Ordered Weighted Average Optimization in Multiobjective Spanning Tree Problems

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Outline

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Introduction: Multiobjective optimization and aggregation functions

- Multiobjective combinatorial optimization deals with problems considering more than one viewpoint or scenario.
- The standard solution concept is the set of Pareto solutions (Ehrgott, 2005). However, the number of Pareto solutions can grow exponentially with the size of the instance and the number of objectives.
- More involved decision criteria have been proposed in the field of multicriteria decision making (Perny and Spanjaard 2003). These include objectives focusing on one particular compromise solution.
- The ordered median (OM) objective function is very useful in this context since it assigns importance weights not to specific objectives but to their sorted values. Ordered median objectives have been successfully used for addressing various types of combinatorial problems (Ogryczak and Tamir, 2003; Nickel and Puerto, 2005; Boland et al., 2006).
- When applied to values of different objective functions in multiobjective problems, the OM operator is called in the literature Ordered Weighted Average (OWA) operator (Yager, 1988; Yager, 1997). It assigns importance weights to the sorted values of the objective function elements in a multiple objective optimization problem.

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Problem definition

Definition **Ordered Weighted Average (OWA) operator**

Given

- $Q \subseteq \mathbb{Z}^n$: a combinatorial object (feasible set),
- p linear objective functions. $(P = \{1, \ldots, p\})$
- C^i: coefficients of i-th objective function. $C \in \mathbb{R}^{p \times n}$.
- $y = Cx \in \mathbb{R}^p$: obj. funct. values for $x \in Q$. $y = (y_1, ..., y_p) \in \mathbb{R}^p$.
- σ: permutation of indices of P such that $y_{\sigma_1} \geq \ldots \geq y_{\sigma_p}$.
- $\omega \in \mathbb{R}^p$ weights vector.

the OWA operator is defined as

$$OWA_{(C,\omega)}(x) = \omega' y_\sigma$$

Definition **OWA Problem (OWAP)**

The OWA optimization problem (OWAP) is to find

$$\min_{x \in Q} OWA_{(C,\omega)}(x).$$

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Example: OWA operator

Example. Given $Q = \{x \in \{0,1\}^3 : x_1 + x_2 + x_3 = 2\}$,

$$C = \left(\begin{array}{ccc} 1 & 4 & 1 \\ \hline 1 & 1 & 3 \\ \hline 5 & 1 & 2 \end{array}\right), \omega' = \left(\begin{array}{ccc} 1 & 2 & 4 \end{array}\right)$$

x	$y = Cx$	y_σ	$OWA_{(C,\omega)}(x) = \omega' y_\sigma$
$\left(\begin{array}{ccc} 1 & 1 & 0 \end{array}\right)'$	$\left(\begin{array}{ccc} 5 & 2 & 6 \end{array}\right)'$	$\left(\begin{array}{ccc} 6 & 5 & 2 \end{array}\right)'$	24
$\left(\begin{array}{ccc} 1 & 0 & 1 \end{array}\right)'$	$\left(\begin{array}{ccc} 2 & 4 & 7 \end{array}\right)'$	$\left(\begin{array}{ccc} 7 & 4 & 2 \end{array}\right)'$	23
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Table: Solutions $x \in Q$, values $y = Cx$, sorted values y_σ and $OWA_{(C,\omega)}(x)$.

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Definition **Ordered Weighted Average Spanning Tree Problem (OWASTP)**
Let $\mathcal{T}$ denote the set of spanning trees defined on G. Then, the OWASTP can be defined as

$$\text{OWASTP:} \quad \min_{x \in \mathcal{T}} OWA_{(C,\omega)}(x).$$

Example. Consider the graph $G = (N, E)$ depicted and the 3-cost vectors on E, whose values are represented next to each edge.

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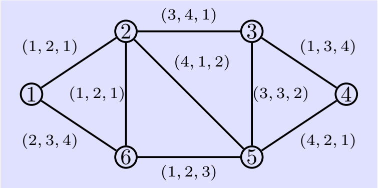

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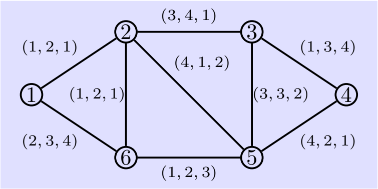

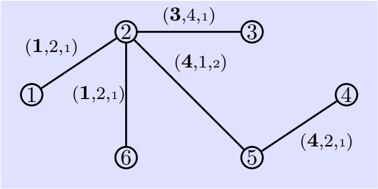

OWASTP solution (value 8.8)
for $\omega' = (0.4, 0, 0.6)$

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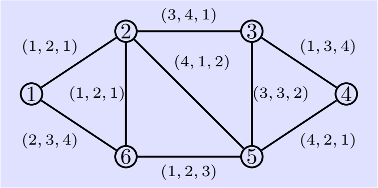

