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VOLTERRA OPERATORS ON SPACES OF ANALYTIC FUNCTIONS - A SURVEY

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Abstract. We give a short and selective account of results known about operators of the form

$$V_g(f)(z) = \frac{1}{z}\int_0^z f(\zeta)g'(\zeta)\,d\zeta,$$

where g is analytic on the disc and the operator $T_g = zV_g$ acts on spaces of analytic functions.

1. Introduction

Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$. For g analytic on the disc consider the linear transformation

$$V_g(f)(z) = \frac{1}{z}\int_0^z f(\zeta)g'(\zeta)\,d\zeta, \tag{1}$$

acting on the space $\mathcal{H}(\mathbb{D})$ of all analytic functions on $\mathbb{D}$. We also consider the modified transformation

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta \tag{2}$$

which maps every $f \in \mathcal{H}(\mathbb{D})$ to a function vanishing at 0. In both cases g is the *symbol* of the transformation, and these transformations have appeared under various names such as Volterra operators, generalized Cesàro operators, Riemann-Stieltjes operators, and integration operators.

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This class of operators includes the integration operator,

$$f(z) \to \frac{1}{z}\int_0^z f(\zeta)\,d\zeta$$

obtained with $g(z) = z$. It also includes the Cesàro operator

$$C(f)(z) = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right) z^n,$$

which is defined on functions $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D})$. Indeed a power series calculation shows

$$\begin{aligned} C(f)(z) &= \frac{1}{z}\int_0^z f(\zeta)\frac{1}{1-\zeta}\,d\zeta \\ &= \frac{1}{z}\int_0^z f(\zeta)\left(\log\left(\frac{1}{1-\zeta}\right)\right)'\,d\zeta \\ &= \frac{1}{z}\int_0^z f(\zeta)g'(\zeta)\,d\zeta, \end{aligned}$$

where $g(z) = \log(1/(1-z))$. With this g we thus have $C = V_g$.

These operators are closely related to the operation of integration on simply connected domains. Consider a simply connected domain $\Omega \subsetneq \mathbb{C}$ with $0 \in \Omega$. Denote by $\mathcal{H}(\Omega)$ the space of analytic functions on Ω, and let $h : \mathbb{D} \to \Omega$ be a Riemann map with $h(0) = 0$. Then the operator

$$C_h : \mathcal{H}(\Omega) \to \mathcal{H}(\mathbb{D}), \quad C_h(f)(z) = f(h(z))$$

is a linear bijection between $\mathcal{H}(\Omega)$ and $\mathcal{H}(\mathbb{D})$. Let

$$I_\Omega(f)(z) = \int_0^z f(\zeta)\,d\zeta,$$

the operator of integration acting on $\mathcal{H}(\Omega)$. Writing $\tilde{I}_\Omega = C_h \circ I_\Omega \circ C_h^{-1}$ we obtain an operator $\tilde{I}_\Omega$ acting on $\mathcal{H}(\mathbb{D})$, and we have

$$\begin{aligned} \tilde{I}_\Omega(f)(z) &= C_h \circ I_\Omega \circ C_h^{-1}(f)(z) \\ &= \int_0^z f(\zeta)h'(\zeta)\,d\zeta, \end{aligned}$$

for each $f \in \mathcal{H}(\mathbb{D})$. Thus $\tilde{I}_\Omega = T_h$ with h the above Riemann map.

Less directly but more interestingly, the averaged integration operator

$$J_\Omega(f)(z) = \frac{1}{z}\int_0^z f(\zeta)\,d\zeta$$

is also related to some V_g. Indeed the counterpart of J_Ω on $\mathcal{H}(\mathbb{D})$ is

$$\begin{aligned} \tilde{J}_\Omega(f)(z) &= C_h \circ J_\Omega \circ C_h^{-1}(f)(z) \\ &= \frac{1}{h(z)}\int_0^z f(\zeta)h'(\zeta)\,d\zeta. \end{aligned}$$

Next, on $\mathcal{H}(\mathbb{D})$ define the transformations

$$M_z(f)(z) = zf(z) \quad \text{(the usual shift)},$$
$$B_h(f)(z) = \frac{1}{zh(z)} \int_0^z f(\zeta)\zeta h'(\zeta)\, d\zeta,$$

and

$$R_h(f)(z) = \frac{1}{z} \int_0^z f(\zeta) \frac{\zeta h'(\zeta)}{h(\zeta)}\, d\zeta.$$

A calculation shows that the following hold

$$\text{(i)}\ M_z \circ B_h = \tilde{J}_\Omega \circ M_z, \quad \text{(ii)}\ R_h = B_h + R_h \circ B_h. \tag{3}$$

From (i) we have $\tilde{J}_\Omega \sim B_h$ and from (ii) that $R_h \sim B_h$, where the symbol $\sim$ is used in a vague manner to mean that the operators have "similar" properties. This can be made precise when the restrictions of these operators are made to act on appropriate Banach spaces of analytic functions that are subspaces of $\mathcal{H}(\mathbb{D})$. For example (ii) says that R_h and B_h belong to the same operator ideals.

