Stackable tessellations

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Abstract

We introduce a class of solids that can be constructed gluing stackable pieces, which has been proven to have advantages in architectural construction. We derive a necessary condition for a solid to belong to this class. This helps to specify a simple sufficient condition for the existence of a stackable tessellation of a given solid. Finally, we show the compatibility of our method with some discretization techniques appearing in the literature.

Introduction

During the last decades, the increasing demand of architectural freeform shapes has motivated the study of surface tesselations in tiles which can be manufactured with a low economic impact. Substantial research has been done on planar tesselations with triangular meshes, quadrilateral meshes [1, 2] and its special cases of spherical and conical meshes [3]. Nevertheless, tessellating methods in non-standard non-planar panels have been rarely considered, since for most construction materials the production of such building components is very expensive. The innovative CASTonCAST project [4] proposes a system which consists of two parts: a geometric method for the construction of double curved shapes by means of stackable tiles and an efficient fabrication technique of curved precast panels which relies on shaping building components in stacks by using the previous component as a mold for the next one. The latter relies on using the top surface of each component as part of the mold for the next one, after applying a chemical barrier. Once the components have been fabricated in stacks, they are detached and assembled by glueing them through their lateral faces, forming a new solid called strip. This procedure features obvious advantages concerning storage and transport as well.

We analyse the kind of solids that can be constructed using stacks under some reasonable assumptions. More precisely, we focus on constructions where the tiles form a face-to-face tessellation whose adjacency graph is a grid both in stack and strip forms, as in Figure 1. We also require the junction faces in the strip configuration to be planar. After stating necessary conditions for a solid to admit a stackable tessellation as desired, we provide sufficient conditions and a procedure to obtain the stacks that generate it. It turns out that the solids for which a stackable tessellation exists are naturally defined, skipping some technical details, as the ones generated sweeping a surface around an axis or in a fixed direction. In addition, we prove that, under certain conditions, a one-parametric family of stackable tessellations exists. Finally, we approach the problem of approximating a target solid using polyhedral tiles.

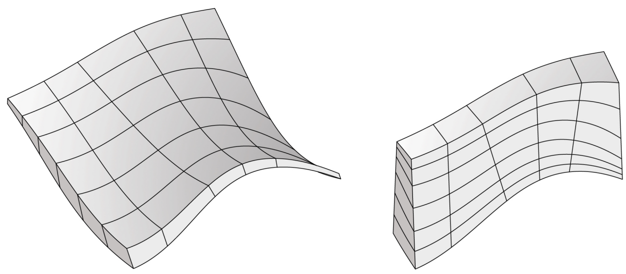

Figure 1: Stacks (right) and associated strips (left)

We normally use capital letters for points, lowercase letters for vectors, lines and rays, and Greek letters for curves. We add a * to the names in order to indicate the correspondences between the 2- and 3-dimensional elements of the constructions in Sections 1 and 2. We will often omit the indices ranges, for ease of reading. The curves and surfaces are considered to have no self-intersections. Abusing notation, we denote the image set of a curve or a surface by its name. Curves are assumed to be parametrized using the unit interval. We define the wedge spanned by two rays to be the cone defined by their supporting lines and containing an unbounded part of both rays.

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1 Planar stacks

Before starting the analysis of the 3-dimensional case, we study a 2-dimensional object we call *planar stack*. Its construction is described in the next subsection and illustrated by Figure 2.

1.1 Construction

Let p and q be two infinite rays emanating from a point K and spanning a wedge W of angle $\alpha \in (0, \pi)$ in $\mathbb{R}^2$. Let $\sigma_0, ..., \sigma_n$ be pairwise disjoint, simple curves contained in W. Assume the indices correspond to the inverse order a ray shot from K would intersect them. Assume further that $\sigma_i \cap p = \sigma_i(0)$ and $\sigma_i \cap q = \sigma_i(1)$ and require that

$$d(\sigma_i(0), \sigma_{i+1}(0)) = d(\sigma_{i-1}(1), \sigma_i(1)), \qquad (1)$$

where d denotes the Euclidean distance in the plane. Let $\mathcal{T}_i$ denote the closed subset of W bounded by σ_i and σ_{i+1}. Each of these simply connected sets will be called a *tile* and each σ_i will be called a *separator*. The whole construction will be referred to as a *planar stack of angle* α. It may also be required that the curves σ_i are *separable*, i.e., that a tile can be moved away from the next one by a rigid motion preventing them to overlap. A simple condition ensuring this property is that each σ_i is monotone in some direction but we will ignore this requirement in this article.

