Improved enumeration of simple topological graphs*

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Abstract

A *simple topological graph* $T = (V(T), E(T))$ is a drawing of a graph in the plane where every two edges have at most one common point (an endpoint or a crossing) and no three edges pass through a single crossing. Topological graphs G and H are *isomorphic* if H can be obtained from G by a homeomorphism of the sphere, and *weakly isomorphic* if G and H have the same set of pairs of crossing edges.

We generalize results of Pach and Tóth and the author's previous results on counting different drawings of a graph under both notions of isomorphism. We prove that for every graph G with n vertices, m edges and no isolated vertices the number of weak isomorphism classes of simple topological graphs that realize G is at most $2^{O(n^2 \log(m/n))}$, and at most $2^{O(mn^{1/2} \log n)}$ if $m \le n^{3/2}$. As a consequence we obtain a new upper bound $2^{O(n^{3/2} \log n)}$ on the number of intersection graphs of n pseudosegments. We improve the upper bound on the number of weak isomorphism classes of simple complete topological graphs with n vertices to $2^{n^2 \cdot \alpha(n)^{O(1)}}$, using an upper bound on the size of a set of permutations with bounded VC-dimension recently proved by Cibulka and the author. We show that the number of isomorphism classes of simple topological graphs that realize G is at most $2^{m^2+O(mn)}$ and at least $2^{\Omega(m^2)}$ for graphs with $m > (6+\varepsilon)n$.

1 Introduction and the results

A *topological graph* $T = (V(T), E(T))$ is a drawing of a graph G in the plane with the following properties. The vertices of G are represented by a set $V(T)$ of distinct points in the plane and the edges of G are represented by a set $E(T)$ of simple curves connecting the corresponding pairs of points. We call the elements of $V(T)$ and $E(T)$ the *vertices* and the *edges* of T. The drawing has to satisfy the following general position conditions: (1) the edges pass through no vertices except their endpoints, (2) every two edges have only a finite number of intersection points, (3) every intersection point of two edges is either a common endpoint or a proper crossing ("touching" of the edges is not allowed), and (4) no three edges pass through the same crossing. A topological graph is *simple* if every two edges have at most one common point, which is either a common endpoint or a crossing. A topological graph is *complete* if it is a drawing of a complete graph.

We use two different notions of isomorphism to enumerate topological graphs.

Topological graphs G and H are *weakly isomorphic* if there exists an incidence preserving one-to-one correspondence between $V(G), E(G)$ and $V(H), E(H)$ such that two edges of G cross if and only if the corresponding two edges of H do.

Note that every topological graph G drawn in the plane induces a drawing G_{S^2} on the sphere, which is obtained by a standard one-point compactification of the plane. Topological graphs G and H are *isomorphic* if there exists a homeomorphism of the sphere which transforms G_{S^2} into H_{S^2}. The isomorphism can be also defined in a combinatorial way.

Unlike the isomorphism, the weak isomorphism can change the faces of the involved topological graphs, the order of crossings along the edges and also the cyclic orders of edges around vertices.

For counting the (weak) isomorphism classes, we consider all the graphs labeled. That is, each vertex is assigned a unique label from the set $\{1, 2, \ldots, n\}$, and we require the (weak) isomorphism to preserve the labels. Mostly it makes no significant difference in the results as we operate with quantities asymptotically larger than $n!$.

For a graph G, let $T_w(G)$ be the number of weak isomorphism classes of simple topological graphs that realize G. Pach and Tóth [13] and the author [6] proved the following lower and upper bounds on $T_w(K_n)$.

Theorem 1 [6, 13] *For the number of weak isomorphism classes of simple drawings of K_n, we have*

$$2^{\Omega(n^2)} \le T_w(K_n) \le ((n-2)!)^n = 2^{O(n^2 \log n)}.$$

*The full version of the paper is available [8].

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We prove generalized upper and lower bounds on $T_{\mathrm{w}}(G)$ for all graphs G.

