# On q-polynomials and some of their applications

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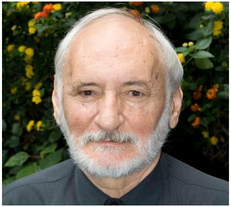

Selected Topics in Mathematical Physics
In honor of Professor Natig Atakishiyev
Instituto de Matemáticas, Cuernavaca, UNAM, 28–30 November 2016

# Why to talk about $q$-polynomials today?

This Conference was organized in honor to Natig Atakishiyev.

Natig was born in Azerbaijan and in 1991 moved to México when he is living now.

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He is very well known for his mathematical works on SF and OP and specially for his important contributions to the **theory of $q$-polynomials** but also for his works related with different kind of harmonic **quantum oscillators**.

Special Functions (SF) appear in (almost) all context of Mathematics and other Sciences.

As Alberto Grunbaum one time said: "*Special functions are to mathematics what pipes are to a house: nobody wants to exhibit them openly but nothing works without them*".

**Definition 1** Given a sequence of normal pol. $(P_n)_n$ we said that $(P_n)_n$ is an OPS w.r.t. $\mu$ if $\forall n \neq m \in \mathbb{N}$,

$$\int_{\mathbb{R}} P_n(x) P_m(x) d\mu(x) = \delta_{n,m} d_n, \qquad d_n \neq 0$$

If $\mu$ is a positive measure $\Rightarrow d_n > 0 \ \forall n$, $\Rightarrow$ we said that SOP is positive definite.

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- If $d\mu(x) = \rho(x)dx \Rightarrow \rho$ is a continuous weight function
- If $d\mu(x) = \sum_k \delta(x - x_k)\rho(x_k)dx \Rightarrow \rho$ is a discrete weight function

If $\displaystyle\int_{\mathbb{R}} P_n(x)P_m(x)d\mu(x) = \delta_{n,m} \quad \Rightarrow \quad \exists\ (a_n)_n$ y $(b_n)_n$ such that

$$xP_n(x) = a_{n+1}P_{n+1}(x) + b_nP_n(x) + a_nP_{n-1}(x), \quad n \geq 0\ ,$$

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¿There exists a converse result?

¿There are any other characterizations?

Sonin (1887): The only OPS $(P_n)_n$ such that their derivatives $(P_n')_n$ also constitute and OPS are the Jacobi, Laguerre, and Hermite polynomials.

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Hahn (1937) also proved and extension of the above characterization: Given and OPS $(P_n)_n$ it is classical iff the sequence $(P_n^{(k)})_n$ is orthogonal for some $k \in \mathbb{N}$

¿What else we can said about classical families?

Bochner (1929): They are the only solution of

$$\sigma(\mathbf{x})\mathbf{y}''(\mathbf{x}) + \tau(\mathbf{x})\mathbf{y}'(\mathbf{x}) + \lambda_{\mathbf{n}}\mathbf{y}(\mathbf{x}) = \mathbf{0}, \quad \deg\sigma \leq 2, \; \deg\tau = 1.$$

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$$P_n(x) = \frac{B_n}{\rho(x)}\frac{d^n}{dx^n}\left[\rho(x)\sigma^n(x)\right], \quad n = 0, 1, 2, \cdots \quad \rho(x) \geq 0 \quad \text{(FR)}$$

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Find, if there exist, the OPS such that:

1. $(\Theta_q^w P_n(x))_n$ is and OPS
2. $\sigma(x)\Theta_q^w\Theta_{q^{-1}}^w P_n(x) + \tau(x)\Theta_q^w P_n(x) + \lambda P_n(x) = 0$ (DE)
3. $\exists \pi \in \mathbb{P}$ y $\rho$ t.q. $\rho(x)P_n(x) = [\Theta_q^w]^n[\pi(x)\rho(x)]$ (RF)

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- If $w = 0$ and $q \to 1 \Rightarrow \Theta_q^w f(x) \to \frac{d}{dx}$: Clasical case!

