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WEAKLY COERCIVE MAPPINGS SHARING A VALUE

J. M. Soriano, Sevilla

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Abstract. Some sufficient conditions are provided that guarantee that the difference of a compact mapping and a proper mapping defined between any two Banach spaces over $\mathbb{K}$ has at least one zero. When conditions are strengthened, this difference has at most a finite number of zeros throughout the entire space. The proof of the result is constructive and is based upon a continuation method.

Keywords: zero point, continuation method, C^1-homotopy, surjerctive implicit function theorem, proper mapping, compact mapping, coercive mapping, Fredholm mapping

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1. Preliminaires

Let X and Y be two Banach spaces. If $u\colon D(F) \subseteq X \to Y$ is a continuous mapping, then one way of solving the equation

$$u(x) = 0 \tag{1}$$

is to embed (1) in a continuum of problems

$$H(x,t) = 0 \quad (0 \leqslant t \leqslant 1), \tag{2}$$

which can easily be resolved when $t = 0$. When $t = 1$, the problem (2) becomes (1). In the case when it is possible to continue the solution for all t in $[0,1]$ then (1) is solved. This method is called continuation with respect to a parameter [1]–[9].

In this paper some sufficient conditions are provided in order to guarantee that the difference of a compact and a proper weakly coercive C^1-mapping has at least

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one zero. If these conditions are conveniently strengthened this difference has at most a finite number of zeros on X. Other conditions, sufficient to guarantee the existence of fixed points, have been given by the author in a finite-dimensional setting, see for example [7], and in an infinite-dimensional setting, see for example [8]. A continuation method was used in the proofs of these papers. The proofs supply the existence of implicitly defined continuous mappings whose ranges reach zero points [1]–[9]. A continuation method is also used here. The key is the use of the surjective implicit function theorem [10], and the properties of proper and Fredholm C^1-mappings (see [9]).

We briefly recall some theorems and concepts to be used.

Definitions [26], [27]**.** Henceforth we will assume that X and Y are Banach spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

A mapping $F\colon D(F) \subseteq X \to Y$ is called *weakly coercive* if and only if $\|F(x)\| \to \infty$ as $\|x\| \to \infty$.

A mapping $F\colon D(F) \subseteq X \to Y$ is said to be *compact* whenever it is continuous and the image $F(B)$ is relatively compact (i.e. its closure $\overline{F(B)}$ is compact in Y) for every bounded subset $B \subset D(F)$.

A mapping F is said to be *proper* whenever the pre-image $F^{-1}(K)$ of every compact subset $K \subset Y$ is also a compact subset of $D(F)$.

The symbol *dim* means dimension, *codim* means codimension, *ker* means kernel, $R(L)$ stands for the range of the mapping L.

That $L\colon X \to Y$ is a *linear Fredholm* mapping means that L is linear and continuous and both the numbers $\dim(\ker(L))$ and $\operatorname{codim}(R(L))$ are finite, and therefore $\ker(L) = X_1$ is a Banach space and has the topological complement X_2, since $\dim(X_1)$ is finite. The integer number $\operatorname{Ind}(L) = \dim(\ker(L)) - \operatorname{codim}(R(L))$ is called the *index* of L.

Let $F\colon D(F) \subseteq X \to Y$. If $D(F)$ is open, then the mapping F is said to be a *Fredholm* mapping if and only if F is a C^1-mapping and $F'(x)\colon X \to Y$ is a Fredholm linear mapping for all $x \in D(F)$. If $\operatorname{Ind}(F'(x))$ is constant with respect to $x \in D(F)$, then we call this number the index of F and write it as $\operatorname{Ind}(F)$.

X, Y are called *isomorphic* if and only if there is a linear homeomorphism (*isomorphism*) $L\colon X \to Y$.

Let $\mathcal{F}(X, Y)$ denote the set of all linear Fredholm mappings $A\colon X \to Y$.

Let $\mathcal{L}(X, Y)$ denote the set of all linear continuous mappings $L\colon X \to Y$.

Let $\operatorname{Isom}(X, Y)$ denote the set of all isomorphisms $L\colon X \to Y$.

