Classifications of evolution algebras over finite fields

Óscar Falcón

(Joint work with Raúl Falcón and Juan Núñez)

Department of Geometry and Topology. University of Seville.

Seville. January 19, 2016.

CONTENTS

Preliminaries

- Evolution algebras.
- Isotopisms of algebras.
- Algebraic Geometry.

Preliminaries

- Evolution algebras.
- Isotopisms of algebras.
- Algebraic Geometry.

Evolution algebras.

Jianjun Paul Tian, 2004

An n-dimensional algebra E over a field $\mathbb{K}$ is said to be an evolution algebra if it admits a basis $\{e_1, \ldots, e_n\}$ such that

(1) $e_i e_j = 0$ if $i \neq j$,
(2) $e_i e_i = \sum_{j \leq n} a_{ij} e_j$, for some $a_{i1}, \ldots, a_{in} \in \mathbb{K}$.

$A = (a_{ij}) \equiv$ Matrix of structure constants.

Evolution algebras.

Jianjun Paul Tian, 2004

An n-dimensional algebra E over a field $\mathbb{K}$ is said to be an evolution algebra if it admits a basis $\{e_1, \ldots, e_n\}$ such that

(1) $e_i e_j = 0$ if $i \neq j$,
(2) $e_i e_i = \sum_{j \leq n} a_{ij} e_j$, for some $a_{i1}, \ldots, a_{in} \in \mathbb{K}$.

$A = (a_{ij}) \equiv$ Matrix of structure constants.

Applications: Non-Mendelian Genetic, Dynamic Systems, Markov Processes, Theory of Knots, Graph Theory and Group Theory.

- Nilpotency and solvability $\Rightarrow$ Disappearance of population in evolution processes.

References (Tian's Web Page):
https://www.math.nmsu.edu/∼jtian/e-algebra/e-alg-index.htm

Jianjun Paul Tian, 2004

An n-dimensional algebra E over a field $\mathbb{K}$ is said to be an evolution algebra if it admits a basis $\{e_1, \ldots, e_n\}$ such that

(1) $e_i e_j = 0$ if $i \neq j$,
(2) $e_i e_i = \sum_{j \leq n} a_{ij} e_j$, for some $a_{i1}, \ldots, a_{in} \in \mathbb{K}$.

$A = (a_{ij}) \equiv$ Matrix of structure constants.

Evolution algebras.

Jianjun Paul Tian, 2004

An n-dimensional algebra E over a field $\mathbb{K}$ is said to be an evolution algebra if it admits a basis $\{e_1, \ldots, e_n\}$ such that

(1) $e_i e_j = 0$ if $i \neq j$,
(2) $e_i e_i = \sum_{j \leq n} a_{ij} e_j$, for some $a_{i1}, \ldots, a_{in} \in \mathbb{K}$.

$A = (a_{ij}) \equiv$ Matrix of structure constants.

Research problem: Distribution of evolution algebras over finite fields into isomorphism and **isotopism** classes according to the matrices of structure constants.

- **Tool**: Computational Algebraic Geometry.

Evolution algebras.

Theorem (Casas et al., 2014)

Every non-zero 2-dimensional complex evolution algebra is isomorphic to exactly one evolution algebra related to one of the next matrix of structure constants

$$E_1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 : \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_3 : \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$E_4 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{5_{a,b}} : \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}, \quad E_{6_c} : \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix},$$

where $ab \neq 1$, $c \neq 0$ *and*

- $E_{5_{a,b}} \cong E_{5_{b,a}}$.
- $E_{6_c} \cong E_{6_{c'}} \Leftrightarrow \frac{c}{c'} = \cos\frac{2k\pi}{3} + i\sin 2k\pi 3$, *for some* $k \in \{0, 1, 2\}$.

Preliminaries

- Evolution algebras.
- Isotopisms of algebras.
- Algebraic Geometry.

