# On the nonexistence of $k$-reptile simplices in $\mathbb{R}^3$ and $\mathbb{R}^4$ *

Jan Kynčl†1 and Zuzana Safernová‡1

[1] Department of Applied Mathematics and Institute for Theoretical Computer Science, Charles University, Faculty of Mathematics and Physics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic

## Abstract

A $d$-dimensional simplex $S$ is called a *$k$-reptile* (or a *$k$-reptile simplex*) if it can be tiled without overlaps by $k$ simplices with disjoint interiors that are all mutually congruent and similar to $S$. For $d = 2$, triangular $k$-reptiles exist for many values of $k$ and they have been completely characterized by Snover, Waiveris, and Williams. On the other hand, the only $k$-reptile simplices that are known for $d \geq 3$, have $k = m^d$, where $m$ is a positive integer. We substantially simplify the proof by Matoušek and the second author that for $d = 3$, $k$-reptile tetrahedra can exist only for $k = m^3$. We also prove a weaker analogue of this result for $d = 4$ by showing that four-dimensional $k$-reptile simplices can exist only for $k = m^2$.

## 1 Introduction

A closed set $X \subset \mathbb{R}^d$ with nonempty interior is called a *$k$-reptile* (or a *$k$-reptile set*) if there are sets $X_1, X_2, \ldots, X_k$ with disjoint interiors and with $X = X_1 \cup X_2 \cup \cdots \cup X_k$ that are all mutually congruent and similar to $X$. Such sets have been studied in connection with fractals and also with crystallography and tilings of $\mathbb{R}^d$ [4, 8, 9, 11].

It easy to see that whenever $S$ is a $d$-dimensional $k$-reptile simplex, then all of $\mathbb{R}^d$ can be tiled by congruent copies of $S$: indeed, using the tiling of $S$ by its smaller copies $S_1, \ldots, S_k$ as a pattern, one can inductively tile larger and larger similar copies of $S$. On the other hand, not all space-filling simplices must be $k$-reptiles for some $k \geq 2$.

Clearly, every triangle tiles $\mathbb{R}^2$. Moreover, every triangle $T$ is a $k$-reptile for $k = m^2$, since $T$ can be tiled in a regular way with $m^2$ congruent tiles, each positively or negatively homothetic to $T$. See e. g. Snover et al. [19] for an illustration.

The question of characterizing the tetrahedra that tile $\mathbb{R}^3$ is still open and apparently rather difficult. The first systematic study of space-filling tetrahedra was made by Sommerville. Sommerville [20] discovered a list of exactly four tilings (up to isometry and rescaling), but he assumed that all tiles are *properly congruent* (that is, congruent by an orientation-preserving isometry) and meet face-to-face. Edmonds [6] noticed a gap in Sommerville's proof and by completing the analysis, he confirmed that Sommerville's classification of proper, face-to-face tilings is complete. In the non-proper and non face-to-face situations there are infinite families of non-similar tetrahedral tilers. Goldberg [10] described three such families, obtained by partitioning a triangular prism. In fact, Goldberg's first family was found by Sommerville [20] before, but he selected only special cases with a certain symmetry. Goldberg [10] noticed that even the general case admits a proper tiling of $\mathbb{R}^3$. Goldberg's first family also coincides with the family of simplices found by Hill [14], whose aim was to classify *rectifiable* simplices, that is, simplices that can be cut by straight cuts into finitely many pieces that can be rearranged to form a cube. The simplices in Goldberg's second and third families are obtained from the simplices in the first family by splitting into two congruent halves. According to Senechal's survey [17], no other space-filling tetrahedra than those described by Sommerville and Goldberg are known.

For general $d$, Debrunner [5] constructed $\lfloor d/2 \rfloor + 2$ one-parameter families and a finite number of additional special types of $d$-dimensional simplices that tile $\mathbb{R}^d$. Smith [18] generalized Goldberg's construction and using Debrunner's ideas, he obtained $(\lfloor d/2 \rfloor + 2)\phi(d)/2$ one-parameter families of space-filling $d$-dimensional simplices; here $\phi(d)$ is the Euler's totient function. It is not known whether for some $d$ there is an acute space-filling simplex or a two-parameter family of space-filling simplices [18].