Let $F\colon D(F) \subseteq X \to Y$ with $D(F)$ open be a C^1-mapping. The point $u \in X$ is called a *regular point* of F if and only if $F'(u) \in \mathcal{L}(X, Y)$ maps onto Y. A point

$v \in Y$ is called a *regular value* of F if and only if the pre-image $F^{-1}(v)$ is empty or consists solely of regular points.

Theorem 1. The Surjective Implicit Function Theorem. [10, Section 4–13, Theorem 4–H]. *Let X, Y, Z be Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let*

$$F\colon U(u_0, v_0) \subseteq X \times Y \to Z$$

be a C^1-mapping on an open neighbourhood of the point (u_0, v_0). Suppose that

(i) *$F(u_0, v_0) = 0$, and*

(ii) *$F_v(u_0, v_0)\colon Y \to Z$ is surjective.*

Then the following assertion is true:

Let $r > 0$. There is a number $\varrho > 0$ such that, for each given $u \in X$ with $\|u - u_0\| < \varrho$, the equation

$$F(u, v) = 0$$

has a solution v such that $\|v - v_0\| < r$.

Theorem 2 [9, Section 7–9, Theorem 7–33]**.** *Let $g\colon D(g) \subseteq X \to Y$ be a compact mapping, where $a \in D(g)$ and $D(g)$ is open. If the derivative $g'(a)$ exists, then $g'(a) \in \mathcal{L}(X, Y)$ is also a compact mapping.*

Theorem 3 [9, Section 8–4, Example 8–16]**.** *Let $S \in \mathcal{F}(X, Y)$. The perturbed mapping $S + C$ verifies $S + C \in \mathcal{F}(X, Y)$ and $\mathrm{Ind}(S + C) = \mathrm{Ind}(S)$ provided $C \in \mathcal{L}(X, Y)$ and C is a compact mapping.*

2. Weakly coercive mappings sharing a value

Clearly, if we define $u := f - g$, then u has a zero if and only if f and g share a value, that is, there is $x \in X$ with $f(x) = g(x)$. We thereby establish our result in terms of f, g.

Theorem 4. *Let $f, g\colon D \subseteq X \to Y$ be two C^1-mappings, where X and Y are two Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and D is open.*

We assume

(i) *f is a compact mapping, g is a proper mapping and $tf(x) - g(x)$ is weakly coercive, jointly in both coordinates.*

(ii) *The mapping g has a zero, x_0.*

(iii) *For any fixed t, belonging to $[0, 1]$, the zero of Y is a regular value of the mapping $tf(x) - g(x)$.*

Then the following assertion holds

(a) *f and g share at least one value, i.e., there is at least one x^* such that $f(x^*) = g(x^*)$.*

If in addition the condition

(iv) *g is a Fredholm mapping of index zero is satisfied, then in addition we have*

(b) *f and g share at most a finite number of values on D, and at least one value.*

Proof.

(a) Conclusion (a) will be proved in this section.

(a1) Let us define a mapping

$$H\colon\ D \times [0,1] \subseteq X \times [0,1] \to Y, \quad \text{where} \quad H(x,t) := tf(x) - g(x).$$

We will prove here that H is a proper mapping, which will imply that $H^{-1}(0)$ is a compact set, since $\{0\} \subset Y$ is a compact set and H is proper.

Let $\mathcal{C}$ be any fixed compact subset of Y, and let a sequence be fixed such that $(H(x_n, t_n))_{n\geqslant 1}$ belongs to $\mathcal{C}$. It suffices to show that the sequence $((x_n, t_n))_{n\geqslant 1}$ contains a convergent subsequence $((x_{n'''}, t_{n'''}))_{n'''\geqslant 1}$, which will imply that $H^{-1}(\mathcal{C})$ is relatively compact, and since

$$(x_{n'''}, t_{n'''}) \to (u,t) \quad \text{as} \quad n''' \to \infty,$$

H is continuous and $\mathcal{C}$ compact, therefore $H(u,t) \in \mathcal{C}$, that is, $(u,t) \in H^{-1}(\mathcal{C})$, and hence $H^{-1}(\mathcal{C})$ is compact.

