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**Applied Nonautonomous and Random Dynamical
Systems
Applied Dynamical Systems**

To our families and friends

Preface

During the past two decades, the theory of nonautonomous dynamical systems and random dynamical systems has made substantial progress in studying the long term dynamics of open systems subject to time-dependent or random forcing. However, most of the existing pertinent literatures are fairly technical and almost impenetrable for a general audience, except the most dedicated specialists. Moreover, the concepts and methods of nonautonomous and random dynamical systems, though well-established, are extremely nontrivial to apply to real-world problems. The aim of this work is to provide an accessible and broad introduction to the theory of non-autonomous and random dynamical systems, with an emphasis on applications of the theory to problems arising in the applied sciences and engineering.

The book starts with basic concepts in the theory of autonomous dynamical systems which are easier to understand, and used as the motivation for the study of non-autonomous and random dynamical systems. Then the framework of non-autonomous dynamical systems is set up, including various approaches to analyze the long time behavior of non-autonomous problems. The major emphasis is given to the novel theory of pullback attractors, as it can be regarded as a natural extension of the autonomous theory and allows a larger variety of time-dependence forcing than other alternatives such as skew-product flows or cocycles. In the end the theory of random dynamical systems and random attractors is introduced and shown to be fairly informative to the study of long term behavior of stochastic systems with random forcing.

Each set of theory is illustrated by being applied to three different models, the chemostat model, the SIR epidemic model, and the Lorenz-84 model, in their autonomous, nonautonomous, and stochastic formulations, respectively. The techniques and methods adopted can be applied to the study of the long term behavior of a wide range of applications arising in applied sciences and engineering.

Sevilla, Spain, October 22, 2016
Auburn, Alabama

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Chapter 1

Introduction

The research on the theories of dynamical systems (autonomous, nonautonomous or stochastic) has been receiving much attention over the last decades, due to the fact that many real world phenomena can be modeled by a system of differential or difference equations which generates one dynamical system. A general theory for dynamical systems allows us to analyze various type of evolutionary problems, but is sometimes over abstract for practitioner research. In particular, while analyzing finite dimensional systems, the underlying theory can be less complicated and more intuitive than the general theory for abstract dynamical systems. In this book, we will set up a unified framework of dynamical systems accessible to researchers with different background, which suffices to analyze finite dimensional systems arising in applied sciences and engineering.

Roughly speaking, a dynamical system is a system that evolves in time through the iterated application of an underlying dynamical transition rule. The application of the transition rule can happen either at discrete time, with the time parameter taking integer values, or infinitesimally with continuous time taking real values as in differential equations. The theoretical framework of dynamical systems allows us to develop a unified theory which can be applied directly to real-life evolutionary events to analyze their long term behavior. To motivate the use of the theory of dynamical systems, we start from considering the evolution in time of a certain quantity, e.g., the position of a particle moving on a plane \mathbb{R}^2 (or in the space \mathbb{R}^3), that can be modeled by the following differential equation in \mathbb{R}^2 (or \mathbb{R}^3),

$$\frac{dx}{dt} = f(x), \quad (1.1)$$

where $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ $d = 2$ (or 3) is a function which ensures existence of solutions to equation(1.1).

Usually, additional information is needed to determine properties of solutions to the system (1.1). In fact, knowing the initial position $x_0 \in \mathbb{R}^d$ of the particle is sufficient to obtain its positions in all future instants. This corresponds to the knowledge of the value of x at an initial instant t_0 . In other words, we can consider an initial

value problem (IVP) associated with equation (1.1):

$$\frac{dx}{dt} = f(x), \quad x(t_0) = x_0 \in \mathbb{R}^d. \quad (1.2)$$

Under a set of suitable assumptions on the function f , one can prove in a mathematically rigorous way that the IVP (1.2) possesses a unique solution which is defined for all future instants of time. The solution is a continuously differentiable mapping $x(\cdot; t_0, x_0) : \mathbb{R} \rightarrow \mathbb{R}^d$ with derivative $x'(t) = f(x(t))$ and satisfies $x(t_0; t_0, x_0) = x_0$. Here and also in the sequel the notation “ $\cdot; t_0, x_0$ ” is used to specify the explicit dependence of the solution on the initial value (t_0, x_0) . Basic properties of the solution mapping include:

- (a) The evolution of x depends on the elapsed time instead of the initial and final times separately.

As the vector field f does not depend on t , a simple shift on the time variable $t \rightarrow t - t_0$ allows us to prove that the position of the particle only depends on the elapsed time $t - t_0$. Mathematically this is described as

$$x(t; t_0, x_0) = x(t - t_0; 0, x_0) \quad \text{for } t \geq t_0,$$

which means that the position of the particle at time $t(\geq t_0)$, starting from the point x_0 at the initial time t_0 , is exactly the same as the position of the particle at time $t - t_0$ if it starts from point x_0 at the initial time zero. Thanks to this property we can always choose the initial time for the IVP (1.2) to be the time instant zero, i.e., $t_0 = 0$.

- (b) Initial value property.

This is a straightforward property meaning that if the particle starts from x_0 at the initial time t_0 , then the solution trajectory of x passes through the point (t_0, x_0) . Mathematically this property reads

$$x(t_0; t_0, x_0) = x_0, \quad \text{for all } x_0 \in \mathbb{R}^d, t_0 \in \mathbb{R}.$$

- (c) Concatenation of solutions.

Assume that the particle starts from the point x_0 at the initial time $t_0 = 0$, arrives at the point $x_1 = x(t_1; 0, x_0)$ after t_1 period of time, and takes off again from x_1 and arrives at the location $x_2 = x(t_2; 0, x_1)$ after another t_2 period of time. Then x_2 will be the same as the location where the particle arrives at if it starts from the point x_0 at the initial time $t_0 = 0$ and travels a total of $t_1 + t_2$ period of time. Mathematically this is described as

$$x(t_2; 0, x(t_1; 0, x_0)) = x(t_1 + t_2; 0, x_0).$$

- (d) Continuity of solutions with respect to initial values.

It happens often that the initial point is different from where it is supposed to be, due to factors such as environmental influences, human errors, deviations in the measurements, etc. It is then necessary to check the difference between the solution starting from x_0 and the solution starting from x_0^* , which is an approximation of x_0 , at the same initial time t_0 . In fact, $x(t; t_0, x_0)$ should be close to $x(t; t_0, x_0^*)$ as long as x_0 is close to x_0^* , as otherwise a small perturbation in initial data can produce a large deviation in their corresponding solutions. This property, usually referred to as the continuous dependence on initial data, can be ensured by some proper assumptions on the vector field function f .

Mathematically, given a state space X (e.g., \mathbb{R}^d), a dynamical system is a mapping $\varphi(t, x_0)$, defined for all $t \in \mathbb{T}$ ($\mathbb{T} = \mathbb{Z}$ for discrete time or $\mathbb{T} = \mathbb{R}$ for continuous time) and $x_0 \in X$, that describes how each $x_0 \in X$ evolves with respect to time. In the previous example, we can define $\varphi(t, x_0) := x(t; 0, x_0)$ and translate the above properties of solutions of the IVP (1.2) into corresponding properties of the mapping $\varphi(\cdot, \cdot)$.

Definition 1.1. Let X be a metric space. A *dynamical system* is a continuous mapping $\varphi : \mathbb{T} \times X \rightarrow X$ with the following properties:

- (i) initial value property

$$\varphi(0, x) = x \quad \text{for all } x \in X; \quad (1.3)$$

- (ii) group property

$$\varphi(t + \tau, x) = \varphi(t, \varphi(\tau, x)) \quad \text{for all } t, \tau \in \mathbb{T} \text{ and } x \in X. \quad (1.4)$$

Note that the continuous dependence on the initial values is implied by the continuity of the mapping φ . Property (1.4) is called a “group property” because the family of mapping $\{\varphi(t, \cdot) : t \in \mathbb{T}\}$ that maps the state space X into itself, forms a group under composition [9]. In some occasions the family of mappings $\{\varphi(t, \cdot) : t \in \mathbb{T}_0^+\}$ of X , where $\mathbb{T}_0^+ = \{t \in \mathbb{T} : t \geq 0\}$, that maps X into itself forms a semigroup under composition rather than a group [9]. While the group property (1.4) is replaced by a semigroup property, we have a semi-dynamical system defined as follows.

Definition 1.2. Let X be a metric space. A *semi-dynamical system* is a continuous function $\varphi : \mathbb{T}_0^+ \times X \rightarrow X$ with the initial value property (1.3) and the following semi-group property

$$\varphi(t + \tau, x) = \varphi(t, \varphi(\tau, x)) \quad \text{for all } t, \tau \in \mathbb{T}_0^+ \text{ and } x \in X. \quad (1.5)$$

When $\mathbb{T} = \mathbb{Z}$, the dynamical (semi-dynamical) system is called a discrete dynamical (semi-dynamical) system, and when $\mathbb{T} = \mathbb{R}$, the dynamical (semi-dynamical) system is called a continuous dynamical (semi-dynamical) system. In this book we will focus on continuous dynamical (semi-dynamical) systems, and omit the phrase “continuous” when the context is clear.

Not every dynamical system is generated by the solutions of a system of ordinary differential equations (ODEs) or partial differential equations (PDEs), but the main focus of this book is to analyze such dynamical systems, due to their broad range of applications in the applied sciences. Furthermore, our goal is to make the theory of dynamical systems accessible to not only mathematicians but also researchers from interdisciplinary fields such as engineers, biologists, physicists, and ecologists. Therefore we will not state the general abstract theory of dynamical systems which includes both ODEs and PDEs. More precisely, we will restrict our theories and examples to finite dimensional dynamical systems, i.e., those generated by solutions to systems of ODEs. The systems under consideration in this book are mainly dissipative systems (will be defined later) which have a broad range of applications in applied sciences and engineering.

In particular, we will first establish some basic results ensuring existence and uniqueness of solutions defined globally in time, i.e., defined for all future times. Such existence is typical in systems in real-world problems and is easier to understand for a broad range of researchers, while as a trade-off it requires stronger assumptions. In addition, we will also mention some weaker but less intuitive assumptions which can ensure similar results so that a reader who may be potentially interested in the formal mathematical aspects beyond basic analysis can benefit from the information. Only fundamental results will be presented; for more details and proofs the reader is referred to [25, 39, 63, 69] and references therein.

Let $f : (a, +\infty) \times \mathbb{R}^d \subseteq \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ be a continuous mapping, and let (t_0, x_0) be a point in $(a, +\infty) \times \mathbb{R}^d$. Then we can formulate the following IVP

$$\frac{dx(t)}{dt} = f(t, x), \quad x(t_0) = x_0. \quad (1.6)$$

Definition 1.3. Let $I \subseteq (a, +\infty)$ be a time interval. A solution to (1.6) on I is a mapping $\varphi : I \rightarrow \mathbb{R}^d$ which is continuously differentiable on I , i.e., $\varphi \in C^1(I; \mathbb{R}^d)$, and satisfies:

- (i) $\frac{d}{dt}\varphi(t) = f(t, \varphi(t))$, for all $t \in I$;
- (ii) $\varphi(t_0) = x_0$.

We are interested in the IVPs of form (1.6) whose solutions are defined on an interval I containing the interval $[t_0, +\infty)$, i.e., IVPs with solutions defined globally in time. There are various results ensuring this fact, and we first present one that is straightforward to understand but still covers many interesting situations.