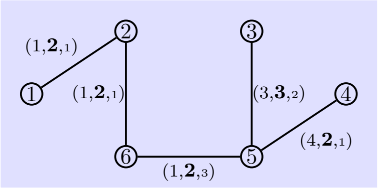

OWASTP solution (value 10.4)
for $\omega' = (0.8, 0, 0.2)$

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Background

STPs

- Kruskal, J. (1956) "On the shortest spanning subtree of a graph and the traveling salesman problem".
- Prim, R. (1957) "Shortest connection networks and some generalizations".
- Edmonds, J. (1970) *Submodular functions, matroids, and certain polyhedra*.
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Multiobjective STPs

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- Kasperski, A. and Zielinski, P. (2013) "Combinatorial optimization problems with uncertain costs and the OWA criterion".
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OWASTP is NP-hard on general graphs

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OWASTP is NP-complete on cactus graphs even when $p = 2$.

Proof. (sketch) The reduction comes from Partition with Disjoint Pairs.

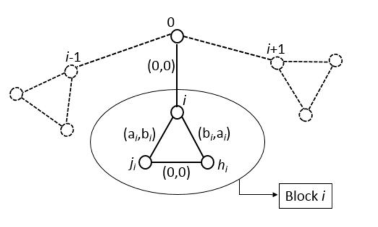

Figure: The Cactus graph used in proof of the NP-completeness claim.

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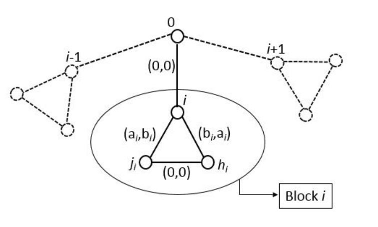

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Formulation: MSTP

Formulation	main constraints	root	# vars	# const.	int
Subtour Edmonds, J. (1970)	$\sum_{e\in E(S)} x_e \leq \|S\|-1,\ \emptyset \neq S \subset V$		$O(\|E\|)$	$Exp(n)$	Y
Kipp Martin Martin (1991)	$\sum_{(u,v)\in\delta^+(u)} q_{kuv} \leq \begin{cases}1, & k\in V, u\in V : u\neq k\\ 0, & k\in V, u=k\end{cases}$	$\forall k$	$O(n\|E\|)$	$O(n\|E\|)$	Y
Miller-Tucker-Zemlim Miller et al. (1960)	$l_v \geq l_u + 1 - n(1-y_{uv}),\ (u,v)\in A$	r	$O(\|E\|)$	$O(\|E\|)$	N
Flow Gavish (1983)	$\sum_{(u,v)\in\delta^+(u)} \varphi_{uv} - \sum_{(v,u)\in\delta^-(u)} \varphi_{vu} = \begin{cases} n-1, & u=r\\ -1, & u\in V\setminus\{r\}\end{cases}$	r	$O(\|E\|)$	$O(\|E\|)$	N
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Corollary. Let $P(\mathcal{T}^{(\cdot)})$ denote the polyhedron associated with the linear programming relaxation of formulation $\mathcal{T}^{(\cdot)}$ and $P_x(\mathcal{T}^{(\cdot)})$ the projected polyhedron associated with formulation $\mathcal{T}^{(\cdot)}$. Then

$$P_x(\mathcal{T}^{sub}) = P_x(\mathcal{T}^{km}) = P_x(\mathcal{T}^{km2}) \subseteq \begin{cases} P_x(\mathcal{T}^{mtz}) \\ \neq \\ P_x(\mathcal{T}^{flow}) \end{cases}$$

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OWA formulation: Decision variables

- z: Defines the permutation (ordering)

$$z_{ij} = \begin{cases} 1 & \text{if cost function } i \text{ occupies position } j \text{ in the permutation,} \\ 0 & \text{otherwise.} \end{cases}$$

Example. (*Permutation*)

$$z = \{z_{ij} : i, j \in P\} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- y_{ij}: real variable equal to the value of the cost function i if it occupies the j-th position in the ordering.
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MILP formulations for the OWAP