Further since

$$\frac{zh'(z)}{h(z)} = 1 + z\left(\log \frac{h(z)}{z}\right)',$$

we find

$$\begin{aligned} R_h(f)(z) &= \frac{1}{z}\int_0^z f(\zeta)\, d\zeta + \frac{1}{z}\int_0^z f(\zeta)\zeta \left(\log \frac{h(\zeta)}{\zeta}\right)' d\zeta \\ &= J_{\mathbb{D}}(f)(z) + (V_g \circ M_z)(f)(z), \end{aligned}$$

where

$$g(z) = \log \frac{h(z)}{z}.$$

Since the integration $J_{\mathbb{D}}$ is a "small" operator, we conclude that with this g, in the vague sense mentioned above we have

$$J_\Omega \sim V_g.$$

In addition to the above connections to integration operators, the Volterra type operators also arise when studying semigroups of composition operators. Indeed they are closely related to the resolvent operators of those semigroups, see [Si3] for details.

2. Some general observations

Before we consider these Volterra type operators to act on specific spaces of analytic functions, let us take a more general point of view. Suppose X and Y are two Banach spaces consisting of analytic functions on $\mathbb{D}$. We may ask:
Question. For what symbols g is

$$T_g : X \to Y$$

a bounded operator? A compact operator? If further $X = Y$ how do the spectral properties of $T_g : X \to X$ depend on g?

Let us see how far we can go in this generality. Assume for simplicity $X = Y$ and define

$$V = V_X := \{g \in \mathcal{H}(\mathbb{D}) : T_g : X \to X \text{ is bounded}\}$$

and

$$V_0 = V_{0,X} := \{g \in \mathcal{H}(\mathbb{D}) : T_g : X \to X \text{ is compact}\}.$$

Because

$$T_{\lambda g} = \lambda T_g, \quad T_{g+h} = T_g + T_h,$$

both V and V_0 are nontrivial vector spaces (both contain the constants) and $V_0 \subseteq V$. We introduce a norm on V

$$\|g\| = \|g\|_V = |g(0)| + \|T_g\|_{X\to X},$$

which is also a norm on $V_0 \subset V$. Then it is easy to show [SiZh] that if the convergence $f_n \to f$ in X implies uniform convergence on compact subsets of $\mathbb{D}$ then V and V_0 are complete under $\|\cdot\|_V$ and are therefore Banach spaces. Further assume the multiplication operator $M_z(f)(z) = zf(z)$ is bounded on X. Then from

$$T_{z^n}(f)(z) = n\int_0^z f(\zeta)\zeta^{n-1}\,d\zeta = n\int_0^z M_z^{n-1}(f)(\zeta)\,d\zeta,$$

we have

$$T_{z^n} = nT_z \circ M_z^{n-1},$$

and it follows that V contains all monomials z^n whenever it contains z. Thus if the integration operator T_z is bounded on X, then the linear space V contains all polynomials. In the same way it follows that if T_z is compact on X, then in fact the smaller space V_0 contains all polynomials.

Now we consider the Möbius invariance of V and V_0. Let

$$\phi_a(z) = \frac{a-z}{1-\bar{a}z}, \qquad a \in \mathbb{D}.$$

be the Möbius maps which map the disc conformally onto itself and exchange a with 0. Let C_a be the composition operator

$$C_a(f) = f \circ \phi_a,$$

induced by ϕ_a on $\mathcal{H}(\mathbb{D})$. For $f \in \mathcal{H}(\mathbb{D})$ write $F(z) = T_g(f)(z)$, then

$$F'(z) = f(z)g'(z).$$

Composing with $\phi_a(z)$ and multiplying by $\phi_a'(z)$ we obtain

$$(F\circ\phi_a)'(z) = (f\circ\phi_a)(z)(g\circ\phi_a)'(z),$$

thus

$$(F\circ\phi_a)(z) - (F\circ\phi_a)(0) = \int_0^z (f\circ\phi_a)(\zeta)(g\circ\phi_a)'(\zeta)\,d\zeta.$$

We write this equation in terms of C_a,

$$C_a \circ T_g(f)(z) - C_a \circ T_g(f)(0) = T_{g\circ\phi_a} \circ C_a(f)(z),$$

equivalently,

$$I_0 \circ C_a \circ T_g \circ C_a = T_{g\circ\phi_a}, \tag{4}$$

where I_0 is the operator $I_0(f)(z) = f(z) - f(0)$ acting on $\mathcal{H}(\mathbb{D})$.

Now consider the restriction of the above to the Banach space $X \subset \mathcal{H}(\mathbb{D})$. If X contains the constant functions, then $I_0(f) \in X$ whenever $f \in X$. Using (4) we conclude that if X contains the constants and is preserved by C_a (that is $f \circ \phi_a \in X$ for each $f \in X$) then V and V_0 are also preserved by C_a.

Suppose again X contains the constants. Then for the function $1 \in X$ and each $g \in V$ we obtain

$$T_g(1)(z) = \int_0^z g'(\zeta)\, d\zeta = g(z) - g(0) \in X$$

so $g \in X$, i.e. $V \subset X$. Further,

$$\begin{aligned}
\|g\|_X &= \|g(0) + g - g(0)\|_X \\
&\leq \|g(0)\|_X + \|g(z) - g(0)\|_X \\
&= |g(0)|\|1\|_X + \|T_g(1)\|_X \\
&\leq |g(0)|\|1\|_X + \|T_g\|_{X\to X}\|1\|_X \\
&= \|1\|_X\|g\|_V,
\end{aligned}$$

so $\|g\|_X \leq C\|g\|_V$ with the constant $C = \|1\|_X$ independent of $g \in V$.

We can now iterate the above construction to obtain a sequence of spaces. We start with a Banach space X with the properties:

(i) X contains the constant functions.
(ii) If $f_n \to f$ in X then $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$.
(iii) If $f \in X$ and $a \in \mathbb{D}$ then $f \to f \circ \phi_a \in X$.

