Some recent results concerning the theoretical and numerical controllability of PDEs

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Outline

CONTROL PROBLEMS

What is usual: analysis and (numerical) resolution of

$$\begin{cases} E(U) = F \\ + \dots \end{cases}$$

Beyond: control, i.e. acting to get good (or the best) results . . .

What is easier? Solving? Controlling?

OPTIMAL CONTROL

The (general) optimal control problem; an Euler's sentence: "Everything in the world obeys to a maximum or minimum principle"

Minimize $J(v, y)$
Subject to $v \in \mathcal{V}_{ad}, \ \ y \in \mathcal{Y}_{ad}, \ (v, y)$ satisfies (S)

with

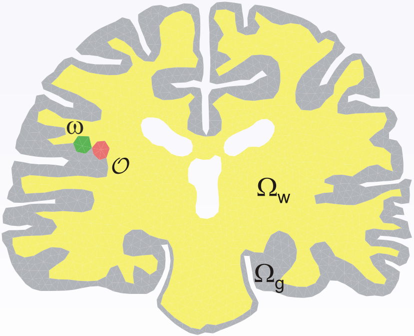
$$E(y) = F(v) \quad + \ \dots \qquad (S)$$

Main questions: $\exists$, uniqueness/multiplicity, characterization, computation, ...

MODELLING AND OPTIMIZING RADIOTHERAPY STRATEGIES

(glioblastoma, results by R Echevarría and others, 2007)

- Brain $\approx$ a two-dimensional crown section
- 2 subdomains



The state equation (a simplified description of the phenomenon):

$$\left\{\begin{array}{l} c_t - \partial_i(D(x)\partial_i c) = (\rho - v 1_\omega)\, c, \quad (x,t) \in \Omega \times (0,T) \\ c|_{t=0} = c_0, \quad x \in \Omega \\ + \dots \end{array}\right. \qquad (E)$$

$c = c(x,t)$ is the state: a cancer cell population density
$v = v(x,t)$ is the control: a radiotherapy administration dose
Glioblastoma [Murray-Swanson, 90's], $D(x) = D_w$ or D_g (white and grey matters)

The optimal control problem:

$$\left\{\begin{array}{l} \text{Minimize } \ J(v,y) = \frac{1}{2}\int_\Omega |c(x,T)|^2 + \frac{1}{2}\iint_{\omega\times(0,T)} |v|^2 \\ \text{Subject to } \ 0 \le v \le M, \ \iint_{\omega\times(0,T)} v \le R, \ \dots, \ (v,y) \ \text{ satisfies } \ (E) \end{array}\right.$$

CONTROLLABILITY

A null controllability problem

Find (v, y)
Such that $v \in \mathcal{V}_{ad}$, (v, y) satisfies (ES), $y(T) = 0$

with

$$E(y) \equiv y_t + A(y) = F(v) \quad + \ \dots \qquad (ES)$$

Main questions: $\exists$, uniqueness/multiplicity, characterization, computation, ...

FIRST EXAMPLE:
1D heat:

$$(H_1) \qquad \begin{cases} y_t - y_{xx} = v 1_\omega, & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = y(1,t) = 0, & t \in (0,T) \\ y(x,0) = y^0(x), & x \in (0,1) \end{cases}$$

We assume: $\omega = (a,b)$, $0 < a < b < 1$
Null controllability problem: For all y^0 find v such that $y(T) = 0$
NC? Yes, for all ω and T

Applications: Heating and cooling, controlling a population, etc.

A HIERARCHICAL CONTROL PROBLEM

Three controls: one leader, two followers

$$(H)\quad \begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} + v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2}, & (x,t) \in (0,1)\times(0,T) \\ y(0,t) = y(1,t) = 0, \quad t \in (0,T) \\ y(x,0) = y^0(x), \quad x \in (0,1) \end{cases}$$

Different intervals $\mathcal{O}$, $\mathcal{O}_i$

Three objectives:

- Get $y(T) = 0$ — Null controllability
- At the same time, $y \approx y_{i,d}$ in $\mathcal{O}_{i,d} \times (0,T)$, $i = 1,2$, reasonable effort:

$$\text{Minimize} \;\; \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 + \mu_i \iint_{\mathcal{O}_i\times(0,T)} |v_i|^2, \quad i = 1,2$$

Bi-objective optimal control

What can we do?

$$(H) \quad \begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} + v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2}, & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = y(1,t) = 0, \quad t \in (0,T) \\ y(x,0) = y^0(x), \quad x \in (0,1) \end{cases}$$

Goal: drive y to rest and keep y close to $y_{i,d}$ in $\mathcal{O}_i \times (0,T)$ for $i = 1,2$

Many applications:

