

## Some results on open edge guarding of polygons

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### Abstract

This paper focuses on a variation of the Art Gallery problem that considers open edge guards. The “open” prefix means the endpoints of an edge where a guard is are not taken into account for visibility purposes. This paper studies the number of open edge guards that are sufficient and sometimes necessary to guard some classes of simple polygons.

### Introduction

The well known Art Gallery problem studies the minimum number of guards that are needed to fully cover a polygon  $P$ , that is, the number of guards from which every point of  $P$  is visible. Ideally, guards may be placed anywhere on  $P$  but usually they are restricted to vertices of the polygon or its edges. In the first case such guards are called vertex guards and in the second edge guards. Moreover, a point guard is a guard that can be placed anywhere on the polygon. Lee et al. [3] proved that finding the minimum number of guards to fully cover a polygon without holes is NP-hard for all three variations of guards. Toussaint conjectured that  $\lfloor \frac{n}{4} \rfloor$  edge guards are sufficient to cover any simple polygon of  $n$  vertices, except for small values of  $n$ . Later, Shermer [4] proved that  $\lfloor \frac{3n}{10} \rfloor$  edge guards are sufficient to cover any simple polygon, except for  $n = 3, 6, 13$  where an extra edge guard might be needed. Shermer actually proved a combinatorial result: any triangulation of a polygon with  $n$  vertices can be dominated by  $\frac{3n}{10}$  edge guards.

In this paper guards are assumed to be placed along open edges of a polygon, that is, the endpoints of any

edge are not taken into account for visibility purposes. Therefore, a point  $p$  is covered by an edge  $e$  if  $p$  is visible from some interior point of  $e$ . As shown in Figure 1, open edge guards can see considerably less polygon area than the usual edge guards, and are therefore an interesting topic of research on their own.

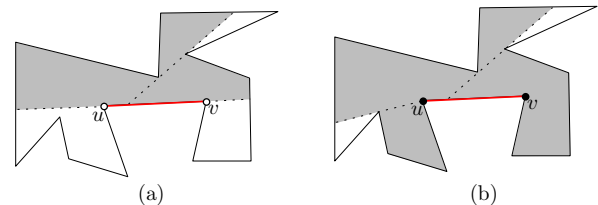


Figure 1: (a) The area covered by the open edge guard  $\overline{uv}$  is shown in grey. (b) The area covered by the closed edge guard  $\overline{uv}$  is shown in grey.

Open edge guarding is a variation of the Art Gallery problem that was first introduced by Viglietta [6] as a way to guard 3D polyhedra. This work was built on by Benbernou et al. [7] and Tóth et al. [5]. The latter studied open edge guards and proved that there are polygons that need at least  $\lfloor \frac{n}{3} \rfloor$  guards to be covered, and that  $\lfloor \frac{n}{2} \rfloor$  are always sufficient.

This article is structured in the following way. Section 1 introduces the concept of open edge guards and presents results on the number of guards that cover orthogonal and spiral polygons. Some results related to the Fortress problem on simple and orthogonal polygons are also presented. The paper concludes with Section 2.

## 1 Open edge guards

Given a simple polygon  $P$ ,  $\mathcal{G}_{OE}(P)$  is the minimum number of open edge guards that fully cover  $P$  and let  $\mathcal{G}_{OE}(n) := \min\{\mathcal{G}_{OE}(P) : P \text{ is a polygon of } n \text{ vertices}\}$ . Consequently, this section is devoted to calculate  $\mathcal{G}_{OE}(n)$  for different classes of polygons.

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## 1.1 Orthogonal polygons

Bjorling-Sachs [1] proved that  $\lfloor \frac{3n+4}{16} \rfloor$  closed edge guards are sufficient and sometimes necessary to fully cover an orthogonal polygon. This section shows that  $\lfloor \frac{n}{4} \rfloor$  open edge guards are sometimes necessary and always sufficient to fully cover an orthogonal polygon.