Theorem 2 *Let G be a graph with n vertices and m edges. Then*

$$T_{\mathrm{w}}(G) \leq 2^{O(n^2 \log(m/n))}.$$

If $m < n^{3/2}$, then

$$T_{\mathrm{w}}(G) \leq 2^{O(mn^{1/2} \log n)}.$$

Let $\varepsilon > 0$. If G is a graph with no isolated vertices and at least one of the conditions $m > (1+\varepsilon)n$ or $\Delta(G) < (1-\varepsilon)n$ is satisfied, then

$$T_{\mathrm{w}}(G) \geq 2^{\Omega(\max(m, n \log n))}.$$

We also improve the upper bound from Theorem 1.

Theorem 3 *We have*

$$T_{\mathrm{w}}(K_n) \leq 2^{n^2 \cdot \alpha(n)^{O(1)}}.$$

Here $\alpha(n)$ is the inverse of the Ackermann function. It is an extremely slowly growing function, which can be defined in the following way [10]. $\alpha(m) := \min\{k : \alpha_k(m) \leq 3\}$ where $\alpha_d(m)$ is the dth function in the *inverse Ackermann hierarchy.* That is, $\alpha_1(m) = \lceil m/2 \rceil$, $\alpha_d(1) = 0$ for $d \geq 2$ and $\alpha_d(m) = 1 + \alpha_d(\alpha_{d-1}(m))$ for $m, d \geq 2$. The constant in the $O(1)$ notation in the exponent is huge (roughly 4^{30^4}), due to a Ramsey-type argument used in the proof.

In the proof of Theorem 3 we use the fact that for simple complete topological graphs, the weak isomorphism class is determined by the rotation system [7, 13]. This is combined with a Ramsey-type theorem by Pach, Solymosi and Tóth [12], which says that a simple complete topological graph with sufficiently many vertices contains a subgraph weakly isomorphic to a *convex* graph or a *twisted* graph of given size; see Figure 1. Once we have a convex graph with 5 vertices or a twisted graph with 6 vertices, we may restrict the set of possible rotations of other vertices in terms of forbidden subpermutations. The last main ingredient is a recent combinatorial result, a slightly superexponential upper bound on the size of a set of permutations with bounded VC-dimension obtained together with Josef Cibulka [4].

The method in the proof of Theorem 2 is more topological, gives a slightly weaker upper bound, but can be generalized to all graphs. Here the main tool is a construction of a *topological spanning tree* $\mathcal{T}$ of $\mathcal{G}$, which is a simply connected subset of the single topological component of $\mathcal{G}$ containing all vertices of $\mathcal{G}$ and satisfying the property that the only nonseparating points of $\mathcal{T}$ are the vertices of $\mathcal{G}$. We find such a tree consisting of $O(n)$ connected portions of edges

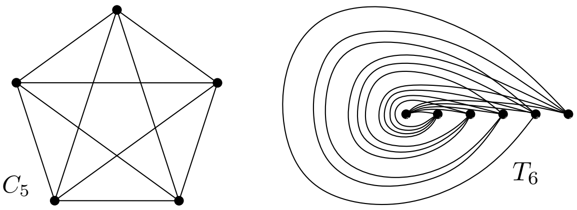

Figure 1: The convex graph C_5 and the twisted graph T_6.

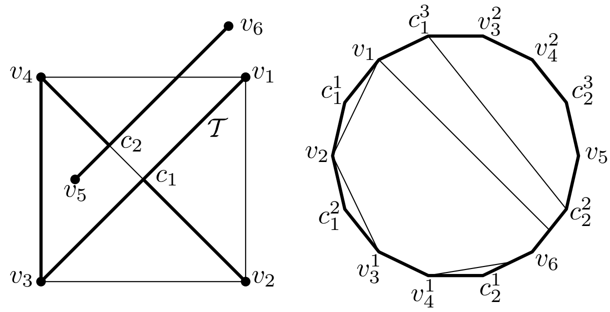

Figure 2: A topological spanning tree $\mathcal{T}$ of a simple topological graph with two components (left) and the corresponding $\mathcal{T}$-representation (right).

of $\mathcal{G}$. By cutting the plane along $\mathcal{T}$, we obtain the *$\mathcal{T}$-representation* of G, which is equivalent to a disc with at most $2mn$ chords, each chord corresponding to a portion of some edge of G. See Figure 2. We give an upper bound on the number of inequivalent $\mathcal{T}$-representations, exploiting the fact that many portions of edges do not cross.