- Heating: Controlling temperatures
  Various heat sources at different locations
  Heat PDE (linear, semilinear, etc.)
- Tumor growth: Controlling tumor cell densities
  Radiotherapy strategies
  Reaction-diffusion systems (linear, semilinear, etc.), bilinear control
- Fluid mechanics: Controlling fluid velocity fields
  Several mechanical actions
  Stokes, Navier-Stokes or similar
- Finance: Controlling the price of an option
  Several agents at different stock prices, etc.
  Backwards in time heat-like PDE

THE STACKELBERG-NASH STRATEGY

Step 1: f is fixed

$$J_i(v_1, v_2) := \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 + \mu_i \iint_{\mathcal{O}_i\times(0,T)} |v_i|^2, \quad i = 1, 2$$

Find a Nash equilibrium $(v_1(f), v_2(f))$ with $v_i(f) \in L^2(\mathcal{O}_i \times (0,T))$:

$$J_1(v_1(f), v_2(f)) \leq J_1(v_1, v_2(f)) \quad \forall v_1 \in L^2(\mathcal{O}_1 \times (0,T))$$
$$J_2(v_1(f), v_2(f)) \leq J_2(v_1(f), v_2) \quad \forall v_2 \in L^2(\mathcal{O}_2 \times (0,T))$$

Equivalent to:

$$(HN) \quad \begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} - \dfrac{1}{\mu_1}\phi_1 1_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi_2 1_{\mathcal{O}_2} \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_i}, \quad i = 1, 2 \\ \phi_i(0,t) = \phi_i(1,t) = 0, \ y(0,t) = y(1,t) = 0, \quad t \in (0,T) \\ y(x,0) = y^0(x), \ \phi_i(x,T) = 0, \quad x \in (0,1) \end{cases}$$

Then: $v_i(f) = -\frac{1}{\mu_i}\phi_i|_{\mathcal{O}_i\times(0,T)}$ (Pontryagin)

THE STACKELBERG-NASH STRATEGY

Step 2: Find f such that

$$(HSN)_1 \quad \begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} - \dfrac{1}{\mu_1}\phi_1 1_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi_2 1_{\mathcal{O}_2} \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_i}, \quad i = 1,2 \\ \phi_i(0,t) = \phi_i(1,t) = 0,\ y(0,t) = y(1,t) = 0, \quad t \in (0,T) \\ y(x,0) = y^0(x),\ \phi_i(x,T) = 0, \quad x \in (0,1) \end{cases}$$

$$(HSN)_2 \qquad y(x,T) = 0, \quad x \in (0,1)$$

with $\|f\|_{L^2(\mathcal{O}\times(0,T))} \leq C\|y^0\|_{L^2}$

For instance, for $y_{i.d} \equiv 0$, equivalent to:
$R(L) \hookrightarrow R(M)$, with $Ly^0 := y(\cdot\,,T)$, $Mf := y(\cdot\,,T)\dots$
In turn, equivalent to: $\|L^*\psi^T\| \leq \|M^*\psi^T\| \quad \forall \psi^T \in L^2(0,1)$
(classical, functional analysis; [Russell, 1973])

Theorem [Araruna-EFC-Santos]

Assume: $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$, $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, large μ_i

$\exists \hat\rho$ such that, if $\iint_{\mathcal{O}_d \times (0,T)} \hat\rho^2 |y_{i,d}|^2 \, dx\, dt < +\infty, \ i = 1, 2$, then:

$\forall y^0 \in L^2(\Omega)$ $\exists$ null controls $f \in L^2(\mathcal{O} \times (0,T))$ & Nash pairs $(v_1(f), v_2(f))$

Idea of the proof:

1 - Large $\mu_i \ \Rightarrow \ \forall f \in L^2(\mathcal{O} \times (0,T))$ $\exists!$ Nash equilibrium $(v_1(f), v_2(f))$

2 - $\|L^*\psi^T\| \leq \|M^*\psi^T\| \quad \forall \psi^T \in L^2(0,1)$ means observability:

$$\|\psi|_{t=0}\|^2 + \sum_{i=1}^{2} \iint_Q \hat\rho^{-2} |\gamma^i|^2 \, dx\, dt \leq C \iint_{\mathcal{O}\times(0,T)} |\psi|^2 \, dx\, dt$$

for all ψ^T, with

$$\begin{cases} -\psi_t - \psi_{xx} = \sum_{i=1}^2 \alpha_i \gamma^i 1_{\mathcal{O}_d}, \quad \gamma^i_t - \gamma^i_{xx} = -\frac{1}{\mu_i} \psi 1_{\mathcal{O}_i} \\ \psi|_{t=T} = \psi^T(x), \quad \gamma^i|_{t=0} = 0, \quad \text{etc.} \end{cases}$$