Abraham Adrian Albert

1905-1972

Two algebras $\mathfrak{a}$ and $\mathfrak{a}'$ are **isotopic** ($\simeq$) if there exist three regular linear transformations f, g and h from $\mathfrak{a}$ to $\mathfrak{a}'$ such that

$$f(u)g(v) = h(uv), \text{ for all } u, v \in \mathfrak{a}.$$

- The triple (f, g, h) is an isotopism between $\mathfrak{a}$ and $\mathfrak{a}'$.
- To be isotopic is an equivalence relation among algebras.
- $f = g = h \Rightarrow$ Isomorphism ($\cong$) of algebras.

Literature: Division algebras (Albert, Benkart, Bruck, Dieterich, Petersson, Sandler), **Lie algebras** (Falcón, Núñez, Jiménez), **Jordan algebras** (McCrimmon, Oehmke, Petersson, Ple, Thakur), **Alternative algebras** (Babikov, McCrimmon), **Absolute valued algebras** (Albert, Cuenca), **Structural algebras** (Allison).

Preliminaries

- Evolution algebras.
- Isotopisms of algebras.
- Algebraic Geometry.

Let $\mathbb{F}_p[\underline{x}]$ be the ring of polynomials in $\underline{x} = \{x_1, \ldots, x_n\}$ over the finite field $\mathbb{F}_p$.

Let $\mathbb{F}_p[\underline{x}]$ be the ring of polynomials in $\underline{x} = \{x_1, \ldots, x_n\}$ over the finite field $\mathbb{F}_p$.

- A term order $<$ on the set of monomials of $\mathbb{F}_p[\underline{x}]$ is a multiplicative well-ordering that has the constant monomial 1 as its smallest element.

Algebraic Geometry

Let $\mathbb{F}_p[\underline{x}]$ be the ring of polynomials in $\underline{x} = \{x_1, \ldots, x_n\}$ over the finite field $\mathbb{F}_p$.

- A term order $<$ on the set of monomials of $\mathbb{F}_p[\underline{x}]$ is a multiplicative well-ordering that has the constant monomial 1 as its smallest element.
- The largest monomial of a polynomial in $\mathbb{F}_p[\underline{x}]$ with respect to the term order $<$ is its leading monomial.

Algebraic Geometry

Let $\mathbb{F}_p[\underline{x}]$ be the ring of polynomials in $\underline{x} = \{x_1, \ldots, x_n\}$ over the finite field $\mathbb{F}_p$.

- A term order $<$ on the set of monomials of $\mathbb{F}_p[\underline{x}]$ is a multiplicative well-ordering that has the constant monomial 1 as its smallest element.
- The largest monomial of a polynomial in $\mathbb{F}_p[\underline{x}]$ with respect to the term order $<$ is its leading monomial.
- The ideal generated by the leading monomials of all the non-zero elements of an ideal is its initial ideal.

Let $\mathbb{F}_p[\underline{x}]$ be the ring of polynomials in $\underline{x} = \{x_1, \ldots, x_n\}$ over the finite field $\mathbb{F}_p$.

- A term order $<$ on the set of monomials of $\mathbb{F}_p[\underline{x}]$ is a multiplicative well-ordering that has the constant monomial 1 as its smallest element.
- The largest monomial of a polynomial in $\mathbb{F}_p[\underline{x}]$ with respect to the term order $<$ is its leading monomial.
- The ideal generated by the leading monomials of all the non-zero elements of an ideal is its initial ideal.
- Those monomials of polynomials in the ideal that are not leading monomials are called standard monomials.

Algebraic Geometry

Let $\mathbb{F}_p[\underline{x}]$ be the ring of polynomials in $\underline{x} = \{x_1, \ldots, x_n\}$ over the finite field $\mathbb{F}_p$.

