to finding equivalent norms in the Bloch space $\mathcal{B}$, to be defined below. Observe that Q_0 is the Dirichlet class $\mathcal{D}$, that $Q_1 = BMOA$ and, as it turned out, cf. [3], $Q_p = \mathcal{B}$ for all $p > 1$. Also, cf. [4], $Q_p \subsetneq Q_q$, $0 \leq p < q \leq 1$.

Dyakonov and Girela [14] proved the following.

Theorem 14. *If* $0 < p < 1$ *and* $f \in \cap_{0<q<\infty} H^q$, *then the following conditions are equivalent.*

(i) $f \in Q_p$.

(ii) $$\sup_{|a|<1} \int_{\mathbb{D}} |f'(z)|^2 \left(\frac{1}{|\varphi_a(z)|^2} - 1\right)^p dA(z) < \infty.$$

(iii) *There exists a function* $F \in C^1(\mathbb{D}_-)$ *satisfying*

$$F(z) = \mathrm{O}(1), \quad \text{as } z \to \infty,$$

$$\lim_{r\to 1^+} F(re^{i\theta}) = f(e^{i\theta}), \quad \text{a.e. and in } L^q([-\pi,\pi]) \text{ for all } q \in [1,\infty)\ ,$$

and

$$\sup_{|a|<1} \int_{\mathbb{D}_-} \left|\overline{\partial} F(z)\right|^2 \left(|\varphi_a(z)|^2 - 1\right)^p dA(z) < \infty.$$

Using this characterization, it was proved in [14] that the spaces Q_p, $0 < p < 1$ satisfy the K-property.

J. Xiao has proved in Theorem 5.4.1 of [56] that if $0 < p < 1$ and $f \in H^1$, $f \not\equiv 0$, then $f \in Q_p$ if and only if f can be factored in the form $f = IO$ where O is an outer function in Q_p and I is an inner function for which

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |O(z)|^2 (1 - |I(z)|^2) \left(\frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}\right)^p dA(z) < \infty.$$

This result can be used to deduce that Q_p $(0 < p < 1)$ has the f-property.

The space $Q_{p,0}$ $(0 < p < \infty)$ consists of those analytic functions on the unit disk such that

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^2 \log^p \left|\frac{1-\overline{a}z}{z-a}\right| dA(z) = 0.$$

For all p, $Q_{p,0}$ is a subspace of Q_p. For $1 < p < \infty$, $Q_{p,0} = \mathcal{B}_0$ and $Q_{1,0} = VMOA$. It is possible to find a substitute for $Q_{p,0}$ $(0 < p < 1)$ of Xiao's factorization theorem. This result implies that the $Q_{p,0}$ spaces $(0 < p < 1)$ also have the f-property. However, a characterization of the $Q_{p,0}$ spaces $(0 < p < 1)$ in terms of pseudoanalytic extension is not known and, in fact, the question of whether the spaces $Q_{p,0}$ $(0 < p < 1)$ have the K-property remains open. This question is closely related to that of determining whether $Q_{p,0}$ $(0 < p < 1)$ is the closure of the polynomials in Q_p. An affirmative answer to this would show that $Q_{p,0}$ has the K-property.

6. Blaschke products

Blaschke products are typical inner functions, and they are used in this subject mainly to prove that a given subspace of H^1 does not have the f-property.

The *Bloch space* $\mathcal{B}$ consists of those functions f analytic in $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}\bigl(1-|z^2|\bigr)\bigl|f'(z)\bigr| < \infty,$$

while the *little Bloch space* $\mathcal{B}_0$ consists of those functions f analytic in $\mathbb{D}$ such that

$$\lim_{|z|\to 1}\bigl(1-|z^2|\bigr)\bigl|f'(z)\bigr| = 0.$$

We mention [2] as a general reference for Bloch functions. Let us point out that $H^\infty \subsetneq BMOA \subsetneq \mathcal{B}$, that $\mathcal{B}_0$ is the closure of the polynomials in $\mathcal{B}$, that $VMOA \subsetneq \mathcal{B}_0$, and that $\mathcal{B}$ and $\mathcal{B}_0$ are not subspaces of H^1 for they contain functions without finite radial limit on sets of positive measures.

In 1979, Anderson [1] showed that $\mathcal{B}_0 \cap H^\infty$ does not have the f-property using results of Shapiro [47] and Kahane [32], from 1968 and 1969 respectively, on the existence of certain positive singular measures on $\partial\mathbb{D}$. These results were also used by Sarason [45] in 1984 to prove that there exist infinite Blaschke products in $\mathcal{B}_0$. It is remarkable to see how Anderson's theorem can be easily deduced from Sarason's result.