Observability $\Leftarrow$ Carleman estimates for ψ, γ^i

$$\iint_Q \rho^{-2} |\psi|^2 \, dx\, dt + \sum_{i=1}^{2} \iint_Q \hat\rho^{-2} |\gamma^i|^2 \, dx\, dt \leq C \iint_{\mathcal{O}\times(0,T)} \rho^{-2} |\psi|^2 \, dx\, dt$$

EXTENSIONS

- More followers, coefficients, non-scalar parabolic systems, other functionals, boundary controls, higher dimensions, etc.
- Semilinear systems, for instance:

$$\left\{ \begin{array}{l} y_t - y_{xx} = F(x,t;y) + f 1_{\mathcal{O}} + \sum_{i=1}^m v_i 1_{\mathcal{O}_i} \\ y(0,t) = y(1,t) = 0, \quad t \in (0,T), \ \text{etc.} \end{array} \right.$$

  OK for Lipschitz-continuous F

- Constraints, for instance:

$$\left\{ \begin{array}{l} y_t - y_{xx} = f 1_{\mathcal{O}} + \sum_{i=1}^m v_i 1_{\mathcal{O}_i} \\ y(0,t) = y(1,t) = 0, \quad t \in (0,T), \ \text{etc.} \end{array} \right.$$

  Find a constrained Nash equilibrium $(v_1(f), v_2(f))$ with $v_i(f) \in \mathcal{U}_{i,ad} \subset L^2(\mathcal{O}_i \times (0,T))$:

$$J_1(v_1(f), v_2(f)) \leq J_1(v_1, v_2(f)) \quad \forall v_1 \in \mathcal{U}_{1,ad}$$
$$J_2(v_1(f), v_2(f)) \leq J_2(v_1(f), v_2) \quad \forall v_2 \in \mathcal{U}_{2,ad}$$

  Then, find f such that $y|_{t=T} = 0$
  OK for local constraints, i.e. $\mathcal{U}_{i,ad} = \{ v_i \in L^2(\mathcal{O}_i \times (0,T)) : v_i(x,t) \in L_i \}$

MORE COMMENTS:

- Previous work: [Guillén et al. 2013]
- The previous proof $\rightarrow$ a method to compute f and $(v_1(f), v_2(f))$
- $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$?
- Other strategies? Stackelberg-Pareto controllability?
- Numerical results?

In progress . . .

CONTROLLING TURBULENCE (I)

The Leray-α model - distributed controls:

$$\left\{ \begin{array}{l} y_t + (z \cdot \nabla) y - \nu_0 \Delta y + \nabla p = v 1_\omega, \quad \nabla \cdot y = 0 \\ z - \alpha^2 \Delta z + \nabla \pi = y, \quad \nabla \cdot z = 0 \\ y(x,t) = z(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y^0(x) \end{array} \right.$$

AC? NC? ECT? OPEN

Fluid regimes: Laminar or turbulent
[Reynolds 1895], [Kolmogorov 1941], [Batchelor 1953]

Main characteristics of turbulence:
- Fast variations in space and time, wide range of length scales (eddy motion)
- Well behavior of (appropriately) averaged variables

Typically: small (resp. large) Re $:= UL/\nu \Rightarrow$ laminar (resp. turbulent) flow

Turbulent flows in waves and tornados

Turbulent smoke rings

To understand something on turbulence: [Schlichting 1968], [Temam 1988], [Lesieur 1997], [Matthieu-Scott 2000]

Turbulence modelling

1 - Start from Navier-Stokes:

$$y_t + (y \cdot \nabla)y - \nu_0 \Delta y + \nabla p = f, \ \ \nabla \cdot y = 0$$

2 - Averages:

$$y = \overline{y} + y', \quad p = \overline{p} + p'$$

For instance, $\overline{y}(x,t) := \lim_{\varepsilon \to 0^+} \iint_{|(x',t')-(x,t)| \leq \varepsilon} y(x',t')\, dx\, dt$

Reynolds (PDE's for $\overline{y}$ and $\overline{p}$?):

$$\overline{y}_t + \nabla \cdot (\overline{y \otimes y}) - \nu_0 \Delta \overline{y} + \nabla \overline{p} = \overline{f}, \ \ \nabla \cdot \overline{y} = 0$$

3 - Closure hypotheses: assumptions relating $\overline{y \otimes y}$ and $\overline{y}$

Reynolds:

$$\overline{y}_t + \nabla\cdot(\overline{y\otimes y}) - \nu_0\Delta\overline{y} + \nabla\overline{p} = \overline{f},\quad \nabla\cdot\overline{y} = 0$$

A particular closure hypothesis:

$$\overline{y\otimes y} \approx z_\alpha\otimes\overline{y},\ \text{with}\ \ z_\alpha = (\mathbf{Id.} + \alpha^2 A)^{-1}\overline{y},\quad \alpha\to 0^+$$