We further generalize Theorem 3 by removing almost all topological aspects of the proof. The resulting theorem is a purely combinatorial statement, involving n-tuples of cyclic permutations avoiding a certain simple substructure.

We also consider the class of simple complete topological graphs with maximum number of crossings and suggest an alternative method for obtaining an upper bound on the number of weak isomorphism classes of such drawings.

An arrangement of *pseudosegments* (or also 1-*strings*) is a set of simple curves in the plane such that any two of the curves cross at most once. An *intersection graph of pseudosegments* (also called a *string graph of rank* 1) is a graph G such that there exists an arrangement of pseudosegments with one pseudosegment for each vertex of G and a pair of pseudosegments crossing if and only if the corresponding pair of vertices forms an edge in G. Using tools from extremal graph theory, Pach and Tóth [13] proved that the number of intersection graphs of n pseudosegments is $2^{o(n^2)}$. As a special case of Theorem 2 we obtain the following upper bound.

Theorem 4 *There are at most $2^{O(n^{3/2}\log n)}$ intersection graphs of n pseudosegments.*

The best known lower bound for the number of (unlabeled) intersection graphs of n pseudosegments is $2^{\Omega(n\log n)}$. This follows by a simple construction or from the the fact that there are $2^{\Theta(n\log n)}$ nonisomorphic permutation graphs with n vertices.

Let $T(G)$ be the number of isomorphism classes of simple topological graphs that realize G. The following theorem generalizes the result $T(K_n) = 2^{\Theta(n^4)}$ from [7].

Theorem 5 *Let G be a graph with n vertices, m edges and no isolated vertices. Then $T(G) \le 2^{m^2+O(mn)}$. More precisely,*

$$\begin{aligned}T(G) &\le \binom{6mn}{2mn}\binom{m^2+6mn}{\frac{m^2}{2}+2mn}\cdot 2^{O(n\log n)}\\ &\le 2^{m^2+2mn(1+3\log_2 3)+O(n\log n)}, \text{ and}\\ T(G) &\le 2^{m^2+4mn}\cdot\binom{2mn+\frac{m^2}{2}}{2mn}\cdot 2^{O(n\log n)}\\ &\le 2^{m^2+2mn(\log(1+\frac{m}{4n})+2+\log_2 e)+O(n\log n)}.\end{aligned}$$

Let $\varepsilon > 0$. For graphs G with $m > (6+\varepsilon)n$ we have

$$T(G) \ge 2^{\Omega(m^2)}.$$

For graphs G with $m > \omega(n)$ we have

$$T(G) \ge 2^{m^2/60} - o(1).$$

The two upper bounds on $T(G)$ come from two essentially different approaches to enumerating isomorphism classes of $\mathcal{T}$-representations. In the first approach, we reduce the problem to enumerating *simple quadrangulations* of the disc [9]. In the second approach, we split the problem into two parts: enumerating *chord diagrams* [14] and enumerating isomorphism classes of *arrangements of pseudochords*. The first method gives better asymptotic results for dense graphs, whereas the second one is better for sparse graphs (roughly, with at most $35n$ edges). For graphs with $m = O(n)$ the second term in the exponent becomes more significant. Since $m \ge n/2$, the exponent in the first upper bound can be bounded by $23.118m^2 + o(1)$, using the entropy bound for the binomial coefficient. Similarly, the exponent in the second upper bound can be bounded by $11.265m^2 + o(1)$. For such very sparse graphs (for example, matchings), however, better upper bounds can be deduced more directly from other known results.