- Case $q = 1$ and $w = 1 \Rightarrow$ "discrete" (Lesky, 1962)

$$\Theta_q^w f(x) = \Delta f(x) := f(x+1) - f(x), \quad \nabla f(x) = \Delta f(x-1)$$

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# “Discrete” polynomials and $q-$polynomials

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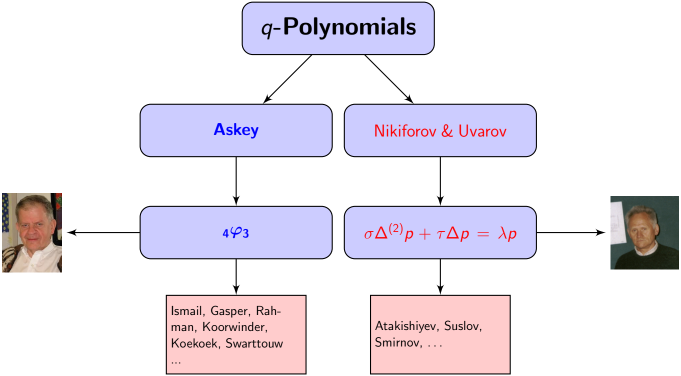
In the next years the $q$-polynomials appeared in several contexts.

# $q-$Polynomials: In the 1980's there were two approaches



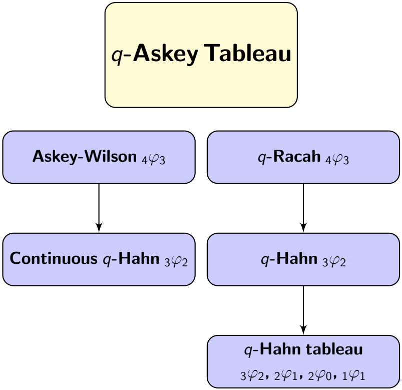

$$
{}_r\phi_p\left(\begin{array}{c} a_1,\dots,a_r \\ b_1,\dots,b_p \end{array}\,\middle|\, q\,;\,z\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_p;q)_k}\frac{z^k}{(q;q)_k}\left[(-1)^k q^{\frac{k}{2}(k-1)}\right]^{p-r+1}
$$

$$
\sigma(s)\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla y(s)}{\nabla x(s)} + \tau(s)\frac{\Delta y(s)}{\Delta x(s)} + \lambda_n y(s) = 0
$$

In 1998 Koekoek and Swarttouw compiled in a report all known families of $q$-polynomials that was called the $q$-Askey Tableu.

All $q$-classical polynomials can be obtained from the Askey-Wilson:

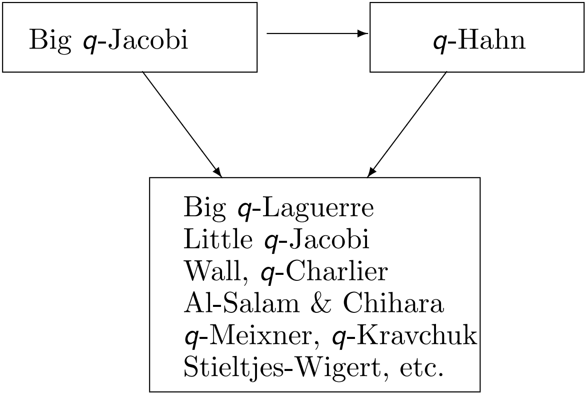
$$p_n(x,a,b,c,d) = {}_4\phi_3\left( \begin{matrix} q^{-n}, q^{n-1}abcd, ae^{-i\theta}, ae^{i\theta} \\ ab, ac, ad \end{matrix} \,\middle|\, q, q \right), \quad x = \cos\theta$$



# The $q$-Hahn Tableau (Koornwinder, 1993)

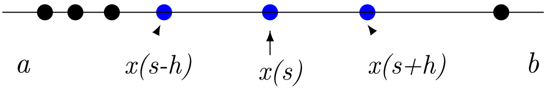
Big $q-$ Jacobi polynomials (if $c = q^{-N-1} \to q$-Hahn)

$$p_n(x; a, b, c; q) = \frac{(aq;q)_n (cq;q)_n}{(abq^{n+1};q)_n} {}_3\varphi_2 \left( \begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \middle| q; q \right).$$