Leray-α model:

$$\begin{cases} \overline{y}_t + (z_\alpha\cdot\nabla)\overline{y} - \nu_0\Delta\overline{y} + \nabla p = \overline{f},\quad \nabla\cdot\overline{y} = 0 \\ z_\alpha - \alpha^2\Delta z_\alpha + \nabla\pi_\alpha = \overline{y},\quad \nabla\cdot z_\alpha = 0 \end{cases}$$

[Leray 1934], [Cheskidov-Holm-Olson-Titi 2005]

The significance of controlling a turbulence model:

$$\overline{y}_t + \nabla \cdot S - \nu_0 \Delta \overline{y} + \nabla \overline{p} = v 1_\omega, \quad \nabla \cdot \overline{y} = 0$$

with $S = S(\overline{y}(\cdot\,,\cdot))$ (an approximation of $\overline{y \otimes y}$)

- We control averaged states
- With averages depending on α, are controls uniformly bounded?
  Do averaged controls converge?
  If yes: controlling the Navier-Stokes system in the limit

Navier-Stokes:

$$\begin{cases} y_t + (y \cdot \nabla) y - \nu_0 \Delta y + \nabla p = v 1_\omega, \quad \nabla \cdot y = 0 \\ y(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y^0(x) \end{cases}$$

AC? NC? ECT? OPEN
What we know: Local ECT

Theorem [EFC-Guerrero-Imnuvilov-Puel 2004]

Fix a solution $(\overline{y}, \overline{p})$, with $\overline{y} \in L^\infty$
$\exists \varepsilon > 0$ such that $\|y^0 - \overline{y}(0)\|_{H_0^1} \leq \varepsilon \Rightarrow \exists$ controls such that $y(T) = \overline{y}(T)$

For the proof:

1. Reduce ECT to NC, (NC) $\cong$ "$F(y,v) = 0$" in an appropriate space
2. Then: apply Liusternik's Theorem (linearized at zero is NC)

Other results, among them:
- Global AC for when $N = 2$, Navier boundary conditions [Coron 1996]
- Global NC with periodicity [Fursikov-Imanuvilov 1999], without boundary [Coron-Fursikov 1996], ...

The Leray-α model - distributed controls:

$$\begin{cases} y_t + (z\cdot\nabla)y - \nu_0\Delta y + \nabla p = v1_\omega, & \nabla\cdot y = 0 \\ z - \alpha^2\Delta z + \nabla\pi = y, & \nabla\cdot z = 0 \\ y(x,t) = z(x,t) = 0, & (x,t)\in\partial\Omega\times(0,T) \\ y(x,0) = y^0(x) \end{cases}$$

AC? NC? ECT? OPEN
What we know: local NC, controls converge as $\alpha\to 0^+$:

Theorem [Araruna, EFC, Souza 2014]

$\exists\varepsilon>0$ such that $y^0\in H$, $\|y^0\|_{L^2}\le\varepsilon\Rightarrow\exists$ controls v_α such that $y(T)=0$
Furthermore, $\|v_\alpha\|_{L^2}\le C$

$$H = \{w\in L^2(\Omega)^N : \nabla\cdot w = 0 \text{ in } \Omega,\ w\cdot n = 0 \text{ on } \partial\Omega\}$$

Idea of the proof (I):

Lemma (regularizing effect)

$\exists \phi = \phi(s) > 0$, with $\phi(s) \to 0$ as $s \to 0^+$:

a) $\exists$ arbitrarily small $t^* \in (0, T/2)$ with $\|y(t^*)\|_{D(A)} \leq \phi(\|y_0\|_{L^2})$

b) The set of these t^* has positive measure

This lemma $\Rightarrow$ we can assume that $\|y_0\|_{D(A)} << 1$

Idea of the proof (II):

- Fixed-Point formulation:

$$\begin{cases} z - \alpha^2 \Delta z + \nabla \pi = \bar{y}, \quad \nabla \cdot z = 0 \\ \quad \text{i.e.} \quad z = (\mathbf{Id.} + \alpha^2 A)^{-1} \bar{y} \\ y_t + (z \cdot \nabla) y - \nu_0 \Delta y + \nabla p = v 1_\omega, \quad \nabla \cdot y = 0, \quad \text{etc.} \end{cases}$$

- $\bar{y} \in L^\infty(0, T; D(A^{s/2}))$, $s > N/2 \Rightarrow z \in L^\infty$ and NC for Oseen uniformly
- $\|v_\alpha\|_{L^\infty(L^2)} \leq C, \ \forall \alpha > 0$
- $y \in$ compact set of $L^\infty(0, T; D(A^{s/2}))$
- $\|y_0\|_{H^2}$ small $\Rightarrow \|y\|_{L^\infty(0,T;D(A^{s/2}))} \leq C$ if $\|\bar{y}\|_{L^\infty(0,T;D(A^{s/2}))} \leq C$