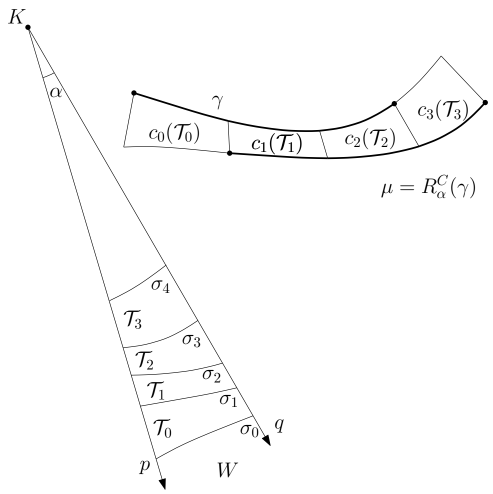

Figure 2: A planar stack and the corresponding strip.

Note that condition (1) ensures that the tiles can be rearranged gluing $\sigma_{i+1}(0)$ with $\sigma_i(1)$ and $\sigma_i(0)$ with $\sigma_{i-1}(1)$. This configuration will be called the *strip* associated to the planar stack.

It will be handy to define $s_i : \mathbb{R}^2 \to \mathbb{R}^2$ to be the congruence that maps $\mathcal{T}_i$ to its position in the strip, assuming $s_0 = Id$. Given a point $C \in \mathbb{R}^2$ and an angle α, we define R_α^C to be the counterclockwise rotation of angle α and center C.

Theorem 1 *If the strip $\bigcup_{i=0}^{n-1} s_i(\mathcal{T}_i)$ associated to a planar stack of angle α is a simply connected set, then its boundary contains the curves $\gamma = \bigcup_{i=0}^{n-1} s_i(\sigma_{i+1})$ and $R_\alpha^C(\gamma)$, for some point $C \in \mathbb{R}^2$.*

Proof. The assumptions that all the σ_i start in p and end in q, are pairwise disjoint and have no self-intersections ensure that the tiles are topologically disks. Provided that $\bigcup_{i=0}^{n-1} s_i(\mathcal{T}_i)$ is simply connected, the tiles must intersect only in the glued segments. In addition, condition (1) ensures that $s_i(\sigma_i(1)) = s_{i+1}(\sigma_{i+1}(0))$ and $s_i(\sigma_{i+1}(1)) = s_{i+1}(\sigma_{i+2}(0))$, for $i \in \{0, ..., n-2\}$. Consequently, it is clear that γ and $\mu = \bigcup_{i=1}^{n} s_i(\sigma_i)$ are contained in the boundary of the strip. Observe now that when $\mathcal{T}_{i+1}$ is glued next to $\mathcal{T}_i$, the two copies of σ_{i+1} involved are congruent via $R_\alpha^{C_i}$, for some point C_i. But $R_\alpha^{C_i}$ maps $s_i(\sigma_{i+1}(1))$ to $s_{i+1}(\sigma_{i+1}(1))$ and $R_\alpha^{C_{i+1}}$ maps $s_{i+1}(\sigma_{i+2}(0))$ to $s_{i+2}(\sigma_{i+2}(0))$, which are the same pair of points. Since for any pair of points $P \neq Q$ and a given α there is only one point O such that $R_\alpha^O(P) = Q$, it has to be $C_i = C$ for all $i \in \{0, ..., n-1\}$. Thus, we have that $\mu = R_\alpha^C(\gamma)$. □

1.2 Tessellation

Constructing a planar stack and developing it can be used as a modelling tool. However, we are now interested in the other direction of the procedure. That is, given a curve γ, an angle α and a center of rotation C, construct a planar stack such that, when reconfigured into strip, contains γ and $R_\alpha^C(\gamma)$ in its boundary. Such a planar stack may not exist. If it does, it may not be unique. Indeed, there is in general a one-parametric family of possibly suitable planar stacks.