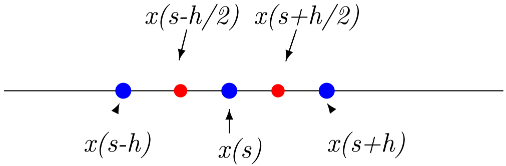


# The 1983 Nikiforov & Uvarov approach

Discretize $\widetilde{\sigma} y''(x) + \widetilde{\tau} y'(x) + \lambda y(x) = 0$ in a *nonuniform lattice*




$$y'(x) \sim \frac{1}{2}\left[\frac{y(x(s+h)) - y(x(s))}{x(s+h) - x(s)} + \frac{y(x(s)) - y(x(s-h))}{x(s) - x(s-h)}\right]$$




$$y''(x) \sim \frac{1}{x(s+\frac{h}{2}) - x(s-\frac{h}{2})}\left[\frac{y(x(s+h)) - y(x(s))}{x(s+h) - x(s)} - \frac{y(x(s)) - y(x(s-h))}{x(s) - x(s-h)}\right]$$

$$\tilde{\sigma} y''(x) + \tilde{\tau} y'(x) + \lambda y(x) = 0$$
$$\Downarrow$$
$$\sigma(s) \frac{\Delta}{\Delta x(s-\frac{1}{2})} \frac{\nabla y(s)}{\nabla x(s)} + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda_n y(s) = 0$$

$$\nabla f(s) = f(s) - f(s-1), \qquad \Delta f(s) = f(s+1) - f(s)$$

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$$x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_3(q) = c_1(q)[q^s + q^{-s-\mu}] + c_3(q)$$

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Sufficient cond. NU (1983).

Necessary cond. Atakishiyev, Rahman y Suslov *Const. Appr.* (1993).

- *The q-analogue of the Rodrigues formula*

$$P_n(s) = \frac{B_n}{\rho(s)} \underbrace{\frac{\nabla}{\nabla x(s+\frac{1}{2})} \cdots \frac{\nabla}{\nabla x(s+\frac{n}{2})}}_{\nabla^{(n)}} \underbrace{\left[\rho(s+n) \prod_{m=1}^{n} \sigma(s+m)\right]}_{\rho_n(s)}$$

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For linear-type lattices $x(s+\alpha) = A(\alpha)x(s) + B(\alpha)$ ($q$-Hahn Tableau) there is a complete study in Medem, et. al. *JCAM* (2001) and RAN, *JCAM* (2006). For the general case see Foupouagnigni et al. *Integral Transforms Spec. Funct.* (2011).

The most general case of the NU Eq. corresponds to the choice:

$$\sigma(s) = q^{-2s}(q^s - q^{s_1})(q^s - q^{s_2})(q^s - q^{s_3})(q^s - q^{s_4}).$$

and the corresponding general polynomial solution can be expressed in term of basic hypergeometric series

$$P_n(s) = {}_4\phi_3 \left( \begin{array}{c} q^{-n}, q^{2\mu+n-1+s_1+s_2+s_3+s_4}, q^{s_1-s}, q^{s_1+s+\mu} \\ q^{s_1+s_2+\mu}, q^{s_1+s_3+\mu}, q^{s_1+s_4+\mu} \end{array} \middle| q, q \right)$$

From the above solution we can obtain Askey-Wilson, $q$-Racah, $q$-duales de Hahn, $q$-Hahn, ... NU *Integral Transforms Spec. Funct.* (1993); Atakishiyev, Rahman y Suslov *Const. Appr.* (1993).

There are a series of interesting papers by Natig (some of then with other people) that further developed the theory initiated by NU:

- The study of the orthogonality of Askey-Wilson polynomials
- The moments of the weight functions of $q$-polynomials
- The study of the continuous orthogonality of the solutions of the NU Eq. including the discrete case.
- etc.

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$q$-Askey $\subset$ Nikiforov-Uvarov Tableau

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## The $q$-hypergemetric Eq. of NU: A final remark

There are a series of interesting papers by Natig (some of then with other people) that further developed the theory initiated by NU:

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In RAN, Medem *JCAM* (2001) we found two new families within the $q$-Hahn tableau. One of then is a positive definite case that has been recently studied by Area et. al. (2016).