Assume $y^0 \in H$, $\|y^0\|_{L^2} \leq \varepsilon$

$$\begin{cases} y_{\alpha\,t} + (z_\alpha \cdot \nabla) y_\alpha - \nu_0 \Delta y_\alpha + \nabla p = v_\alpha 1_\omega, \quad \nabla \cdot y_\alpha = 0 \\ z_\alpha - \alpha^2 \Delta z_\alpha + \nabla \pi_\alpha = y_\alpha, \quad \nabla \cdot z = 0 \\ y_\alpha(x,t) = z_\alpha(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \\ y_\alpha(x,0) = y^0(x), \quad y_\alpha(x,T) = 0 \end{cases}$$

Then, at least for a subsequence

- $v_\alpha \to v$ weakly in $L^2(\omega \times (0,T))$
- $y_\alpha \to y$ and $z_\alpha \to y$ strongly in $L^2(\Omega \times (0,T))$ etc.

$$\begin{cases} y_t + (y \cdot \nabla) y - \nu_0 \Delta y + \nabla p = v 1_\omega, \quad \nabla \cdot y = 0 \\ y(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y^0(x), \quad y(x,T) = 0 \end{cases}$$

The Leray-α model - boundary controls:
More natural, but how?

The good boundary control problem:

$$\begin{cases} y_t + (z\cdot\nabla)y - \nu_0\Delta y + \nabla p = 0, \quad \nabla\cdot y = 0 \\ z - \alpha^2\Delta z + \nabla\pi = y, \quad \nabla\cdot z = 0 \\ y(x,t) = z(x,t) = h1_\gamma, \quad (x,t)\in\partial\Omega\times(0,T) \\ y(x,0) = y^0(x) \end{cases}$$

Again, AC, NC, ECT are open and we get uniform local NC:

Theorem [Araruna, EFC, Souza 2014]

$\exists\varepsilon > 0$ such that $y^0 \in V$, $\|y^0\|_{H_0^1} \le \varepsilon \Rightarrow \exists h_\alpha$ with $\int_\gamma h_\alpha\cdot n\,d\Gamma = 0$, $y(T) = 0$
Furthermore, $\|h_\alpha\|_{L^\infty(0,T;H^{1/2}(\gamma))} \le C$

$$V = \{w \in H_0^1(\Omega)^N : \nabla\cdot w = 0 \ \text{ in } \ \Omega\}$$

Idea of the proof (I): An auxiliary extension $\widetilde{\Omega}$, a fictitious ω

Lemma (modified regularizing effect)

$\exists \psi = \psi(s) > 0$, with $\psi(s) \to 0$ as $s \to 0^+$:

a) $\exists T_0 \in (0, T)$, $h_\alpha \in L^\infty(0, T_0; H^{1/2}(\gamma))$, $(y_\alpha, p_\alpha, z_\alpha, \pi_\alpha)$ and arbitrarily small t^* such that
y_α can be extended to $\widetilde{\Omega} \times (0, T_0)$, with $\|\tilde{y}_\alpha(t^*)\|_{D(\widetilde{A})} \le \psi(\|y_0\|_{H_0^1})$

b) The set of these t^* has positive measure

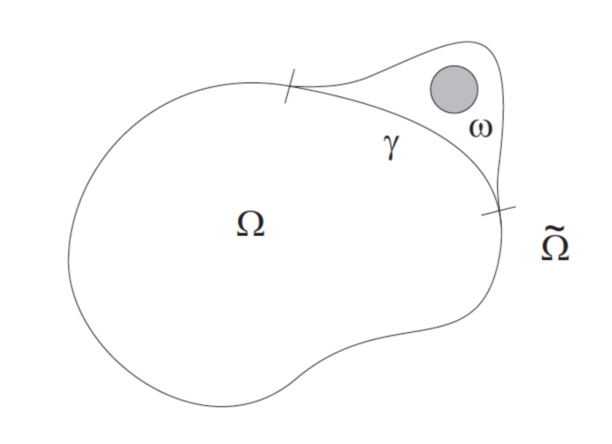
c) h_α is uniformly bounded in $L^\infty(0, T_0; H^{1/2}(\gamma))$

This lemma $\Rightarrow$ we can work in $\widetilde{\Omega} \times (0, T)$ assuming $\|\tilde{y}_0\|_{D(\widetilde{A})} << 1$

Idea of the proof (II): Solve

$$
\begin{cases}
\tilde{y}_t + (\tilde{z} \cdot \nabla)\tilde{y} - \nu_0 \Delta \tilde{y} + \nabla \tilde{p} = v 1_\omega, \ \ \nabla \cdot \tilde{y} = 0, & \widetilde{\Omega} \times (0, T) \\
z - \alpha^2 \Delta z + \nabla \pi = \tilde{y}, \ \ \nabla \cdot z = 0, & \Omega \times (0, T) \\
\tilde{y}(x, t) = 0, & \partial\widetilde{\Omega} \times (0, T) \\
z(x, t) = \tilde{y}(x, t), & \partial\Omega \times (0, T) \\
\tilde{y}(x, 0) = \tilde{y}^0(x), \ \ \tilde{y}(x, T) = 0, & \widetilde{\Omega}
\end{cases}
$$

Again: Fixed-Point argument works ...