In recent years the subject of tilings has received a certain impulse from computer graphics and other computer applications. In fact, our original motivation for studying simplices that are $k$-reptiles comes from a problem of probabilistic marking of Internet

*The authors were supported by the project CE-ITI (GACR P202/12/G061) of the Czech Science Foundation, by the grant SVV-2013-267313 (Discrete Models and Algorithms) and by project GAUK 52410. The research was partly conducted during the Special Semester on Discrete and Computational Geometry at École Polytechnique Féderale de Lausanne, organized and supported by the CIB (Centre Interfacultaire Bernoulli) and the SNSF (Swiss National Science Foundation).

†Email: kyncl@kam.mff.cuni.cz

‡Email: zuzka@kam.mff.cuni.cz

packets for IP traceback [1, 2]. See [15] for a brief summary of the ideas of this method. For this application, it would be interesting to find a $d$-dimensional simplex that is a $k$-reptile with $k$ as small as possible.

For dimension 2 there are several possible types of $k$-reptile triangles, and they have been completely classified by Snover et al. [19]. In particular, $k$-reptile triangles exist for all $k$ of the form $a^2 + b^2$, $a^2$ or $3a^2$ for arbitrary integers $a, b$. In contrast, for $d \geq 3$, reptile simplices seem to be much more rare. The only known constructions of higher-dimensional $k$-reptile simplices have $k = m^d$. The best known examples are the *Hill simplices* (or the *Hadwiger–Hill simplices*) [5, 12, 14]. A $d$-dimensional Hill simplex is the convex hull of vectors $0$, $b_1$, $b_1 + b_2$, $\ldots$, $b_1 + \cdots + b_d$, where $b_1, b_2, \ldots, b_d$ are vectors of equal length such that the angle between every two of them is the same and lies in the interval $(0, \frac{\pi}{2} + \arcsin \frac{1}{d-1})$.

Concerning nonexistence of $k$-reptile simplices in dimension $d \geq 3$, Hertel [13] proved that a 3-dimensional simplex is an $m^3$-reptile using a "standard" way of dissection (which we will not define here) if and only if it is a Hill simplex. He conjectured that Hill simplices are the only 3-dimensional reptile simplices. Herman Haverkort recently pointed us to an example of a $k$-reptile tetrahedron which is not Hill, which contradicts Hertel's conjecture. In fact, except for the one-parameter family of Hill tetrahedra, three other space-filling tetrahedra described by Sommerville [20] and Goldberg [10] are also $k$-reptiles for every $k = m^3$. The simplices and their tiling are based on the barycentric subdivision of the cube. The construction can be naturally extended to find similar examples of $d$-dimensional $k$-reptile simplices for $d \geq 4$ and $k = m^d$. Matoušek [15] showed that there are no 2-reptile simplices of dimension 3 or larger. For dimension $d = 3$ Matoušek and the second author [16] proved the following theorem.

**Theorem 1** [16] *In $\mathbb{R}^3$, $k$-reptile simplices (tetrahedra) exist only for $k$ of the form $m^3$, where $m$ is a positive integer.*

We give a new, simple proof of Theorem 1 in Section 3.

Matoušek and the second author [16] conjectured that a $d$-dimensional $k$-reptile simplex can exist only for $k$ of the form $m^d$ for some positive integer $m$. We prove a weaker version of this conjecture for four-dimensional simplices.

**Theorem 2** *Four-dimensional $k$-reptile simplices can exist only for $k$ of the form $m^2$, where $m$ is a positive integer.*

Four-dimensional Hill simplices are examples of $k$-reptile simplices for $k = m^4$. Whether there exists a four-dimensional $m^2$-reptile simplex for $m$ non-square remains an open question.

## 2 Angles in simplices and Coxeter diagrams

Given a $d$-dimensional simplex $S$ with vertices $v_0, \ldots, v_d$, let $F_i$ be the facet opposite to $v_i$. A *dihedral angle* $\beta_{i,j}$ of $S$ is the internal angle of the facets $F_i$ and $F_j$, that meet at the $(d-2)$-face $F_i \cap F_j$.