# Discrete oscillators

*There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.*

N.I. Lobachevsky

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$$\mathfrak{H}\varphi_n = \lambda_n\varphi_n, \qquad \mathfrak{H} \quad \text{de orden 2}$$

To find 1° order diff operators $a$ and $a^+$ such that

$$\mathfrak{H} = a^+a, \quad a^+\varphi_n = \alpha_n\varphi_{n+1}, \quad a\varphi_n = \beta_n\varphi_{n-1}, \quad (a^+)^* = a, \; a^* = a^+.$$

Interest: Solving $a\varphi_0 = 0$, one gets $\varphi_0$, and $a^+\varphi_n$ generate the others

# The typical example is the quantum harmonic oscillator

$$\mathfrak{H}\Psi_n(x) := -\Psi_n''(x) + x^2\Psi_n(x) = (H_- H_+ + I)\Psi_n(x) = \lambda_n\Psi_n(x).$$

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$$H_+\Psi_0 = 0, \quad H_+\Psi_n = \sqrt{2n}\,\Psi_{n-1}, \quad H_-\Psi_n = \sqrt{2n+2}\,\Psi_{n+1}.$$

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How it works?

$$x\Psi_0(x) + \Psi_0'(x) = 0 \quad \Rightarrow \quad \Psi_0(x) = \frac{1}{\sqrt[4]{\pi}} e^{-x^2/2},$$

and $[H_-]^n\Psi_0(x) = \sqrt{(2n)!!}\Psi_n(x)$, thus

$$\Psi_n(x) = \frac{1}{\pi^{\frac{1}{4}}\sqrt{(2n)!!}}[H_-]^n e^{-x^2/2} = \frac{1}{\pi^{\frac{1}{4}}\sqrt{(2n)!!}}\left[xI - \frac{d}{dx}\right]^n e^{-x^2/2}.$$

This the classical algebraic realization of the quantum oscillator.

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$$\varphi_n(s) = \sqrt{\frac{\rho(s)}{d_n^2}}\, P_n(x(s))_q, \qquad \mathfrak{H}(s,n)\varphi_n(s) = 0$$

$$\mathfrak{H}(s,n) \equiv \frac{\sqrt{\sigma(-s-\mu+1)\sigma(s)}}{\nabla x(s)} e^{-\partial_s} + \frac{\sqrt{\sigma(-s-\mu)\sigma(s+1)}}{\Delta x(s)} e^{\partial_s} - \left(\frac{\sigma(-s-\mu)}{\Delta x(s)} + \frac{\sigma(s)}{\nabla x(s)} - \lambda_n \Delta x(s-1/2)\right) I.$$

**Main properties:**

1. The orthonormal functions $\varphi_n$ satisfy a 2º diff Eq. & TTRR
2. There exist tow ladder oerators: $L^+(s,n)\varphi_n(s) = A_n\varphi_{n+1}(s)$ and $L^-(s,n)\varphi_n(s) = B_n\varphi_{n-1}(s)$

$$H(s,n) \equiv \sqrt{\sigma(-s-\mu+1)\sigma(s)}\frac{1}{\nabla x(s)}e^{-\partial_s} + \sqrt{\Theta(s)\sigma(s+1)}\frac{1}{\Delta x(s)}e^{\partial_s} -$$

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**Theorem:** The operator $H(s,n)$ admits the following factorization

$$u(s+1,n)H(s,n) = L^-(s,n+1)L^+(s,n) - h^{\mp}(n)I,$$

$$u(s,n)H(s,n+1) = L^+(s,n)L^-(s,n+1) - h^{\mp}(n)I,$$

respectively, where

$$h^{\pm}(n) = \frac{\lambda_{2n-2}}{[2n-2]_q}\frac{\lambda_{2n}}{[2n]_q}\alpha_{n-1}\gamma_n, \quad u(s,n) = \frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'} - \frac{\sigma(s)}{\nabla x(s)}$$

where $\alpha$ and $\gamma$ are the coeff. of the TTRR.

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This is not a good solution to the problem. Why?

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THE DYNAMICAL ALGEBRA of the HO and the SODE

The problem of build such dynamical algebra from the NU Eq. was proposed by Atakishiyev in 2002 motivated by the previous works of MacFarlane 1989, Biedenharn 1989, Atakishiyev et at 1991, 1994, 1996, ...