The extended domain and the fictitious control region

Assume $y^0 \in V$, $\|y^0\|_{H^1_0} \leq \varepsilon$

$$\left\{ \begin{array}{l} y_{\alpha\,t} + (z_\alpha \cdot \nabla) y_\alpha - \nu_0 \Delta y_\alpha + \nabla p = 0, \;\; \nabla \cdot y_\alpha = 0 \\ z_\alpha - \alpha^2 \Delta z_\alpha + \nabla \pi_\alpha = y_\alpha, \;\; \nabla \cdot z_\alpha = 0 \\ y_\alpha(x,t) = z_\alpha(x,t) = h_\alpha 1_\gamma, \quad (x,t) \in \partial\Omega \times (0,T) \\ y_\alpha(x,0) = y^0(x), \quad y_\alpha(x,T) = 0 \end{array} \right.$$

Then, at least for a subsequence

- $h_\alpha \to h$ weakly-$*$ in $L^\infty(0,T;H^{1/2}(\gamma))$
- $y_\alpha \to y$ and $z_\alpha \to y$ strongly in $L^2(\Omega \times (0,T))$ etc.

$$\left\{ \begin{array}{l} y_t + (y \cdot \nabla) y - \nu_0 \Delta y + \nabla p = 0, \;\; \nabla \cdot y = 0 \\ y(x,t) = z(x,t) = h 1_\gamma, \quad (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y^0(x), \quad y(x,T) = 0 \end{array} \right.$$

Simplified models: the Burgers and Burgers-α systems
$L > 0$, $T > 0$

Burgers:

$$\begin{cases} y_t - \nu_0 y_{xx} + y y_x = f, & (x,t) \in (0,L) \times (0,T) \\ y(0,\cdot) = y(L,\cdot) = 0, & t \in (0,T) \\ y(\cdot,0) = y_0, & x \in (0,L) \end{cases}$$

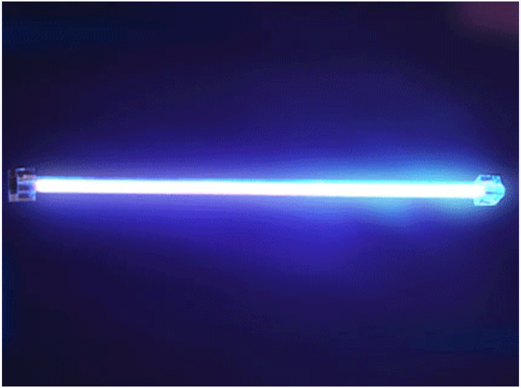
Burgers-α:

$$\begin{cases} y_t - \nu_0 y_{xx} + z_\alpha y_x = f, & (x,t) \in (0,L) \times (0,T) \\ z_\alpha - \alpha^2 (z_\alpha)_{xx} = y, & (x,t) \in (0,L) \times (0,T) \\ y(0,\cdot) = y(L,\cdot) = z_\alpha(0,\cdot) = z_\alpha(L,\cdot) = 0, & t \in (0,T) \\ y(\cdot,0) = y_0, & x \in (0,L) \end{cases}$$

Motivations:

- A “toy model” for Leray-α
- Applications to the description of 1D motion

Similar results



1D motion in a neon tube

Traffic motion

For small y_0, again:

- NC
- $\|v_\alpha\|_{L^\infty(\omega\times(0,T))}$ is uniformly bounded

Remarks:

- Comparison (maximum) principle, easier to get z_α bounded in L^∞
- Burgers is not globally NC.
  Therefore: for large y^0, at most, $\|v_\alpha\|_{L^\infty(\omega\times(0,T))}$ is unbounded

CONTROLLING TURBULENCE (II)

The Ladyzhenskaya-Smagorinsky model:
Coming back to turbulence modelling - Reynolds:

$$\overline{y}_t + \nabla \cdot (\overline{y \otimes y}) - \nu_0 \Delta \overline{y} + \nabla \overline{p} = \overline{f}, \quad \nabla \cdot \overline{y} = 0$$

How to relate $\overline{y \otimes y}$ and $\overline{y}$?
Boussinesq-like closure hypotheses:

$$\overline{y \otimes y} \approx \overline{y} \otimes \overline{y} - R, \text{ with } \; R = \nu_T(\nabla \overline{y}(\cdot\,,\cdot)) D\overline{y}$$