Theorem 1.1. Assume that $f : (a, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is continuously differentiable, i.e., its partial derivatives of first order are continuous functions, and there exist non-negative continuous mappings $h, k : (a, +\infty) \rightarrow \mathbb{R}$ such that

$$|f(t, x)| \leq h(t)|x| + k(t), \text{ for all } (t, x) \in (a, +\infty) \times \mathbb{R}^d. \quad (1.7)$$

Then, there exists a unique solution to (1.6) which is defined globally in time.

Remark 1.1. The existence and uniqueness of a local solution to (1.6), that is defined on a finite time interval, can be proved by using a fixed point theorem. This local solution may only be defined on a small time interval but can always be extended to a larger time interval as long as it remains bounded. As a consequence, if the local solution does not blow up within finite time, then it can be defined globally in time.

Two examples are provided below to illustrate Theorem 1.1 and Remark 1.1. First consider the following IVP which satisfies the growth condition (1.7) in Theorem 1.1:

$$\frac{dx(t)}{dt} = \frac{3t^2 x^3(t)}{1+x^2(t)} + x(t), \quad x(t_0) = x_0. \quad (1.8)$$

Multiplying the equation by $2x(t)$ results in

$$2x(t) \frac{dx(t)}{dt} = \frac{6t^2 x^3(t)}{1+x^2(t)} + 2x^2(t)$$

which implies that

$$\frac{d}{dt} x(t)^2 \leq 2(3t^2 + 1)x^2(t). \quad (1.9)$$

Integrating (1.9) as a linear differential inequality gives

$$x(t)^2 \leq x_0^2 e^{2(t^3+t)-2(t_0^3+t_0)},$$

and as a result Theorem 1.1, the solution to the IVP (1.8) is defined globally in time.

We next consider another example which does not fulfill the linear growth condition (1.7) in Theorem 1.1:

$$\frac{dx(t)}{dt} = x^2(t), \quad x(t_0) = x_0. \quad (1.10)$$

The IVP (1.10) can be solved directly by standard methods to obtain explicitly

$$x(t) = \frac{x_0}{1 - x_0(t - t_0)}.$$

It is clear that when $x_0 \leq 0$, the corresponding solution is defined for all $t \geq t_0$. In fact, it is well defined on the interval $(t_0 + \frac{1}{x_0}, +\infty)$, including formally the case $x_0 = 0$ which provides the null solution defined on the whole real line. However, if $x_0 > 0$, the solution is defined only on the interval $(-\infty, t_0 + \frac{1}{x_0})$ because the solution blows up at the finite time instant $t = t_0 + \frac{1}{x_0}$.

Another condition that also ensures the existence of solutions defined globally in time is the so-called *dissipativity condition*. For a simple explanation of this condition, we consider the autonomous version of (1.6) with $f(t, x) = f(x)$. Assume that there exist two constants α, β with $\beta > 0$ such that

$$f(x) \cdot x \leq \alpha|x|^2 + \beta. \quad (1.11)$$

Taking the scalar product of the equation in (1.6) with x results in

$$\frac{d}{dt}|x(t)|^2 \leq 2\alpha|x(t)|^2 + 2\beta,$$

which implies that

$$|x(t)|^2 \leq |x_0|^2 e^{2\alpha(t-t_0)} - \frac{\beta}{\alpha} (1 - e^{2\alpha(t-t_0)}).$$

Therefore, if $\alpha > 0$, then

$$|x(t)|^2 \leq \left(|x_0|^2 + \frac{\beta}{\alpha}\right) e^{2\alpha(t-t_0)},$$

and solutions are globally defined in time. On the other hand, if $\alpha < 0$, then

$$|x(t)|^2 \leq |x_0|^2 e^{2\alpha(t-t_0)} - \frac{\beta}{\alpha},$$

and every solution enters the ball centered at zero with radius $-\beta/\alpha + 1$, at a certain time T (dependent on x_0) and remains inside the ball forever, i.e., for all $t \geq T$. More precisely, there exists an absorbing ball for all solutions, and this is due to the dissipativity condition (1.11).

Remark 1.2. For autonomous IVPs, the growth condition (1.7) with constant h and k implies the dissipativity condition (1.11).

Note that Theorem 1.1 is for the special case where f is defined on the whole \mathbb{R}^d . Moreover, from a mathematical point of view, the growth condition (1.7) and the dissipativity condition (1.11) introduced above, though straightforward to understand, are fairly restrictive. In fact, existence of solutions to the IVP (1.6) can be guaranteed by assuming much weaker assumptions on a subspace of \mathbb{R}^d . In what follows we will include a brief summary on weaker and more typical conditions for the existence of solutions to the IVP (1.6). The contents, however, are more abstract and are mainly for the readers who are interested in gaining a deeper understanding of the underlying mathematical foundation.

Consider a nonempty, open, and connected set $O \subseteq \mathbb{R}^d$ with $d \in \mathbb{Z}$ and $d \geq 1$ and an open time interval I , and consider the IVP (1.6) with a vector field function $f(\cdot, \cdot)$ defined in the domain $I \times O$. Now we provide the definition of solution to this IVP.

Definition 1.4. Let $J \subseteq I$ be an interval of time such that t_0 belongs to the interior of J . A mapping $\varphi : J \rightarrow O$ is said to be a *local solution* of the IVP (1.6) on J , if φ is continuously differentiable on J , i.e., $\varphi \in C^1(J; \mathbb{R}^d)$, and satisfies the following conditions:

- (i) $\varphi(t) \in O$ for all $t \in J$;
- (ii) $\frac{d}{dt}\varphi(t) = f(t, \varphi(t))$, for all $t \in J$;
- (iii) $\varphi(t_0) = x_0$.

Definition 1.5. A mapping $f(t, x)$ from $I \times \mathcal{O} \subset \mathbb{R}^{d+1}$ to \mathbb{R}^d is said to be *locally Lipschitz with respect to x* on $I \times \mathcal{O}$ if for each $(t_0, x_0) \in I \times \mathcal{O}$ there exist $\varepsilon > 0$ and $L \geq 0$ (both depending on the point (t_0, x_0)) such that

- (i) the closure of the ball centered at (t_0, x_0) with radius ε is contained in $I \times \mathcal{O}$, i.e.,

$$\overline{\mathbb{B}}((t_0, x_0); \varepsilon) \subset I \times \mathcal{O};$$

- (ii) for any $(t, x_1), (t, x_2) \in \overline{\mathbb{B}}((t_0, x_0); \varepsilon)$, f satisfies

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|. \quad (1.12)$$

When (1.12) holds true for all $(t, x_1), (t, x_2) \in I \times \mathcal{O}$ and the constant L does not depend on the point (t_0, x_0) , then we say that f is *globally Lipschitz with respect to the variable x* on $I \times \mathcal{O}$.

The following Picard theorem allows one to prove the existence and uniqueness of local solutions to the IVP (1.6).

Theorem 1.2. (*Picard's theorem*) Assume that f is continuous with respect to t and locally Lipschitz with respect to x on $I \times \mathcal{O}$. Then, for each $(t_0, x_0) \in I \times \mathcal{O}$, there exists $\delta > 0$ such that the IVP (1.6) possesses a unique solution φ on the interval $I_\delta = [t_0 - \delta, t_0 + \delta]$.

Remark 1.3. The Picard's theorem ensures a unique solution on each single "existence interval" I_δ , but does not ensure the uniqueness of solutions on two or more intersecting existence intervals. In fact, the solutions of (1.6) coincide on any intersection of existence intervals, as stated in the following theorem.

Theorem 1.3. (*Uniqueness of solution*) Assume that f is continuous with respect to t and locally Lipschitz with respect to x on $I \times \mathcal{O}$. Given any $(t_0, x_0) \in I \times \mathcal{O}$, let $\varphi_1(t)$ be the solution to IVP (1.6) on interval I_1 and $\varphi_2(t)$ be the solution to IVP (1.6) on interval I_2 . Then

$$\varphi_1(t) \equiv \varphi_2(t), \quad \forall t \in I_1 \cap I_2.$$

We are interested in studying the long time behavior of the solutions, and thus it is important to ensure that the IVP has solutions defined globally in time, i.e., for all $t \geq t_0$. To this end, we need to introduce the concept of the *maximal solution* to (1.6), a solution which is defined on the largest possible interval $[t_0, T_{max})$ such that there is no other solutions defined on a bigger interval. Hence if $T_{max} = +\infty$ then the solution to (1.6) is defined globally in time. Before defining the maximal solution, we first recall the continuation of local solutions to IVP (1.6) as follows.

Definition 1.6. Consider $(t_0, x_0) \in I \times \mathcal{O}$ and let $\varphi(t)$ be the solution to the IVP (1.6) on interval $J \subset I$. It is said that solution $\varphi(t)$

- (a) can be *continued on the right*, if there exists another solution $\phi(t)$ to IVP (1.6) on interval J_1 , such that $J \subset J_1$ and $\sup J$ belongs to the interior of J_1 .

(b) can be *continued on the left*, if there exists another solution $\phi(t)$ to IVP (1.6) on interval J_2 , such that $J \subset J_2$ and $\inf J$ belongs to the interior of J_2 .

(c) is *continuable*, if it can be continued on the right or on the left, or both.

Definition 1.7. A solution to the IVP (1.6) is called a *maximal solution* if it is not continuable.

Remark 1.4. Let $\varphi(t)$ be a solution to IVP (1.6) on interval J that can be continued on the right, and let $\phi(t)$ be another solution to IVP (1.6) on interval J_1 such that $J \subset J_1$ and $\sup J$ belongs to the interior of J_1 . Then, as a direct consequence of Theorem 1.3, φ and ϕ coincide on the intersection of both intervals, i.e. on J . This implies that ϕ is merely a continuation of φ on the right end of J . Similar observation can be done on the left side.

The following theorem presents the existence and uniqueness of a maximal solution to the IVP (1.6).

Theorem 1.4. (*Existence and uniqueness of a maximal solution*) Let O be an open, nonempty, and connected subset of \mathbb{R}^d and let $f : I \times O \rightarrow \mathbb{R}^d$ be continuous with respect to t and locally Lipschitz with respect to x . Then, for each $(t_0, x_0) \in O$, there exists a unique maximal solution to the IVP (1.6). Moreover, the interval of definition of such a maximal solution, denoted by $I_{\max} = I_{\max}(t_0, x_0)$, is open.

Denote by (m_1, m_2) the maximal interval of existence. Then the ideal situation would be $m_2 = +\infty$, in which case the solution exists for all time t . The following theorem provides a method for determining when a solution is global, i.e., defined for all time t . In particular, if $f(t, x)$ is defined on all of $\mathbb{R} \times \mathbb{R}^d$, then a solution is global if it does not blow up in finite time, which is consistent with Remark 1.1 stated earlier without detailed justification.

Theorem 1.5. Let (m_1, m_2) denote the maximal interval of existence for the IVP (1.6). If $|m_2| < \infty$, then $\lim_{t \rightarrow m_2^-} |\varphi(t)| = \infty$. Similarly for m_1 .

We next discuss the dependence of solutions on the initial data of IVP (1.6). Notice that a maximal solution of (1.6) can also be considered as a function of the initial value (t_0, x_0) on $I \times O$. In fact, define the set

$$\Theta := \{(t, t_0, x_0) \in \mathbb{R}^{d+2} : (t_0, x_0) \in O \text{ and } t \in I_{\max}(t_0, x_0)\}. \quad (1.13)$$

Then we can formulate the solution mapping φ as

$$\begin{aligned} \varphi : \Theta \subset \mathbb{R}^{d+2} &\rightarrow \mathbb{R}^d \\ (t, t_0, x_0) \in \Theta &\mapsto \varphi(t; t_0, x_0) \in \mathbb{R}^d. \end{aligned}$$

Such a formulation of the solution mapping is called the *maximal solution of the IVP (1.6) expressed in terms of the initial values*.