Write X_1 for the space V defined above. Thus X_1 contains the constants, and is preserved by composition with ϕ_a. Further $X_1 \subset X$ and $\|f\|_X \leq C\|f\|_{X_1}$ for each $f \in X_1$. It follows that convergence in X_1 implies convergence in X and by (ii) this implies uniform convergence on compact subsets of $\mathbb{D}$. We see therefore that X_1 is a Banach space which also satisfies the properties (i), (ii) and (iii). Proceeding inductively, suppose X_{k-1} has been defined. Define X_k by

$$X_k =: \{g \in \mathcal{H}(\mathbb{D}) : T_g : X_{k-1} \to X_{k-1} \text{ is bounded}\},$$

and equip it with the norm

$$\|g\|_{X_k} = |g(0)| + \|T_g\|_{X_{k-1}\to X_{k-1}}.$$

By induction, X_k is a Banach space of analytic functions which satisfies properties (i), (ii) and (iii). We have

$$X \supseteq X_1 \supseteq X_2 \supseteq \cdots X_k \cdots ,$$

and

$$\|g\|_{X_{k-1}} \le \|g\|_{X_k}, \qquad k > 1, g \in X_k.$$

We may ask various questions about these spaces X_k. For example if we start with a specific space for X, what are the subsequent spaces X_k? Do they all contain non constant functions? Are they strictly smaller from one step to the next? Does their infinite intersection contain non constant functions?

3. Spaces of functions

Below we will survey some known results about Volterra type operators acting on some classical spaces of analytic functions. We give the definitions and basic properties of the spaces first.

Hardy spaces and $BMOA$. For $0 < p < \infty$ the Hardy space H^p contains all analytic functions $f : \mathbb{D} \to \mathbb{C}$ for which

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

With this norm, H^p is a complete metric linear space and a Banach space for $p \ge 1$. It is a Hilbert space for $p = 2$. For $p = \infty$ the Banach space H^∞ consists of the bounded analytic functions with norm

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

The space $BMOA$ consists of all $f \in H^2$ such that

$$\|f\|_* = |f(0)| + \sup_{a \in \mathbb{D}} \|f \circ \phi_a - f(a)\|_{H^2} < \infty,$$

where $\phi_a(z)$ are the Möbius automorphisms of $\mathbb{D}$. The space $VMOA$ consists of the functions $f \in BMOA$ such that

$$\lim_{|a| \to 1} \|f \circ \phi_a - f(a)\|_{H^2} = 0.$$

Equivalently, $VMOA$ is the closure of polynomials in $BMOA$. Both $BMOA$ and $VMOA$ can be described in terms of Carleson measures.

Recall that a positive measure μ on the disc is a Carleson measure if

$$\sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|} < \infty, \tag{5}$$

where $S(I) = \{z : 1 - |I| \le |z| < 1,\, z/|z| \in I\}$ is the Carleson box based on the arc $I \subset \partial\mathbb{D}$ of length $|I|$. A vanishing Carleson measure μ is one for which $\mu(S(I)) = o(|I|)$ as $|I| \to 0$. It is a basic theorem that condition (5) is equivalent to

$$\|f\|_{L^2(\mathbb{D},\mu)} \leq C\|f\|_{H^2} \qquad f \in H^2, \tag{6}$$

i.e. the inclusion operator $i : H^2 \hookrightarrow L^2(\mathbb{D}, \mu)$ is bounded. Vanishing Carleson measures are those for which this inclusion is compact.

The Carleson measure characterization of $BMOA$ is as follows. A function $f \in H^2$ is in $BMOA$ if and only if the measure

$$d\mu(z) = |f'(z)|^2 \log(z^{-1})dm(z)$$

is a Carleson measure on $\mathbb{D}$ (here $dm(z)$ is the Lebesgue area measure). And a function $f \in BMOA$ is in $VMOA$ if and only if this measure is a vanishing Carleson measure.

Functions $f \in H^p$ have radial limits almost everywhere on the boundary and a boundary function, denoted by $f(e^{i\theta})$, is well defined. For $p \geq 1$ the boundary function is in $L^p(\partial\mathbb{D})$, and H^p embeds in $L^p(\partial\mathbb{D})$ by this correspondence. H^p then can alternatively be described as the subspace of functions in $L^p(\partial\mathbb{D})$ whose Fourier series have all coefficients of negative index equal to zero. For $1 < p < \infty$ the pairing

$$\langle f, h\rangle = \int_0^{2\pi} f(e^{i\theta})\overline{h(e^{i\theta})}\frac{d\theta}{2\pi}$$

establishes a duality between H^p and H^q with $\frac{1}{p} + \frac{1}{q} = 1$. Interpreted in a wider sense, $\langle f, h\rangle = \lim_{r\to 1}\langle f_r, h_r\rangle$, where $f_r(z) = f(rz)$, this pairing gives also the dualities $VMOA^* = H^1$ and $(H^1)^* = BMOA$. This pairing can be written by the Littlewood-Paley formula as

$$\langle f, h\rangle = f(0)\overline{h(0)} + 2\int_{\mathbb{D}} f'(z)\overline{h'(z)} \log\left(\frac{1}{z}\right) dm(z).$$

More information on Hardy spaces and BMOA can be found in [Du], [Bae] and in [Sar].

Bergman and Bloch spaces. Suppose $w : [0, 1) \to (0,\infty)$ is a weight function which is integrable on $[0, 1)$. We extend w on $\mathbb{D}$ by $w(z) = w(|z|)$ and assume that w is normalized so that $\int_{\mathbb{D}} w(z)dm(z) = 1$.