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$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_0 \qquad \text{SU(2)}$$

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$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = -2J_0 \qquad \text{SU(1,1)}$$

# The $q$-wave functions $\varphi_n$ and the $q$-Hamiltonian $\mathfrak{H}_q$

$$\mathfrak{H}_q(s)\varphi_n(s) = \lambda_n\varphi_n(s),$$

$$\mathfrak{H}_q(s) := \frac{1}{\nabla x_1(s)} A(s) H_q(s) \frac{1}{A(s)}, \quad \varphi_n(s) = \frac{A(s)\sqrt{\rho(s)}}{d_n} P_n(s;q),$$

where

$$H_q(s) := -\frac{\sqrt{\sigma(-s-\mu+1)\sigma(s)}}{\nabla x(s)} e^{-\partial_s} - \frac{\sqrt{\sigma(-s-\mu)\sigma(s+1)}}{\Delta x(s)} e^{\partial_s} + \left(\frac{\sigma(-s-\mu)}{\Delta x(s)} + \frac{\sigma(s)}{\nabla x(s)}\right) I,$$

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The next step is to find two operators $a(s)$ and $b(s)$ such that

$$\mathfrak{H}_q(s) = b(s)a(s)$$

We will follow an original idea by Atakishiyev:

## The $\alpha$-operators

*In order to factorize an arbitrary difference equation, one should express it explicitly in terms of the shift operators* $\exp(a\,\frac{d}{ds})$, *defined as* $\exp(a\,\frac{d}{ds})\,f(s) = f(s+a)$, $a \in \mathbb{C}$.

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Let $\alpha \in \mathbb{R}$ and $A(s)$ and $B(s)$ are continuous functions. We define a family of $\alpha$-down and $\alpha$-up operators by

$$\mathfrak{a}_\alpha^{\downarrow}(s) := \frac{B(s)}{\sqrt{\nabla x_1(s)}} e^{-\alpha\partial_s}\left( e^{\partial_s}\sqrt{\frac{\sigma(s)}{\nabla x(s)}} - \sqrt{\frac{\sigma(-s-\mu)}{\Delta x(s)}} \right)\frac{1}{A(s)},$$

$$\mathfrak{a}_\alpha^{\uparrow}(s) := \frac{A(s)}{\nabla x_1(s)}\left( \sqrt{\frac{\sigma(s)}{\nabla x(s)}} e^{-\partial_s} - \sqrt{\frac{\sigma(-s-\mu)}{\Delta x(s)}} \right) e^{\alpha\partial_s}\frac{\sqrt{\nabla x_1(s)}}{B(s)}.$$

$$\mathfrak{H}_q(s) = \mathfrak{a}_\alpha^{\uparrow}(s)\mathfrak{a}_\alpha^{\downarrow}(s), \qquad \forall\alpha \in \mathbb{R},\ \text{and}\ B(s).$$

**Definition:** Let $\varsigma$ be a complex number, and let $a(s)$ and $b(s)$ be two operators. We define the $\varsigma$-*commutator of a and b* as

$$[a(s), b(s)]_{\varsigma} = a(s)b(s) - \varsigma b(s)a(s), \qquad \varsigma = q^{\gamma} \neq 0.$$

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**Proposition:** Let $\mathfrak{H}_q(s)$ be an operator, such that $\exists a(s)$, $b(s)$ and $\varsigma, \Lambda \in \mathbb{C}$, that $\mathfrak{H}_q(s) = b(s)a(s)$, and $[a(s), b(s)]_\varsigma = \Lambda$.

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- $\mathfrak{H}_q(s)\{b(s)\Phi(s)\} = (\Lambda + \varsigma\lambda)\{b(s)\Phi(s)\}.$ $b(s)$ raising op.

Let us assume that $[a(s), b(s)]_\varsigma = I$, $\varsigma = q^2$, and $b(s) = a^+(s)$.

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Then one can rewrite the $q^2$-commutator as follows

$$[a(s), a^{+}(s)] = I - (1 - q^2)a^{+}(s)\, a(s) \equiv q^{2N(s)},$$

where, $N(s) = \ln[I - (1 - q^2)\, a^{+}(s)\, a(s)]/ \ln q^2$.

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i.e., $N(s)$ is the "number" operator. Next we introduce

$$\widetilde{b}(s) := q^{-N(s)/2}\, a(s), \quad \widetilde{b}^+(s) := a^+(s)\, q^{-N(s)/2},$$

which satisfy $\widetilde{b}(s)\, \widetilde{b}^+(s) - q\, \widetilde{b}^+(s)\, \widetilde{b}(s) = q^{-N(s)}$.