Galand and Spanjaard (2012) formulation

$$F^{GS}: \min \sum_{j \in P} \omega_j \sum_{i \in P} y_{ij} \quad (1a)$$

$$s.t. \sum_{i \in P} z_{ij} = 1 \qquad j \in P \quad (1b)$$

$$\sum_{j \in P} z_{ij} = 1 \qquad i \in P \quad (1c)$$

$$\sum_{i \in P} y_{ij} \geq \sum_{i \in P} y_{ij+1} \qquad j \in P : j < p \quad (1d)$$

$$y_{ij} \leq M z_{ij} \qquad i, j \in P \quad (1e)$$

$$\sum_{j \in P} y_{ij} = C^i x \qquad i \in P \quad (1f)$$

$$x \in \mathcal{T} \quad (1g)$$

$$y_{ij} \geq 0 \qquad i, j \in P \quad (1h)$$

$$z \in \{0, 1\}^{p \times p} \quad (1i)$$

Fernández et al. (2014) formulation

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$$C^i x \leq \theta_j + M(1 - \sum_{k \geq j} z_{ik}) \qquad i, j \in P \quad (2d)$$

$$\theta_j \geq \theta_{j+1} \qquad j \in P : j < p \quad (2e)$$

$$x \in \mathcal{T} \quad (2f)$$

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$$\theta_j \geq \theta_{j+1} \qquad j \in P : j < p \quad (4e)$$

$$x \in \mathcal{T} \quad (4f)$$

$$\theta_j \geq 0 \qquad j \in P \quad (4g)$$

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Property. Every optimal solution to F^{GS} is also optimal to F^{θ} and conversely

Property. $\Omega_{LR}^{GS} \subset \Omega_{LR}^{\theta}$.

MILP formulations for the OWAP

Galand and Spanjaard (2012) formulation

$$F^{GS}: \min \sum_{j\in P} \omega_j \sum_{i\in P} y_{ij} \quad (3a)$$

$$s.t. \sum_{i\in P} z_{ij} = 1 \qquad j \in P \quad (3b)$$

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$$\sum_{i\in P} y_{ij} \geq \sum_{i\in P} y_{ij+1} \qquad j \in P : j < p \quad (3d)$$

$$y_{ij} \leq M z_{ij} \qquad i,j \in P \quad (3e)$$

$$\sum_{j\in P} y_{ij} = C^i x \qquad i \in P \quad (3f)$$

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Fernández et al. (2014) formulation

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Enhancements

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Fernández et al. (2014) formulation

$$F^z_{R2}: \min \sum_{j \in P} \omega_j \theta_j \quad (6a)$$

$$s.t. \quad (6b)$$

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$$C^i x \leq \theta_j + M(1 - \sum_{k \geq j} z_{ik}) \; i, j \in P$$

(6d)

(6e)

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$$\sum_{j \in P : j > 1} z_{ij} = 1 \qquad i \in P \quad (6c)$$

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$$(6e)$$

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Enhancements

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Valid inequalities

Constraints related to bounds of cost functions values

- l_i: minimum objective value relative to cost function $i \in P$
- u_i: maximum objective value relative to cost function $i \in P$

$$l_i \leq C^i x \leq u_i \qquad i \in P \qquad (14)$$

Constraints related to bounds of specific positions values

- l_j^π: j-th lowest value of l_i,
- u_j^π: j-th largest value of u_i.

$$l_j^\pi \leq \theta_j \leq u_j^\pi \qquad j \in P \qquad (15)$$

Constraints related to bounds of cost functions in specific positions

$$\sum_{j \in P} \max\{l_i, l_j^\pi\} z_{ij} \leq C^i x \leq \sum_{j \in P} \min\{u_i, u_j^\pi\} z_{ij} \qquad i \in P \qquad (16)$$

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Outline

Computational experience

- Hurwicz criterion: $\alpha \max_{i \in P} y_i + (1-\alpha)\min_{i \in P} y_i$ with $\alpha \in \{0.4, 0.6, 0.8\}$ and $p \in \{5, 8, 10\}$.
- Graph sizes of $|V| \in \{40, 50, 60, 80, 100\}$ nodes for complete graphs.
- For each selection of the parameters $(|V|, p, \alpha)$, 10 instances were randomly generated.
- All instances were solved with the MIP Xpress optimizer, under a Windows 7 environment in an Intel(R) Core(TM)i7 CPU 2.93 GHz processor and 16 GB RAM. A CPU time limit of 3600 seconds was set.