- A term order $<$ on the set of monomials of $\mathbb{F}_p[\underline{x}]$ is a multiplicative well-ordering that has the constant monomial 1 as its smallest element.
- The largest monomial of a polynomial in $\mathbb{F}_p[\underline{x}]$ with respect to the term order $<$ is its leading monomial.
- The ideal generated by the leading monomials of all the non-zero elements of an ideal is its initial ideal.
- Those monomials of polynomials in the ideal that are not leading monomials are called standard monomials.
- A Gröbner basis of an ideal I is any subset G of polynomials in I whose leading monomials generate the initial ideal.

Algebraic Geometry

Let $\mathbb{F}_p[\underline{x}]$ be the ring of polynomials in $\underline{x} = \{x_1, \ldots, x_n\}$ over the finite field $\mathbb{F}_p$.

- A term order $<$ on the set of monomials of $\mathbb{F}_p[\underline{x}]$ is a multiplicative well-ordering that has the constant monomial 1 as its smallest element.
- The largest monomial of a polynomial in $\mathbb{F}_p[\underline{x}]$ with respect to the term order $<$ is its leading monomial.
- The ideal generated by the leading monomials of all the non-zero elements of an ideal is its initial ideal.
- Those monomials of polynomials in the ideal that are not leading monomials are called standard monomials.
- A Gröbner basis of an ideal I is any subset G of polynomials in I whose leading monomials generate the initial ideal.
- It is reduced if all its polynomials are monic and no monomial of a polynomial in G is generated by the leading monomials.

Let I be an ideal in $\mathbb{F}_p[\underline{x}]$.

Algebraic Geometry

Let I be an ideal in $\mathbb{F}_p[\underline{x}]$.

- The algebraic set defined by I is the set

$$\mathcal{V}(I) = \{\underline{a} \in \mathbb{F}_p^n \colon\ f(\underline{a}) = 0 \text{ for all } f \in I\}.$$

Algebraic Geometry

Let I be an ideal in $\mathbb{F}_p[\underline{x}]$.

- The algebraic set defined by I is the set

$$\mathcal{V}(I) = \{\underline{a} \in \mathbb{F}_p^n \colon \ f(\underline{a}) = 0 \text{ for all } f \in I\}.$$

- I is zero-dimensional if $\mathcal{V}(I)$ is finite. In particular,

$$|\mathcal{V}(I)| \leq \dim_{\mathbb{F}_p} \mathbb{F}_p[\underline{x}]/I.$$

Let I be an ideal in $\mathbb{F}_p[\underline{x}]$.

- The algebraic set defined by I is the set

$$\mathcal{V}(I) = \{\underline{a} \in \mathbb{F}_p^n : \ f(\underline{a}) = 0 \text{ for all } f \in I\}.$$

- I is zero-dimensional if $\mathcal{V}(I)$ is finite. In particular,

$$|\mathcal{V}(I)| \leq \dim_{\mathbb{F}_p} \mathbb{F}_p[\underline{x}]/I.$$

- I is radical if

$$\{f^m \in I \Rightarrow f \in I\}, \text{ for all } f \in \mathbb{F}_p[\underline{x}] \text{ and } m \in \mathbb{N}.$$

Algebraic Geometry

Let I be an ideal in $\mathbb{F}_p[\underline{x}]$.

- The algebraic set defined by I is the set

$$\mathcal{V}(I) = \{\underline{a} \in \mathbb{F}_p^n : \; f(\underline{a}) = 0 \text{ for all } f \in I\}.$$

- I is zero-dimensional if $\mathcal{V}(I)$ is finite. In particular,

$$|\mathcal{V}(I)| \leq \dim_{\mathbb{F}_p} \mathbb{F}_p[\underline{x}]/I.$$

- I is radical if

$$\{f^m \in I \Rightarrow f \in I\}, \text{ for all } f \in \mathbb{F}_p[\underline{x}] \text{ and } m \in \mathbb{N}.$$

Theorem

If I is zero-dimensional and radical, then

$$|\mathcal{V}(I)| = \dim_{\mathbb{F}_p} \mathbb{F}_p[\underline{x}]/I$$

and coincides with the number of standard monomials of I.