The operators $\widetilde{b}(s)$, $\widetilde{b}^{+}(s)$, and $N(s)$ lead to the dynamical algebra $su_q(1,1)$ with the generators ($\beta^{-1} = q + q^{-1}$.)

$$K_0(s) = \frac{1}{2}\left(N(s) + 1/2\right), \quad K_+(s) = \beta\,(\widetilde{b}^{+}(s))^2, \quad K_-(s) = \beta\,\widetilde{b}^{2}(s),$$

$$[K_0(s), K_\pm(s)] = \pm K_\pm(s), \quad [K_-(s), K_+(s)] = [2K_0(s)]_{q^2},$$

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Similarly we can derive the dynamical algebra $su_q(2)$.

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For the first question we already have the answer: the operators $b(s) = \mathfrak{a}^{\uparrow}_{\alpha}(s)$ and $a(s) = \mathfrak{a}^{\downarrow}_{\alpha}(s)$ factorize the Hamiltonian

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In the following we assume that $A(s) = B(s)$. $\Rightarrow$

**Theorem 1:** NECESSARY CONDITION

Let $(\varphi_n)_n$ the eigenfunctions of $\mathfrak{H}_q(s)$ corresponding to the eigenvalues $(\lambda_n)_n$ and suppose that the problem 1 has a solution for $\Lambda \neq 0$. Then, the eigenvalues $\lambda_n$ of the NU $q$-equation are $q$-linear or $q^{-1}$-linear functions of $n$, i.e.,

$$\lambda_n = C_1 q^n + C_3 \quad \text{or} \quad \lambda_n = C_2 q^{-n} + C_3,$$

respectively.

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Let $(\varphi_n)_n$ the eigenfunctions of $\mathfrak{H}_q(s)$ corresponding to the eigenvalues $(\lambda_n)_n$ and suppose that the problem 1 has a solution for $\Lambda \neq 0$. Then, the eigenvalues $\lambda_n$ of the NU $q$-equation are $q$-linear or $q^{-1}$-linear functions of $n$, i.e.,

$$\lambda_n = C_1 q^n + C_3 \quad \text{or} \quad \lambda_n = C_2 q^{-n} + C_3,$$

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Impossible to solve: general Askey-Wilson, $q$-Racah, big and little $q$-Jacobi polynomials.

**Theorem 2:** NECESSARY AND SUFFICIENT CONDITION

Let be $\mathfrak{H}_q(s)$ the $q$-Hamiltonian defined from the NU Eq. The operators $\mathfrak{a}^{\uparrow}_{\alpha}(s)$ and $\mathfrak{a}^{\downarrow}_{\alpha}(s)$ factorize the Hamiltonian $\mathfrak{H}_q(s)$ and satisfy the relation $[\mathfrak{a}^{\downarrow}_{\alpha}(s), \mathfrak{a}^{\uparrow}_{\alpha}(s)]_{\varsigma} = \Lambda$ for $\varsigma \in \mathbb{C}$ iff the following two conditions hold:

$$\frac{\nabla x(s)}{\nabla x_1(s-\alpha)}\sqrt{\frac{\nabla x_1(s-1)\nabla x_1(s)}{\nabla x(s-\alpha)\Delta x(s-\alpha)}}\sqrt{\frac{\sigma(s-\alpha)\sigma(-s-\mu+\alpha)}{\sigma(s)\sigma(-s-\mu+1)}} = \varsigma, \quad \text{and}$$

$$\frac{1}{\Delta x(s-\alpha)}\left(\frac{\sigma(s-\alpha+1)}{\nabla x_1(s-\alpha+1)} + \frac{\sigma(-s-\mu+\alpha)}{\nabla x_1(s-\alpha)}\right) - \varsigma\frac{1}{\nabla x_1(s)}\left(\frac{\sigma(s)}{\nabla x(s)} + \frac{\sigma(-s-\mu)}{\Delta x(s)}\right) = \Lambda.$$

The values $\varsigma$ and $\Lambda$ are uniquely determined!

*To find two operators $a(s)$ and $b(s)$ and a constant $\varsigma$ such that the Hamiltonian $\mathfrak{H}_q(s) = b(s)a(s)$ and $[a(s), b(s)]_\varsigma = I$ and such that $a(s)$ and $b(s)$ are the lowering and raising operators, i.e.,*

$$a(s)\varphi_n(s) = D_n\Phi_{n-1}(s) \quad \text{and} \quad b(s)\varphi_n(s) = U_n\Phi_{n+1}(s).$$

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When the $\alpha$-operators are mutually adjoint?