Since the set $\mathcal{C}$ is bounded and the mapping H is weakly coercive, $((x_n, t_n))_{n\geqslant 1}$ is a bounded sequence. Consequently $(x_n)_{n\geqslant 1}$ and $(t_n)_{n\geqslant 1}$ are bounded sequences. Since f is a compact mapping and $(x_n)_{n\geqslant 1}$ a bounded sequence, there exists a subsequence $(x_{n'})_{n'\geqslant 1}$ such that

$$f(x_{n'}) \to w' \quad \text{as} \quad n' \to \infty$$

for some $w' \in Y$, and furthermore, since $(t_n)_{n\geqslant 1}$ is a bounded sequence of real numbers, there is $t \in \mathbb{R}$ with

$$t_{n'} \to t \quad \text{as} \quad n' \to \infty.$$

Therefore

$$t_{n'} f(x_{n'}) \to tw' := w \quad \text{as} \quad n' \to \infty$$

for some $w \in Y$.

All $H(x_{n'}, t_{n'})$, $n' \geqslant 1$ lie in the compact set $\mathcal{C}$ and therefore, there is a subsequence $(H(x_{n''}, t_{n''}))_{n''\geqslant 1}$ such that

$$H(x_{n''}, t_{n''}) \to v \quad \text{as} \quad n'' \to \infty$$

for some $v \in \mathcal{C}$. Therefore

$$g(x_{n''}) \to w - v \quad \text{as} \quad n'' \to \infty.$$

Since g is proper, $(x_{n''})$ has a convergent subsequence

$$x_{n'''} \to u \quad \text{as} \quad n'' \to \infty.$$

On the other hand,

$$t_{n'''} \to t \quad \text{as} \quad n''' \to \infty,$$

hence

$$(x_{n'''}, t_{n'''}) \to (u, t) \quad \text{as} \quad n''' \to \infty$$

as required.

(a2) Let us suppose that

$$H(x_a, t_a) = 0 \tag{3}$$

for a fixed (x_a, t_a). We will prove that there is $\varrho > 0$ such that for any $t \in (t_a - \varrho, t_a + \varrho)$, there exists $x = x(t)$ such that $H(x(t), t) = 0$.

Since zero is a regular value for the mappings $\{tf(x) - g(x)\}$ by hypothesis (iii),

$$H_x(x, t) = tf'(x) - g'(x) \in \mathcal{L}(X, Y)$$

is surjective for every pair (x, t) such that $H(x, t) = 0$. In particular, the mapping

$$H_x(x_a, t_a) = t_a f'(x_a) - g'(x_a) \in \mathcal{L}(X, Y)$$

is surjective, which together with identity (3) and Theorem 1 implies the existence of $\varrho > 0, r > 0$ such that for any $t \in (t_a - \varrho, t_a + \varrho)$ there is $x(t)$ with $H(x(t), t) = 0$ and $\|x(t) - x_a\| < r$.

(a3) We will prove that for every t in $[0, 1]$ there exists $x(t)$ such that $H(x(t), t) = 0$.

Let M denote the set of all t such that there is a solution $x(t)$. By assumption (ii) this set is not empty. By (a2), the set is relatively open. Finally, since the set is the projection of $H^{-1}(0)$ into the second component (i.e. the t component), and since the set is the image of a compact set by the continuous function projection, it is therefore compact, and hence also closed. However, a relatively open, closed, non-empty subset of $[0,1]$ is the whole interval, since $[0,1]$ is connected.

Thus

$$H(x(1),1) = f(x(1)) - g(x(1)) = 0 \Rightarrow f(x(1)) = g(x(1)),$$

which is conclusion (a) of the theorem, where $x^* := x(1)$.

(b) We include the hypothesis (iv) in Section (b).

(b1) We will see that the C^1-mapping $u\colon D \subseteq X \to Y$, $u := f - g$ is a proper Fredholm mapping of index zero.

Since f is a compact mapping, Theorem 2 implies that for each $x \in X$, the mapping $f'(x) \in \mathcal{L}(X,Y)$ is also a compact mapping.

Since $f'(x)$ is a linear compact mapping and given that $g'(x)$ is a linear Fredholm mapping of index zero, Theorem 3 implies that

$$u'(x) = f'(x) - g'(x) \in \mathcal{L}(X,Y), \quad \forall x \in D$$

is a Fredholm linear mapping and $\mathrm{Ind}(u'(x)) = \mathrm{Ind}(g'(x)) = 0, \forall\, x \in D$. Therefore the non-linear C^1-mapping u is a Fredholm mapping of index zero.