For $1 \leq p < \infty$ the weighted Bergman space A^p_w is the space of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^p_{p,w} = \int_{\mathbb{D}} |f(z)|^p w(z)dm(z) < \infty.$$

These are Banach spaces. We write simply A^p when $w \equiv 1$.

The Bloch space $\mathcal{B}$ consists of all f analytic on $\mathbb{D}$ such that the Bloch norm

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)|$$

is finite, while the little Bloch space $\mathcal{B}_0$ contains those $f \in \mathcal{B}$ for which $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$. Equivalently, $\mathcal{B}_0$ is the closure of polynomials in the Bloch norm. Both $\mathcal{B}$ and $\mathcal{B}_0$ are Banach spaces under the Bloch norm. Bloch spaces are the area measure versions of $BMOA$ and $VMOA$. In this connection, a useful characterization is the following. Suppose $1 \le p < \infty$ and let ϕ_a denote the usual Möbius automorphisms of $\mathbb{D}$. Then we have, see [Axl],

$$g \in \mathcal{B} \Leftrightarrow \sup_{a\in\mathbb{D}} \|g \circ \phi_a - g(a)\|_{A^p} < \infty, \tag{7}$$

and

$$g \in \mathcal{B}_0 \Leftrightarrow \lim_{|a|\to 1} \|g \circ \phi_a - g(a)\|_{A^p} = 0.$$

Besov spaces. For $1 < p < \infty$ the Besov space B_p consists of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{B_p}^p = \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2}\, dm(z) < \infty.$$

The space B_p is a Banach space for $p > 1$. For $p = 2$, B_2 is a Hilbert space, also known as Dirichlet space and denoted by $\mathcal{D}$. More information about Bergman, Bloch and Besov spaces can be found in [Zhu].

4. Boundedness and compactness

In this section we will present the main theorems that characterize those g for which T_g is bounded or compact. The setting is on Hardy spaces, Bergman spaces and on $BMOA$.

We consider Hardy spaces first. It was proved by Pommerenke in [Pom, Lemma 1] that T_g is bounded on H^2 if and only if $g \in BMOA$. To see this assume f is a polynomial and use the Littlewood-Paley formula to write

$$\begin{aligned} \|T_g(f)\|_{H^2}^2 &= \langle T_g(f), T_g(f)\rangle \\ &= 2\int_{\mathbb{D}} |f(z)|^2 |g'(z)|^2 \log(z^{-1})\, dm(z) \\ &= 2\|f\|_{L^2(\mathbb{D},\, \mu_g)}^2, \end{aligned}$$

where

$$\mu_g(z) = |g'(z)|^2 \log(z^{-1}) dm(z). \tag{8}$$

Since polynomials are dense in H^2, we see that $T_g : H^2 \to H^2$ is bounded if and only if the inclusion operator

$$i : H^2 \hookrightarrow L^2(\mathbb{D}, \mu_g)$$

is bounded, and this is equivalent to that μ_g is a Carleson measure. In other words, T_g is bounded on H^2 if and only if $g \in BMOA$. The same argument shows that T_g is compact on H^2 if and only if $g \in VMOA$.

The same characterization of boundedness and compactness of T_g on H^p is valid for any $0 < p < \infty$, the proofs however are more difficult. The case $p \geq 1$ was proved in [AleSi1] while the extension for $0 < p < 1$ was obtained in [AleCi]. Thus we have

Theorem 4.1. *Let $0 < p < \infty$. Then $T_g : H^p \to H^p$ is bounded if and only if $g \in BMOA$.*

In [AleCi] the authors in fact consider any pair of indices $p, q \in (0, \infty)$ and characterize the symbols g for which T_g maps H^p into H^q (the closed graph theorem then implies that for those g, T_g maps H^p boundedly into H^q). We will not state the details of all cases but we remark that [AleCi, Theorem 1(iii)] is a substantial strengthening of a classical result of Hardy and Littlewood. Indeed they prove: *If $p < q$ and $\frac{1}{p} - \frac{1}{q} \leq 1$, then T_g maps H^p into H^q if and only if $g \in \Lambda_{\frac{1}{p}-\frac{1}{q}}$.* Here the Lipschitz class Λ_α, $0 < \alpha \leq 1$, consists of all analytic functions g on $\mathbb{D}$ such that

$$|g'(z)| = O((1 - |z|)^{\alpha - 1}), \qquad |z| \to 1.$$

Now apply this with $\frac{1}{p} - \frac{1}{q} = 1$ and $g(z) = z \in \Lambda_1$, to obtain the following theorem of Hardy and Littlewood. *If $0 < p < 1$, $f \in H^p$ and $F' = f$ then $F \in H^q$ and $q = p/(1-p)$.* Theorem 1(iii) of [AleCi] in its full generality as stated above is a nontrivial strengthened form of the Hardy-Littlewood theorem, see [AleCi] for details.

As a byproduct in the proof of the boundedness of T_g one obtains the following corollary for compactness which was proved for $p \geq 1$ in [AleSi1] and for all $p > 0$ in [AleCi].