Given an orthogonal polygon  $P$  with  $n$  vertices, the edges of  $P$  can be divided into four categories: north, south, west and east edges. North edges see the interior of the polygon below them, south edges see it above them, east edges see it to their left and west edges see it to their right. Each of these four sets represents a group of open edge guards that completely covers  $P$ . In order to see this, choose any point  $p$  of  $P$ . From  $p$ , it is always possible to draw vertical segments that will hit a north edge if it goes up from  $p$  and a south edge if it goes down. The reasoning for the horizontal directions is similar. Therefore, the smallest of these four sets of edges proves the upper bound: any orthogonal polygon can be covered by  $\lfloor \frac{n}{4} \rfloor$  open edge guards. Furthermore, there is an example of an orthogonal polygon that needs  $\lfloor \frac{n}{4} \rfloor$  open edge guards to be fully covered. Such polygon is very similar to the one depicted in Figure 2, but where each spike only hides one point since there are no holes. These two bounds prove the following theorem.

**Theorem 1** *Any orthogonal polygon with  $n$  vertices can be covered by  $\lfloor \frac{n}{4} \rfloor$  open edge guards, and in some cases this number is necessary.*

Observe that this result essentially holds for orthogonal polygons with holes, and the upper bound can be obtained in the same way it was explained above. In Figure 2 there is an example of a polygon with holes that needs  $\lfloor \frac{n}{4} \rfloor - 1$  open edge guards to be fully covered, since each marked point is seen by a different open edge guard.

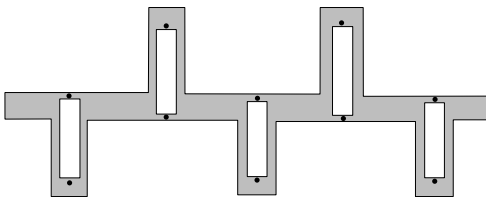


Figure 2: A polygon with holes that needs  $\lfloor \frac{n}{4} \rfloor - 1$  open edge guards to cover it,  $n = 44$ .

## 1.2 Spiral polygons

This section studies open edge guarding of spiral polygons, which will also be called spirals when it eases the reading of the text. According to a previous work,  $\lfloor \frac{n+2}{5} \rfloor$  closed edge guards are sufficient and sometimes necessary to cover spiral polygons [8].

### 1.2.1 Tight bound on the number of open edge guards

In the example in Figure 3(a), each point marked on the polygon needs a different open edge guard to cover it. Since this spiral has only one possible triangulation and there is one of these points per four triangles, this polygon needs  $\lceil \frac{n-2}{4} \rceil$  open edge guards in order to be fully covered. Therefore,  $\mathcal{G}_{OE}(n) \geq \lceil \frac{n-2}{4} \rceil$ . This lower bound can be rewritten as  $\lfloor \frac{n+1}{4} \rfloor$ , and it is proven below that this bound is tight.

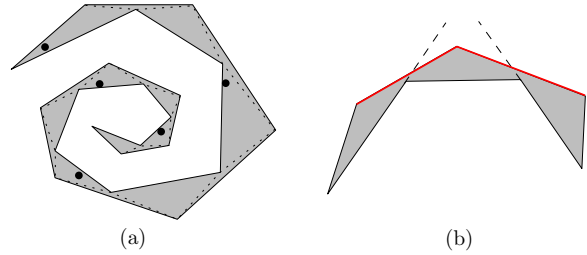


Figure 3: (a) No two of the marked points can be covered by the same open edge guard. (b) The two red open edge guards cover the whole spiral.

The boundary of each spiral can be decomposed into a sequence of consecutive reflex vertices (called a reflex chain) and a sequence of consecutive convex vertices (called a convex chain). The proof that any spiral can be covered by  $\lfloor \frac{n+1}{4} \rfloor$  open edge guards uses induction on the number of edges of the polygon.