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Computational results 1/4

			F^{GS}				F^{km}				F^{cut}			
$\|P\|$	$\|V\|$	α	$gapLR$	$t/gap(\#)$	t^*/gap^*	nod	$gapLR$	$t/gap(\#)$	t^*/gap^*	nod	$gapLR$	$t/gap(\#)$	t^*/gap^*	nod
5	40	0.4	**41.47**	3303(1)	14.68%	4530253	47.45	3101.6(2)	48.85%	13309	48.49	350.3	1629.4	2809
5	40	0.6	**26.93**	2432.1(4)	13.52%	3656732	35.95	2365.9(4)	36.1%	12585	37.37	140.3	490.8	1233
5	40	0.8	**8.23**	2300.5(4)	3.61%	2028372	20.29	1481.3(7)	19.68%	5383	22.24	685	2806.6	5846
5	50	0.4	**40.38**	3206.6(2)	19.38%	3294056	47.95	3113.2(2)	49.63%	4628	48.83	428.7	945	**1552**
5	50	0.6	**27.15**	3275.6(1)	13.29%	3557911	36.53	3336.6(1)	37.75%	4955	37.53	820(9)	0.12%	**2817**
5	50	0.8	**7.35**	3309.7(2)	1.62%	2043011	20.25	2541.9(5)	20%	**4469**	22.2	1661.1(8)	0.36%	5810
5	60	0.4	**41.59**	13.31%(0)	18.25%	1753859	48.3	3085.5(4)	50.31%	2721	49.18	1728.3(7)	0.5%	**2524**
5	60	0.6	**27.43**	3311.6(1)	12.23%	1713348	36.42	3340.6(2)	37.27%	**2209**	37.78	1881.4(6)	0.29%	2714
5	60	0.8	**8.63**	1.6%(0)	4.01%	1662827	18.91	3571.7(1)	20.1%	163433	22.25	3382.7(2)	0.88%	**5277**
8	40	0.4	**35.42**	6.44%(0)	13.75%	3794485	41.74	41.52%(0)	43.13%	14468	39.44	1589.7	3187.7	13602
8	40	0.6	**26.54**	10.08%(0)	31.17%	3497335	32.16	31.92%(0)	34.12%	18424	30.25	1727.2(9)	0.3%	15099
8	40	0.8	**14.71**	7.5%(0)	10.68%	3094546	19.84	19.53%(0)	20.12%	**20398**	19.83	2749.9(5)	0.91%	21788
8	50	0.4	**36.45**	9.44%(0)	12.86%	1646653	42.08	41.98%(0)	46.24%	**4591**	40.43	3021.4(3)	1.19%	9660
8	50	0.6	**27.22**	11.47%(0)	25.27%	1772306	32.14	32.05%(0)	33.87%	**5413**	31.33	0.48%(0)	1.05%	11358
8	50	0.8	**14.85**	8.77%(0)	12.64%	1952442	20.13	20.01%(0)	20.42%	**5976**	20.12	0.41%(0)	0.85%	12947
8	60	0.4	**36.47**	12.54%(0)	14.84%	1417232	41.29	41.24%(0)	48.98%	**1889**	39.56	0.58%(0)	1.02%	5801
8	60	0.6	**27.21**	13.42%(0)	25.98%	1472262	32.09	32.04%(0)	34.81%	**2279**	30.77	0.51%(0)	0.9%	6144
8	60	0.8	**15.74**	10.37%(0)	13.74%	1347127	19.84	19.79%(0)	20.68%	**2555**	20.14	0.46%(0)	0.74%	6767

			F^{flow}				F^{mtz}				F^{km2}			
$\|P\|$	$\|V\|$	α	$gapLR$	$t/gap(\#)$	t^*/gap^*	nod	$gapLR$	$t/gap(\#)$	t^*/gap^*	nod	$gapLR$	$t/gap(\#)$	t^*/gap^*	nod
5	40	0.4	48.47	108	707.3	132275	47.25	**16.1**	**66.3**	**1025**	47.25	20.9	114.1	2645
5	40	0.6	37.34	19.1	46.1	27210	35.85	17	64.6	1218	35.85	**12.5**	**40**	**1062**
5	40	0.8	22.21	82.8	371	79107	20.37	**16.4**	65.4	**1794**	20.37	21.6	**47**	2620
5	50	0.4	48.77	239	815.5	186992	47.58	**33**	**70.2**	2431	47.62	49.4	163	3179
5	50	0.6	37.47	772.1(9)	0.06%	598456	36.02	**34.2**	**72.7**	5594	36.06	48.3	191.2	3107
5	50	0.8	22.11	511.8	2560	263337	20.45	**55.2**	**168.9**	15687	20.36	97.3	354.4	9116
5	60	0.4	49.19	1469.8(7)	0.98%	689822	47.94	**80.6**	**249.2**	17061	47.97	163.3	703.4	4396
5	60	0.6	37.75	1334.1(7)	0.48%	562257	36.27	**58.1**	**81.6**	3829	36.3	170.2	617.2	6058
5	60	0.8	22.22	2822.4(4)	0.89%	998333	20.32	**361**	**2875.4**	183986	20.37	787.4(9)	0.26%	44705
8	40	0.4	39.43	679.2(9)	1.16%	493769	38.54	51.1	201.5	33678	38.54	**41.8**	**165.3**	**13374**
8	40	0.6	30.22	717.7(9)	0.42%	513069	29.21	70.4	356.1	53909	29.21	**32**	**52.2**	**12598**
8	40	0.8	19.78	1366.7(8)	0.86%	848285	18.61	64.1	212.2	49536	18.61	**57.2**	**146.2**	24701
8	50	0.4	40.52	3021.2(2)	1.93%	1312489	39.59	216.5	1078.5	98764	39.65	**179**	**461.1**	45894
8	50	0.6	31.43	2897.2(4)	2.37%	1505621	30.34	**276.7**	**1225.8**	146400	30.41	288.3	1233.3	131315
8	50	0.8	20.08	2352.8(5)	0.83%	851597	19.02	**363.2**	**1418.4**	202370	19.09	495.9	2625.2	209851
8	60	0.4	39.8	3310.1(1)	3.54%	1350196	38.49	**295.6**	**516.8**	92452	38.5	357.6	2050.1	86194
8	60	0.6	30.96	0.87%(0)	2.28%	1316337	29.59	765.9(9)	0.08%	232349	29.61	**526.6**	**1469.6**	153895
8	60	0.8	20.09	0.4%(0)	0.69%	1086016	18.81	1574.7(**9**)	**0.22%**	634672	18.84	**1293(9)**	0.32%	338859