Reduced Gröbner bases play a fundamental role in the computation of $|\mathcal{V}(I)|$.

Theorem (Lakshman and Lazard, 1991)

The complexity of computing the reduced Gröbner basis of a zero-dimensional ideal is $d^{O(n)}$, where

- *d is the maximal degree of the polynomials of the ideal.*
- *n is the number of variables.*

Isotopisms of evolution algebras.

Lemma

Let E and E' be two isotopic evolution algebras of respective matrices of structure constants $A = (a_{ij})$ and $A' = (a'_{ij})$. Let (f, g, h) be an isotopism between both algebras related, respectively, to the matrices $F = (f_{ij})$, $G = (g_{ij})$ and $H = (h_{ij})$. Then,

a) $\sum_{j\leq n} f_{ij} g_{ij} a'_{jk} = \sum_{j\leq n} a_{ij} h_{jk}$ *for all* $i, k \leq n$.

b) $\sum_{k\leq n} f_{ik} g_{jk} a'_{kl} = 0$, *for all* $i, j, l \leq n$.

Lemma

Let E and E' be two isotopic evolution algebras of respective matrices of structure constants $A = (a_{ij})$ and $A' = (a'_{ij})$. Let (f, g, h) be an isotopism between both algebras related, respectively, to the matrices $F = (f_{ij})$, $G = (g_{ij})$ and $H = (h_{ij})$. Then,

a) $\sum_{j \leq n} f_{ij} g_{ij} a'_{jk} = \sum_{j \leq n} a_{ij} h_{jk}$ *for all* $i, k \leq n$.

b) $\sum_{k \leq n} f_{ik} g_{jk} a'_{kl} = 0$, *for all* $i, j, l \leq n$.

Proof.

a) Let $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ be respective bases of E and E'. Let $i \leq n$. Then,

$$\sum_{j,k \leq n} f_{ij} g_{ij} a'_{jk} e'_k = \sum_{j \leq n} f_{ij} e'_j \cdot \sum_{j \leq n} g_{ij} e'_j = f(e_i) g(e_i) =$$

$$= h(e_i e_i) = h\left(\sum_{j \leq n} a_{ij} e_j\right) = \sum_{j,k \leq n} a_{ij} h_{jk} e'_k$$

□

Lemma

Let E and E' be two isotopic evolution algebras of respective matrices of structure constants $A = (a_{ij})$ and $A' = (a'_{ij})$. Let (f, g, h) be an isotopism between both algebras related, respectively, to the matrices $F = (f_{ij})$, $G = (g_{ij})$ and $H = (h_{ij})$. Then,

a) $\sum_{j \leq n} f_{ij} g_{ij} a'_{jk} = \sum_{j \leq n} a_{ij} h_{jk}$ *for all* $i, k \leq n$.

b) $\sum_{k \leq n} f_{ik} g_{jk} a'_{kl} = 0$, *for all* $i, j, l \leq n$.

Proof.

b) Let $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ be respective bases of E and E'. Let $i, j \leq n$ be such that $i \neq j$. Then,

$$\sum_{k,l \leq n} f_{ik} g_{jk} a'_{kl} e'_l = \sum_{k \leq n} f_{ik} e'_k \cdot \sum_{k \leq n} g_{jk} e'_k = f(e_i) g(e_j) =$$

$$= h(e_i e_j) = 0 \qquad \square$$

Let E be an evolution algebra. The annihilator of E is

$$\mathrm{Ann}(E) = \{x \in E | xE = 0\}.$$

Lemma

Let E and E' be two isotopic evolution algebras and let (f, g, h) be an isotopism between them. Then,

$$f(\mathrm{Ann}(E)) = g(\mathrm{Ann}(E)) = \mathrm{Ann}(E').$$

Let E be an evolution algebra. The annihilator of E is

$$\mathrm{Ann}(E) = \{x \in E | xE = 0\}.$$

Lemma

Let E and E' be two isotopic evolution algebras and let (f, g, h) be an isotopism between them. Then,

$$f(\mathrm{Ann}(E)) = g(\mathrm{Ann}(E)) = \mathrm{Ann}(E').$$

Proof.