It is known that the maximal solution $\varphi(\cdot; t_0, x_0)$ is continuously differentiable with respect to the time variable t on the interval $I_{\max}(t_0, x_0)$. A natural question would

then be whether or not the maximal solution $\varphi(t; \cdot, \cdot)$ expressed in terms of the initial values is also continuous with respect to the initial value (t_0, x_0) . The answer is YES as stated in the following theorem.

Theorem 1.6. (*Continuous dependence on the initial values*) Assume that O is a nonempty, open and connected subset of \mathbb{R}^d and $f : I \times O \rightarrow \mathbb{R}^d$ is continuous with respect to t and locally Lipschitz with respect to x . Let Θ be defined in (1.13). Then the global solution $\varphi(\cdot; \cdot, \cdot)$ of the IVP (1.6) expressed in terms of the initial values is continuous in Θ , i.e. $\varphi \in C^0(\Theta; \mathbb{R}^d)$.

Existence and uniqueness of global solutions, along with the continuous dependence of the global solutions on initial conditions, are the essential prerequisites of the study of dynamical systems. In the theories of dynamical systems, we will assume these hold, while in the applications we always need to prove these before using the theories of dynamical systems.

The rest of the book is organized as follows. In Chapter 2 we will introduce the basic concepts from the theory of autonomous dynamical systems which are easier to understand, and can serve as the motivation for the study of non-autonomous and random dynamical systems. In Chapter 3 we will set up the framework of non-autonomous dynamical systems, that can be regarded as a natural extension of the autonomous theory, but allows a larger variety of time dependence forces than other alternatives such as skew-product flows or cocycles. In particular we will describe various approaches to analyze the long time behavior of non-autonomous dynamical systems, with an emphasis on the novel theory of pullback attractors. Then in Chapter 4 we will introduce the theory of random dynamical systems and random pullback attractors and show how suitable is to analyze real-world problems with randomness. In each chapter after establishing the main concepts, we will apply them to analyze the long term behavior of several interesting models arising in applied sciences and engineering. In particular, the chemostat model, the SIR epidemic model and the Lorenze-84 model are studied in their autonomous, nonautonomous and stochastic/random formulations.

Chapter 2

Autonomous dynamical systems

The theory of autonomous dynamical systems is now well established after being studied intensively over the past years. In this chapter we will provide a brief review of autonomous dynamical systems, as the background and motivation to introduce nonautonomous and random dynamical systems which are the major topics of the book.

An ordinary differential equation (ODE) is said to be an *autonomous differential equation* if the right hand side does not depend on time explicitly, i.e., it can be formulated as

$$\frac{dx}{dt} = g(x), \quad (2.1)$$

where $g : O \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a mapping from an open subset O of \mathbb{R}^d to \mathbb{R}^d . This is a particular case of the system in (1.6) where $I = \mathbb{R}$ and the vector field g does not depend on the time t . Associating Equation (2.1) with an initial datum $x(t_0) = x_0 \in O$ gives the following IVP

$$\frac{dx}{dt} = g(x), \quad x(t_0) = x_0. \quad (2.2)$$

To apply the general results presented in Chapter 1 to obtain the existence and uniqueness of a maximal solution to equation (2.2), we only need to impose a locally Lipschitz assumption on function g . Note that if a function is locally Lipschitz with respect to all its variables, then it is continuous. But in general, the continuity of g guarantees only the existence of solutions to (2.1) (see e.g., [25]), but not the uniqueness.

Remark 2.1. One effective way to check if a function satisfies a Lipschitz condition is to check if it is continuously differentiable. A continuously differentiable function is always locally Lipschitz (see, e.g., [66]), hence every IVP problem (2.2) with $g \in C^1(O)$ possesses a unique maximal solution. Moreover, if the domain O is convex, then a continuously differentiable function is globally Lipschitz if and only if the partial derivatives $\frac{\partial g_i}{\partial x_j}(x), i, j = 1, 2, \dots, d$, are globally bounded.

A general theorem on existence and uniqueness of a maximal solution to the IVP (2.2) is stated below.

Theorem 2.1. (*Existence and uniqueness of a maximal solution*) Let \mathcal{O} be an open subset of \mathbb{R}^d and assume that g is continuously differentiable on \mathcal{O} . Then for any $t_0 \in \mathbb{R}$ and any $x_0 \in \mathcal{O}$ the initial value problem (2.2) has a unique maximal solution $\varphi(\cdot; t_0, x_0)$ defined in its maximal open interval $I_{max} = I_{max}(t_0, x_0)$.

Remark 2.2. As our main interest is the long term behavior of solutions to (2.2), we will focus on the cases when I_{max} contains the interval $[t_0, +\infty)$. This means that the solution $\varphi(\cdot; t_0, x_0)$ is globally defined in all future times, i.e., is a *global solution*. However, note that the existence of a global solution is not free; it requires conditions such as in Theorem 1.1.

An important property of autonomous ODEs, which can be easily verified, is that the solution mapping depends only on the elapsed time $t - t_0$ but not separately on the initial time t_0 and current time t (see, e.g., [51]), i.e.,

$$\varphi(t - t_0; 0, x_0) = \varphi(t; t_0, x_0), \quad \text{and} \quad I_{max}(t_0, x_0) = t_0 + I_{max}(0, x_0),$$

for all $x_0 \in \mathcal{O}$, $t_0 \in \mathbb{R}$, and $t \in I_{max}(t_0, x_0)$. Therefore for autonomous ODEs we can always focus on $t_0 = 0$. With $t_0 = 0$ the solution can be written as $\varphi(t; x_0)$ and the existence interval of maximal solution can be written as $I_{max}(x_0)$. It is straightforward to check that this solution mapping $\varphi(\cdot; \cdot) : \mathbb{R} \times \mathcal{O} \rightarrow \mathbb{R}^d$ satisfies the initial value property (1.3) and the group property (1.4) (when $I_{max} = \mathbb{R}$), and hence defines a dynamical system, namely an *autonomous dynamical system*.

Next we will provide a survey on the long term behavior for autonomous dynamical systems. In particular, we will start from the stability theory of linear ODE systems, followed by the stability of nonlinear ODE systems by the first approximation method. Then we will introduce basic Lyapunov theory for stability and asymptotic stability. Some comments on globally attracting sets will be provided via the LaSalle theorem and the Poincaré–Bendixson theorem in dimension $d = 2$, and serve as a motivation for the analysis of global asymptotic behavior in higher dimensions. In the end we will introduce the general concept of attractors and their main properties for autonomous dynamical systems.

2.1 Basic Stability Theory

For ease of understanding, we start from the stability of equilibrium points of system (2.1). Recall that an *equilibrium point* (or *steady state*) x^* of system (2.1) is a constant solution to (2.1) satisfying $g(x^*) = 0$.

Definition 2.1. An equilibrium x^* is said to be

- *stable* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|x_0 - x^*| < \delta(\varepsilon)$ then $I_{max}(x_0) \supseteq [0, +\infty)$ and

$$|\varphi(t; x_0) - x^*| < \varepsilon, \quad \forall t \geq 0.$$

(solutions starting close to one equilibrium point remain close to it in the future.)

- *convergent or attractive* if there exists $\delta > 0$ such that if $|x_0 - x^*| < \delta$ then $I_{max}(x_0) \supseteq [0, +\infty)$ and

$$\lim_{t \rightarrow \infty} \varphi(t; x_0) = x^*.$$

(solutions starting close to one equilibrium point will converge to it when time goes to infinity.)

- *asymptotically stable* if it is both stable and convergent.
- *exponentially stable* if there exist $\delta > 0$ and $\alpha, \lambda > 0$ such that if $|x_0 - x^*| < \delta$ then $I_{max}(x_0) \supseteq [0, +\infty)$ and

$$|\varphi(t; x_0) - x^*| < \alpha |x_0 - x^*| e^{-\lambda t}, \quad \forall t \geq 0.$$

Remark 2.3. Definition 2.1 includes only the stability for an equilibrium point, i.e., x^* is constant, but can be easily generalized to any non-constant particular solution of equation (2.1). More precisely, a particular solution $x^*(t)$ to (2.1) is said to be stable if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|x_0 - x^*(0)| < \delta(\varepsilon)$ then $I_{max} \supseteq [0, +\infty)$ and

$$|\varphi(t; x_0) - x^*(t)| < \varepsilon, \quad \forall t \geq 0.$$

Following a similar manner all the other concepts in Definition 2.1 can be generalized to any particular solution of (2.1).

It is worth mentioning that exponential stability implies asymptotic stability, and asymptotic stability implies stability and convergence. However, stability and convergence are independent properties. Convergence implies stability for linear ODEs (see, e.g., [81]), but does not imply stability in general. For example, the autonomous ODE system

$$\frac{dx(t)}{dt} = \frac{x^2(y-x) + y^5}{(x^2 + y^2)(1 + (x^2 + y^2)^2)}; \quad \frac{dx(t)}{dt} = 0 \text{ for } x = 0, y = 0, \quad (2.3)$$

$$\frac{dy(t)}{dt} = \frac{y^2(y-2x)}{(x^2 + y^2)(1 + (x^2 + y^2)^2)}; \quad \frac{dy(t)}{dt} = 0 \text{ for } x = 0, y = 0. \quad (2.4)$$

has an isolated equilibrium at $(0, 0)$ that is convergent but unstable (see Fig. 2.1).

Remark 2.4. In the autonomous framework, all the stability concepts in Definition 2.1 are uniform in time, i.e., the choice of δ does not depend on time. But this does not hold true for nonautonomous ODEs, which requires new definitions to distinguish uniform and non-uniform type of stability (see, e.g., [71]).

Notice that a very simple change of variable can reduce the problem of analyzing the stability of any solution to (2.1), to the problem of analyzing the stability of the zero (or trivial) solution $\varphi_0(t) \equiv 0$ (for all $t \in \mathbb{R}$) of a corresponding system of ODEs.

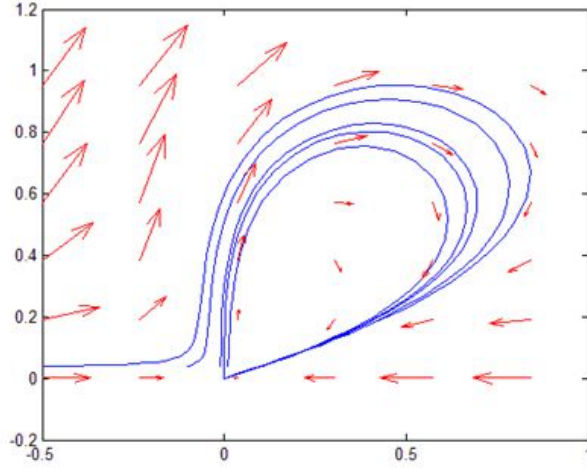


Fig. 2.1: Attractivity and convergence of example (2.3) – (2.4).

More precisely, assume that $x^*(t)$ is a non-zero particular solution of system (2.1), and consider the change of variables $y(t) = x(t) - x^*(t)$, then it is easy to check that

$$\frac{dy(t)}{dt} = \tilde{g}(t, y(t)), \quad \text{where } \tilde{g}(t, y) = g(y + x^*(t)) - g(x^*(t)),$$

in which the right hand side satisfies $\tilde{g}(t, 0) = 0$ for all $t \in \mathbb{R}$. In the special case that $x^*(t) = x^*$ is a constant for all $t \in \mathbb{R}$, $\tilde{g}(t, y) = \tilde{g}(y) = g(y + x^*) - g(x^*)$. This observation allows us to focus the stability analysis to the zero solution by assuming that function g in (2.1) satisfies $g(0) = 0$.