The base case comprehends four cases. When the spiral has four, five or six edges it is easy to see that one open edge guard covers the whole polygon. Spirals with seven edges can be covered by two open edge guards, since it suffices to place the guards on the edges of the convex chain that are intersected by the extensions of the first and last edges of the reflex chain (see Figure 3(b)).

For the inductive step, suppose  $\lfloor \frac{n+1}{4} \rfloor$  open edge guards are sufficient to cover any spiral of  $n'$  vertices with  $n' < n$  edges,  $n > 7$ . Let  $P$  be a spiral with  $n > 7$  vertices, whose reflex chain is formed by the vertices  $\{r_1, r_2, \dots, r_k\}$ . Now extend the edge  $\overline{r_1 r_2}$  until it intersects some edge of the convex chain. Let  $v$  be the rightmost endpoint of the edge of the convex chain just intersected as shown in Figure 4(a). The proof is now divided into four cases depending on the number of edges of the convex chain from  $c_1$  to  $v$ : (a) five or more than five edges, (b) four edges, (c) three edges and (d) two edges. For case (a), suppose the convex chain from  $c_1$  to  $v$  has at least five edges. Then draw the diagonal between  $r_1$  and  $c_5$ , the fifth vertex of the convex chain. In this way, the spiral is partitioned into two spiral polygons:  $P'$  that has six edges and so can be guarded with one open edge guard and  $P''$  with  $n - 4$  edges. For case (b), suppose the convex chain from  $c_1$  to  $v$  has four edges as shown

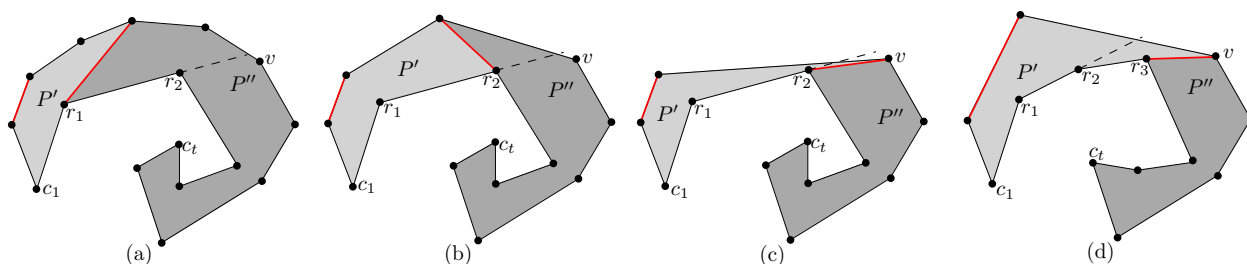


Figure 4: (a) The convex chain from  $c_1$  to  $v$  is formed by six edges. (b) The convex chain from  $c_1$  to  $v$  is formed by four edges. (c) The convex chain from  $c_1$  to  $v$  is formed by three edges: the first reflex vertex visible from  $v$  is  $r_2$ ; (d) the first reflex vertex visible from  $v$  is  $r_3$ .

in Figure 4(b). Then draw the diagonal between  $r_2$  and  $c_4$ , the fourth vertex of the convex chain. This partitions the spiral into two spiral polygons:  $P'$  that has six edges and therefore can be guarded with one open edge guard and  $P''$  with  $n - 4$  edges.

In case (c) the convex chain from  $c_1$  to  $v$  has three edges and the situation is slightly different. As shown in Figures 4(c) and 4(d), draw the diagonal between  $v$  and the first visible reflex vertex starting from  $r_2$  (note that  $r_2$  can be such a vertex). This procedure partitions the spiral into two polygons:  $P'$  that can be guarded with one open edge guard and  $P''$  with at most  $n - 4$  edges.

