

A Special Type of Triangulations in Numerical Nonlinear Analysis¹

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1. INTRODUCCION

In the following \mathbb{R} denotes the set of real numbers, D an open subset of \mathbb{R}^{2n} and $C^\omega(D)$ the set of all mappings from D into \mathbb{R}^{2n} with derivatives of every order. To solve

$$F(u) = 0, \quad F: D \subseteq \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}, \quad F \in C^\omega(D)$$

with a finite numbers of zeros, let G be a mapping from $D \subseteq \mathbb{R}^{2n}$ into \mathbb{R}^{2n} , $G \in C^\omega(D)$ all whose zeros are known. We construct the homotopy

$$H: D \times [0,1] \longrightarrow \mathbb{R}^{2n}; \quad H(u,t) = (1-t) \times G(u) + t \times F(u).$$

1.1. We suppose that zero is a regular value for H , which is not a substantial restriction according to Brown's corollary of Sard's theorem [4]. We study the inverse image of zero, using the following theorems (with $m = 2n + 1$, $p = 2n$, $M = D \times [0,1]$ and $N = \mathbb{R}^{2n}$).

THEOREM 1.1.1. ([4]) *If H is a smooth map $H: M \longrightarrow N$ from a m -manifold M with boundary to a p -manifold N , where $m > p$, and Y is a regular value for H and for the restriction $H/\delta M$, then $H^{-1}(Y)$ is a smooth $(m-p)$ -manifold with boundary. Furthermore the boundary $\delta(H^{-1}(Y))$ is the intersection of $H^{-1}(Y)$ with δM , and its dimension is $m-n-1$.*

THEOREM 1.1.2. ([4]) *Any smooth connected 1-manifold is diffeomorphic either to the circle S^1 or to some real interval; that is, it is either a loop or a path.*

To calculate the zeros of F we consider the connected component of $H^{-1}(0)$ from $t = 0$ to $t = 1$; $H^{-1}(0)$ is a union of paths and loops, according to theorems 1.1.1 and 1.1.2. By differentiation of $H(u,t) = 0$ with respect to the variables s , s being the arc and $\{(u,t)\}$ lying in a connected component of $H^{-1}(0)$, we obtain:

$$\sum_{i=0}^{2n} (\partial H / \partial u_i) \times (du_i / ds) + (\partial H / \partial t) \times (dt / ds) = 0$$

It follows (see [2]) that

$$du_i / ds = (-1)^i \times \det(H'_{-i}(u,t)), \quad i = 1, \dots, 2n, \quad (1)$$

$$dt / ds = (-1)^{2n+1} \times \det(H'_{-2n+1}(u,t)), \quad (2)$$

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where H'_i is the jacobian of H with the i -th column deleted; so if $\text{sign}\{\det(H'_{2n-1}(u,t))\}$ is constant, (2) implies that $t = t(s)$ is monotonous.

1.1.3. We want to solve $F^*(z) = 0$, where $F^*: D \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^n$ is holomorphic with a finite number of zeros (\mathbb{C} denotes the set of complex numbers). Let G^* be an holomorphic mapping from $D \subseteq \mathbb{C}^n$ into \mathbb{C}^n whose zeros are known. We construct the homotopy

$$H^*: D \times [0,1] \rightarrow \mathbb{C}^n, \quad H^* = (H_1^*, \dots, H_n^*),$$

$$H^*(z,t) = (1-t) \times G^*(z) + t \times F^*(z).$$

Let $H: D \times [0,1] \rightarrow \mathbb{R}^{2n}$ be the mapping defined by $H = (\text{Re } H_1^*, \text{Im } H_1^*, \dots, \text{Im } H_n^*)$, and analogously

$$F: D \subseteq \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}, \quad \text{with } F = (\text{Re } F_1^*, \text{Im } F_1^*, \dots, \text{Im } F_n^*)$$

$$G: D \subseteq \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}, \quad \text{with } G = (\text{Re } G_1^*, \text{Im } G_1^*, \dots, \text{Im } G_n^*).$$

It is obvious that the equations $H^*(z,t) = 0$ and $H(u,t) = 0$ are equivalent.