Corollary 4.2. *Let $0 < p < \infty$. Then $T_g : H^p \to H^p$ is compact if and only if $g \in VMOA$.*

Analogous characterization for the symbols g hold for T_g to be bounded on Bergman spaces. In the unweighted case, for $1 \leq p < \infty$ and $f \in A^p$ it is well known that one can write

$$\begin{aligned} \|f\|_{A^p}^p &= \int_{\mathbb{D}} |f(z)|^p \, dm(z) \\ &\simeq |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^p \, dm(z), \end{aligned}$$

where $\simeq$ means the two sides are comparable. Thus we have

$$\|T_g(f)\|_{A^p}^p \simeq \int_{\mathbb{D}} |f(z)|^p |g'(z)|^p (1 - |z|^2)^p \, dm(z).$$

We see that if g is in the Bloch space $\mathcal{B}$ then $|g'(z)|(1 - |z|^2) \leq \|g\|_{\mathcal{B}}$ for all $z \in \mathbb{D}$, and we have

$$\|T_g(f)\|_{A^p} \leq C \|f\|_{A^p} \|g\|_{\mathcal{B}},$$

so $T_g : A^p \to A^p$ is bounded.

This last inequality together with the fact that integration in A^p is a compact operator (and therefore T_P is compact on A^p for any polynomial P) and the fact that the little Bloch space $\mathcal{B}_0$ is the closure of polynomials in $\mathcal{B}$, imply that if $g \in \mathcal{B}_0$ then $T_g : A^p \to A^p$ is compact.

The converse of this is also true. To prove it we will make use of the characterization of Bloch and little Bloch functions given in (7). Indeed assume $1 < p < \infty$ and $T_g : A^p \to A^p$ is bounded. We will show that $g \in \mathcal{B}$, by showing that $\sup_{a\in\mathbb{D}} \|g \circ \phi_a - g(a)\|_{A^1} < \infty$. We have,

$$\begin{aligned}
\|g \circ \phi_a - g(a)\|_{A^1} &\simeq \int_{\mathbb{D}} |(g \circ \phi_a)'(z)|(1 - |z|^2)\, dm(z) \\
&= \int_{\mathbb{D}} |(g'(\phi_a(z)))||\phi_a'(z)|(1 - |z|^2)\, dm(z) \\
&= \int_{\mathbb{D}} |(g'(\phi_a(z)))|(1 - |\phi_a(z)|^2)\, dm(z) \\
&= \int_{\mathbb{D}} |g'(z)|(1 - |z|^2)|\phi_a'(z)|^2\, dm(z) \\
&= \int_{\mathbb{D}} |g'(z)|(1 - |z|^2)\frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4}\, dm(z) \\
&= \int_{\mathbb{D}} |g'(z)|(1 - |z|^2)\frac{(1 - |a|^2)^{2/p}}{|1 - \bar{a}z|^2}\,\frac{(1 - |a|^2)^{2/q}}{|1 - \bar{a}z|^2}\, dm(z),
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Write $k_a(z) = \frac{(1-|a|^2)^{2/p}}{(1-\bar{a}z)^2}$, $h_a(z) = \frac{(1-|a|^2)^{2/q}}{(1-\bar{a}z)^2}$ and use Hölder's inequality to obtain

$$\begin{aligned}
\|g \circ \phi_a - g(a)\|_{A^1} &\leq ||k_a||_{L^q}\||g'(z)|(1 - |z|^2)|h_a(z)|\|_{L^p} \\
&\simeq \|k_a\|_{L^q}\|T_g(h_a)\|_{A^p} \\
&\leq \|k_a\|_{L^q}\|T_g\|_{A^p\to A^p}\|h_a\|_{A^p} \\
&\leq C\|T_g\|_{A^p\to A^p}
\end{aligned}$$

because $\|k_a\|_{L^q} \simeq \|h_a\|_{A^p} \simeq 1$ for each $a \in \mathbb{D}$. By assumption T_g is bounded on A^p. Thus

$$\sup_{a\in\mathbb{D}} \|g \circ \phi_a - g(a)\|_{A^1} < \infty$$

so $g \in \mathcal{B}$. The case $p = 1$ can be handled similarly.

An analogous argument, based on weak convergence of test functions gives that if $T_g : A^p \to A^p$ is compact then $g \in \mathcal{B}_0$. We summarize

Theorem 4.3. *Let $1 \leq p < \infty$. Then*
(i) $T_g : A^p \to A^p$ *is bounded if and only if* $g \in \mathcal{B}$.
(ii) $T_g : A^p \to A^p$ *is compact if and only if* $g \in \mathcal{B}_0$.

This theorem can be extended to a class of weighted Bergman spaces for some general weights. The details can be found in [AleSi2]. We will only describe here the weights w for which the above theorem is valid on A^p_w.

Theorem 4.4. *Suppose $w : [0,1) \to (0,\infty)$ is a weight function which is integrable on $[0,1)$ and satisfies the conditions*
(P_1) *There is a constant C such that*

$$w(r) \geq \frac{C}{1-r}\int_r^1 w(u)\,du, \qquad 0<r<1.$$

(P_2) *There is an $s \in (0,1)$ and a constant D such that*

$$w(sr+1-s) \geq Dw(r), \qquad 0<r<1.$$

Then:
(i) *T_g is bounded on A^p_w if and only if $g \in \mathcal{B}$.*
(ii) *T_g is compact on A^p_w if and only if $g \in \mathcal{B}_0$.*

Condition (P_1) in the above theorem is used to prove an inequality of the form

$$\int_{\mathbb{D}} |f(z)|^p w(z)\,dm(z) \leq C\left(|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p(1-|z|)^p w(z)\,dm(z)\right),$$

for f analytic on $\mathbb{D}$, which clearly gives the sufficiency assertions for T_g. Condition (P_2) which is a generalized version of $w(\frac{1+r}{2}) \geq Dw(r)$ $(s=1/2)$ is used to prove the necessity. All standard weights $w(r) = (1-r)^\alpha$, $\alpha > -1$, satisfy conditions (P_1) and (P_2). Other weights that satisfy them are,

$$w(r) = (1-r)^\alpha\left(\log\frac{e}{1-r}\right)^\beta, \quad \alpha > -1, \beta \in \mathbb{R},$$

$$w(r) = \exp\left(-\beta\left(\log\frac{e}{1-r}\right)^\alpha\right), \quad 0<\alpha\leq 1, \beta>0.$$

However the exponential weight

$$w(r) = \exp\left(\frac{-\beta}{(1-r)^\alpha}\right), \quad \alpha,\beta>0$$

does not satisfy condition (P_2), see [AleSi2] for details.