Furthermore, Section (a1) implies that u is a proper mapping. In fact, this is the particular case in which $t = 1$, i.e. $u(x) = H(x,1)$.

(b2) We will see that, if any $x \in D$ exists which verifies $u(x) = 0$, then u is a local C^1-diffeomorphism at x.

Let $x \in D$ exist such that $u(x) = 0$. Since zero is a regular value of u, the linear Fredhom mapping $u'(x)$ maps onto Y, and since $\mathrm{Ind}(u'(x)) = 0$, therefore $\dim(\ker(u'(x))) = 0$. Thus $u'(x) \in \mathrm{Isom}(X,Y)$. By the Local Inverse Theorem [10], u is a local diffeomorphism at x.

(b3) We will prove here that f and g share at most a finite number of values on D.

Since u is proper, $u^{-1}(0)$ is a compact set. If there were an infinite sequence

$$(x_n)_{n\geqslant 1} \subset D \qquad \text{with } x_n \neq x_m \text{ when } n \neq m$$

verifying $u(x_n) = 0, \forall n \in \mathbb{N}$, there would be a subsequence $(x_{n'})_{n'\geqslant 1} \subset u^{-1}(0)$, which would converge at a point $x \in u^{-1}(0)$, and x would be an non-isolated zero of u. However, since u is a local diffeomorphism at x, given in Section (b2), x is an isolated zero of u. This is a contradiction. Hence there is not an infinite number of zeros of u on D. Thus f and g *share at most a finite number of values on* D. □

Example. Let us consider an integral mapping

$$(Au)(x) = \int_a^b F(x,y,u(y))\,\mathrm{d}y, \quad \forall x \in [a,b],$$

where $-\infty < a < b < +\infty$. Define

$$Q := \{(x, y, u) \in \mathbb{R}^3\colon\ x, y \in [a, b] \text{ and } |u| < r \text{ for fixed } r > 0\}.$$

Suppose that the function

$$F\colon\ \{(x, y, u) \in \mathbb{R}^3\colon\ x, y \in [a, b] \text{ and } |u| \leqslant r\} \to \mathbb{R}$$

is twice continuously differentiable. Define $X := C[a, b]$ and $M := \{u \in X\colon \|u\| < r\}$, where $\|u\| = \max_{a \leqslant y \leqslant b} |u(y)|$. It can be easily proved that the mapping $A\colon\ M \to X$ is compact, and it is twice cotinuously differentiable.

Let B be the mapping $B\colon\ M \times [0, 1] \to X,\ B(u, t) = u$ which is $C^\infty(M \times [0, 1], X)$, is proper, has a zero, and since it is defined only on a bounded set, it is trivially weakly coercive.

The mapping $H(u, t) = t(Au) - (Bu)\colon\ M \times [0, 1] \to X$ is weakly coercive, since it is defined only on a bounded set. Theorem 4 implies that if zero is a regular value, then there is $u \in M$ such that

$$(Au)(x) = (Bu)(x),\ \forall x \in [a, b].$$

If we do not know that zero is a regular value, it is possible to prove the existence of $u \in M$ with $(Au)(x)$ as near to $u(x), \forall x \in [a, b]$ as wanted in the following way:

Since $B'(u, t) \in \mathcal{L}(X \times \mathbb{R}, X)$ is surjective and $\dim(\ker(B'(u, t))) = 1$ for $\forall (u, t) \in M \times [0, 1]$, B is a Fredholm mapping of index one.

Define a mapping $A^*\colon\ M \times [0, 1] \to X,\ A^*(u, t) := t(Au)$ that is differentiable and compact. Theorem 2 implies that $(A^*)'(u, t)$ is compact. Theorem 3 implies that $H'(u, t)$ is a linear Fredholm mapping of index one for $\forall (u, t) \in M \times [0, 1]$, and hence $H(u, t)$ is a Fredholm mapping of index one. Thus we obtain from the Sard-Smale theorem [9, Theorem 4.K] that the set of regular values of the proper mapping H is open and dense in X.

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Author's address: J. M. Soriano, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Sevilla, Aptdo. 1160, Sevilla 41080, Spain, e-mail: soriano@us.es.