Table: OWASTP results for the different formulations.

Computational results 2/4

			F^{km2}				$F^{km2}+(14)$				$F^{km2}+(15)$				$F^{km2}+(16)$			
$\|P\|$	$\|V\|$	α	$gapLR$	$t(\#)$	t^*/gap^*	nod	$gapLR$	$t(\#)$	t^*/gap^*	nod	$gapLR$	$t(\#)$	t^*/gap^*	nod	$gapLR$	$t(\#)$	t^*/gap^*	nod
5	40	0.4	47.25	20.9	114.1	2645	47.25	18.6	98.4	**2188**	**36.55**	24	129.2	2638	47.25	**15.4**	**83.6**	2470
5	40	0.6	35.85	12.5	40	1062	35.85	10.1	18.2	870	**30.07**	11.9	26.5	1234	35.85	**7.2**	**14.6**	**707**
5	40	0.8	20.37	21.6	47	2620	20.37	22.3	50.3	3430	**17.67**	22.9	43.3	**2233**	20.37	**14.4**	**27.5**	2237
5	50	0.4	47.62	49.4	163	3179	47.62	49.3	139.3	2787	**37.72**	76.7	263.5	4058	47.62	**29.2**	**70.2**	**1983**
5	50	0.6	36.06	48.3	191.2	3107	36.06	36.9	**92.7**	2455	**30.69**	51.4	179.7	2662	36.06	**24.6**	99.1	**1554**
5	50	0.8	20.36	97.3	354.4	9116	20.36	109.5	354.3	17686	**17.85**	76.9	**205.4**	**6544**	20.36	**65.8**	250.6	7878
5	60	0.4	47.97	163.3	703.4	**4396**	47.97	165.9	**408.6**	5725	**38.89**	182.7	700	7411	47.97	**148.4**	733.3	4990
5	60	0.6	36.3	170.2	617.2	6058	36.3	**101.3**	**532.5**	6624	**31.37**	156.4	783.7	5730	36.3	115.4	747.9	**4837**
5	60	0.8	20.37	787.4(9)	0.26%	**44705**	20.36	849.8(9)	0.11%	108904	**18.05**	824.8(9)	0.07%	96309	20.36	**526.2**	**2720.2**	58259
8	40	0.4	38.54	41.8	165.3	13374	38.54	34.3	128.4	11951	**29.35**	68.2	202.8	11791	38.54	**20.5**	**59.4**	**9118**
8	40	0.6	29.21	32	52.2	**12598**	29.21	31.6	78.6	15065	**24.5**	51.6	137.3	16363	29.21	**23**	**49.7**	14471
8	40	0.8	18.61	57.2	146.2	24701	18.61	45.8	94.2	25522	**16.58**	66.2	129.5	25040	18.61	**32**	**53.3**	**23646**
8	50	0.4	39.65	179	461.1	45894	39.65	182.2	462.4	65715	**31.36**	537.2	3578.1	49015	39.65	**121.9**	**345**	**43489**
8	50	0.6	30.41	288.3	**1233.3**	131315	30.41	343.5	2168.3	159662	**26.16**	397.4	1310	**88529**	30.41	**249.1**	1493.2	139265
8	50	0.8	19.09	495.9	2625.2	209851	19.09	679(9)	0.19%	208494	**17.24**	729.5(9)	0.21%	**143540**	19.09	**379.1**	**2262.3**	207310
8	60	0.4	38.5	357.6	2050.1	86194	38.5	270.7	1564.3	86715	**30.74**	599	3528.7	98769	38.5	**224.9**	**939.8**	**69979**
8	60	0.6	29.61	526.6	1469.6	153895	29.67	757.1(9)	17.05%	256279	**25.66**	1207.1	2819.4	193414	29.61	**367.9**	**711.6**	**145116**
8	60	0.8	18.84	1293(**9**)	0.32%	338859	18.84	1536.5(8)	**0.18%**	466837	**17.13**	1779.9(8)	0.35%	**311132**	18.84	**1023.7(9)**	0.38%	360528