We now consider T_g acting on $BMOA$. If T_g is bounded on $BMOA$ then $T_g(1) = g(z) - g(0) \in BMOA$ so $g \in BMOA$. On the other hand, for $g(z) = \log(\frac{1}{1-z}) \in BMOA$ we have

$$T_g(g)(z) = \int_0^z g(\zeta)g'(\zeta)\,d\zeta = \frac{1}{2}\log^2\left(\frac{1}{1-z}\right)$$

and this function is not in $BMOA$. Thus the space of g's for which T_g is bounded is a proper subspace of $BMOA$. This space was determined in [SiZh] and is a space of functions that are of *bounded logarithmically weighted mean oscillation* as described in the following

Theorem 4.5. *The operator T_g maps $BMOA$ boundedly into itself if and only if*

$$\sup_{I\subset\partial\mathbb{D}}\left\{\frac{\left(\log\frac{2}{|I|}\right)^2}{|I|}\int_{S(I)}|g'(z)|^2(1-|z|^2)\,dm(z)\right\}<\infty. \tag{9}$$

We now show that this condition also characterizes those g for which T_g is bounded on $VMOA$. First observe that under this condition $T_g(\text{VMOA})\subset$ VMOA. Indeed suppose g satisfies (9) then clearly $g\in VMOA$. Since $T_g(1)=g-g(0)$, the constant functions are mapped into $VMOA$. Let n be a positive integer. An integration by parts gives

$$T_g(z^n)=z^ng(z)-n\int_0^z\zeta^{n-1}g(\zeta)\,d\zeta.$$

Since multiplication by z and the integration operator are bounded on $VMOA$ we see that $T_g(z^n)\in VMOA$, and the same is true for $T_g(p)$ for any polynomial p. Next let $f\in VMOA$. There is a sequence (p_n) of polynomials such that $\|f-p_n\|_*\to 0$ and we have

$$\|T_g(f)-T_g(p_n)\|_*=\|T_g(f-p_n)\|_*\le\|T_g\|\|f-p_n\|_*.$$

This shows that $T_g(f)$ can be approximated in the $\|\cdot\|_*$ norm by $VMOA$ functions. Since $VMOA$ is closed in this norm the assertion follows.

Now assume $T_g: VMOA\to VMOA$ is bounded and recall the dualities between $VMOA, H^1$ and $BMOA$. Let

$$A_g=T_g^*:H^1\to H^1$$

be the adjoint of T_g acting on $H^1=(VMOA)^*$, and let

$$A_g^*:BMOA\to BMOA$$

be the adjoint of A_g. Because $VMOA$ is weak* dense in $BMOA$ and since $T_g(\text{VMOA})\subset$ VMOA, we see that $A_g^*=T_g$. It follows that T_g is bounded on $VMOA$ if and only if it is bounded on $BMOA$.

The following theorem from [SiZh] asserts that the little oh condition corresponding to (9) characterizes those g for which T_g is a compact operator on $BMOA$.

Theorem 4.6. *T_g is compact on $BMOA$ if and only if*

$$\lim_{|I|\to 0}\left\{\frac{\left(\log\frac{2}{|I|}\right)^2}{|I|}\int_{S(I)}|g'(z)|^2(1-|z|^2)\,dm(z)\right\}=0. \tag{10}$$

5. Schatten ideals

Let H be a Hilbert space and $T : H \to H$ a bounded operator. The singular numbers of T are defined by

$$\lambda_n = \inf\{\|T - F\| : F : H \to H \text{ is an operator of rank} \le n\}.$$

The operator T is compact if and only if $\lambda_n \to 0$, and of finite rank if and only if $\lambda_n = 0$ for all sufficiently large n. For $0 < p < \infty$ the Schatten classes $\mathcal{S}^p$ contain those bounded linear operators on H for which $(\lambda_n) \in l^p$, the space of p-summable sequences. For $p \ge 1$ the Schatten norm in $\mathcal{S}^p$ is defined by $|T|_{\mathcal{S}^p} = \|(\lambda_n)\|_{l^p}$. Each $\mathcal{S}^p$ is a two sided self-adjoint ideal in the space of all bounded operators on H and is a Banach space under the Schatten norm. Further $T \in \mathcal{S}^p$ if and only if $T^*T \in \mathcal{S}^{p/2}$. The classes $\mathcal{S}^2$ and $\mathcal{S}^1$ are the Hilbert-Schmidt and trace class respectively. An operator T is Hilbert-Schmidt if and only if $\sum_n \|T(e_n)\|^2 < \infty$ for some orthonormal basis $\{e_n\}$ of H.