Theorem 1 indicates that we need to construct a stack of angle α if we aim to obtain a shape with γ and $R_\alpha^C(\gamma)$ forming part of its boundary. Consider the locus of points that see a counterclockwise angle α between $\gamma(0)$ and $R_\alpha^C(\gamma(0))$, i.e., the set

$$\kappa = \{P \in \mathbb{R}^2 : R_\alpha^P(\overrightarrow{P\gamma(0)}) = \overrightarrow{PR_\alpha^P(\gamma(0))}\}, \qquad (2)$$

where $\overrightarrow{XY}$ is the ray starting at X and going through Y. It is well known that κ is a circumference arc going from $\gamma(0)$ to $R_\alpha^C(\gamma(0))$. It is not hard to see that $C \in \kappa$. Obviously, any candidate K to be the vertex of the stack must belong to κ and there is a possibly valid planar stack leading to our target for each of such points.

Consider a fixed $K \in \kappa$ and let r_0 be the ray containing K, passing through $\gamma(0)$ and such that $R_\alpha^C(r_0)$ touches but does not cross r_0. We define $r_i = R_{i\alpha}^C(r_0)$ for $i \in \mathbb{N}$. Let W_i be the wedge of angle α spanned by r_i and r_{i+1} and define also $\mathcal{U}_i$ to be the region in W_i bounded by γ and $R_\alpha^C(\gamma)$. For simplicity, we assume that γ does not intersect $R_{-\alpha}^C(r_0)$. This last restriction is not necessary, but simplifies the technicalities.

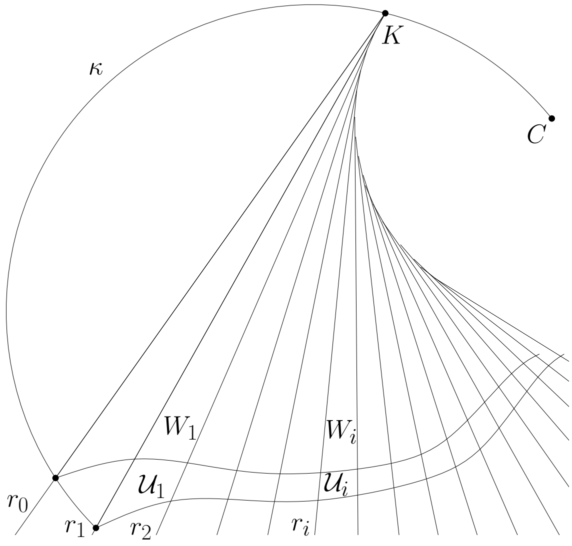

Figure 3: Tessellation procedure.

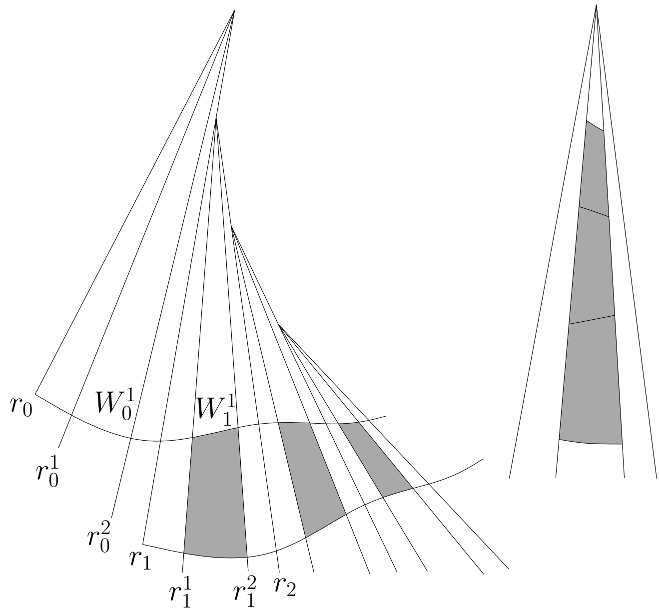

Figure 4: Shaded tiles can be arranged in stack form.