			F^{mtz}				$F^{mtz}+(14)$				$F^{mtz}+(15)$				$F^{mtz}+(16)$			
$\|P\|$	$\|V\|$	α	$gapLR$	$t(\#)$	t^*/gap^*	nod	$gapLR$	$t(\#)$	t^*/gap^*	nod	$gapLR$	$t(\#)$	t^*/gap^*	nod	$gapLR$	$t(\#)$	t^*/gap^*	nod
5	40	0.4	47.25	16.1	66.3	1025	47.25	**15.6**	**63.7**	1525	**37.48**	16.9	66.1	**839**	47.25	15.8	65.6	1028
5	40	0.6	35.85	**17**	64.6	**1218**	35.85	19.1	**63.7**	2751	**30.07**	18.7	65.4	1384	35.85	**17**	64	**1218**
5	40	0.8	20.37	16.4	65.4	**1794**	20.37	17.3	**64.2**	2684	**17.93**	18.3	65.8	2051	20.37	**16.1**	65	**1794**
5	50	0.4	47.58	33	70.2	**2431**	47.58	34.6	**68.9**	3646	**37.68**	36.1	73.8	2819	47.58	**32.9**	69.8	**2431**
5	50	0.6	36.02	**34.2**	**72.7**	**5594**	36.02	55.8	176.3	15449	**30.64**	42.2	116.1	8635	36.02	**34.2**	73.2	**5594**
5	50	0.8	20.45	55.2	168.9	15687	20.31	63.5	278.7	15408	**17.8**	53.4	132.5	13190	20.31	**39**	**81.5**	**7858**
5	60	0.4	47.94	**80.6**	249.2	17061	47.94	104.5	317.5	17720	**38.87**	89.4	**160.4**	**12878**	47.94	81.5	253	17159
5	60	0.6	36.27	**58.1**	81.6	3829	36.27	77.8	162.6	7190	**31.34**	61.4	**81.1**	**3520**	36.27	58.9	84.8	3829
5	60	0.8	20.32	361	2875.4	183986	20.33	481.5(9)	0.14%	**112809**	**18.01**	**345**	**2758.7**	132584	20.32	366.2	2919.2	183986
8	40	0.4	38.54	51.1	201.5	33678	38.54	**41.8**	**121**	**24403**	**29.35**	84.2	300	37295	38.54	51.5	203.9	33746
8	40	0.6	29.21	**70.4**	**356.1**	**53909**	29.21	93.4	424.2	71499	**24.5**	160.3	969.2	88534	29.21	82.7	476.8	66164
8	40	0.8	18.61	**64.1**	**212.2**	49536	18.61	127.1	723.2	68204	**16.58**	92.6	297.1	**41549**	18.61	64.5	214.2	49505
8	50	0.4	39.59	**216.5**	**1078.5**	98764	39.59	455.2	1707.6	148924	**32.16**	445.4	1841.1	144083	39.59	216.9	1093.4	**96896**
8	50	0.6	30.34	**276.7**	**1225.8**	146400	30.34	493.6	2858.9	167886	**26.09**	483.3	2259.5	**140059**	30.34	280.2	1243.5	146400
8	50	0.8	19.02	**363.2**	**1418.4**	**202370**	19.02	632.8	2668.5	235793	**17.17**	700	2745.7	244777	19.02	372.7	1499.1	205833
8	60	0.4	38.49	295.6	**516.8**	92452	38.49	672.7(9)	0.07%	141778	**30.72**	1110.6(9)	0.06%	200233	38.49	**290.7**	524.8	**88878**
8	60	0.6	29.59	765.9(9)	0.08%	232349	29.59	**700.7**	**2927.8**	252104	**25.64**	1242.8(8)	0.15%	226052	29.59	741.8(9)	0.05%	**224952**
8	60	0.8	18.81	**1574.7(9)**	**0.22%**	634672	18.81	2070.1(7)	0.31%	746909	**17.65**	2225(6)	1.42%	**459314**	18.81	1595.1(**9**)	0.23%	632596

Table: OWASTP results for the different reinforced formulations.