An *edge-angle* of $S$ is the internal $(d-1)$-dimensional angle incident to an edge and can be represented by a $(d-2)$-dimensional spherical simplex.

The *Coxeter diagram* of $S$ is a graph $c(S)$ with labeled edges such that the vertices of $c(S)$ represent the facets of $S$ and for every pair of facets $F_i$ and $F_j$, there is an edge $e_{i,j}$ labeled by the dihedral angle $\beta_{i,j}$.

**Observation 3** *The edge-angles of a four-dimensional simplex $S$ can be represented by spherical triangles, whose angles are dihedral angles in $S$. Therefore, an edge-angle in $S$ represented by a spherical triangle with angles $\alpha, \beta, \gamma$ corresponds to a triangle in the Coxeter diagram with edges labeled by $\alpha, \beta, \gamma$.* □

The most important tool we use is Debrunner's lemma [5, Lemma 1], which connects the symmetries of a $d$-simplex with the symmetries of its Coxeter diagram (which represents the "arrangement" of the dihedral angles). This lemma allows us to substantially simplify the proof of Theorem 1 and enables us to step up by one dimension and prove Theorem 2, which seemed unmanageable before.

**Lemma 4 (Debrunner's lemma [5])** *Let $S$ be a $d$-dimensional simplex. The symmetries of $S$ are in one-to-one correspondence with the symmetries of its Coxeter diagram $c(S)$ in the following sense: each symmetry $\varphi$ of $S$ induces a symmetry $\Phi$ of $c(S)$ so that $\varphi(v_i) = v_j \Leftrightarrow \Phi(F_i) = F_j$, and vice versa.*

## 3 A simple proof of Theorem 1

We proceed as in the original proof, but instead of using the theory of scissor congruence, Jahnel's theorem about values of rational angles and Fiedler's theorem, we only use Debrunner's lemma (Lemma 4).

Assume for contradiction that $S$ is a $k$-reptile tetrahedron where $k$ is not a third power of a positive integer. A dihedral angle $\alpha$ is called *indivisible* if it cannot be written as a linear combination of other dihedral angles in $S$ with nonnegative integer coefficients.

The following lemmas are proved in [16].

**Lemma 5** [16, Lemma 3.1] *If $\alpha$ is an indivisible dihedral angle in $S$, then the edges of $S$ with dihedral angle $\alpha$ have at least three different lengths.*

**Lemma 6** [16, Lemma 3.3] *One of the following two possibilities occur:*

(i) *All the dihedral angles of $S$ are integer multiples of the minimal dihedral angle $\alpha$, which has the form $\frac{\pi}{n}$ for an integer $n \geq 3$.*

(ii) *There are exactly two distinct dihedral angles $\beta_1$ and $\beta_2$, each of them occurring three times in $S$.*

First we exclude case (ii) of Lemma 6. If $S$ has two distinct dihedral angles $\beta_1 \neq \beta_2$, each occurring at three edges, then they can be placed in $S$ in two essentially different ways; see Fig. 1. In both cases, for each $i \in \{1, 2\}$, the Coxeter diagram of $S$ has at least one nontrivial symmetry which swaps two distinct edges with label $\beta_i$. By Debrunner's lemma, the corresponding symmetry of $S$ swaps two distinct edges with dihedral angle $\beta_i$, which thus have the same length. But then the edges with dihedral angle $\beta_i$ have at most 2 different lengths and this contradicts Lemma 5, since the smaller of the two angles $\beta_1, \beta_2$ is indivisible.

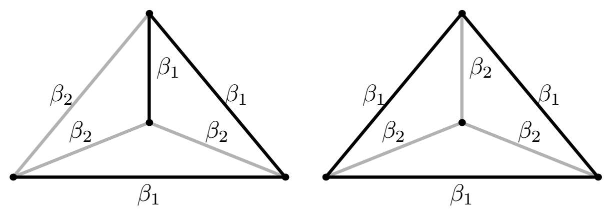


Figure 1: Two possible configurations of two dihedral angles.