Stability for linear ODEs

One important fact of linear ODE systems is that all solutions possess the same type of stability. Moreover, the type of stability can be characterized by the asymptotic behavior of its fundamental matrix. For the autonomous system (2.1), the fundamental matrix can be determined by the eigenvalues of the matrix on the right hand side. More specifically, consider the following linear system of ODEs

$$\frac{dx}{dt} = Ax + b(t), \quad (2.5)$$

where $A = (a_{ij})_{i,j=1,\dots,d}$ is a $d \times d$ matrix with real coefficients $a_{ij} \in \mathbb{R}$. Clearly if $x^*(t)$ is a solution to (2.5), then $y(t) = x(t) - x^*(t)$ is solution to the ODE system

$$\frac{dy}{dt} = Ay. \quad (2.6)$$

The same holds true if $x^*(t)$ is a solution to the homogeneous counterpart of equation (2.5), $x' = Ax$. Consequently, the stability of any solution of the linear ODE system (2.5) is equivalent to the stability of the zero solution to (2.6), regardless whether or not $b(t)$ is zero. The theorem presents the stability of linear ODEs.

Theorem 2.2. *Let $\{\lambda_j\}_{1 \leq j \leq d} \subset \mathbb{C}$ be the set of eigenvalues for the matrix A . Then,*

- (i) *any solution to (2.6) is exponentially stable if and only if the real parts of all the eigenvalues are negative, i.e., $\Re(\lambda_j) < 0$ for all $1 \leq j \leq d$.*
- (ii) *any solution to (2.6) is (uniformly) stable if and only if $\Re(\lambda_j) \leq 0$ for all $1 \leq j \leq d$ and, if for those eigenvalues λ_j ($1 \leq j \leq d$) such that $\Re(\lambda_j) = 0$, the dimension of the Jordan boxes associated to them in their canonical forms is 1 (in other words, their algebraic and geometric multiplicity coincide).*

Stability for nonlinear ODEs

Theorem 2.2 can characterize completely the stability of linear ODE systems with constant coefficients. It can also be used to analyze the stability of nonlinear differential equations, by the so called *first approximation method* which can be briefly described as follows. Assume that the function g in (2.1) is continuously differentiable and satisfies $g(0) = 0$. Then according to the Taylor formula, g can be written as

$$g(x) = Jx + T_1(x), \quad J = \left(\frac{\partial g_i}{\partial x_j}(0) \right)_{i,j=1,\dots,d} \quad (2.7)$$

where the higher order term $T_1(\cdot)$ is sufficiently small for small values of x in the sense that

$$\lim_{x \rightarrow 0} \frac{|T_1(x)|}{|x|} = 0.$$

We now state the following result on stability and instability of solutions to system (2.1).

Theorem 2.3. *(Stability in first approximation) Assume that $g \in C^1(\mathcal{O})$ and $g(0) = 0$. Let J be the Jacobian matrix defined in (2.7). Then,*

- (i) *if all the eigenvalues of matrix J have negative real parts, the trivial solution of (2.1) is (locally) exponentially stable.*
- (ii) *if one of the eigenvalues of matrix J has positive real part, the trivial solution of (2.1) is (locally) unstable.*

Stability by the Lyapunov theory

Theorem 2.3 is based on a spectrum analysis of the linearization of system (2.1), and provides only local stability. For nonautonomous systems, such linearization

requires additional justification (see, e.g., [55]). We next introduce the Lyapunov theory, that allows us to determine the stability of a system without either explicitly solving the differential equation (2.1) or approximating it by its first approximation linear system. Practically, the theory is a generalization of the idea that if there exists some “measure of energy” in one dynamical system, then we can study the rate of change of the energy to ascertain stability. In fact this “measure of energy” can be characterized by the so-called *Lyapunov function*. Roughly, if there exists a function $V : X \rightarrow \mathbb{R}$ satisfying certain conditions on V and \dot{V} (the derivative along solution trajectories) that proves the Lyapunov stability of a system, we call it a Lyapunov function.

Definition 2.2. A continuous function $V : \mathcal{O} \in \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be

- *positive definite* around $x = 0$ if $V(0) = 0$ and $V(x) > 0$ for all $x \in \mathcal{O} \setminus \{0\}$.
- *positive semi-definite* around $x = 0$ if $V(0) = 0$ and $V(x) \geq 0$ for all $x \in \mathcal{O} \setminus \{0\}$.
- *negative definite* or *negative semi-definite* if $-V$ is positive definite or positive semi-definite, respectively.

The following basic theorems provide sufficient conditions for the stability and instability of the origin of an autonomous dynamical system. First we recall that given a continuously differentiable function $V : \mathcal{O} \rightarrow \mathbb{R}$, the derivative of V along the trajectories of system (2.1), $\dot{V} : X \rightarrow \mathbb{R}$ is defined by

$$\dot{V}(x) = \sum_{i=1}^d \frac{\partial V}{\partial x_i}(x) g_i(x), \quad x \in \mathcal{O}.$$

It is straightforward to check that if $x(\cdot)$ is a solution to (2.1), then

$$\frac{d}{dt} V(x(t)) = \dot{V}(x(t)), \quad \forall t \in \mathbb{R}.$$

Next we present the Lyapunov theorem on stability and the Tchetaev theorem on instability based on the Lyapunov functions.

Theorem 2.4. (*Lyapunov’s stability theorem*) Let $V : \mathcal{O} \rightarrow \mathbb{R}$ be a continuously differentiable function with derivative \dot{V} along the trajectories of the system (2.1).

1. If V is positive definite and \dot{V} is negative semi-definite, then the zero solution is stable.
2. If V is positive definite and \dot{V} is negative definite, then the zero solution is asymptotically stable.
3. If there exist some positive constants a_1, a_2, a_3 and k such that

$$a_1|x|^k \leq V(x) \leq a_2|x|^k \quad \text{and} \quad \dot{V}(x) \leq -a_3|x|^k, \quad \forall x \in \mathcal{O},$$

then the zero solution is exponentially stable.

Denote by $\mathbb{B}(x_0; r)$ the ball centered at x_0 with radius r .

Theorem 2.5. (*Tchetaev's instability theorem*) Assume that there exists $\rho > 0$ and $V \in C^1(\mathbb{B}(0; \rho))$ such that $\mathbb{B}(0; \rho) \subseteq \mathcal{O}$ and

- (i) $V(0) = 0$,
- (ii) \dot{V} is positive definite in $\mathbb{B}(0; \rho)$,
- (iii) for any $\sigma \in (0, \rho)$ there exists $y_\sigma \in \mathbb{B}(0; \sigma)$ such that $V(y_\sigma) > 0$.

Then the zero solution is unstable.

Sometimes an equilibrium point can be asymptotically stable even if \dot{V} is not negative definite. In fact, if we can find a Lyapunov function whose derivative along the trajectories of the system is only negative semi-definite, but we can further establish that no trajectory can stay identically at points where \dot{V} vanishes, then the equilibrium must be asymptotically stable. This is the idea of LaSalle's invariance principle [54]. Before stating the principle we first introduce the definition of ω -limit set and invariant set, which is needed to state the LaSalle theorem.

Let $\varphi(t; x_0)$ be the autonomous dynamical system generated by the solutions of IVP (2.2).

Definition 2.3. A set $S \subset \mathbb{R}^d$ is said to be the ω -limit set of $\varphi(t; x_0)$ if for every $x \in S$, there exists a strictly increasing sequence of times t_n such that

$$\varphi(t_n; x_0) \rightarrow x \quad \text{as } t_n \rightarrow \infty.$$

It is usual to write $S = \omega(x_0)$. In a similar way, it is defined the omega limit of a set $A \subset \mathcal{O}$, and it is denoted as $\omega(A)$, as the set of points $x \in \mathcal{O}$ such that there exist two sequences $\{x_n\} \subset A$, $t_n \rightarrow +\infty$ such that

$$\varphi(t_n; x_n) \rightarrow x, \quad \text{as } n \rightarrow +\infty.$$

Definition 2.4. A set $M \subset \mathbb{R}^d$ is said to be (*positively*) *invariant* if for all $x \in M$ we have

$$\varphi(t; x) \in M, \quad \forall t \geq 0.$$

Remark 2.5. The positive invariance means that as long as a solution passes a point inside M it will remain inside M forever, although the solution may have been outside of M in some previous instants of time.

Theorem 2.6. (*LaSalle's Invariance Principle*) Let $K \subset X$ be a compact and positively invariant set. Let $V : K \subset \mathbb{R}^d \rightarrow \mathbb{R}$ be continuously differentiable with $\dot{V} \leq 0$ on K . Let M be the largest invariant set in $E = \{x \in K : \dot{V} = 0\}$. Then $\varphi(t; x_0)$ approaches M as $t \rightarrow \infty$ for every $x_0 \in K$.

Remark 2.6. LaSalle's Invariance principle requires V to be continuously differentiable but not necessarily positive definite. It is applicable to any equilibrium set, rather than just an isolated equilibrium point. But when M is just a single point, it provides additional information about the type of stability of the equilibrium point. Indeed, when M is just a single point, and we are able to find a Lyapunov function

which is only negative semi-definite, we can then ensure that this equilibrium is stable (thanks to Theorem 2.4) and also convergent as a consequence of LaSalle's principle, and hence is asymptotically stable.

To illustrate how the LaSalle invariance principle work, consider the following second order differential equation describing the movement of a pendulum with friction

$$\frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + k \sin x = 0,$$

where k and β are positive constants. First we transform the equation into an equivalent first order system by setting $x = y_1$, $\frac{dx}{dt} = y_2$ to obtain

$$\begin{aligned} \frac{dy_1}{dt} &= y_2, \\ \frac{dy_2}{dt} &= -\beta y_2 - k \sin y_1. \end{aligned}$$

Consider the function

$$V(y_1, y_2) = \frac{1}{2}y_2^2 + k(1 - \cos y_1).$$

It is easy to prove that V is positive definite on $\mathbb{B}(0; \pi)$ and satisfies

$$\dot{V}(y) = (k \sin y_1) y_2 + y_2(-\beta y_2 - k \sin y_1) = -\beta y_2^2 \leq 0, \quad \forall (y_1, y_2) \in \mathbb{B}(0; \pi).$$

Therefore, \dot{V} is negative semi-definite and, consequently, the trivial solution is stable. To prove that the trivial solution is also attractive, we will use LaSalle's invariance principle. Denote by

$$E := \{y \in \mathbb{B}(0; \pi) : \dot{V}(y) = 0\} \equiv \{(y_1, 0) : y_1 \in (-\pi, \pi)\}.$$

If we prove that $\{(0, 0)\}$ is the only positively invariant subset of E , then it will be also attractive, and thus the trivial solution will be uniformly asymptotically stable. To this end, pick $y_0 = (y_{01}, 0) \in E$, with $y_{01} \in (-\pi, \pi) \setminus \{0\}$, then the solution $\varphi(t; 0, y_0) = (\varphi_1(t), \varphi_2(t))$ satisfies the differential system as well as the initial condition $(\varphi_1(0), \varphi_2(0)) = y_0 = (y_{01}, 0)$. Notice that

$$\begin{aligned} \varphi'(0) &= \varphi_2(0) = 0, \\ \varphi_2'(0) &= -k\varphi_2(0) - k \sin \varphi_1(0) = -k \sin y_{01} \neq 0 \quad (y_{01} \in (-\pi, \pi) \setminus \{0\}). \end{aligned}$$

Therefore, the function φ_2 is strictly monotone in a neighborhood of $t = 0$, and since $\varphi_2(0) = 0$, there exists $\tilde{t} \in I_{\max}(y_0)$ such that $\varphi_2(\tilde{t}) \neq 0$. Thus, solutions starting from points of $E \setminus \{(0, 0)\}$ leave this set and, therefore, the unique invariant subset is $(0, 0)$.

