Phase transitions in the Ramsey-Turán theory

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Abstract

Let $f(n)$ be a function and L be a graph. Denote by $\mathbf{RT}(n, L, f(n))$ the maximum number of edges of an L-free graph on n vertices with independence number less than $f(n)$. Erdős and Sós [7] asked if $\mathbf{RT}\left(n, K_5, c\sqrt{n}\right) = o\left(n^2\right)$ for some constant c. We answer this question by proving the stronger $\mathbf{RT}\left(n, K_5, o\left(\sqrt{n \log n}\right)\right) = o\left(n^2\right)$. It is known that $\mathbf{RT}\left(n, K_5, c\sqrt{n \log n}\right) = n^2/4 + o\left(n^2\right)$ for $c > 1$, so one can say that K_5 has a Ramsey-Turán-*phase transition* at $c\sqrt{n \log n}$. We extend this result to several other K_p's and functions $f(n)$, determining many more phase transitions. We shall formulate several open problems, in particular, whether variants of the Bollobás-Erdős graph, which is a geometric construction, exist to give good lower bounds on $\mathbf{RT}\left(n, K_p, f(n)\right)$ for various pairs of p and $f(n)$. These problems are studied in depth by Balogh-Hu-Simonovits [1], where among others, the Szemerédi's Regularity Lemma and the Hypergraph Dependent Random Choice Lemma are used.

Notation

Definition 1 Denote by $\mathbf{RT}(n, L, m)$ the maximum number of edges of an L-free graph on n vertices with independence number less than m.

We are interested in the asymptotic behavior of $\mathbf{RT}(n, L, f(n))$ and its "phase transitions", i.e., in the question, when and how the asymptotic behavior of $\mathbf{RT}(n, L, f)$ changes sharply when we replace f by a slightly smaller g.

Definition 2 Let

$$\overline{\rho\tau}(L, f) = \limsup_{n\to\infty} \frac{\mathbf{RT}(n, L, f(n))}{n^2}$$

and

$$\underline{\rho\tau}(L, f) = \liminf_{n\to\infty} \frac{\mathbf{RT}(n, L, f(n))}{n^2}.$$

If $\overline{\rho\tau}(L, f) = \underline{\rho\tau}(L, f)$, then we write $\mathbf{RT}(L, f) = \overline{\rho\tau}(L, f) = \underline{\rho\tau}(L, f)$, and call $\mathbf{RT}$ the *Ramsey-Turán density* of L with respect to f, $\overline{\rho\tau}$ the upper, $\underline{\rho\tau}$ the lower *Ramsey-Turán densities*, respectively.

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1 Introduction

Szemerédi [10], using his regularity lemma [11], proved $\overline{\rho\tau}(K_4, o(n)) \le 1/8$. Bollobás and Erdős [4] constructed the so-called Bollobás-Erdős graph, one of the most important constructions in this area, which shows that $\underline{\rho\tau}(K_4, o(n)) \ge 1/8$. Indeed, the Bollobás-Erdős graph on n vertices is K_4-free, with $(\frac{1}{8} + o(1))n^2$ edges and independence number $o(n)$. Later, Erdős, Hajnal, Sós and Szemerédi [5] extended these results, determining $\mathbf{RT}(K_{2r}, o(n))$.

Our focus here is somewhat different; we are interested exploring the situation when the independence number is really small.

In [7], Erdős and Sós proved that $\mathbf{RT}(n, K_5, c\sqrt{n}) \le \frac{1}{8}n^2 + o\left(n^2\right)$ for every $c > 0$. They also asked if $\mathbf{RT}(n, K_5, c\sqrt{n}) = o\left(n^2\right)$ for some $c > 0$. They also suggested the following construction, which with the current state of the art yields $\mathbf{RT}\left(n, K_5, c\sqrt{n \log n}\right) = n^2/4 + o\left(n^2\right)$ for $c > 1$: Into each of the parts of a complete balanced bipartite graph place a triangle-free graph with as small independence number as possible. Then we obtain a K_5-free graph with $(1/4 + o(1))n^2$ edges, with low independence number. One of our main results answers the question of Erdős and Sós; practically saying that this construction is optimal.

Theorem 3

$$\mathbf{RT}\left(n, K_5, o\left(\sqrt{n \log n}\right)\right) = o(n^2).$$

We finish this section with an observation, a result and an open question on K_6.