R is the Reynolds tensor, ν_T is the turbulent viscosity
[Launder-Spalding 1972], [Cebeci-Smith 1974]

A simple assumption: $\nu_T = \nu_1(\|\nabla \overline{y}(\cdot\,, t)\|^2)$

$$\overline{y}_t + (\overline{y} \cdot \nabla)\overline{y} - \nu(\int_\Omega |\nabla \overline{y}|^2)\Delta \overline{y} + \nabla \overline{p} = \overline{y}, \quad \nabla \cdot \overline{y} = 0$$

[Ladyzhenskaya 1961], [Smagorinsky 1963]

The Ladyzhenskaya-Smagorinsky model:

$$\begin{cases} y_t + (y \cdot \nabla) y - \nu(\int_\Omega |\nabla y|^2) \Delta y + \nabla p = v 1_\omega, \quad \nabla \cdot y = 0 \\ y(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y^0(x) \end{cases}$$

We assume: $\nu_T \in C_b^1$, $\nu_T \geq \nu_0 > 0$
AC? NC? ECT? OPEN - What we know: local NC

Theorem [EFC-Limaco-Menezes 2014]

$\exists \varepsilon > 0$ such that $\|y^0\|_{L^2} \leq \varepsilon \Rightarrow \exists$ null controls

Arguments similar to those for Navier-Stokes:

1. Rewrite NC in the form (NC) $\cong$ "$F(y,v) = 0$" in an appropriate space X
   Key point: Choose X to have
   - $F : X \mapsto Z$ well defined and C^1 (small)
   - $F'(0,0) \in \mathcal{L}(X;Z)$ onto (large)
2. Then: apply Liusternik's Theorem (linearized at zero is Stokes, NC)

Attention: local ECT is also open!

ADDITIONAL COMMENTS:

- Many open questions remain:
  - Other similar α-models (LANS-α, Cannasa-Holm model, etc.). NC?
  - Global control results?
  - Reducing the number of controls? Specially difficult in the boundary case!
  - Control results of other kinds? In particular, Lagrangian controllability? [Glass-Horsin 2010 . . .]
- Numerical analysis and convergence results for these and other problems: in progress . . .

Similar results for nonlinear-nonlocal parabolic systems:

$$(NN) \quad \left\{ \begin{array}{l} y_t - a(\int_\Omega y, \int_\Omega z)\Delta y = f(y,z) + v 1_\omega, \quad (x,t) \in \Omega \times (0,T) \\ z_t - b(\int_\Omega y, \int_\Omega z)\Delta y = g(y,z), \quad (x,t) \in \Omega \times (0,T) \\ y(x,t) = z(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y^0(x), \;\; z(x,0) = z^0(x), \quad x \in \Omega \end{array} \right.$$

Several difficulties, mainly:

- Nonlinear a, b, f, g
- Only one control

[EFC-Límaco-Menezes 2013]

Applications: Controlling reacting media, interacting populations, among others

An experiment, nonlinear-nonlocal system:

$$(NN) \quad \begin{cases} y_t - a(\int_\Omega y, \int_\Omega z)\Delta y = f(y,z) + v 1_\omega, & (x,t) \in \Omega \times (0,T) \\ z_t - b(\int_\Omega y, \int_\Omega z)\Delta y = g(y,z), & (x,t) \in \Omega \times (0,T) \\ y(x,t) = z(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y^0(x), \ \ z(x,0) = z^0(x), & x \in \Omega \end{cases}$$

$a, b, f, g \in C^1_b$, $a \geq a_0 > 0$, $b \geq b_0 > 0$, $\partial_y g(0,0) \neq 0$

$\Omega = (0,1)$, $\omega = (0.2, 0.8)$, $T = 0.5$, $y_0(x) \equiv \sin(\pi x)$, $z_0(x) \equiv \sin(2\pi x)$,
$f \equiv A_1(1 + \sin y)\, y + B_1(1 + \sin z)\, z$, $g \equiv A_2(1 + \sin y)\, y + B_2(1 + \sin z)\, z$
$a \equiv a_0(1 + (1 + r^2 + s^2)^{-1})$, $b \equiv b_0(1 + (1 + r^2 + s^2)^{-1})$.