The upper bound $T(G) \le 2^{O(m^2)}$ is trivially obtained from the upper bound on the number of unlabeled plane graphs (or planar maps). Indeed, every drawing $\mathcal{G}$ of G can be transformed into a plane graph H by subdividing the edges of $\mathcal{G}$ by its crossings and regarding the crossings of $\mathcal{G}$ as new 4-valent vertices in H. The graph H has thus at most $n + \binom{m}{2}$ vertices, at most $m + 2\binom{m}{2} = m^2$ edges, no loops and no multiple edges. Tutte [17] showed that there are

$$\frac{2(2M)!3^M}{M!(M+2)!} = 2^{(\log_2(12)+o(1))M}$$

rooted connected planar maps with M edges (see also [2, 3, 5]). Walsh and Lehman [18] showed that the number of rooted connected planar loopless maps with M edges is

$$\frac{6(4M+1)!}{M!(3M+3)!} = 2^{(\log_2(256/27)+o(1))M}.$$

This implies the upper bound $T(G) \le 2^{(\log_2(256/27)+o(1))m^2}$. Somewhat better estimates could be obtained by reducing the problem to counting 4-regular planar maps [15, 16], since typically almost all vertices in H are the 4-valent vertices obtained from the crossings of $\mathcal{G}$. But such a reduction would be less straightforward and the resulting upper bound $2^{(\frac{1}{2}\log_2(196/27)+o(1))m^2}$ still relatively high for dense graphs (for graphs with more than $27n$ edges the two upper bounds from Theorem 5 are better).

The proof in [7] implies the upper bound $T(K_n) \le 2^{(1/12+o(1))(n^4)}$, although it is not explicitly stated there. However, the key Proposition 7 in [7] is incorrect. We prove a correct version in the full paper.

Note that by the reduction to counting planar maps, for every fixed constant k, we also obtain the upper bound $2^{O(km^2)}$ on the number of isomorphism classes of connected topological graphs with m edges where all pairs of edges are allowed to cross k times.

2 A few open problems

The problem of counting the asymptotic number of "nonequivalent" simple drawings of a graph in the plane is answered only partially. Many open questions remain.

The gap between the lower and upper bounds on $T_{\mathrm{w}}(G)$ proved in Theorem 2 is wide open, especially for graphs with low density. For graphs with cn^2 edges, the lower and upper bounds on $\log T_{\mathrm{w}}(G)$ differ by a logarithmic factor. We conjecture that the correct answer is closer to the lower bound.

We do not even know whether $T_{\mathrm{w}}(G)$ is a monotone function with respect to the subgraph relation, since there are simple topological graphs that cannot be extended to simple complete topological graphs. Due to somewhat "rigid" properties of simple complete topological graphs, we have a much better upper bound for the complete graph than, say, for the complete bipartite graph on the same number of vertices.

Problem 1 *Does the complete graph K_n maximize the value $T_w(G)$ among the graphs G with n vertices? More generally, is it true that $T_w(H) \le T_w(G)$ if $H \subseteq G$?*

Our methods for proving upper bounds on the number of weak isomorphism classes of simple topological graphs do not generalize to the case of topological graphs with two crossings per pair of edges allowed.

Problem 2 *What is the number of weak isomorphism classes of drawings of a graph G where every two independent edges are allowed to cross at most twice and every two adjacent edges at most once?*

For the complete graph with n vertices, Pach and Tóth [13] proved the lower bound $2^{\Omega(n^2 \log n)}$ and the upper bound $2^{o(n^4)}$.

A nontrivial lower bound can be proved also in the case when G is a matching. Ackerman et al. [1] constructed a system of n x-monotone curves where every pair of curves intersect in at most one point where they either cross or touch, with $\Omega(n^{4/3})$ pairs of touching curves. Eyal Ackerman (personal communication) noted that this also follows from an earlier result by Pach and Sharir [11], who constructed an arrangement of n segments with $\Omega(n^{4/3})$ vertically visible pairs of disjoint segments. By changing the drawing in the neighborhood of every touching point, we obtain $2^{\Omega(n^{4/3})}$ different intersection graphs of 2-intersecting curves, also called *string graphs of rank* 2 [13]. This improves the trivial lower bound observed by Pach and Tóth [13].

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