For any $c > 1$,

$$\underline{\rho\tau}\left(K_6, c\sqrt{n \log n}\right) \ge \underline{\rho\tau}\left(K_5, c\sqrt{n \log n}\right) \ge 1/4. \tag{1}$$

Sudakov [9] using the dependent random choice proved that $\sqrt{n}$ is the proper range for the phase-transition for K_6.

Theorem 4 **RT** $\left(K_6, \sqrt{n}e^{-c\sqrt{\log n}}\right) = o(n^2)$.

Problem 1 *Is* $\underline{\rho\tau}\left(K_6, 0.1\sqrt{n\log n}\right) = 0$*?*

Balogh and Lenz [2, 3] using some variants of the Bollobás-Erdős graph solved some longstanding open problems in this field. Therefore there is a chance that using discrete geometry, someone can solve Problem 1. Here we describe the Bollobás-Erdős graph, and provide the argument for the K_5 case. These problems are studied in depth by Balogh-Hu-Simonovits [1], where among others, the Szemerédi's Regularity Lemma and the Hypergraph Dependent Random Choice Lemma are used.

2 Proof of Theorem 3

The next lemma is taken from the survey of Fox and Sudakov [8].

Lemma 5 *(Dependent Random Choice Lemma) Let* a, d, m, n, r *be positive integers. Let* $G = (V, E)$ *be a graph with* n *vertices and average degree* $d = 2e(G)/n$*. If there is a positive integer* t *such that*

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a, \qquad (2)$$

then G *contains a subset* U *of at least* a *vertices such that every* r *vertices in* U *have at least* m *common neighbors.*

We actually prove the following quantitive version of the theorem.

Theorem 6 *For every* $k > 3$*, if* $\omega(n) \to \infty$ *then*

$$\mathbf{RT}\left(n, K_5, \frac{\sqrt{n\log n}}{\omega(n)}\right) \leq \frac{n^2}{(\omega(n))^{1/k}} = o\left(n^2\right). \qquad (3)$$

Proof. In Theorem 6, we have fixed a $k > 3$ and an $\omega \to \infty$. If n is large enough, then $\varepsilon \geq \omega(n)^{-1/k}$. Assume that there is a K_5-free graph G_n with

$$e(G_n) \geq \varepsilon n^2 \text{ and } \alpha(G_n) < \frac{\sqrt{n\log n}}{\omega(n)}. \qquad (4)$$

We apply Lemma 5 with $n, a = \frac{4n}{\omega(n)^2}$, $r = 3$, $d = 2\varepsilon n$, $m = \sqrt{n\log n}$, and $t = k + 3$.
Now the condition of Lemma 5, (2) is satisfied as

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq (2\varepsilon)^t n - n^3\left(\frac{\log n}{n}\right)^{t/2} > \varepsilon^t n$$
$$\geq \frac{n}{\omega(n)^{1+3/k}} > a.$$

So there exists a vertex subset U of G with $|U| = a = 4n/\omega(n)^2$ such that all subsets of U of size 3 have at least m common neighbors. Either U has an independent set of size at least $\left(\frac{1}{\sqrt{2}} - o(1)\right)\sqrt{\frac{4n}{\omega(n)^2}\log\left(\frac{4n}{\omega(n)^2}\right)} > \alpha(G_n)$, or $G_n[U]$ contains a triangle. In the latter case, denote by W the common neighborhood of the vertices of the triangle. It follows that $|W| \geq m = \sqrt{n\log n} > \alpha(G_n)$, so $G_n[W]$ contains an edge, and this edge forms a K_5 with the triangle. □

3 The Bollobás-Erdős Graph

The Bollobás-Erdős Graph [4] is a surprising construction, which we describe in this section. First we list some properties of the high dimensional sphere $\mathbb{S}^k$. Denote $\mu()$ the 'normalized' measure on $\mathbb{S}^k$, i.e., $\mu(\mathbb{S}^k) = 1$. The following properties of the high dimensional sphere is crucial: Given any $\alpha, \beta > 0$, it is possible to select $\epsilon > 0$ small enough and then k large enough so that Properties (P1), (P2), and (P3) below are satisfied.

(P1) Let C be a spherical cap in $\mathbb{S}^k$ with height h, where $2h = \left(\sqrt{2} - \epsilon/\sqrt{k}\right)^2$ (this means that all points of the spherical cap are within distance $\sqrt{2} - \epsilon/\sqrt{k}$ of the center). Then $\mu(C) \geq \frac{1}{2} - \alpha$.