Formulation $F(y,z,v) = 0$ + Quasi-Newton method — Only $F'(0,0,0)$!
Convergence is ensured

At every step: NC for a linear parabolic system (1 control)

Approximation: P_1 in (x,t) + multipliers (mixed formulation), C^0 in (x,t)
freeFem++ & mesh adaptation

MESH ADAPTATION

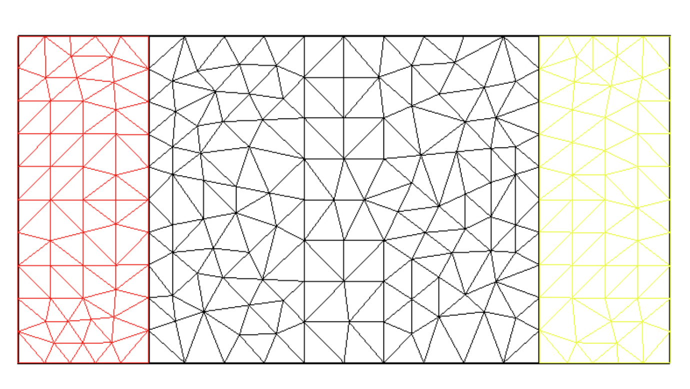


Figure: The initial mesh. Number of vertices: 232. Number of triangles: 402. Total number of unknowns: $6 \times 232 = 1392$.

MESH ADAPTATION

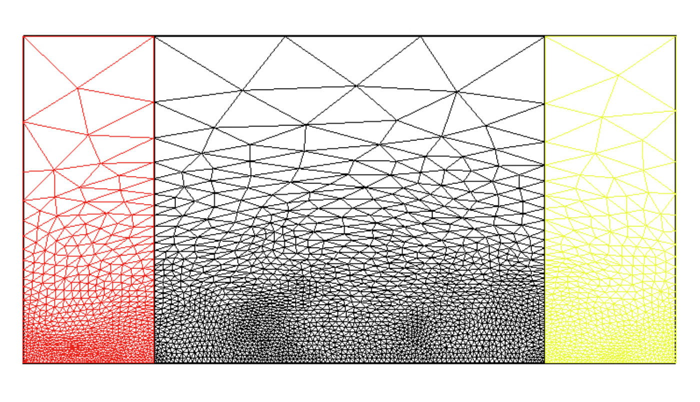


Figure: The final adapted mesh. Number of vertices: 2903. Number of triangles: 5594. Total number of unknowns: $6 \times 2903 = 17418$.

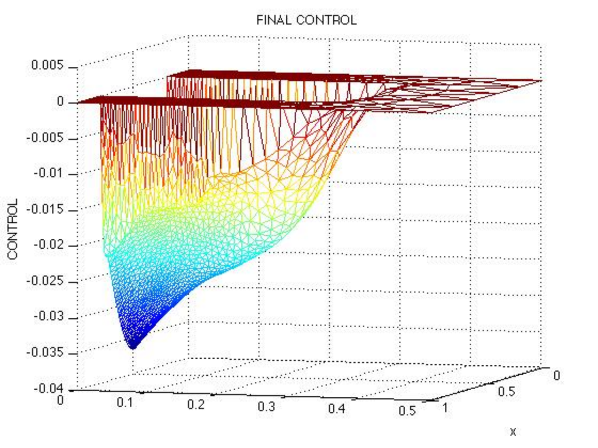


Figure: The computed null control.

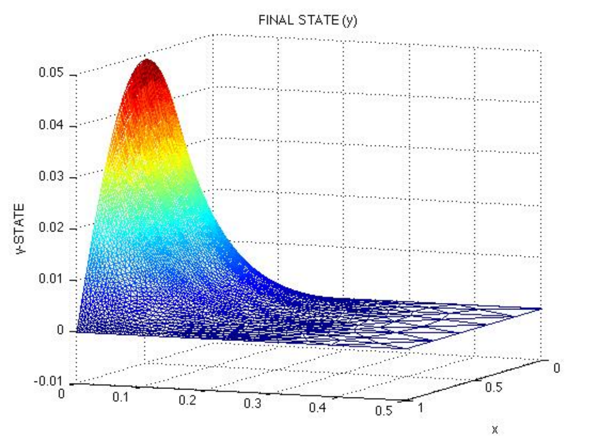


Figure: The computed state y.

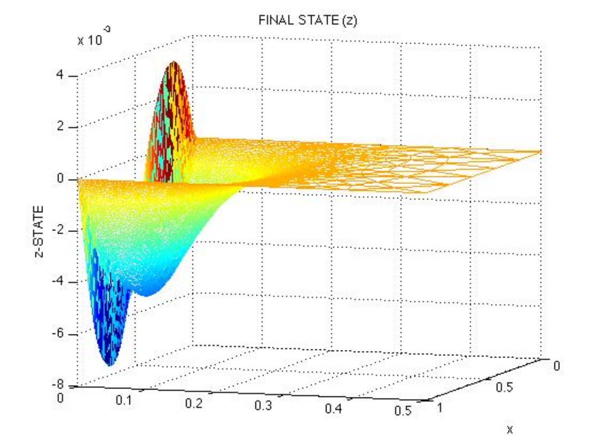


Figure: The computed state z.

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THANK YOU VERY MUCH . . .