THEOREM 1.1.4. ([2]) *If H^* is holomorphic, then $\det(H'(u,t)) \geq 0$.*

It follows that the paths of $H^{-1}(0)$ are monotonous with respect to s and there are no loops. If we choose G such that for $0 \leq t \leq 1$, each path starting in $(X^*, 0)$ with $G(X^*) = 0$, finishes in $(X^{**}, 1)$, $H(X^{**}, 1) = 0$. So, solving the problem of initial values (1), (2), $G(X^*) = 0$ we deduce that for each solution of $G(u) = 0$ we obtain a solution of $F(u) = 0$, and perhaps some paths which diverge to infinity. For a sufficient condition of non divergence of the paths see [2, pg. 349]. In the section 1.2 we relate $H^{-1}(0)$ with θ , θ being a piecewise linear approximation of H .

1.2. Now we denote by δs the border of the simplex s ,

$$\|H\|_1 = \sup_{X \in \delta s} \|DH(X)\| + \sup_{X \in \delta s} \|H(X)\|.$$

So if θ and H are sufficiently closed there is a tube for each connected component of $H^{-1}(0)$ that contains a connected component of $\theta^{-1}(0)$.

Our next theorem gives a sufficient condition to secure that θ is sufficiently closed to H ; (i.e. the triangulation is fine enough).

THEOREM 1.2.1. *Let $f: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a $C^2(s)$ function and $\theta: D \subseteq \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ a piecewise linear linear approximation of f relative to a regular triangulation of the domain of f . Then, if θ has a zero in the simplex s , f has also a zero in s , if*

$$2M\epsilon^2 < d_w(0, \theta(\delta s));$$

M being a upper bound for the second derivative of f in s , and ϵ the diameter of s .

Remarks. 1) A n -simplex s of $\mathbb{R}^n \times [0,1]$ is contained in an hyperplane; it follows that $H|_s, \theta|_s$ are mappings from \mathbb{R}^n to \mathbb{R}^n .

2) If for each simplex s of the triangulation, $\theta(s) \subset H(s)$ is verified, then the condition of theorem 1.2.1 is necessary and sufficient.

1.2.2. PIECEWISE LINEAR APPROXIMATION. We need a domain of $\mathbb{R}^{2n} \times [0,1]$, whose closure contains the connected components of $H^{-1}(0)$, zero being a regular value for H ; so that the intersection of the closure of these domains with the hiperplane $X_{2n+1} = 1$ are neighborhoods of the solutions of our problem. To obtain it, we construct a regular triangulation K of $\mathbb{R}^{2n} \times [0,1]$, so that the point X^* satisfying $G(X^*) = 0$, is contained in the interior of one of its $2n$ -simplices s ; and the diameter of the $2n$ -simplices is less than the admissible error. We construct a piecewise-linear approximation θ of H relative to the triangulation K .

$$\theta : D \times [0,1] \longrightarrow \mathbb{R}^{2n};$$

$$\theta(X) = \begin{cases} H(u, t) & \text{for } (u, t) \in \{ \text{vertices } K \} \\ \sum_{i=0}^k \lambda_i \times H(Y^i) & \text{for } X = (u, t) \in \langle Y^0, \dots, Y^k \rangle \end{cases}$$

λ_i being the barycentric coordinates of X concerning $\{Y^0, \dots, Y^k\}$. The study of $\theta^{-1}(0)$ is the usual in this class of algorithms.

2. REGULAR TRIANGULATIONS

Define an open simplex $\langle s \rangle = \{ \alpha \in |K| \text{ so that } \alpha(Y) \neq 0 \Leftrightarrow Y \in s \}$.

Let i be the linear extension to $|K|$ of the identity vertex mapping, and let K be a simplicial complex with the following properties:

- 1) The vertices of K are points of \mathbb{R}^n .
- 2) Each simplex is a subsimplex of a n -simplex.
- 3) Each $(n-1)$ -simplex is contained in two n -simplices.
- 4) The $(n+1)$ -points that establish a n -simplex are affinely independent.
- 5) The diameter of the convex hull of the vertices of a simplex is fixed.