Finally, case (d) in which the convex chain from  $c_1$  to  $v$  has only two edges. In this case, draw the diagonal from  $v$  to the first visible reflex vertex after  $r_2$ . If there are no visible reflex vertices left, then the reflex chain is over and an open edge guard placed on the second edge of the convex chain covers the whole spiral (see Figure 5(a)). If there is one visible reflex vertex then draw the diagonal as before, which will partition the spiral into two polygons:  $P'$  that can be guarded with one open edge guard and  $P''$  with at most  $n - 4$  edges (see Figure 5(b)).

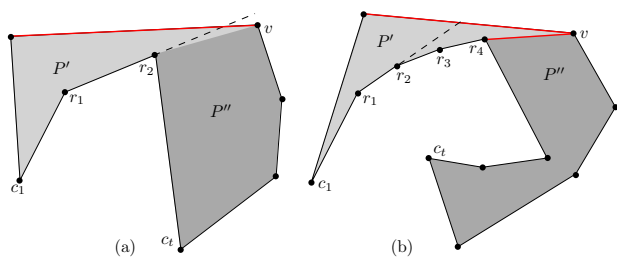


Figure 5: The convex chain from  $c_1$  to  $v$  has two edges. (a) There is no reflex vertex visible from  $v$  besides  $r_2$ . (b) The first reflex vertex visible from  $v$  is  $r_4$ .

All the four cases described above end with a polygon  $P''$  that has at most  $n - 4$  edges, which means the inductive hypothesis can be applied. Therefore,  $P''$  can be covered by  $\lfloor \frac{(n-4)+1}{4} \rfloor = \lfloor \frac{n+1}{4} \rfloor - 1$  open edge guards. Since polygon  $P'$  is covered exactly by one open edge guard, the whole spiral is covered by

$\lfloor \frac{n+1}{4} \rfloor$  open edge guards and this concludes the proof.

**Theorem 2** Any spiral polygon with  $n$  vertices can be covered by  $\lfloor \frac{n+1}{4} \rfloor$  open edge guards, and in some cases this number is necessary.

### 1.2.2 Placing the minimum number of open edge guards

This section presents an algorithm to place the minimum number of open edge guards that cover a spiral polygon  $P$ . The main idea of the algorithm is to build two sets simultaneously:  $G$ , which is the set of open edge guards, and  $H = \{h_1, h_2, \dots\}$  that is the set of points that guarantees  $G$  is of minimum size. The points that form set  $H$  are placed on the polygon in such a way that each open edge guard can see only one of them and is therefore associated with it. Consequently,  $|G| = |H|$ . Let  $\{r_1, r_2, \dots, r_k\}$  be the reflex chain and  $\{c_1, c_2, \dots, c_{n-k}\}$  the convex chain of  $P$ . Moreover, let  $\{c_1, c_2, \dots, c_{n-k}, r_k, r_{k-1}, \dots, r_1\}$  be the sequence of  $n$  vertices of a spiral polygon  $P$ . The steps of the algorithm to place the minimum number of open edge guards to cover  $P$  are depicted in Figure 6 and explained in the following.

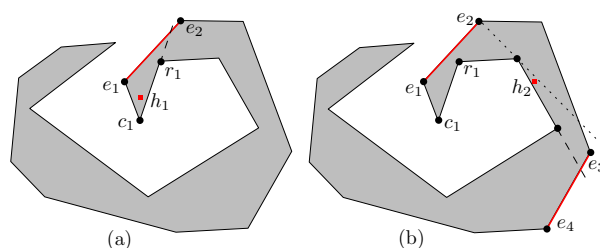


Figure 6: (a) Finding an edge of the convex chain that covers point  $h_1$ . (b) Finding an edge of the convex chain that covers point  $h_2$ .