Theorem 2 *If $r_0, ..., r_n$ intersect γ in a single point, the tiles $\mathcal{U}_1, ..., \mathcal{U}_{n-1}$ can be arranged as a planar stack.*

Proof. We will prove that $\{R^C_{-i\alpha}(\mathcal{U}_i)\}_{i=1}^{n-1}$ is a set of cells tessellating a simply connected subset of W_0. Observe first that $W_{i+1} = R^C_\alpha(W_i)$, since $r_{j+1} = R^C_\alpha(r_j)$ for all $j \in \mathbb{N}$. For the same reason, $r_1, ..., r_{n+1}$ intersect $R^C_\alpha(\gamma)$ in a single point. Since γ and $R^C_\alpha(\gamma)$ do not intersect, the boundary of $\mathcal{U}_i$ is formed by a subcurve of γ and a subcurve of $R^C_\alpha(\gamma)$ which are mutually disjoint, one segment contained in r_i and one contained in r_{i+1}. It remains to be proven that each tile matches the next one and they all fit in W_0. This can be derived from the facts that $R^C_\alpha(\mathcal{U}_i \cap \gamma) = \mathcal{U}_{i+1} \cap R^C_\alpha(\gamma)$ and that $R^C_{-i\alpha}(W_i) = W_0$. $\square$

Using the previous theorem, it is not hard to see that under an additional technical condition, one can construct a stack containing the whole γ and $R^C_\alpha(\gamma)$ in its boundary. We state this in the following corollary, omitting its simple proof.

Corollary 3 *If $r_0, ..., r_l$ intersect γ in a single point and r_{l+1} does not intersect γ, then there exists a planar stack containing γ and $R^C_\alpha(\gamma)$ in its boundary.*

1.3 Refinement

For obvious reasons, it can be useful to refine a stackable tessellation. This feature will be essential in the discussion concerning approximation in Section 3.

Adopting the notation used in the previous section, consider the angularly ordered rays $r_0 = r_0^0, ..., r_0^m = r_1$ emanating from K and contained in W_0. Define also r_i^j to be $R^C_{i\alpha}(r_0^j)$ for $i \in \{1, ..., n\}$ and $j \in \{0, ..., m\}$. Let W_i^j be the wedge spanned by r_i^j and r_i^{j+1}. Define also $\mathcal{U}_i^j$ to be the space in W_i^j bounded by γ and $R^C_\alpha(\gamma)$.

Proposition 4 *If $r_0^0, ..., r_0^m = r_1^0, r_1^1, ..., r_l^m$ intersect γ in a single point, the tiles $\mathcal{U}_1^j, ..., \mathcal{U}_{n-1}^j$ can be arranged as a planar stack, for $j \in \{0, ..., m-1\}$.*

Proof. The way we defined r_i^j ensures that $R^C_{-i\alpha}(W_i^j) = W_0^j$. Therefore, the arguments in the proof of Theorem 2 apply to each j individually. $\square$

Note that, in addition, the m stacks obtained from the refined tessellation can be obtained by cutting the original stack by straight radial cuts.

The results presented in this section can easily be extended to the case where K is thought of as a point at infinity. That is, we consider the wedge W to be the space between two parallel lines. Then, the two copies of γ appearing in the boundary of the strip are related by a simple translation. Conversely, given a curve and a translated copy of it, a one-parametric family of possible stacks can be again evaluated. In this case, we may consider κ to be the set of directions of the plane, except for the direction of W. We can then cut the curves by a set of equally spaced lines in any direction $u \in \gamma$. This case is specially interesting because it generates offset shapes. Moreover, requiring that γ is monotone in u is sufficient to ensure the existence of the corresponding stack with the additional separability property.

2 Solid stacks

We consider now a 3-dimensional extension of the previous results, which is the actual target of this work. In this way, we model a meaningful subset of the solids constructible using the CASTonCAST fabrication technique.