Remark 2.7. The LaSalle invariance principle is applicable to autonomous or periodic systems and can be extended to some specific nonautonomous systems (see, e.g., [65]), but not to general nonautonomous systems.

Remark 2.8. The Lasalle invariance principle provides a natural connection between the Lyapunov stability and the concept of attractors, to be introduced in the next section.

The largest invariant set M in Theorem 2.6 is the union of all invariant sets in the compact set K . It contains critical information on the asymptotic behavior of the system, as any solution has to approach this set as time goes on. In fact, the asymptotic dynamics of an autonomous dynamical system can be fully characterized by its invariant sets [51]. An invariant set possesses an independent dynamics inside itself and can also determine if any other trajectory outside the invariant set is approaching it (attractor) or being repelled from it (repeller). Next we will introduce in more details the concept of attractor, which is an invariant compact set that attracts all trajectories of a dynamical system starting either in a neighborhood (local attractor) or in the entire state space (global attractor).

2.2 Attractors

First we generalize the definition of invariance established in the previous section.

Definition 2.5. Let $\varphi : \mathbb{R} \times \mathcal{O} \rightarrow \mathcal{O}$ be a dynamical system on \mathcal{O} . A subset M of \mathcal{O} is said to be

- invariant under φ (or φ -invariant), if

$$\varphi(t, M) = M \quad \text{for all } t \in \mathbb{R}.$$

- positively invariant under φ if

$$\varphi(t, M) \subset M \quad \text{for all } t \in \mathbb{R}.$$

- negatively invariant under φ if

$$\varphi(t, M) \supset M \quad \text{for all } t \in \mathbb{R}.$$

For any $x \in \mathcal{O}$, the function $\varphi(\cdot, x) : \mathbb{R} \rightarrow \mathcal{O}$ defines a solution curve, trajectory, or orbit of (2.2) passing through the point x_0 in \mathcal{O} . Graphically, the function $\varphi(\cdot, x)$ can be thought as an object moving along the curve

$$\gamma(x_0) := \{x \in \mathcal{O} \mid x = \varphi(t; x_0), t \in \mathbb{R}\}$$

defined by (2.2), as well as a possible parametrization of the orbit $\gamma(x_0)$ passing through the point x_0 . A local attractor of a dynamical system is a compact invariant set that attracts all trajectories starting in some neighborhood of the attractor as $t \rightarrow \infty$, and a global attractor is such a compact invariant set that attracts not only

trajectories in a neighborhood but trajectories in the entire state space. In this book we will focus on global attractor whose precise definition is given below.

Definition 2.6. A nonempty compact subset \mathcal{A} is called a global attractor for a dynamical system φ on O if

- (i) \mathcal{A} is φ -invariant;
- (ii) \mathcal{A} attracts all bounded sets of O , i.e.,

$$\lim_{t \rightarrow \infty} \text{dist}(\varphi(t, B), \mathcal{A}) = 0 \quad \text{for any bounded subset } B \subset O,$$

where dist denotes the Hausdorff semi-distance given by

$$\text{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b|.$$

The existence of attractors is mostly due to some dissipativity property (a loss of energy) of the dynamical system. The mathematical formulation for this concept is given in the following definition.

Definition 2.7. A nonempty bounded subset K of O is called an *absorbing set* of a dynamical system φ on O if for every bounded subset $B \subset O$, there exists a time $T_B = T(B) \in \mathbb{R}$ such that $\varphi(t, B) \subset K$ for all $t > T_B$.

The following theorem stated in [51] provides the existence of a unique global attractor.

Theorem 2.7. *Assume that a dynamical system φ on O possesses a compact absorbing set K . Then, there exists a unique global attractor $\mathcal{A} \subset K$ which is given by*

$$\mathcal{A} = \omega(K).$$

If, in addition, K is positively invariant, then the unique global attractor is given by

$$\mathcal{A} = \bigcap_{t \geq 0} \varphi(t, K).$$

Global attractors are crucial for the analysis of dynamical systems, as they can characterize their asymptotic behavior. On the one hand, any trajectory starting from inside the global attractor is not allowed to leave it (because its invariance), and the original dynamical system restricted to the global attractor forms another dynamical system. On the other hand, any trajectory starting from a point outside the global attractor has to approach it, but can never touch it. Due to the complexity of the trajectories, the global attractor may eventually exhibit a strange and chaotic structure. Therefore, in addition to the general existence and continuity properties of the global attractors, geometrical structures of the global attractors can provide more detailed information about the long term dynamics of a dynamical system. We will not elaborate this point in this section, but will include below a few more comments

including one example which can help the reader to understand the importance of the global attractor.

There are some particular dynamical systems, such as gradient dynamical systems, for which the internal geometrical structure of the global attractor is well known. These are the dynamical systems for which a “Lyapunov function” exists, but in this context is a continuous function V that (i) is nonincreasing along the trajectories of the dynamical system, i.e., the mapping $t \mapsto V(\varphi(t, x_0))$ is nonincreasing for any $x_0 \in \mathcal{O}$, and (ii) if it is constant along one trajectory, then this trajectory must be an equilibrium point. In fact, if $V(\varphi(t, x_0)) = V(x_0)$ for some $t > 0$, then x_0 is an equilibrium point. This concept is slightly different for the concept of Lyapunov function used in the stability analysis, but we will still adopt the same terminology while the context is clear.

One simple example of a gradient ordinary differential equation is

$$\frac{dx}{dt} = -\nabla h(x), \quad (2.8)$$

where $h : \mathbb{R}^d \rightarrow \mathbb{R}$ is at least continuously differentiable. It is straightforward to verify that $V(x) := h(x)$ is a Lyapunov function for the dynamical system generated by (2.8). Let $\varphi(t, x_0)$ be the solution to (2.8) with initial condition $x(0) = x_0$, and let $\omega(x_0)$ be the ω -limit set of the orbit through x_0 . Then $\omega(x_0)$ is a compact invariant set in \mathbb{R}^d . According to the LaSalle invariance principle, for any $x \in \omega(x_0)$, the solution $\varphi(t, x)$ belongs to $\omega(x_0)$ and $V(\varphi(t, x)) = V(x)$ for all $t \in \mathbb{R}$. As a consequence, $\dot{V}(\varphi(t, x)) = 0$, which implies that $\nabla h(\varphi(t, x)) = 0$ for all $t \in \mathbb{R}$, i.e., $\omega(x_0)$ belongs to the set of equilibria of (2.8). More details on gradient dynamical systems can be found in [38].

The most interesting property of gradient dynamical systems is that their attractors are formed by the union of the unstable manifold of the equilibrium points (see, e.g. [53]). Briefly, given an equilibrium point x^* for the dynamical system φ , its unstable manifold is defined by

$$\mathcal{W}^u(x^*) = \{x \in \mathcal{O} : \varphi(t, x) \text{ is defined for } t \in \mathbb{R}, \text{ and } \varphi(t, x) \rightarrow x^*, \text{ as } t \rightarrow -\infty\}.$$

Then, if φ is a gradient dynamical system, it holds that the global attractor \mathcal{A} is given by

$$\mathcal{A} = \bigcup_{x^* \in \mathcal{E}} \mathcal{W}^u(x^*),$$

where \mathcal{E} denotes the set of all the equilibrium points.

Another interesting aspect of global attractors is related to *how* the global attractor determines the asymptotic dynamics of the system. According to the definition of the global attractor, we can say that any trajectory outside the global attractor can be tracked by some trajectories (or pieces of trajectories) inside the attractor. In other words, any external trajectory has a “target” on the attractor, that is getting closer to the trajectory as time passes. This property is known as the “tracking property” reads as follows. Given a trajectory $\varphi(t, x_0)$ with x_0 not necessarily inside the

global attractor \mathcal{A} , and given any $\varepsilon > 0$ and $T > 0$, there exists a time $\tau = \tau(\varepsilon, T)$ and a point $v_0 \in \mathcal{A}$ such that

$$|\varphi(t + \tau, x_0) - \varphi(t, v_0)| \leq \varepsilon, \text{ for all } 0 \leq t \leq T.$$

If we wish to follow a trajectory for a time longer than T , then we may need to use more than one trajectory of \mathcal{A} (see e.g., [68]).

Remark 2.9. There are several other interesting and important properties of the global attractor which can also help in understanding the dynamics of a dynamical system. The reader is referred to the monograph [68] for more details.

2.3 Applications

In this section we will introduce three applications of autonomous systems arising from different areas of the applied sciences. In particular we will discuss the long term dynamics including stability and existence of attractors for (1) a chemostat ecological model, (2) an SIR epidemic model and (3) a Lorenz-84 climate model. Later we will study the nonautonomous and random counterparts of these systems in Chapter 3 and 4, respectively. To simplify the content and to avoid unnecessary repeated calculations, we will assume in this section that given any positive initial condition each of these systems possesses a continuous positive global solution. The proofs will be provided for their corresponding nonautonomous versions in Section 3.

2.3.1 Application to ecology: a chemostat model

A chemostat is associated with a laboratory device which consists of three interconnected vessels and is used to grow microorganisms in a cultured environment (see Fig. 2.2). In its basic form, the outlet of the first vessel is the inlet for the second vessel and the outlet of the second vessel is the inlet for the third. The first vessel is called a feed bottle, which contains all the nutrients required to grow the microorganisms. All nutrients are assumed to be abundantly supplied except one, which is called a *limiting nutrient*. The contents of the first vessel are pumped into the second vessel, which is called the culture vessel, at a constant rate. The microorganisms feed on nutrients from the feed bottle and grow in the culture vessel. The culture vessel is continuously stirred so that all the organisms have equal access to the nutrients. The contents of the culture vessel are then pumped into the third vessel, which is called a collection vessel. Naturally it contains nutrients, microorganisms and the products produced by the microorganisms [76].

As the best laboratory idealization of nature for population studies, the chemostat plays an important role in ecological studies [8, 13, 14, 32, 34, 41, 79, 82, 83, 84].

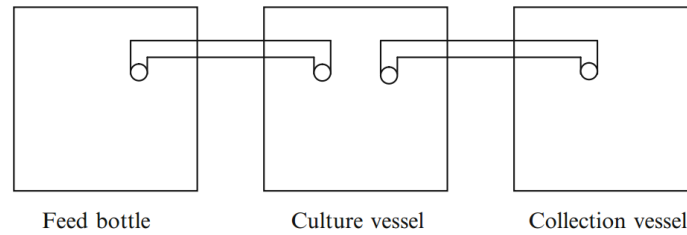


Fig. 2.2: A basic chemostat

With some modifications it is also used as the model for waste-water treatment process [33, 52]. The chemostat model can be considered as the starting point for many variations that yield more realistic biological models, e.g., the recombinant problem in genetically altered organisms [56, 77] and the model of mammalian large intestine [36, 37]. More literature on the derivation and analysis of chemostat-like models can be found in [74, 75, 83] and the references therein.