Now we exclude case (i) of Lemma 6. Call the edges of $S$ (and of $c(S)$) with dihedral angle $\alpha$ the *$\alpha$-edges*. Since there are at least three $\alpha$-edges in $S$, there is a vertex $v$ of $S$ where two $\alpha$-edges meet. Let $\beta$ be the dihedral angle of the third edge incident to $v$ (possibly $\beta$ can be equal to $\alpha$).

We have $2\alpha + \beta > \pi$, using a well-known fact that the sum of the three dihedral angles occurring at a vertex of $S$ exceeds $\pi$. Writing $\beta = m\alpha = \frac{m}{n} \cdot \pi$, we have $2 \cdot \frac{\pi}{n} + \frac{m}{n} \cdot \pi > \pi$, which implies $m > n - 2$. Since $m < n$, we have $m = n - 1$ and hence $\beta = \pi - \alpha$.

Now we distinguish several cases depending on the subgraph $H_\alpha$ of $c(S)$ formed by the $\alpha$-edges.

- $H_\alpha$ contains three edges incident to a common vertex (which correspond to a triangle in $S$). Then all the other edges must have the angle $\beta$ and we get the configuration as in Fig. 1 (right), which we excluded earlier.

- $H_\alpha$ contains a triangle (the corresponding edges in $S$ meet at a single vertex). Then $\beta = \alpha$, and thus $\alpha = \frac{\pi}{2}$, which contradicts the condition $n \geq 3$ from Lemma 6 (i).

- $H_\alpha$ is a path of length three (this corresponds to a path in $S$, too). Then two edges have the angle $\beta > \alpha$ and the remaining edge has some angle $\gamma \neq \alpha$. See Figure. 2 (left). The resulting Coxeter diagram has a nontrivial involution swapping two $\alpha$-edges. By Debrunner's lemma, this contradicts Lemma 5.

- It remains to deal with the case where $H_\alpha$ is a four-cycle (which corresponds to a four-cycle in $S$). In this case the remaining two edges have dihedral angle $\beta$, so the Coxeter diagram has a dihedral symmetry group $D_4$ acting transitively on the $\alpha$-edges. By Debrunner's lemma, all the $\alpha$-edges have the same length. This again contradicts Lemma 5.

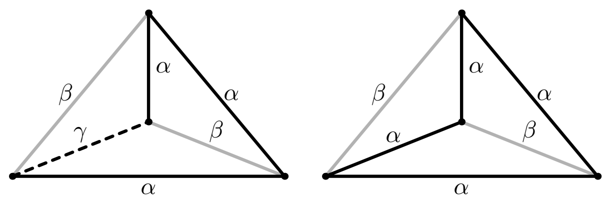


Figure 2: The $\alpha$-edges form a path (left) or a four-cycle (right) in $c(S)$

We obtained a contradiction in each of the cases, hence the proof of Theorem 1 is finished.

# 4 The proof of Theorem 2

The method of the proof is similar to the three-dimensional case.

Assume for contradiction that $S$ is a four-dimensional $k$-reptile simplex where $k$ is not a square of a positive integer. Let $S_1, \ldots, S_k$ be mutually congruent simplices similar to $S$ that tile $S$. Then each $S_i$ has volume $k$-times smaller than $S$, and thus $S_i$ is scaled by the ratio $\rho := k^{-1/4}$ compared to $S$. For $k$ non-square, $\rho$ is an irrational number of algebraic degree 4 over $\mathbb{Q}$.

Similarly to [16] we define an *indivisible* edge-angle (spherical triangle) as a spherical triangle which cannot be tiled with smaller spherical triangles representing the other edge-angles of $S$ or their mirror images. Clearly, the edge-angle with the smallest surface area is indivisible. We consider a spherical triangle and its mirror image as the same spherical triangle.

We obtain a result similar to Lemma 5: if $\mathcal{T}_0$ is an indivisible edge-angle in $S$, then the edges of $S$ with edge-angle $\mathcal{T}_0$ have at least four different lengths (and in particular, there are at least four such edges).

The strategy of the proof is now the following. First we exclude the case of two indivisible edge-angles, using only elementary combinatorial arguments and Debrunner's lemma.