**Answer:** It depends of the "scalar product". E.g. Discrete case: $\alpha = 0$ is a sufficient condition

The $q$-Hamiltonian is, in this case, $(x = q^s)$.

$$\mathfrak{H}_q(s) = -\frac{q^2\sqrt{a(x-1)(x-a)}}{(q-1)^2x^2}e^{-\partial_s} - \frac{\sqrt{a(1-qx)(a-qx)}}{x^2}e^{\partial_s} + \left(\frac{\sqrt{q}(q(x-1)x+a(1+q-qx))}{(q-1)^2x^2}\right)I$$

Then, $\mathfrak{H}_q(s)\varphi_n(s) = q^{\frac{3}{2}}\frac{1-q^{-n}}{(1-q)^2}\varphi_n(s)$ and the operators

$$\mathfrak{a}^{\downarrow}(s) \equiv \mathfrak{a}_0^{\downarrow}(s) = \frac{q^{\frac{1}{4}}x^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\left(\sqrt{(x-1/q)\,(x-a/q)}\,e^{\partial_s} - \sqrt{a}\;I\right),$$

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are such that

$$\mathfrak{a}^{\uparrow}(s)\mathfrak{a}^{\downarrow}(s) = \mathfrak{H}_q(s), \quad \text{and} \quad [\mathfrak{a}^{\downarrow}(s), \mathfrak{a}^{\uparrow}(s)]_{q^{-1}} = \frac{1}{k_q}.$$

Notice that since this is a discrete case when $\alpha = 0$ the operators $\mathfrak{a}^{\uparrow}(s)$ and $\mathfrak{a}^{\downarrow}(s)$ are mutually adjoint.

Other related cases:

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## Example 1: The Al-Salam & Carlitz I

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- Changing $q$ by $q^{-1}$, we obtain the factorization and the dynamical algebra for the Al-Salam & Carlitz functions II
- from where putting $a = -1$ and $x \to ix$ follows the solution for the discrete Hermite $q$-polynomials $\widetilde{h}_n(x; q)$

# Other examples in the $q$-Askey Tableau

| $x(s)$ | $P_n(s)_q$ | $\sigma(s)+\tau(s)\nabla x_1(s)$ | $\sigma(s)$ | $\lambda_n$ |
|---|---|---|---|---|
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| $q^s$ | $S_n(x;q)$ | $x^2$ | $q^{-1}x$ | $q^{\frac{1}{2}}\frac{1-q^{n}}{(1-q)^2}$ |
| $q^s$ | $p_n(x;a\vert q)$ | $-ax$ | $q^{-1}x(x-1)$ | $q^{\frac{1}{2}}\frac{1-q^{-n}}{(1-q)^2}$ |
| $q^s$ | $L_n^{\alpha}(x;q)$ | $ax(x+1)$ | $q^{-1}x$ | $q^{\frac{1}{2}}a\frac{1-q^{n}}{(1-q)^2}$ |
| $q^{-s}$ | $C_n(x;a;q)$ | $x(x-1)$ | $q^{-1}ax$ | $q^{\frac{1}{2}}\frac{1-q^{n}}{(1-q)^2}$ |

Al-Salam & Carlitz I, II, discrete $q$-Hermite I, II, $q$-Charlier-type, Stieltjes-Wigert,

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The Askey-Wilson case: Only for some special cases. continuous $q$-Laguerre and continuous $q$-Hermite polynomials.

# Open problems

- What about the lowering and raising properties of the $\alpha$-down and $\alpha$-up operators?
- What happen for the general Askey-Wilson case and for the general big $q$-Jacobi?
- There exists a more general dynamical algebra? Of which kind?

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**Last but not least ...**

Some other relevant results related with the $q$-polynomials by Natig: Classical-type integral transform formulas: Mellin transforms, Fourier-Gauss transforms, etc.

# That's all folks ... thanks for your attention!

Leganés, June 1996

München, July 2005

**Natig Atakishiyev's secret:** "Los problemas matemáticos hacen que canalicemos nuestros esfuerzos y nos ayudan a sobrevivir. Solucionar un enigma es lo más gratificante que hay. Si no tuviéramos este tipo de incógnitas esperándonos al día siguiente, los matemáticos no viviríamos tanto"