Before we consider the Schatten classes of T_g on specific Hilbert spaces let us consider a general case. Suppose the Hilbert space H consists of analytic functions on the disc $\mathbb{D}$. For each $p \ge 1$ define

$$W_p = W_{p,H} = \{g \in \mathcal{H}(\mathbb{D}) : T_g \in \mathcal{S}^p\}.$$

Since each $\mathcal{S}^p$ is a Banach space under the Schatten norm, each W_p is a vector space always containing the constants. We can give it a norm

$$\|g\|_{W_p} = |g(0)| + \|T_g\|_{\mathcal{S}^p},$$

under which W_p is a Banach space. This follows from the fact that a Cauchy sequence of operators in the Schatten norm is also Cauchy in the operator norm. For $p = 2$ the inner product

$$\langle g, h\rangle_{W_2} = g(0)\overline{h(0)} + \langle T_g, T_h\rangle_{\mathcal{S}^2}$$

makes W_2 a Hilbert space.

Further if H contains the constants and the compositions $C_a(f) = f \circ \phi_a$ with Möbius automorphisms of $\mathbb{D}$ are bounded operators on H, using arguments similar to those in section 2 we find that W_p are preserved by composition with ϕ_a.

We will now consider the Schatten classes of T_g on the Hardy space H^2. For the basis $\{e_n(z) = z^n : n = 0, 1, 2, \dots\}$ of H^2 we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \|T_g(e_n)\|_{H^2}^2 &= \sum_{n=0}^{\infty} \left\| \int_0^z \zeta^n g'(\zeta)\, d\zeta \right\|_{H^2}^2 \\
&\simeq \sum_{n=0}^{\infty} \int_{\mathbb{D}} |z|^{2n} |g'(z)|^2 (1-|z|)\, dm(z) \\
&= \int_{\mathbb{D}} \left(\sum_{n=0}^{\infty} |z|^{2n} \right) |g'(z)|^2 (1-|z|)\, dm(z) \\
&= \int_{\mathbb{D}} \frac{1}{1-|z|^2} |g'(z)|^2 (1-|z|)\, dm(z) \\
&\simeq \int_{\mathbb{D}} |g'(z)|^2\, dm(z).
\end{aligned}$$

This says that T_g is Hilbert-Schmidt on H^2 if and only if g is in the Dirichlet space $\mathcal{D}$. To determine the other Schatten classes observe that for $f, h \in H^2$ we have by the Littlewood-Paley formula

$$\langle T_g^* T_g(f), h \rangle = \langle T_g(f), T_g(h) \rangle = 2 \int_{\mathbb{D}} f(z) \overline{h(z)} |g'(z)|^2 \log(z^{-1})\, dm(z).$$

Choosing $h(z) = k_w(z) = (1 - \bar{w} z)^{-1}$, the reproducing kernel of H^2, the above equation becomes

$$T_g^* T_g(f)(w) = 2 \int_{\mathbb{D}} f(z) \frac{1}{1 - \bar{z} w}\, d\mu(z),$$

with

$$d\mu(z) = |g'(z)|^2 \log(z^{-1})\, dm(z).$$

Thus the operator $T_g^* T_g$ is a generalized Toeplitz operator and the Schatten ideals for those operators have been determined in [Lu]. Applying the main theorem of [Lu] we can determine the Schatten class of $T_g^* T_g$ and therefore also that of T_g, see [AleSi1] for details.

Theorem 5.1. *Suppose $1 < p < \infty$. Then $T_g \in \mathcal{S}^p$ on H^2 if and only if g is in the Besov space B_p. Further T_g is not in $\mathcal{S}^1$ unless g is a constant.*

We now consider the Schatten classes of T_g on the Bergman space A^2. Taking $\{e_n(z) = \sqrt{n+1}z^n : n = 0, 1, \cdots\}$ as an orthonormal basis we have

$$\begin{aligned}\sum_{n=0}^{\infty} \|T_g(e_n)\|_{A^2}^2 &\simeq \sum_{n=0}^{\infty} \int_{\mathbb{D}} (n+1)|z|^{2n}|g'(z)|^2(1-|z|)^2\,dm(z) \\ &= \int_{\mathbb{D}} \left(\sum_{n=0}^{\infty}(n+1)|z|^{2n}\right)|g'(z)|^2(1-|z|)^2\,dm(z) \\ &= \int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2}|g'(z)|^2(1-|z|)^2\,dm(z) \\ &\simeq \int_{\mathbb{D}} |g'(z)|^2\,dm(z),\end{aligned}$$

thus T_g is Hilbert-Schmidt on A^2 if and only if $g \in \mathcal{D}$.

More generally consider the weighted Bergman spaces A^2_α with weights

$$w(r) = (\alpha+1)(1-r^2)^\alpha, \quad \alpha > -1.$$

The reproducing kernel here is

$$k_w(z) = \frac{1}{(1-\bar{w}z)^{\alpha+2}},$$

and the inner product can be written

$$\begin{aligned}\langle f, h\rangle &= (\alpha+1)\int_{\mathbb{D}} f(z)\overline{h(z)}(1-|z|^2)^\alpha\,dm(z) \\ &= f(0)\overline{h(0)} + \int_{\mathbb{D}} f'(z)\overline{h'(z)}v(|z|)\,dm(z),\end{aligned}$$

where

$$v(r) = 2\int_r^1 \frac{(1-u^2)^{\alpha+1}}{u}\,du.$$

We can now work as in the Hardy space case to obtain the operator $T_g^*T_g$ in the form

$$T_g^*T_g(f)(w) = \int_{\mathbb{D}} f(z)\frac{1}{(1-\bar{z}w)^{\alpha+2}}\,d\mu(z)$$

with $d\mu(z) = |g'(z)|^2v(|z|)dm(z)$. Applying again the work of [Lu] we obtain

Theorem 5.2. *Let $\alpha > -1$ and $1 < p < \infty$. The operator T_g is in $\mathcal{S}^p$ on A^2_α if and only if $g \in B_p$. Further T_g is not in $\mathcal{S}^1$ unless g is a constant.*

6. Some final remarks

Volterra type operators and some variations of them have been also studied on various spaces in [Hu], [Xi], [AMN], [You1], [You2], [You3]. We will not recount the results of these papers here. We will make however some remarks about such operators.