Denote by x the growth-limiting nutrient and by y the microorganism feeding on the nutrient x . Assume that all other nutrients, except x , are abundantly available, i.e., we are interested only in the study of the effect of this essential limiting nutrient x on the species y . Under the standard assumptions of a chemostat, a list of basic parameters and functional relations in the system includes [76]:

- D , the rate at which the nutrient is supplied and also the rate at which the contents of the growth medium are removed.
- I , the input nutrient concentration which describes the quantity of nutrient available with the system at any time.
- a , the maximal consumption rate of the nutrient and also the maximum specific growth rate of microorganisms – a positive constant.
- U , the functional response of the microorganism describing how the nutrient is consumed by the species. It is known in literature as consumption function or uptake function. Basic assumptions on $U : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are given by

1. $U(0) = 0$, $U(x) > 0$ for all $x > 0$.
2. $\lim_{x \rightarrow \infty} U(x) = L_1$, where $L_1 < \infty$.
3. U is continuously differentiable.
4. U is monotonically increasing.

Note that conditions 1 and 2 of the uptake function U ensure the existence of a positive constant $L > 0$ such that

$$U(x) \leq L \quad \text{for all } x \in [0, \infty).$$

Throughout this book, when solid computation is needed, we assume that the consumption function follows the Michaelis-Menten or Holling type-II form:

$$U(x) = \frac{x}{\lambda + x}, \quad (2.9)$$

where $\lambda > 0$ is the half-saturation constant [76].

Denote by $x(t)$ and $y(t)$ the concentrations of the nutrient and the microorganism at any specific time t . In the simplest model where I and D are both constants, the limited resource-consumer dynamics can be described by the following growth equations:

$$\frac{dx}{dt} = D(I - x) - a \frac{x(t)}{\lambda + x(t)} y(t), \quad (2.10)$$

$$\frac{dy}{dt} = -Dy(t) + a \frac{x(t)}{\lambda + x(t)} y(t). \quad (2.11)$$

Stability analysis

We start from the behavior of equilibrium solutions. The equilibrium solutions to system (2.10) – (2.11) can be found by solving

$$\begin{aligned} D(I - x^*) - a \frac{x^*}{\lambda + x^*} y^* &= 0, \\ -Dy^* + a \frac{x^*}{\lambda + x^*} y^* &= 0, \end{aligned}$$

which yields

$$(x^*, y^*) = (I, 0) \quad \text{or} \quad (x^*, y^*) = \left(\frac{D\lambda}{a - D}, I - \frac{D\lambda}{a - D} \right).$$

1. When $a < D$, system (2.10) – (2.11) has only one axial equilibrium $(I, 0)$, which is globally asymptotically stable. This means that the microorganisms y become extinct.

Sketch of proof: First, it is straightforward to check that the positive quadrant is positively invariant. In fact, first notice that

$$\left. \frac{dx}{dt} \right|_{x=0} = DI > 0.$$

On the other hand, the set $\{(x, 0) : x > 0\}$ is formed by three orbits: (1) the equilibrium point $(I, 0)$, (2) the segment $\{(x, 0) : 0 < x < I\}$ which is parameterized by the solution $\{(I + (x_0 - I)e^{-Dt}, 0) : t \in (-D \log I(I - x_0)^{-1}, +\infty)\}$ for any fixed $x_0 \in (0, I)$, and (3) the unbounded segment $\{(x, 0) : x > I\}$ which is parameterized by the solution $\{(I + (x_0 - I)e^{-Dt}, 0) : t \in \mathbb{R}\}$. We now investigate the stability in first approximation by Theorem 2.3. The corresponding Jacobian of system (2.10) – (2.11) and its value at the point $(I, 0)$ are, respectively,

$$J(x, y) = \begin{pmatrix} -D - \frac{a\lambda y}{(\lambda + x)^2} & -\frac{ax}{\lambda + x} \\ \frac{a\lambda y}{(\lambda + x)^2} & -D + \frac{ax}{\lambda + x} \end{pmatrix}, \quad J(I, 0) = \begin{pmatrix} -D & -\frac{aI}{\lambda + I} \\ 0 & -D + \frac{aI}{\lambda + I} \end{pmatrix}.$$

The eigenvalues of $J(I, 0)$ are $-D$ and $-D + \frac{aI}{\lambda + I}$, which are both negative because $a < D$ is assumed. The equilibrium $(I, 0)$ is then asymptotically exponentially stable.

2. When $a > D$ and $\frac{\lambda D}{a - D} < I$, system (2.10) – (2.11) has two equilibria, among which the positive equilibrium $(x^*, y^*) = \left(\frac{D\lambda}{a - D}, I - \frac{\lambda D}{a - D}\right)$ is globally asymptotically stable. This means that the microorganisms y and the nutrient x co-exist.

Sketch of proof: In this case, the axial equilibrium becomes unstable because one of the eigenvalues of $J(I, 0)$, $-D + \frac{aI}{\lambda + I}$, is positive due to the condition $\frac{\lambda D}{a - D} < I$. On the other hand, the positive equilibrium point (x^*, y^*) is now globally asymptotically (exponentially) stable, since the Jacobian evaluated at (x^*, y^*)

$$J(x^*, y^*) = \begin{pmatrix} -D - \frac{a\lambda x^*}{(\lambda + x^*)^2} & -D \\ \frac{a\lambda y^*}{(\lambda + x^*)^2} & 0 \end{pmatrix},$$

has two negative eigenvalues $-D$ and $-\frac{a\lambda x^*}{(\lambda + x^*)^2}$.

Global attractor

As each IVP associated to (2.10) – (2.11) corresponding to positive initial values has a positive global solution, this system generates an autonomous dynamical system $\varphi(t, x_0, y_0)$. Adding (2.10) – (2.11) we obtain immediately that

$$\frac{d(x + y)}{dt} = DI - D(x + y),$$

and given $x(0) = x_0, y(0) = y_0$ we have

$$x(t) + y(t) = I + (x_0 + y_0 - I)e^{-Dt}.$$

This implies that

$$K_\varepsilon := \{(x, y) \in \mathbb{R}_+^2 : x + y \leq I + \varepsilon\}$$

is a bounded absorbing set for the dynamical system φ generated by solutions of (2.10)– (2.11). Hence due to Theorem 2.7 φ possesses a global attractor \mathcal{A} inside

the nonnegative quadrant \mathbb{R}_+^2 . Moreover, we obtain the geometric structure of the global attractor as follows.

1. When $a < D$, the attractor \mathcal{A} has a single point $(I, 0)$.
2. When $a > D$ and $\frac{\lambda D}{a-D} < I$, the attractor \mathcal{A} consists of two points, $(I, 0)$ and $(\frac{D\lambda}{a-D}, I - \frac{D\lambda}{a-D})$, and heteroclinic solutions between them (solutions that converge in one time direction to a steady state and in the other direction to different steady state).

2.3.2 Application to epidemiology: the SIR model

The modeling of infectious diseases and their spread is crucial in the field of mathematical epidemiology, as it is an important and power tool for gauging the impact of different vaccination programs on the control or eradication of diseases. Here we will only introduce a simple autonomous model, which does not take into account age structure nor environmental fluctuation. More sophisticated models will be discussed in later chapters.

The classical work on epidemics is due to Kermack and McKendrick [42, 43, 44]. The Kermack-McKendrick model is essentially a compartmental model based on relatively simple assumptions on the rates of flow between different classes of members of the population. The population is divided into three classes labeled S , I and R , where $S(t)$ denote the number of individuals who are not yet infected but susceptible to the disease, $I(t)$ denotes the number of infected individuals, assumed infectious and able to spread the disease by contact with susceptible, and $R(t)$ denotes the number of individuals who have been infected and then removed from the possibility of being infected again or of spreading infection. Removal is carried out through isolation from the rest of the population, through immunization against infection, recovery from the disease with full immunity against reinfection, or through death caused by the disease.

The terminology ‘‘SIR’’ is used to describe a disease that confers immunity against reinfection, to indicate that the passage of individuals is from the susceptible class S to the infective class I to the removed class R . Epidemics are usually diseases of this type. The terminology ‘‘SIS’’ is used to describe a disease with no immunity against re-infection, to indicate that the passage of individuals is from the susceptible class to the infective class and then back to the susceptible class. Usually, diseases caused by a virus are of SIR type, while diseases caused by bacteria are of SIS type.

In this book we will use SIR as an example and investigate dynamics of autonomous, nonautonomous and random SIR models. The simplest SIR model assumes that the total population size is held constant, i.e.,

$$S(t) + I(t) + R(t) = N,$$

by making birth and death rates equal. Denote by ν the birth and death rate of all population, and assume that all new-born are susceptible to disease, we have the following basic SIR model.

$$\frac{dS}{dt} = \nu(S + I + R) - \frac{\beta}{N}SI - \nu S, \quad (2.12)$$

$$\frac{dI}{dt} = \frac{\beta}{N}SI - \nu I - \gamma I, \quad (2.13)$$

$$\frac{dR}{dt} = \gamma I - \nu R, \quad (2.14)$$

where γ is the fraction of infected individuals being removed per unit of time, and β is the contact rate of susceptible and infected individuals (see Fig. 2.3).

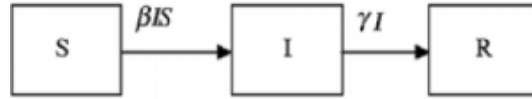


Fig. 2.3: Transition between SIR compartments

The vector field of system (2.12)-(2.14) is continuously differentiable, and hence ensures the maximal solution to the corresponding IVP exists and is unique. Furthermore the solutions are defined globally in time (see Chapter 3 for more details). Notice that assumption of constant total population, $S(t) + I(t) + R(t) = N$, reduces system (2.12) – (2.14) to the following two dimensional system.

$$\frac{dS}{dt} = \nu N - \frac{\beta}{N}SI - \nu S, \quad (2.15)$$

$$\frac{dI}{dt} = \frac{\beta}{N}SI - \nu I - \gamma I. \quad (2.16)$$

Note that such a reduction needs the positiveness of solutions which will be discussed later in Chapter 3.

Stability analysis

The equilibrium solutions to system (2.15) – (2.16) can be found by solving

$$\nu N - \frac{\beta}{N}S^*I^* - \nu S^* = 0,$$

$$\frac{\beta}{N}S^*I^* - \nu I^* - \gamma I^* = 0,$$

which yields two equilibrium solutions

$$(S^*, I^*) = (N, 0), \quad (S^*, I^*) = \left(\frac{N(\nu + \gamma)}{\beta}, \nu N \left(\frac{1}{\nu + \gamma} - \frac{1}{\beta} \right) \right).$$

1. When $\nu + \gamma > \beta$, system (2.15) – (2.16) has only one axial equilibrium, $(N, 0)$, in the nonnegative quadrant, which is globally asymptotically stable. This means that all infected and removed individuals are cleared, the system is fully immunized.

Sketch of proof: The corresponding Jacobian of system (2.15) – (2.16) and its value at the equilibrium point $(N, 0)$ are, respectively,

$$J(S, I) = \begin{pmatrix} -\frac{\beta}{N}I - \nu & \frac{\beta}{N}S \\ \frac{\beta}{N}I & \frac{\beta}{N}S - \nu - \gamma \end{pmatrix}, \quad J(N, 0) = \begin{pmatrix} -\nu & -\beta \\ 0 & \beta - \nu - \gamma \end{pmatrix}.$$

The eigenvalues of $J(N, 0)$ are negative, and hence the axial equilibrium $(N, 0)$ is globally asymptotically stable.