Then we have

THEOREM 2.1. *The open simplex of vertices $\langle Y^0, \dots, Y^j \rangle$ may be identified with $i(\langle s \rangle)$, $\langle s \rangle \in |K|$.*

THEOREM 2.2. *A regular triangulation of \mathbb{R}^n consists of the set $\{i(\langle s \rangle)\}$, from the pair (K, i) .*

2.1. FIRST TRIANGULATION.

THEOREM 2.3. *Given a point $X \in \mathbb{R}^n$ let $\{Y^0, \dots, Y^{n+1}\}$, let*

$$Y_i^0 = X_i - ((n-i+1) \times \delta / (n+1)), \quad i = 1, \dots, n,$$

$$Y_{n+1}^0 = X_{n+1}, \quad Y^i = Y^{i-1} + \delta \times u^i; \quad u^i \text{ being the } i\text{-th unit vector.}$$

The interior of the convex hull of $\{Y^0, \dots, Y^{n+1}\}$ is a $(n+1)$ -simplex s of finite diameter δ , and X belongs to the interior of the subsimplex $\langle Y^0, \dots, Y^n \rangle$.

PROPOSITION 2.4. Given a point $Y^* \in \mathbb{R}^{n+1}$, let K^0 be the set

$$K^0 = \{ Y \in \mathbb{R}^{n+1} / Y = Y^* + \delta \sum_{i=1}^{n+1} k_i w^i, k_i \in \mathbb{Z}, i = 1, \dots, n+1, \}$$

where $w^i = i$ -th unit vector, $\delta \in \mathbb{R}^+$.

The relative interior of the convex hull of $(n+1)$ -points of K^0 that verify $Y^i = Y^{i-1} + \delta \times u^{\pi(i)}$, (2), with π a permutation of $\{1, \dots, n+1\}$, is a simplex $s^{n+1} = \langle Y^0, \dots, Y^{n+1} \rangle$. The set of all the simplices so constructed is a triangulation of \mathbb{R}^{n+1} .

Remark. We construct the 1-triangulation from Y^* to Y^0 defined in 2.3; Y^0 is calculated from a certain diameter of mesh δ and any point X .

2.2. SECOND TRIANGULATION. Given a point $X \in \mathbb{R}^{n+1}$, let us construct the simplex s^n that contains X , given in Theorem 2.3. We now write $Y^i = Y^{i-1} + \delta \times u^{\pi(i)}$, $i > 1$ in the form $Y^i = Y^{i-1} + s_{\pi(i)} \times u^{\pi(i)}$, with s the vector sign $(+1, \dots, +1)$. Let be $Y^* = Y^0$.

THEOREM 2.5. Given a point Y^* , let K^0 be the set

$$K^0 = \{ Y \in \mathbb{R}^{n+1} / Y = Y^* + \delta \sum_{i=1}^{n+1} k_i w^i, k_i \in \mathbb{Z}, i = 1, \dots, n+1; \}$$

$w^i = i$ -th unit vector, $\delta \in \mathbb{R}^+$.

Let $\{Y^0, \dots, Y^{n+1}\}$; $Y^i \in K^0$, $i = 1, \dots, n+1$ be the points

$$Y^0 = Y^* + \delta (k_1, \dots, k_{n+1})^t, \quad k_i \in \mathbb{Z}, \quad i = 1, \dots, n+1$$

$$Y^i = Y^{i-1} + \delta s_{\pi(i)} u^{\pi(i)}, \quad i = 1, \dots, n+1$$

with π a permutation of $1, \dots, n+1$; and s a $(n+1)$ -vector sign. The relative interior of the convex hull of these $(n+2)$ -points is a $(n+1)$ -simplex s^{n+1} . The set of all the $(n+1)$ -simplices so constructed is a triangulation of \mathbb{R}^{n+1} .

Remark. There exists for each diameter of mesh and starting point one and only one affinity transforming the first triangulation in the Freudental's triangulation. Also there exists for each diameter of mesh and starting point one and only one affinity transforming the second triangulation in the Tucker's triangulation.

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