(P2) Let $C_1, \ldots, C_t$ be spherical caps in $\mathbb{S}^k$ with height h, where $2h = \left(\sqrt{2} - \epsilon/\sqrt{k}\right)^2$. Let z_i be the center of C_i. Assume for all $1 \leq i < j \leq t$ that $d(z_i, z_j) \leq \sqrt{2}$. Then $\mu(C_1 \cap \ldots \cap C_t) \geq \frac{1}{2^t} - t\alpha$.

(P3) Let C be a spherical cap with diameter $2 - \epsilon/(2\sqrt{k})$. Then $\mu(C) \leq \beta$.

We also use the following properties of high dimensional spheres.

(P4) For any $0 < \gamma < \frac{1}{4}$, it is impossible to have $p_1, p_2, q_1, q_2 \in \mathbb{S}^k$ such that $d(p_1, p_2) \geq 2 - \gamma$, $d(q_1, q_2) \geq 2 - \gamma$, and $d(p_i, q_j) \leq \sqrt{2} - \gamma$ for all $1 \leq i, j \leq 2$.

(P5) Let $A \subseteq \mathbb{S}^k$ and let C be a spherical cap of the same measure. Then $\mathrm{diam}(A) \geq \mathrm{diam}(C)$.

(P6) Let $A, B \subseteq \mathbb{S}^k$ with equal measure and let C be a cap of the same measure. Then $d_{\max}(A, B) \geq \mathrm{diam}(C)$.

Properties (P1) and (P2) follow directly from the formula for the measure of a spherical cap, Properties (P3), (P5), and (P6) are all folklore results that are easy corollaries of the isoperimetric inequality on the sphere, and Property (P4) can be proved by examining distances and using the triangle inequality.

In order to prove that $\mathbf{RT}(n, K_4, o(n)) \geq \frac{n^2}{8}$, we need to construct, for every $\alpha, \beta > 0$, a K_4-free graph G with n vertices, independence number at most βn, and at least $\frac{n^2}{8}(1-\alpha)$ edges. Given $\alpha, \beta \geq 0$, take ϵ small enough and k large enough so that Properties (P1) and (P3) hold. Divide the k-dimensional unit sphere $\mathbb{S}^k$ into $n/2$ domains having equal measure and diameter at most $\frac{\epsilon}{10\sqrt{k}}$. Choose a point from each domain and let P be the set of these points. Let $\phi : P \to \mathcal{P}(\mathbb{S}^k)$ map points of P to the corresponding domain of the sphere. Take as vertex set of G the disjoint union of two sets V_1 and V_2 both isomorphic to P. For $x, y \in V_i$ we make xy an edge of G if $d(x, y) \geq 2 - \epsilon/\sqrt{k}$. For $x \in V_1, y \in V_2$ we make xy an edge of G if $d(x, y) \leq \sqrt{2} - \epsilon/\sqrt{k}$. Then Property (P1) shows that every vertex in V_1 has at least $\frac{1}{2}|V_2|(1-\alpha)$ neighbors in V_2 so the total number of edges is at least $\frac{1}{8}n^2(1-\alpha)$. If I is a set in V_1 with $|I| \geq \beta |V_1| = \beta \frac{n}{2}$, then $\mu(\phi(I)) = |I| / |P| \geq \beta$. Let C be a spherical cap of measure $\mu(\phi(I))$. Properties (P3) and (P5) show that $2 - \epsilon/(2\sqrt{k}) \leq \text{diam}(C) \leq \text{diam}(\phi(I))$. For $p \in I$, each $\phi(p)$ has diameter at most $\epsilon/(10\sqrt{k})$ so we can find two points $p_1, p_2 \in I$ with $d(p_1, p_2) \geq 2 - \epsilon/\sqrt{k}$, showing that I is not independent. Finally, Property (P4) shows this graph has no K_4 as a subgraph since any K_4 must take two vertices from V_1 and two vertices from V_2 (the graph spanned by V_i is triangle-free). To summarize, we have constructed a K_4-free graph G on n vertices with independence number at most βn and at least $\frac{1}{8}n^2(1-\alpha)$ edges. Since this construction holds for any $\alpha, \beta > 0$, we have proved that $\overline{\rho\tau}(K_4, o(n)) \leq 1/8$.

4 Conclusion

The aim of this note was to raise awareness for a problem in extremal graph theory, where a solution could come by geometric tools.

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