2. When $\nu + \gamma < \beta$, system (2.15) – (2.16) has two equilibria, among which the positive equilibrium $(S^*, I^*) = \left(\frac{N(\nu + \gamma)}{\beta}, \nu N \left(\frac{1}{\nu + \gamma} - \frac{1}{\beta} \right) \right)$ is globally asymptotically stable and the axial equilibrium $(N, 0)$ is unstable. This means that the system is at endemic state, where infected, removed and susceptible individuals co-exist.

Sketch of proof: The instability of the axial equilibrium is obvious as $J(N, 0)$ now has a positive eigenvalue since $\nu + \gamma < \beta$. The Jacobian evaluated at the positive equilibrium (S^*, I^*) is

$$J(S^*, I^*) = \begin{pmatrix} -\beta\nu \left(\frac{1}{\nu + \gamma} - \frac{1}{\beta} \right) - \nu & -(\nu + \gamma) \\ \beta\nu \left(\frac{1}{\nu + \gamma} - \frac{1}{\beta} \right) & 0 \end{pmatrix},$$

whose eigenvalues are the roots of the quadratic equation

$$r^2 + \left(\beta\nu \left(\frac{1}{\nu + \gamma} - \frac{1}{\beta} \right) + \nu \right) r + \beta(\nu + \gamma) \left(\frac{1}{\nu + \gamma} - \frac{1}{\beta} \right) = 0,$$

both of which have negative real part. Hence the positive equilibrium (S^*, I^*) is asymptotic exponential stable.

Attractors

Any IVP of (2.15) – (2.16) associated with positive initial values has positive global solutions (see Chapter 3 for detailed proof). Hence system (2.15) – (2.16) generates an autonomous dynamical system $\varphi(t, S_0, I_0)$. By adding (2.15) – (2.16) we obtain immediately that

$$\frac{d(S+I)}{dt} = \nu N - \nu(S+I) - \gamma I \leq \nu N - \nu(S+I),$$

and given $S(0) = S_0, I(0) = I_0$ we have

$$S(t) + I(t) \leq N + (S_0 + I_0 - N)e^{-\nu t}.$$

This implies that

$$K_\varepsilon := \{(S, I) \in \mathbb{R}_+^2 : S + I \leq N + \varepsilon\}$$

is a bounded absorbing set for the dynamical system φ generated by solutions of (2.15)–(2.16). Hence due to Theorem 2.7, φ possesses a global attractor \mathcal{A} inside the nonnegative quadrant \mathbb{R}_+^2 . Moreover, at light of the stability properties of the equilibrium points, we obtain the geometric structure of the global attractor as follows.

1. When $\nu + \gamma > \beta$, the attractor \mathcal{A} has a single point $(N, 0)$.
2. When $\nu + \gamma < \beta$, the attractor \mathcal{A} consists of two points, $(N, 0)$, and $(\frac{N(\nu+\gamma)}{\beta}, \nu N(\frac{1}{\nu+\gamma} - \frac{1}{\beta}))$, and heteroclinic solutions between them.

2.3.3 Application to climate change: the Lorenz-84 model

A model of atmospheric circulation was introduced by Lorenz in 1984, defined by a system of three nonlinear autonomous differential equations [58, 59, 80]. Let x represents the pole ward temperature gradient or the intensity of the westerly wind current, y and z are the strengths of cosine and sine phases of a chain of superposed waves transporting heat poleward, respectively. Then a modified Hadley circulation can be modeled by

$$\frac{dx}{dt} = -ax - y^2 - z^2 + aF, \quad (2.17)$$

$$\frac{dy}{dt} = -y + xy - bxz + G, \quad (2.18)$$

$$\frac{dz}{dt} = -z + bxy + xz, \quad (2.19)$$

where the coefficient a , if less than 1, allows the westerly wind current to damp less rapidly than the waves, the terms in b represent the displacement of the waves due to interaction with the westerly wind, the terms in F and G are thermal forcings: F represents the symmetric cross-latitude heating contrast and G accounts for the asymmetric heating contrast between oceans and continents.

Despite the simplicity of the Lorenz-84 model, it addresses many key applications in climate studies such as how the coexistence of two possible climates combined with variations of the solar heating causes seasons with inter-annual variability [6, 58, 59, 67], how the climate is affected by the interactions atmosphere and

the oceans [7, 70], how the asymmetry between oceans and continents may result in complex behaviors of the system [62], etc. In addition to applications, the Lorenz-84 model has also attracted much attentions from mathematicians because of certain interesting and subtle mathematical aspects of its underlying differential equations such as multistability, intransitivity and bifurcation [35]. Here we will focus on the Laypunov stability and existence of attractors for the Lorenz-84 model.

Notice that the vector field of system (2.17)-(2.19) is continuously differentiable, which ensures existence and uniqueness of maximal solutions to the corresponding IVP. Moreover, once we prove that the solutions are absorbed by a ball centered at zero, then it follows immediately from Theorem 1.5 that all solutions are defined globally in time (more details will be provided in Chapter 3).

Stability analysis

The equilibrium points of system (2.17) – (2.19) can be calculated by solving

$$a(F - x^*)(1 - 2x^* + (1 + b^2)(x^*)^2) = G^2, \quad (2.20)$$

$$\frac{(1 - x^*)G}{1 - 2x^* + (1 + b^2)(x^*)^2} = y^*, \quad (2.21)$$

$$\frac{bx^*G}{1 - 2x^* + (1 + b^2)(x^*)^2} = z^*. \quad (2.22)$$

When $G = 0$, system (2.20) – (2.22) has only one solution, $(F, 0, 0)$, and hence (2.17) – (2.19) possesses only one equilibrium, $(F, 0, 0)$. To analyze its stability, we first shift the point $(F, 0, 0)$ to the origin $(0, 0, 0)$ by performing the change of variable $\tilde{x} = x - F$. The resulting system reads

$$\frac{d\tilde{x}}{dt} = -a\tilde{x} - y^2 - z^2, \quad (2.23)$$

$$\frac{dy}{dt} = -y + (\tilde{x} + F)y - b(\tilde{x} + F)z, \quad (2.24)$$

$$\frac{dz}{dt} = -z + b(\tilde{x} + F)y + (\tilde{x} + F)z. \quad (2.25)$$

Though we can use again the first approximation method to analyze the stability of $(0, 0, 0)$, to present alternative methods we study the stability of the $(0, 0, 0)$ for this system, by the Lypunov theory and Tchetaev's theorem.

1. When $F < 1$, the equilibrium $(0, 0, 0)$ is globally asymptotically stable.

Sketch of proof: Define the function

$$V(\tilde{x}, y, z) := \frac{1}{2}(\tilde{x}^2 + y^2 + z^2). \quad (2.26)$$

Then $V(0,0,0) = 0$ and $V(\tilde{x}, y, z) > 0$ for all $(\tilde{x}, y, z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, what implies that V is positive definite in any ball centered in $(0,0,0)$ (in particular, in the entire \mathbb{R}^3). Moreover, the derivative of V along trajectories of system (2.17) – (2.19) satisfies

$$\dot{V} = -a\tilde{x}^2 - (1-F)y^2 - (1-F)z^2$$

which implies that V is negative definite when $F < 1$. The conclusion follows immediately from Theorem 2.4.

2. When $F > 1$, the equilibrium $(0,0,0)$ is unstable.

Sketch of proof: The previous Lyapunov function defined in (2.26) is no longer valid, but we can apply Tchetaev's theorem. To this end, let us consider the function

$$V(\tilde{x}, y, z) := \frac{1}{2}(-\tilde{x}^2 + y^2 + z^2). \quad (2.27)$$

Then, straightforward computations give

$$\dot{V} = a\tilde{x}^2 + (F-1)y^2 + (F-1)z^2 + 2\tilde{x}(y^2 + z^2).$$

Observing that

$$\lim_{(\tilde{x}, y, z) \rightarrow (0,0,0)} \frac{2\tilde{x}(y^2 + z^2)}{a\tilde{x}^2 + (F-1)y^2 + (F-1)z^2} = 0,$$

we can find a positive ρ such that

$$\left| \frac{2\tilde{x}(y^2 + z^2)}{a\tilde{x}^2 + (F-1)y^2 + (F-1)z^2} \right| \leq \frac{1}{2}, \text{ for } (\tilde{x}, y, z) \in \mathbb{B}((0,0,0); \rho).$$

Hence

$$2\tilde{x}(y^2 + z^2) \geq -\frac{1}{2}(a\tilde{x}^2 + (F-1)y^2 + (F-1)z^2),$$

which implies that

$$\begin{aligned} \dot{V} &= a\tilde{x}^2 + (F-1)y^2 + (F-1)z^2 + 2\tilde{x}(y^2 + z^2) \\ &\geq \frac{1}{2}(a\tilde{x}^2 + (F-1)y^2 + (F-1)z^2) \end{aligned}$$

for all $(\tilde{x}, y, z) \in \mathbb{B}((0,0,0); \rho)$. The instability of $(0,0,0)$ then follows immediately from Tchetaev's theorem.

3. When $F = 1$, we cannot deduce any information from Theorem 2.3. However, since the Lyapunov function defined in (2.26) has a negative semi-definite \dot{V} , the equilibrium point $(0,0,0)$ is at least stable. Next we will apply the LaSalle invariance principle to prove that the equilibrium point is also attractive, and hence asymptotically stable. In fact, $\dot{V} = -a\tilde{x}^2 \leq 0$ when $F = 1$. Denote by E the set

$$E := \{(\tilde{x}, y, z) \in \mathbb{R}^3 : \dot{V}(\tilde{x}, y, z) = 0\} = \{(0, y_0, z_0) : y_0, z_0 \in \mathbb{R}\}.$$

Let us prove that $(0,0,0)$ is the unique invariant subset of E . To this end, pick $(0, y_0, z_0) \in E \setminus \{(0,0,0)\}$, and denote by $\varphi(\cdot)$ the corresponding solution of the IVP. Then we only need to observe that

$$\frac{d}{dt}\varphi_1(0) = -y_0^2 - z_0^2 < 0.$$

Since $\varphi_1(0) = 0$, then there exists $\varepsilon > 0$ such that $\varphi_1(t) < 0$ for all $t \in (0, \varepsilon)$. This proves that $(0,0,0)$ is the only invariant subset of E , and therefore, the equilibrium point $(0,0,0)$ is asymptotically stable.

When $G \neq 0$, the dynamical behavior of system (2.17) – (2.19) becomes complicated and exhibits chaotic attractors. Note that from (2.20) – (2.22) it is difficult to obtain an explicit expression of the equilibrium points on parameters a, b, F and G . It is also difficult to determine how many equilibrium points there exist. Hence it is impossible to apply the Lyapunov method to obtain stability properties. However, we can still investigate the existence of global attractor.

Global attractor

System (2.17) – (2.19) possesses a global solution that generates an autonomous dynamical system $\varphi(t, x_0, y_0, z_0)$ (see Chapter 3 for a detailed proof). To prove the existence of an attractor, we first prove the existence of a compact absorbing set. In fact, it is sufficient to prove the existence of a bounded absorbing set, as its closure will give us a compact absorbing set.