Let $x \in \mathrm{Ann}(E)$. Then,

$$f(x)E' = f(x)g(E) = h(xE) = h(0) = 0.$$

Thus, $f(\mathrm{Ann}(E)) \subseteq \mathrm{Ann}(E')$. The reciprocal holds similarly. The identity with g also holds analogously. $\square$

2-dimensional complex evolution algebras

$$E_1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 : \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_3 : \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$E_4 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \underset{ab \neq 1}{E_{5_{a,b}}} : \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}, \qquad \underset{c \neq 0}{E_{6_c}} : \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}.$$

Proposition

There are three isotopism classes in the set of 2-dimensional complex evolution algebras:

a) $E_1 \simeq E_4$.

b) $E_2 \simeq E_3$.

c) $E_{5_{a,b}} \simeq E_{6_c}$, *for all* a, b, c.

2-dimensional complex evolution algebras

$$E_1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 : \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_3 : \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$E_4 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \underset{ab \neq 1}{E_{5_{a,b}}} : \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}, \qquad \underset{c \neq 0}{E_{6_c}} : \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}.$$

Proposition

There are three isotopism classes in the set of 2-dimensional complex evolution algebras:

a) $E_1 \simeq E_4$.

b) $E_2 \simeq E_3$.

c) $E_{5_{a,b}} \simeq E_{6_c}$, *for all* a, b, c.

Proof.

a) $F = G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

2-dimensional complex evolution algebras

$$E_1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 : \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_3 : \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$E_4 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \underset{ab \neq 1}{E_{5_{a,b}}} : \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}, \qquad \underset{c \neq 0}{E_{6_c}} : \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}.$$

Proposition

There are three isotopism classes in the set of 2-dimensional complex evolution algebras:

a) $E_1 \simeq E_4$.

b) $E_2 \simeq E_3$.

c) $E_{5_{a,b}} \simeq E_{6_c}$, *for all* a, b, c.

Proof.

b) $F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$

2-dimensional complex evolution algebras

$$E_1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 : \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_3 : \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$E_4 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \underset{ab \neq 1}{E_{5_{a,b}}} : \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}, \qquad \underset{c \neq 0}{E_{6_c}} : \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}.$$

Proposition

There are three isotopism classes in the set of 2-dimensional complex evolution algebras:

a) $E_1 \simeq E_4$.

b) $E_2 \simeq E_3$.

c) $E_{5_{a,b}} \simeq E_{6_c}$, *for all* a, b, c.

Proof.

c) $F = G = \begin{pmatrix} \sqrt{c-a} & 1 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} \frac{1}{1-ab} & 0 \\ \frac{b}{ab-1} & 1 \end{pmatrix}.$

$$E_1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 : \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_3 : \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$E_4 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \underset{ab \neq 1}{E_{5_{a,b}}} : \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}, \qquad \underset{c \neq 0}{E_{6_c}} : \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}.$$

Proposition

There are three isotopism classes in the set of 2-dimensional complex evolution algebras:

a) $E_1 \simeq E_4$.

b) $E_2 \simeq E_3$.

c) $E_{5_{a,b}} \simeq E_{6_c}$, *for all* a, b, c.

Proof.

d) $E_2 \not\simeq E_1 \not\simeq E_5$.

$$\mathrm{Ann}(E_1) = \langle\, e_2 \,\rangle, \qquad \mathrm{Ann}(E_2) = \mathrm{Ann}(E_5) = \emptyset.$$

2-dimensional complex evolution algebras

$$E_1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 : \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_3 : \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$E_4 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \underset{ab \neq 1}{E_{5_{a,b}}} : \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}, \qquad \underset{c \neq 0}{E_{6_c}} : \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}.$$

Proposition

There are three isotopism classes in the set of 2-dimensional complex evolution algebras:

a) $E_1 \simeq E_4$.

b) $E_2 \simeq E_3$.

c) $E_{5_{a,b}} \simeq E_{6_c}$, *for all* a, b, c.