1. **Cyclicity.** Consider T_g acting on H^2 and without loss of generality assume $g(0) = 0$. Then

$$T_g(1)(z) = \int_0^z g'(\zeta)d\zeta = g(z),$$

$$T_g^2(1)(z) = \int_0^z g(\zeta)g'(\zeta)\,d\zeta = \frac{1}{2}g(z)^2$$

and inductively

$$T_g^n(1)(z) = \frac{1}{n!}g(z)^n, \quad n = 1, 2, \ldots$$

Set also $T_g^0(1)(z) = 1$. We see that the linear span of the orbit of 1 under T_g is dense in H^2 if and only if the polynomials in $g(z)$ are dense in H^2. Observe that this can happen only if g is univalent. Indeed if there are $z, w \in \mathbb{D}$ such that $g(z) = g(w)$ then for any polynomial P we will have $P(g(z)) = P(g(w))$ and any limit f of a sequence of such polynomials must satisfy $f(z) = f(w)$, so the set of polynomials in g cannot be dense in H^2.

Let $\Omega = g(\mathbb{D})$, and let $H^2(\Omega)$ to be the Hardy space on Ω defined through harmonic majorants i.e. $f : \Omega \to \mathbb{C}$ belongs to $H^2(\Omega)$ if and only if there is a harmonic function $u(z)$ on Ω such that $|f(z)|^2 \leq u(z)$ for each $z \in \Omega$. In this situation polynomials in g are dense in H^2 if and only if the polynomials in z are dense in $H^2(\Omega)$. We therefore have

Proposition 6.1. *The vector* 1 *is a cyclic vector for* T_g *on* H^2 *if and only if polynomials are dense on* $H^2(\Omega)$ *with* $\Omega = g(\mathbb{D})$.

It would be interesting to see if the role of the vector 1 can be removed in the above proposition. In other words to prove that T_g is cyclic on H^2 if and only if polynomials are dense in $H^2(\Omega)$.

2. **A new proof of an old result**. It is well known that if g is a function in $BMOA$ then e^g is in H^p for some $p > 0$. And if $g \in VMOA$ then $e^g \in H^p$ for all $p < \infty$ [CiSc]. We are going to give a proof of these facts using the operators T_g.

Indeed let $g \in BMOA$. Then T_g is bounded on H^2. Assume without loss of generality $g(0) = 0$. Applying T_g repeatedly to the constant function 1 we have

$$T_g^n(1)(z) = \frac{1}{n!}g(z)^n, \qquad n = 0, 1, 2, \ldots$$

Let r_g denote the spectral radius of T_g and let s be a number such that $0 < s < 1/r_g$. Then the series $\sum_{n=0}^{\infty} s^n T_g^n$ converges in the operator norm topology, thus the series

$$\sum_{n=0}^{\infty} \frac{(sg(z))^n}{n!} = \sum_{n=0}^{\infty} s^n T_g^n(1)(z)$$

converges in H^2. The sum of this series coincides with the pointwise sum which is $e^{sg(z)}$. Thus $e^{sg(z)}$ is in H^2 and $e^g(z)$ is in H^{2s}.

If $g \in VMOA$ then T_g is compact in H^2, and it can be checked by hand that T_g has no eigenvalues so its spectrum is $\{0\}$, and the spectral radius $r_g = 0$. Thus we can choose the number s in the previous argument to be any number in $(0, \infty)$. It follows that $e^{sg(z)}$ is in H^2 for all $s \in (0, \infty)$ so $e^{g(z)}$ is in every H^p, $p < \infty$.

The same reasoning proves the analogous statements for g in the Bloch or little Bloch space. We have

Theorem 6.2. *Let g be analytic on $\mathbb{D}$, then*
(i) *If $g \in BMOA$, then $e^g \in H^p$ for some $p > 0$.*
(ii) *If $g \in VMOA$, then $e^g \in H^p$ for all $p > 0$.*
(iii) *If $g \in \mathcal{B}$, then $e^g \in A^p$ for some $p > 0$.*
(iv) *If $g \in \mathcal{B}_0$, then $e^g \in A^p$ for all $p > 0$.*

This theorem can be extended to the spaces X_k of section 2. The same reasoning can be applied to prove the following. Suppose X is a Banach space of analytic functions on $\mathbb{D}$ which contains the constant functions, $M_z : X \to X$ is bounded, and the point evaluations are bounded on X. Then for the Cesàro operator to be bounded on X it is necessary that X contains the functions $1/(1 - z)^s$ for all sufficiently small s, see [Si2]. This is because the Cesàro operator is obtained as T_g with $g = \log(1/(1 - z))$ and $e^{sg(z)} = 1/(1 - z)^s$.

3. **Spectral properties**. The spectrum of the Cesàro operator has been determined by using its relation to a semigroup of weighted composition operators, see [Si1]. This method can not be applied to the general Volterra operators T_g. The spectral properties of these operators remain unknown even though there have been some papers with sporadic results see [You1], [You2], [AMN], [MMN] and [Mil].

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