The initial object is now the space W^* between two non-parallel planes p^* and q^*, which we further bound by two planes orthogonal to $K^* = p^* \cap q^*$. Cutting W^* with pairwise-disjoint surfaces $\sigma_0^*, ..., \sigma_n^*$, each of them dividing W^* in two parts, a sequence of 3-dimensional tiles $\mathcal{T}_i^*$ is defined. If we require that $R_\alpha^{K^*}(\mathcal{T}_i^* \cap p^*) = \mathcal{T}_{i+1}^* \cap q^*$ (the condition analogous to (1)), the resulting object will be called a *solid stack*. The tiles can be rearranged by glueing congruent faces contained in p^* and q^* in order to form the associated *solid strip*. We can repeat the arguments in Section 1.1 and derive that if the resulting solid strip is simply connected, its boundary contains two congruent copies of a surface γ^*. The congruency relating them is a rotation around a linear axis C^* parallel to K^* by the amount corresponding to the dihedral angle between p^* and q^*. The results in Section 1.2 also extend to the 3-dimensional case if the conditions are properly adapted.

In addition to the refinement possibilities analogous to the planar case, we can now split the tessellation of a solid stack in order to obtain multiple stacks that can be then converted into strips and arranged one next to the other. To ensure that the tessellation is face-to-face in the strip form, we assume here that the solid stack is refined only cutting by planes $h_1^*, ..., h_l^*$ orthogonal to K^*. However, this is not a necessary condition, as can be appreciated in Figure 1.

3 Discretization

This section studies how polyhedral tiles can be used to approximate the stacks constructed in previous sections. This can be thought of as simplifying the tiles one would get in the continuous case while preserving their *stackability* and congruency relations.

We assume here that a solid stack is given and that it has been refined by means of the planes $(r_0^j)^*$ through K^* and the planes h_k^* orthogonal to K^*. A possible way to discretize the tiles is to substitute each separator σ_i^* by the lower convex hull of the points resulting of its intersection with the rays $r_0^{jk} = (r_0^j)^* \cap h_k^*$. It is an easy exercise to prove that this construction preserves the topology and the congruency relations of the tiles. However, the tiles obtained in this way may not be convex, a property that can be useful in some practical applications. This restriction forces the separators to be planar within each of the stacks. This is equivalent to approximate the surface γ^* by a planar-quadrilateral mesh (PQ-mesh) $\widetilde{\gamma}$, whose vertices are constrained to lie on prescribed rays $r_i^{jk} = R_{i\alpha}^{C^*}(r_0^{jk})$. Then, the vertices V_i^{jk} of $\widetilde{\gamma}$ can be expressed as $V_i^{jk} = O_i^{jk} + \lambda_i^{jk} u_i^{jk}$, for some $\lambda_i^{jk} \in \mathbb{R}^+$, where O_i^{jk} is the initial point of r_i^{jk} and u_i^{jk} is its direction. Therefore, the vertices of $R_\alpha^{C^*}(\widetilde{\gamma})$ will also lie on these rays and their positions can be retrieved from the variables λ_i^{jk}. However, one should ensure that the perturbed versions of $\widetilde{\gamma}$ and $R_\alpha^{C^*}(\widetilde{\gamma})$ do not intersect. But provided that $\widetilde{\gamma}$ and $R_\alpha^{C^*}(\widetilde{\gamma})$ are PQ-meshes, these conditions translate into the simple linear inequalities $\lambda_i^{jl} < \lambda_{i-1}^{jl}$.

Existing algorithms, like the PQ perturbation detailed in [3], implement the planarity constraints and minimize functions measuring the distance to the surface and the quality of the mesh. Since the specific requirements of our method are, as hinted before, translated into linear equations and inequalities in the space of coordinates of the vertices, the mentioned algorithm, based on sequential quadratic programming [5], can be applied to our construction.

The results on 3-space generalize to case where p^* and q^* are parallel as well. In this situation, the vertices are constrained to lie in rays that have all the same direction. Therefore, the planarity constraints can be expressed as linear equations and optimizing with respect to linear functions can be done very efficiently by linear programming.

4 Conclusion

We give the first geometric analysis of the solids constructible with the fabrication method CASTonCAST. We restrict the study imposing some constraints to the tiles of the tessellation, which lead to simple and geometrically meaningful constructions. We also provide a scheme to approximate the tiles using polyhedra while preserving their stackability. More general cases can be studied using the introduced framework and are left to future work.

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