Then we consider the case of one indivisible edge-angle. Here we need more involved arguments. We study the problem of tiling spherical triangles by congruent triangular tiles, which might be of independent interest. We give a partial classification of such tilings, which might be possible to extend to a full classification using a reasonable amount of effort. A related question, a classification of edge-to-edge tilings of the sphere by congruent triangles, has been completely solved by Agaoka and Ueno [3].

To rule out several cases, we use a characterization of $\binom{d+1}{2}$-tuples of dihedral angles by Fiedler [7]. The characterization implies, in particular, that if $\beta_{i,j}, i, j = 1, 2, \ldots, d+1$, are the dihedral angles of some $d$-dimensional simplex, then the matrix $(a_{i,j}; i, j = 1, 2, \ldots, n+1)$, where $a_{i,i} = 0$ and $a_{i,j} = \cos\beta_{i,j}$ for $i \neq j$, is singular.

## References


[1] M. Adler, Tradeoffs in probabilistic packet marking for IP traceback, *Proc. 34th Annu. ACM Symposium on Theory of Computing* (2002) 407–418.

[2] M. Adler, J. Edmonds and J. Matoušek, Towards asymptotic optimality in probabilistic packet marking, *Proc. 37th Annu. ACM Symposium on Theory of Computing* (2005) 450–459.

[3] Y. Agaoka and Y. Ueno, Classification of tilings of the 2-dimensional sphere by congruent triangles, *Hiroshima Math. J.* **32**(3) (2002), 463–540.

[4] C. Bandt, Self-similar sets. V. Integer matrices and fractal tilings of $\mathbf{R}^n$, *Proc. Amer. Math. Soc.* **112**(2) (1991), 549–562.

[5] H. E. Debrunner, Tiling Euclidean $d$-space with congruent simplexes, *Discrete geometry and convexity (New York, 1982)*, vol. 440 of *Ann. New York Acad. Sci.*, New York Acad. Sci., New York (1985) 230–261.

[6] A. L. Edmonds, Sommerville's missing tetrahedra, *Discrete Comput. Geom.* **37**(2) (2007), 287–296.

[7] M. Fiedler, Geometry of the simplex in $E_n$ (in Czech, with English and Russian summary), *Časopis pro pěstování matematiky* **79** (1954), 297–320.

[8] M. Gardner, *The unexpected hanging and other mathematical diversions*, The University of Chicago Press (1991).

[9] G. Gelbrich, Crystallographic reptiles, *Geom. Dedicata* **51**(3) (1994), 235–256.

[10] M. Goldberg, Three infinite families of tetrahedral space-fillers, *J. Comb. Theory, Ser. A* **16** (1974), 348–354.

[11] S. W. Golomb, Replicating figures in the plane, *Mathematical Gazette* **48** (1964), 403–412.

[12] H. Hadwiger, Hillsche Hypertetraeder (in German), *Gaz. Mat. (Lisboa)* **12**(50) (1951), 47–48.

[13] E. Hertel, Self-similar simplices, *Beiträge Algebra Geom.* **41**(2) (2000), 589–595.

[14] M. J. M. Hill, Determination of the volumes of certain species of tetrahedra without employment of the method of limits, *Proc. London Math. Soc.* **27** (1896), 39–52.

[15] J. Matoušek, Nonexistence of 2-reptile simplices, *Discrete and Computational Geometry: Japanese Conference, JCDCG 2004*, Lecture Notes in Computer Science 3742, Springer, Berlin (2005) 151–160, erratum at `http://kam.mff.cuni.cz/~matousek/no2r-err.pdf`.

[16] J. Matoušek and Z. Safernová, On the nonexistence of $k$-reptile tetrahedra, *Discrete Comput. Geom.* **46**(3) (2011), 599–609.

[17] M. Senechal, Which tetrahedra fill space?, *Math. Magazine* **54**(5) (1981), 227–243.

[18] W. D. Smith, Pythagorean triples, rational angles, and space-filling simplices (2003), manuscript, `http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.124.6579`.

[19] S. L. Snover, C. Waiveris and J. K. Williams, Rep-tiling for triangles, *Discrete Math.* **91**(2) (1991), 193–200.

[20] D. M. Y. Sommerville, Division of space by congruent triangles and tetrahedra, *Proc. Roy. Soc. Edinburgh* **4** (1923), 85–116.