Theorem 15. *$B_0 \cap H^p$, $p \geq 1$ does not have the f-property.*

Proof. Fix $p \geq 1$. Let B an infinite Blaschke product in $\mathcal{B}_0$, whose infinite sequence of zeros is $\{a_n\}$. Let $\{a_{n_j}\}$ be a subsequence of $\{a_n\}$ which is uniformly separated (see [10, Chapter 9] for the definition and properties of uniformly separated sequences). If B_1 is the Blaschke product with zeros $\{a_{n_j}\}$ then there exists $\delta > 0$ such that

$$(1-|a_{n_j}|^2)|B_1'(a_{n_j})| > \delta, \qquad \text{for all } j,$$

which implies that $B/(B/B_1) = B_1 \notin \mathcal{B}_0$. Consequently, we have that $B \in \mathcal{B}_0 \cap H^p$, $I = B/B_1$ is inner, $B_1 = B/I \in H^1$, and, however, $B_1 \notin \mathcal{B}_0$. This proves that $\mathcal{B}_0 \cap H^p$ does not have the f-property. □

In view of this result it is natural to formulate the following question: *Does $\mathcal{B} \cap H^p$ $(1 \leq p \leq \infty)$ have the f- or K-property?*

Since $H^\infty \subset \mathcal{B}$, we see that $\mathcal{B} \cap H^\infty = H^\infty$ has the f-property but does not have the K-property. However, using that $\mathcal{B}_0 \cap H^\infty$ does not have the f-property and an argument by polynomial approximation, the authors [21] have proved the following result.

Theorem 16. *$\mathcal{B}\cap H^p$ $(1\le p\le\infty)$ does not have the K-property.*

Actually, adapting the proof of Theorem 10, more can be said.

Theorem 17. *If X is a subspace of $\mathcal{B}\cap H^1$ containing $\mathcal{B}_0\cap H^\infty$, then it fails to have the K-property.*

Proof. Assume by contradiction that X has the K-property. Then for any $\psi\in H^\infty$, $T_{\overline{\psi}}(X)\subseteq X$, and hence, by the Closed Graph Theorem they are continuous, and by the hypothesis,

$$T_{\overline{\psi}}(\mathcal{B}_0\cap H^\infty)\subseteq T_{\overline{\psi}}(X)\subseteq X\subseteq \mathcal{B}\cap H^1.$$

Next using the same notation as in the proof of Theorem 15, we have that $T_{\overline{I}}B = B/I = B_1\notin\mathcal{B}_0$. However, since $B\in\mathcal{B}_0$, there exists a sequence of polynomials $\{q_n\}$ that approximate B in the Bloch norm. So, bearing in mind that $T_{\overline{I}}$ transforms polynomials into polynomials and that $T_{\overline{I}}$ is continuous from $\mathcal{B}_0\cap H^\infty$ to $\mathcal{B}\cap H^1$, we obtain that $T_{\overline{I}}q_n$ is a sequence of polynomials approaching to $T_{\overline{I}}B = B_1$ in the $\mathcal{B}\cap H^1$-norm. This implies that $B_1\in\mathcal{B}_0$, which is false. □

In the same paper [21] a result is proved which answers the above question.

Theorem 18. *If $1\le p<\infty$ then $\mathcal{B}\cap H^p$ does not have the f-property.*

Sketch of the proof. It goes as follows. Let B be an infinite Blaschke product in $\mathcal{B}_0$, whose sequence of zeros have a subsequence accumulating at 1. Then

$$\mu(r)=\sup_{r\le|z|<1}\left(1-|z|^2\right)\left|B'(z)\right|\searrow 0,\qquad \text{as } r\nearrow 1.$$

Let B_1 be an interpolating Blaschke product, subproduct of B, whose sequence of zeros accumalate at 1.

Construct now a conformal mapping F from $\mathbb{D}$ onto a circularly symmetric and starlike domain with respect to 0 satisfying the following conditions (see [7, Section 3] for the construction of such mappings):

$$F\in BMOA;\qquad F(z)\xrightarrow[z\to 1]{}\infty;\qquad M_\infty(r,F)=\mathrm{O}\Big(\frac{1}{\mu(r)}\Big),\ \text{as } r\to 1.$$

From this, obtain that $f = B\cdot F\in\mathcal{B}\cap H^p$, $(p<\infty)$, $I = B/B_1$ is inner, $f/I = B_1\cdot F\in H^1$, but $f/I\notin\mathcal{B}$. □

7. Final comments

First of all, we have to remark that, in addition to those that we have considered, many other results have been obtained in this subject. Let us simply mention the works of F.A. Šamojan [42, 43, 44], R.F. Shamoyan [46], N.A. Širokov [49, 50, 51, 52, 53, 54] and S.A. Vinogradov [55], as well as, those of Axler and Gorkin [5], K. Izuchi and Y. Izuchi [30, 31] and Gorkin and Mortini [22, 23, 24] where these and other related questions are studied in the setting of Douglas algebras, for further reading.

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