Computational results 3/4

			F^{GS}			F^{GS}_{P1}			F^{GS}_{P2}			F^{mtz}			$F^{km2} + (16)$		
$\|P\|$	$\|V\|$	α	t_*	t	t^*	t_*	t	t^*	t_*	t	t^*	t_*	t	t^*	t_*	t	t^*
5	20	0.4	**0.3**	1.2	2.3	0.5	1.4	2.6	**0.3**	1.1	2.3	0.9	2	9.4	**0.3**	**0.5**	**0.8**
5	20	0.6	0.9	1.8	3.4	1.1	2.1	3.4	0.6	1.7	2.8	0.9	2.9	8.2	**0.4**	**0.6**	**1.1**
5	20	0.8	0.6	1.7	4.3	0.5	1.5	3.4	0.5	1.5	**3.1**	0.7	3.3	9.8	**0.3**	**0.9**	3.9
5	30	0.4	2.4	4.6	10.1	3.6	6.1	11.8	2.2	4.6	10.1	2.2	5.5	16.5	**1.3**	**2.4**	**4.3**
5	30	0.6	1.9	8.9	41.7	3.1	10.3	44.3	1.6	9.1	43.8	2.3	4.8	**9.8**	**0.9**	**3.7**	22.8
5	30	0.8	1.5	28.1	104.7	1.2	18.6	60.7	**0.5**	18.4	90.5	1.9	**7.2**	**34.4**	1.4	7.9	54
5	40	0.4	6.5	18	50.8	11.5	23.9	57	6.6	18.1	**45.3**	4	11.5	66.3	**1.9**	**9.6**	90.3
5	40	0.6	7.9	46.2	155.8	13.3	51.9	163.6	7.7	46.1	155.6	3.8	11.6	64.6	**3.5**	**8.7**	**37.8**
5	40	0.8	6.5	70.5	211.7	7.1	53.4	184.8	4.1	51.7	226	2.9	23.5	153.7	**1.2**	**21.6**	**141.6**
5	50	0.4	21.7	123.9	323.6	38.8	143.4	352.7	21.5	124.6	335.8	**6.4**	**22.4**	**70.2**	10.2	47	632.8
5	50	0.6	26.3	367.3	2374.1	41.7	384.8	2404.1	26	368	2394.3	**5.8**	**25.5**	**155.1**	8.1	99	2363.3
5	50	0.8	9.7	297.8	3664.5	23.3	217.6	1972.1	14.5	225.3	1978.9	**9.2**	**52.7**	338.9	10	59.4	**303**
5	60	0.4	41	460.9	4131.3	86.9	511.7	4174.9	40.5	461.1	4092.9	**8.7**	**38.3**	**249.2**	20.3	113.2	739.8
5	60	0.6	-	-	-	-	-	-	-	-	-	**7.8**	**29.1**	**81.6**	20.7	140.5	1470
5	60	0.8	6.4	1725.3	16778.8	25.2	866.9	9426.5	**2.6**	1586.2	29575.1	5.8	**234.8**	**2875.4**	32.9	386.8	3131.4

Table: Comparison among the results obtained by Galand and Spanjaard (2012)