Set  $G$  is empty to start with. Let  $h_1 \in P$  be a point very close to  $c_1$ , which has to be covered. Draw the ray  $\overrightarrow{c_1 r_1}$  that will intersect some edge of the convex chain that sees point  $h_1$ . Let such edge be denoted by  $(e_1, e_2)$  and assign  $G \leftarrow \{(e_1, e_2)\}$ . Secondly, find

the last reflex vertex  $r_j$  that can be seen from  $e_2$  and consider point  $h_2$ , which is very close to  $r_j$  along the edge  $(r_j, r_{j+1})$ . Then draw the ray  $\overrightarrow{r_j r_{j+1}}$  that will intersect some edge of the convex chain that sees point  $h_2$ . Let such an edge be denoted by  $(e_3, e_4)$  and assign  $G \leftarrow G \cup \{(e_3, e_4)\}$ . Repeat this last step until all reflex vertices and  $c_{n-k}$  are guarded.

This algorithm selects the edges  $\overline{e_j e_{j+1}}$  of the convex chain that will be part of set  $G$ , which fully covers any spiral polygon since it totally covers its convex chain.

**Lemma 3** *The algorithm described in this section builds a set  $H$  of points interior to  $P$  such that  $\mathcal{G}_{OE}(P) \geq |H|$ .*

**Theorem 4** *The algorithm described in this section places the minimum number of open edge guards needed to cover a spiral polygon in  $\mathcal{O}(n)$  time.*

**Proof.** Let  $G$  be the set of open edge guards built by the algorithm to cover spiral polygon  $P$  and  $H$  the set of points of  $P$  in which each point is covered by a different guard. That is, the set of points that are placed in such way that the According to Lemma 3,  $\mathcal{G}_{OE}(P) \geq |H|$  but since  $|H| = |G|$  then  $\mathcal{G}_{OE}(P) \geq |G|$  and therefore  $G$  is a minimum set of open edge guards. Regarding the time complexity, each edge of the convex chain is only processed once whilst analysing the rays  $\overrightarrow{r_j r_{j+1}}$ . In the same way, each edge of the reflex chain is checked once to find the last reflex vertex that is visible from the chosen edges. Consequently, each vertex of the spiral polygon is analysed just once by the algorithm and therefore it runs in linear time.  $\square$

### 1.3 Fortress problem

This section is devoted to another variation of the Art Gallery problem called the *Fortress Problem*. Instead of guarding the interior of a simple polygon, the Fortress Problem variation focuses on monitoring the exterior of a polygon. Choi et al. [2] proved that the exterior of any simple polygon can be covered by  $\lceil \frac{n}{3} \rceil$  closed edge guards and that these guards are necessary to cover the exterior of convex polygons. In the case of open edge guards, this problem is trivial since it is easy to see that every edge will be needed as a guard to cover the exterior of a convex polygon.

The natural following step is to study orthogonal polygons. Again, Choi et al. [2] proved that the exterior of any orthogonal polygon can be covered by  $\lfloor \frac{n}{4} \rfloor + 1$  edge guards and that this number can be necessary. The proof of the following theorem is omitted, but it is based on the technique of dividing the edges according to their orientation, as introduced in Section 1.1. The lower bound is given by the orthoconvex polygon depicted in Figure 7.

**Theorem 5** *The exterior of any orthogonal polygon with  $n$  vertices can be covered by  $\frac{n}{2} + 2$  open edge guards, and in some cases this number is necessary.*

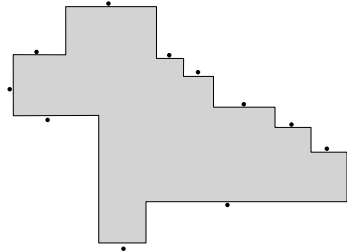


Figure 7: Orthoconvex polygon that needs  $\frac{n}{2} + 2$  open edge guards to be covered.

## 2 Final remarks

We studied other classes of polygons, such as monotone polygons, as well as other geometric configurations. There are monotone polygons that need  $\lfloor \frac{n}{3} \rfloor$  open edge guards in order to be fully covered and we believe this bound is tight. Furthermore, this bound is proved for open mobile guards, which can patrol edges and diagonals of a polygon. This type of coverage has also been studied for other polygons.

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