Proof. d) $E_2 \not\simeq E_{6_c}$.

$$\begin{cases} f_{12}g_{22} = f_{22}g_{12} = f_{11}g_{21} = f_{21}g_{11} = 0, \\ h_{11} = f_{12}g_{12} = f_{22}g_{22}, \\ h_{12} = f_{11}g_{11} = f_{21}g_{21} \end{cases} \Rightarrow h_{11} = h_{12} = 0 \Rightarrow |H| = 0\,!!!$$

Procedures

- Field: $\mathbb{K}$.
- Sets of n^2 variables:

$$\mathfrak{A}_n = \{\mathfrak{a}_{ij} \colon\ i, j \leq n\},$$
$$\mathfrak{F}_n = \{\mathfrak{f}_{ij} \colon\ i, j \leq n\},$$
$$\mathfrak{G}_n = \{\mathfrak{g}_{ij} \colon\ i, j \leq n\},$$
$$\mathfrak{H}_n = \{\mathfrak{h}_{ij} \colon\ i, j \leq n\}.$$

- Multivariate polynomial rings:

$$\mathbb{K}[\mathfrak{A}_n \cup \mathfrak{F}_n] \qquad \text{and} \qquad \mathbb{K}[\mathfrak{A}_n \cup \mathfrak{F}_n \cup \mathfrak{G}_n \cup \mathfrak{H}_n].$$

- Matrices:

$$F = (\mathfrak{f}_{ij}), \qquad G = (\mathfrak{g}_{ij}), \qquad H = (\mathfrak{h}_{ij}).$$

- Let $\mathfrak{E}_n^{\mathbb{K}}$ be the n-dimensional algebra with basis $\beta_n = \{e_1, \ldots, e_n\}$ such that

$$e_i e_i = \sum_{k=1}^{n} \mathfrak{a}_{ij} e_j, \text{ for all } i, j \leq n.$$

Isomorphisms between two n-dimensional evolution algebras over $\mathbb{F}_p$.

procedure $\textsc{Isom}(n, p, A, A')$
  for $i \leftarrow 1, n$ **do**
    for $k \leftarrow 1, n$ **do**
      $I = I + (\mathfrak{f}_{ik}^p - \mathfrak{f}_{ik}) + (\mathfrak{a}_{ik}^p - \mathfrak{a}_{ik}) + (\mathfrak{a}'^p_{ik} - \mathfrak{a}_{ik});$
      $\mathrm{pol}_1 = 0$
      for $j \leftarrow 1, n$ **do**
        $\mathrm{pol}_1 = \mathrm{pol}_1 + (\mathfrak{f}_{ij}^2 \mathfrak{a}'_{jk} - \mathfrak{a}_{ij} \mathfrak{f}_{jk});$
        $\mathrm{pol}_2 = 0$
        for $l \leftarrow 1, n$ **do**
          $\mathrm{pol}_2 = \mathrm{pol}_2 + \mathfrak{f}_{il} \mathfrak{f}_{jl} \mathfrak{a}'_{lk}$
        end for
        $I = I + \mathrm{pol}_2;$
      end for
      $I = I + \mathrm{pol}_1;$
    end for
  end for
  for $i \leftarrow 1, \mathrm{size}(A)$ **do**
    $I = I + (\mathfrak{a}_{A[i][1]A[i][2]} - A[i][3]);$
  end for
  for $i \leftarrow 1, \mathrm{size}(A')$ **do**
    $I = I + (\mathfrak{a}'_{A'[i][1]A'[i][2]} - A'[i][3]);$
  end for
  $I = I + (det(F)^{p-1} - 1);$
  $I = \text{Gröbner}(I);$
  return $|\mathcal{V}(I)|;$
end procedure