Computational results 4/4

$\|P\|$	$\|V\|$	α	$gapLR$	$t/gap(\#)$	t^*/gap^*	nod	$gapLR$	$t/gap(\#)$	t^*/gap^*	nod
			F^{mtz}				$F^{km2} + (16)$			
5	40	0.4	**47.25**	16.1	**66.3**	**1025**	**47.25**	**15.4**	83.6	2470
5	40	0.6	**35.85**	17	64.6	1218	**35.85**	**7.2**	**14.6**	**707**
5	40	0.8	**20.37**	16.4	65.4	**1794**	**20.37**	**14.4**	**27.5**	2237
5	50	0.4	**47.58**	33	**70.2**	2431	47.62	**29.2**	**70.2**	**1983**
5	50	0.6	**36.02**	34.2	**72.7**	5594	36.06	**24.6**	99.1	**1554**
5	50	0.8	**20.31**	398.3(9)	14.88%	148802	20.36	**65.8**	**250.6**	**7878**
5	60	0.4	**47.94**	**80.6**	**249.2**	17061	47.97	148.4	733.3	**4990**
5	60	0.6	**36.27**	**58.1**	**81.6**	**3829**	36.3	115.4	747.9	4837
5	60	0.8	**20.32**	**361**	2875.4	183986	20.36	526.2	**2720.2**	**58259**
5	80	0.4	**47.42**	**295.2**	**1201.3**	52052	47.66	711.1(9)	0.42%	**8982**
5	80	0.6	**32.51**	**148.3**	**307.4**	**12645**	32.54	480.2	2446.3	13898
5	80	0.8	**20.19**	**258.9**	**1164**	30330	20.23	1065.6(8)	0.24%	**29978**
5	100	0.4	**47.8**	**825(9)**	**0.17%**	102871	**47.8**	945.4(**9**)	0.2%	**8248**
5	100	0.6	**36.2**	**195.9**	**531.8**	12745	36.21	1076.3(8)	0.14%	**5759**
5	100	0.8	**20.12**	**954.1(8)**	**0.04%**	111674	20.13	1713.2(7)	0.25%	**15144**
8	40	0.4	**38.54**	51.1	201.5	33678	**38.54**	**20.5**	**59.4**	**9118**
8	40	0.6	**29.21**	70.4	356.1	53909	**29.21**	**23**	**49.7**	**14471**
8	40	0.8	**18.61**	64.1	212.2	49536	**18.61**	**32**	**53.3**	**23646**
8	50	0.4	**39.59**	216.5	1078.5	98764	39.65	**121.9**	**345**	**43489**
8	50	0.6	**30.34**	276.7	**1225.8**	146400	30.41	**249.1**	1493.2	**139265**
8	50	0.8	**19.02**	**363.2**	**1418.4**	**202370**	19.09	379.1	2262.3	207310
8	60	0.4	**38.49**	295.6	**516.8**	92452	38.5	**224.9**	939.8	**69979**
8	60	0.6	**29.59**	765.9(9)	0.08%	232349	29.61	**370.5**	**711.6**	**144495**
8	60	0.8	**18.81**	1574.7(**9**)	0.22%	634672	18.84	**874.2(9)**	**0.2%**	**282782**
8	80	0.4	**38.52**	1823.1(7)	**0.44%**	310488	38.55	**1375(8)**	0.67%	**167755**
8	80	0.6	**29.6**	1760.8(7)	0.17%	**300635**	29.62	**1513.5(8)**	**0.09%**	380514
8	80	0.8	**18.78**	1610.8	3031.7	293955	18.81	**1119.4**	**2356.6**	**282151**
8	100	0.4	41.03	2325.6(5)	52.44%	**145535**	**40.91**	**1995.7(6)**	**15.55%**	174704
8	100	0.6	29.74	2733.3(**4**)	0.46%	**213661**	**29.71**	**2653(4)**	**0.22%**	271702
8	100	0.8	**18.9**	3448.7(1)	**0.28%**	**329944**	18.99	**3004.8(4)**	0.7%	392475
10	40	0.4	**35.8**	153.9	786	115039	35.82	**81.7**	**351.5**	**62760**
10	40	0.6	**27.19**	320.2	1028.4	249844	27.21	**149.9**	**442.8**	**124073**
10	40	0.8	**17.47**	545.3	1935.6	464815	17.49	**435.3**	**1158.2**	**364270**
10	50	0.4	**35.97**	1012(8)	0.24%	495899	36.01	**454.1**	**2960.6**	**195997**
10	50	0.6	**27.46**	1541(7)	0.22%	711540	28.86	**705.4**	**2694.6**	**443840**
10	50	0.8	**17.99**	2618.3(5)	0.37%	1298372	18.04	**2191.1(6)**	**0.31%**	**1232379**
10	60	0.4	**35.68**	1847.3(7)	**0.19%**	586485	35.72	**1361.3(8)**	0.2%	**559878**
10	60	0.6	**27.12**	2622.3(5)	0.47%	**803544**	27.16	**2247.4(9)**	**0.44%**	838333
10	60	0.8	**17.7**	3543.5(1)	0.45%	**1029643**	17.72	**3438.9(2)**	**0.27%**	1537301
10	80	0.4	35	3387.7(2)	0.87%	**409111**	**34.95**	**2743.4(4)**	**0.37%**	582164
10	80	0.6	**27.01**	3448.3(1)	**0.33%**	**433129**	**27.01**	**3122.9(3)**	0.36%	648178
10	80	0.8	17.7	0.35%(**0**)	0.79%	**407926**	**17.65**	**0.24%(0)**	**0.58%**	752602
10	100	0.4	34.97	0.28%(**0**)	0.7%	**189952**	**34.93**	**0.15%(0)**	**0.35%**	437191
10	100	0.6	26.86	0.3%(**0**)	0.82%	**227787**	**26.82**	**0.2%(0)**	**0.41%**	441767
10	100	0.8	17.59	0.31%(**0**)	0.54%	**224362**	**17.55**	**0.24%(0)**	**0.44%**	378484

Thanks for your attention

Questions, comments, suggestions... are welcome.

All details available at:

Fernández, E.; Pozo, M.A. & Puerto, J. (2014). *A modeling framework for Ordered Weighted Average Combinatorial Optimization*. Discrete Applied Mathematics, (169): 97-118.

Fernández, E.; Pozo, M.A. & Puerto, J. (2015). *Ordered Weighted Average Optimization in multiobjective spanning tree problems*. Submitted.

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