Isotopisms between two n-dimensional evolution algebras over $\mathbb{F}_p$.

procedure $\textsc{Isot}(n, p, A, A')$
 for $i \leftarrow 1, n$ **do**
  for $k \leftarrow 1, n$ **do**
   $I = I + (\mathfrak{f}_{ik}^p - \mathfrak{f}_{ik}) + (\mathfrak{g}_{ik}^p - \mathfrak{g}_{ik}) + (\mathfrak{h}_{ik}^p - \mathfrak{h}_{ik}) + (\mathfrak{a}_{ik}^p - \mathfrak{a}_{ik}) + (\mathfrak{a}'^p_{ik} - \mathfrak{a}_{ik})$;
   $\mathrm{pol}_1 = 0$
   for $j \leftarrow 1, n$ **do**
    $\mathrm{pol}_1 = \mathrm{pol}_1 + (\mathfrak{f}_{ij}\mathfrak{g}_{ij}\mathfrak{a}'_{jk} - \mathfrak{a}_{ij}\mathfrak{h}_{jk})$;
    $\mathrm{pol}_2 = 0$
    for $l \leftarrow 1, n$ **do**
     $\mathrm{pol}_2 = \mathrm{pol}_2 + \mathfrak{f}_{il}\mathfrak{g}_{jl}\mathfrak{a}'_{lk}$
    end for
    $I = I + \mathrm{pol}_2$;
   end for
   $I = I + \mathrm{pol}_1$;
  end for
 end for
 for $i \leftarrow 1, \mathrm{size}(A)$ **do**
  $I = I + (\mathfrak{a}_{A[i][1]A[i][2]} - A[i][3])$;
 end for
 for $i \leftarrow 1, \mathrm{size}(A')$ **do**
  $I = I + (\mathfrak{a}'_{A'[i][1]A'[i][2]} - A'[i][3])$;
 end for
 $I = I + (det(F)^{p-1} - 1) + (det(G)^{p-1} - 1) + (det(H)^{p-1} - 1)$;
 $I = \text{Gröbner}(I)$;
 return $|\mathcal{V}(I)|$;
end procedure

Finite dimensional evolution algebras.

2-dimensional evolution algebras.

Lemma

Let A and A' be the matrix of structure constants of two isotopic 2-dimensional evolution algebras and let (f, g, h) be an isotopism between them. Let $F = (f_{ij})$ and $G = (g_{ij})$ be the matrices related, respectively, to f and g. If $|A'| \neq 0$, then exactly one of the next two assertions holds.

a) $f_{11} = f_{22} = g_{11} = g_{22} = 0.$

b) $f_{12} = f_{21} = g_{12} = g_{21} = 0.$

Proof. The reduced Gröbner basis of the ideal related to our isotopism contains the next two generators

$$\begin{cases} f_{22} \cdot g_{12} \cdot |A'| = 0, \\ f_{12} \cdot g_{22} \cdot |A'| = 0. \end{cases}$$

Hence, $f_{22} = g_{22} = 0$ or $f_{12} = g_{12} = 0$.

2-dimensional evolution algebras.

Lemma

Let A and A' be the matrix of structure constants of two isotopic 2-dimensional evolution algebras and let (f, g, h) be an isotopism between them. Let $F = (f_{ij})$ and $G = (g_{ij})$ be the matrices related, respectively, to f and g. If $|A'| \neq 0$, then exactly one of the next two assertions holds.

a) $f_{11} = f_{22} = g_{11} = g_{22} = 0$.

b) $f_{12} = f_{21} = g_{12} = g_{21} = 0$.

Proof. In both cases,

$$\begin{cases} a'_{11} f_{11} g_{21} = 0, \\ a'_{12} f_{11} g_{21} = 0, \\ a'_{11} f_{21} g_{11} = 0, \\ a'_{12} f_{21} g_{11} = 0. \end{cases} \Rightarrow \begin{cases} f_{22} = g_{22} = 0 \Rightarrow f_{11} = g_{11} = 0, \\ f_{12} = g_{12} = 0 \Rightarrow f_{21} = g_{21} = 0. \end{cases}$$

□

2-dimensional evolution algebras.

Theorem

There are seven non-zero 2-dimensional evolution algebras over $\mathbb{F}_2$ *up to isomorphisms:*

$$E_1^2 : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2^2 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3^2 : \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad E_4^2 : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$E_5^2 : \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad E_6^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_7^2 : \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Theorem

There are three non-zero 2-dimensional evolution algebras over $\mathbb{F}_2$ *up to isotopisms:*

a) $E_1^2 \simeq E_2^2$.

b) $E_3^2 \simeq E_7^2$.

c) $E_4^2 \simeq E_5^2 \simeq E_6^2$.

2-dimensional evolution algebras.

Theorem

Let $p \in \{3,5\}$. There are six non-zero 2-dimensional evolution algebras over $\mathbb{F}_p$ up to isomorphisms:

$$E_1^p : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2^p : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3^p : \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad E_4^p : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$E_5^p : \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad E_6^p : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorem

Let $p \in \{3,5\}$. There are three non-zero 2-dimensional evolution algebras over $\mathbb{F}_p$ up to isotopisms:

a) $E_1^p \simeq E_2^p$.

b) E_3^p.

c) $E_4^p \simeq E_5^p \simeq E_6^p$.

Theorem

Let $A = (a_{ij})$ be the matrix of structure constants of a 2-dimensional evolution algebra E over a field $\mathbb{K}$. If $|A| \neq 0$, then, E is isotopic to the 2-dimensional evolution algebra over $\mathbb{K}$ of related matrix of structure constants

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Proof. It is enough to consider the isotopism (f, g, h) of matrices

$$F = G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad H = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}.$$

□

References

- Albert, A. A. Non-associative algebras: I. Fundamental Concepts and Isotopy, *Ann. of Math., Second Series* **43** (1942), 685-707.
- Camacho, L. M., Gómez, J. R., Omirov, B. A. and Turdibaev, R. M. Some properties of evolution algebras. *Bull. Korean Math. Soc.* **50**:5 (2013), 1481–1494. DOI: 10.4134/BKMS.2013.50.5.1481.
- Casas, J. M., Ladra, M., Omirov, B. A. and Rozikov, U. A. On Evolution Algebras, *Algebra Colloq.* **21** (2014), 331-342. DOI 10.1142/S1005386714000285.
- Cox DA., Little JB. O'Shea D. *Using Algebraic Geometry*. Springer-Verlag: New York; 1998.
- Cox DA., Little JB. O'Shea D. *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra*. Springer: New York; 2007.
- Etherington I. M. H., Non-associative algebra and the symbolism of genetics, *Proc. Roy. Soc. Edinburgh. Sect. B.* **61** (1941), 24–42.
- Khudoyberdiyev, A. Kh., Omirov, B. A. and Qaralleh, I. Few remarks on evolution algebras, *J. Algebra Appl.* **14**:4 (2015), 1550053, 16 pp. DOI: 10.1142/S021949881550053X.
- Tian, J. P. *Evolution Algebras and their Applications*, Lecture Notes in Mathematics **1921**. Springer-Verlag, Berlin, 2008. DOI: 10.1007/978-3-540-74284-5.
- Tian, J. P. and Vojtechovsky, P. Mathematical concepts of evolution algebras in non-mendelian genetics, *Quasigroups Related Systems* **14**:1 (2006), 111–122.

Many thanks!!

Classifications of evolution algebras over finite fields

Óscar Falcón

(Joint work with Raúl Falcón and Juan Núñez)

Department of Geometry and Topology. University of Seville.

Seville. January 19, 2016.

View publication stats