Let $D \subset \mathbb{R}^3$ be a bounded set. Then, there exists $k > 0$ such that $|u_0| \leq k$ for all $d \in D$. Then, for any $u_0 = (x_0, y_0, z_0) \in D$ we have

$$\begin{aligned} \frac{d}{dt}(x^2 + y^2 + z^2) &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} \\ &= 2(-ax^2 - y^2 - z^2 + aFx + Gy) \\ &\leq -ax^2 - y^2 - 2z^2 + aF^2 + G^2 \\ &\leq -\mu_1(x^2 + y^2 + z^2) + aF^2 + G^2, \end{aligned}$$

where $\mu_1 = \min\{a, 1\}$, and hence

$$\begin{aligned} x^2(t) + y^2(t) + z^2(t) &\leq (x_0^2 + y_0^2 + z_0^2)|u_0|^2 e^{-\mu_1 t} + \frac{aF^2 + G^2}{\mu_1}(1 - e^{-\mu_1 t}) \\ &\leq k^2 e^{-\mu_1 t} + \frac{aF^2 + G^2}{\mu_1}. \end{aligned}$$

This implies that given any fixed positive ε , there exists $T = T(\varepsilon, D) >$ such that

$$x^2(t) + y^2(t) + z^2(t) \leq \frac{aF^2 + G^2}{\mu_1} + \varepsilon, \quad \text{for all } t \geq T. \quad (2.28)$$

In fact, simple calculations yield that we can take $T(\varepsilon, D) = -\frac{1}{\mu_1} \log \frac{\varepsilon}{k^2}$. Note that we can always take k such that $k^2 > \varepsilon$, so that $-\frac{1}{\mu_1} \log \frac{\varepsilon}{k^2} > 0$.

The inequality (2.28) implies that

$$K_\varepsilon := \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq \frac{aF^2 + G^2}{\mu_1} + \varepsilon \right\}$$

is a bounded absorbing set for φ . Hence by Theorem 2.7, we conclude immediately that the dynamical system generated by solutions to (2.17) – (2.18) possesses a global attractor \mathcal{A} . When $G = 0$, this attractor \mathcal{A} consists of a single point, $(F, 0, 0)$. When $G \neq 0$, the geometric structure of \mathcal{A} is difficult to obtain. In fact, numerical simulation shows that the attractor \mathcal{A} exhibits chaotic behavior.

Chapter 3

Nonautonomous dynamical systems

It was emphasized in Chapter 2 that the formulation of an autonomous dynamical system as a group (or semigroup) of mappings is based on the fact that such systems depend only on the elapsed time $t - t_0$ instead of the starting time t_0 or current time t independently. The key difference between nonautonomous and autonomous dynamical systems lies in that the nonautonomous systems depend on both the starting time t_0 and the current time t .

There are several ways to formulate the concept of nonautonomous dynamical systems in the literature. In this work we will only introduce the two most typical formulation of nonautonomous dynamical systems: the *process* formulation and the *skew product flow* formulation. The process formulation is also known as the two-parameter semigroup formulation, where both t_0 and t are the parameters. The skew product flow formulation is induced by an autonomous dynamical system as a driving mechanism which is responsible for the temporal change of the vector field of a dynamical system. The driving mechanism can be considered either on a metric space, which leads to the concept of nonautonomous dynamical systems, or on a probability space, which leads to the concept of random dynamical systems (which will be studied in more details in Chapter 4). For this reason, notation of skew product flow can be used for both nonautonomous and random dynamical systems. We will highlight the main differences between nonautonomous and random dynamical systems in Chapter 4.

3.1 Formulations of nonautonomous dynamical systems

The main motivation to study nonautonomous dynamical systems comes from the interest in studying phenomena which can be modeled by nonautonomous ODEs. Let $\mathcal{O} \subset \mathbb{R}^d$ be an open connected set, and let $f : \mathbb{R} \times \mathcal{O} \rightarrow \mathbb{R}^d$ be a continuous map. Recall that an initial value problem for a nonautonomous ordinary differential equation in \mathbb{R}^d is given by

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0. \quad (3.1)$$

Unlike the solutions to autonomous ODEs that depend only on the elapsed time $t - t_0$, solutions to (3.1) depend separately on the starting time t_0 and the current time t in general. Assuming that the vector field f in (3.1) satisfies appropriate hypotheses (such as stated in Chapter 1) ensuring the existence and uniqueness of a global solution $\varphi(t, t_0, x_0)$ of the IVP (3.1), i.e., a solution defined globally in time such that $I_{\max}(t_0, x_0) \supset [t_0, +\infty)$, we next present the two different formulations of nonautonomous dynamical systems mentioned above.

3.1.1 Process formulation

Thanks to properties of the global solution to the IVP (3.1), in particular the uniqueness and the continuous dependence of solutions on initial values and parameters, it is straightforward to verify that the global solution φ of (3.1) satisfies:

- (i) the *initial value property*

$$\varphi(t_0, t_0, x_0) = x_0,$$

where $\varphi(t_0, t_0, x_0)$ denotes the value of the solution of (3.1) at time t_0 ;

- (ii) the *two-parameter semigroup evolution property*

$$\varphi(t_2, t_0, x_0) = \varphi(t_2, t_1, \varphi(t_1, t_0, x_0)), \quad t_0 \leq t_1 \leq t_2,$$

which essentially means the concatenation of solutions. Let $\varphi(t_1, t_0, x_0)$ be the value of the solution to (3.1) at time t_1 . If we take $\varphi(t_1, t_0, x_0)$ as the initial value for another IVP starting at time t_1 , then the value of the solution of this new IVP at time t_2 is the same as the value of the solution of the original IVP (3.1) at time t_2 ;

- (iii) *continuity* of the map $(t, t_0, x_0) \mapsto \varphi(t, t_0, x_0)$ on the state space \mathbb{R}^d , which is ensured by the continuous dependence of the solutions of (3.1) on the initial values.

The above properties of the solution mapping of nonautonomous ODEs give rise to the motivation for the *process* formulation of a nonautonomous dynamical system on a state space \mathbb{R}^d (or, more generally, a metric space (X, d_X)) and time set \mathbb{R} for a continuous-time process. Define

$$\mathbb{R}_{\geq}^2 := \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0\}.$$

Definition 3.1. A process φ on space \mathbb{R}^d is a family of mappings

$$\varphi(t, t_0, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad (t, t_0) \in \mathbb{R}_{\geq}^2,$$

which satisfy

- (i) initial value property: $\varphi(t_0, t_0, x) = x$, for all $x \in \mathbb{R}^d$ and any $t_0 \in \mathbb{R}$;
- (ii) two-parameter semigroup property: for all $x \in \mathbb{R}^d$ and $(t_2, t_1), (t_1, t_0) \in \mathbb{R}_{\geq}^2$ it holds

$$\varphi(t_2, t_0, x) = \varphi(t_2, t_1, \varphi(t_1, t_0, x)),$$

- (iii) continuity property: the mapping $(t, t_0, x) \mapsto \varphi(t, t_0, x)$ is continuous on $\mathbb{R}_{\geq}^2 \times \mathbb{R}^d$.

A process is often called a two-parameter semigroup on X , in contrast with the one-parameter semigroup of an autonomous semi-dynamical systems since it depends on both the initial time t_0 and the current time t .

3.1.2 Skew product flow formulation

To motivate the concept of a skew product flow, we first consider the following triangular system of ODEs (also called a master–slave system):

$$\frac{dx}{dt} = f(q, x), \quad \frac{dq}{dt} = g(q), \quad x \in \mathbb{R}^d, \quad q \in \mathbb{R}^n, \quad (3.2)$$

in which the uncoupled component q can be considered as the driving force for x .

Assuming global existence and uniqueness of solutions to (3.2) forward in time, system (3.2) generates an autonomous semi-dynamical system π on \mathbb{R}^{n+d} , which can be written in the component form as

$$\pi(t, q_0, x_0) = (q(t, q_0), x(t, q_0, x_0)), \quad (3.3)$$

and satisfies the initial value property $\pi(0, q_0, x_0) = (q_0, x_0)$, and the semigroup property on \mathbb{R}^{n+d} ,

$$\pi(t + \tau, q_0, x_0) = \pi(t, \pi(\tau, q_0, x_0)). \quad (3.4)$$

Notice that the q -component of the system is independent, with its solution mapping $q = q(t, q_0)$ generating an autonomous semi-dynamical system on \mathbb{R}^n and satisfies the semigroup property

$$q(t + \tau, q_0) = q(t, q(\tau, q_0)) \quad \forall t, \tau \geq 0. \quad (3.5)$$

Then relations (3.3), (3.4) and (3.5) together imply that

$$\pi(t + \tau, q_0, x_0) = (q(t + \tau, q_0), x(t + \tau, q_0, x_0)) = (q(t, q(\tau, q_0)), x(t + \tau, q_0, x_0)). \quad (3.6)$$

On the other hand by (3.3)

$$\begin{aligned} \pi(t, \pi(\tau, q_0, x_0)) &= \pi(t, q(\tau, q_0), x(\tau, q_0, x_0)) \\ &= \pi(q(t, q(\tau, q_0)), x(t, q(\tau, q_0), x(\tau, q_0, x_0))). \end{aligned} \quad (3.7)$$

According to (3.4), the right-hand side of (3.6) and (3.7) should be the same, i.e.,

$$(q(t, q(\tau, q_0)), x(t + \tau, q_0, x_0)) = (q(t, q(\tau, q_0)), x(t, q(\tau, q_0), x(\tau, q_0, x_0))).$$

This implies that

$$x(t + \tau, q_0, x_0) = x(t, q(\tau, q_0), x(\tau, q_0, x_0)),$$

which is a generalization of the semigroup property and known as the *cocycle* property.

Given a solution $q = q(t, q_0)$ of the q -component of the triangular system (3.2), the x -component becomes a nonautonomous ODE in the x variable on \mathbb{R}^d of the form

$$\frac{dx}{dt} = f(q(t, q_0), x), \quad t \geq 0, \quad x \in \mathbb{R}^d. \quad (3.8)$$

The function $q = q(t, q_0)$ can be considered the “driving” system that is responsible for the changes in the vector field with the passage of time. The solution mapping $x(t) = x(t, q_0, x_0)$ with initial value $x(0) = x_0$ then satisfies

- (i) (initial condition): $x(0, q_0, x_0) = x_0$;
- (ii) (cocycle property): $x(t + \tau, q_0, x_0) = x(t, q(t, q_0), x(\tau, q_0, x_0))$;
- (iii) (continuity): $(t, q_0, x_0) \mapsto x(t, q_0, x_0)$ is continuous.

Such a mapping $x(\cdot, \cdot, \cdot) : \mathbb{R}_0^+ \times \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is called a *cocycle mapping*. It describes the evolution of the solution of the nonautonomous differential equation (3.8) with respect to the driving system q . Note that the variable t here is the time since starting at the state x_0 , with the driving system at state q_0 .

The product system π on $\mathbb{R}^n \times \mathbb{R}^d$ is an autonomous semi-dynamical system and is known as a *skew product flow* due to the asymmetrical roles of the two component systems. This motivates the *skew product flow* formulation for a nonautonomous dynamical system, which is based on a driving dynamical system and a cocycle defined below.

Definition 3.2. Let P be a base or parameter space. Let $\theta = \{\theta_t\}_{t \in \mathbb{T}}$ be a group of homeomorphisms under composition on P that satisfy

- (i) $\theta_0(p) = p$ for all $p \in P$;
- (ii) $\theta_{t+\tau}(p) = \theta_t(\theta_\tau(p))$ for all $t, \tau \in \mathbb{T}$;
- (iii) the mapping $(t, p) \mapsto \theta_t(p)$ is continuous.

Then θ is called a *driving dynamical system*.

Definition 3.3. Let (X, d_X) be a metric space. A cocycle mapping, associated to the driving dynamical system (P, θ) , is a mapping $\psi : \mathbb{T}_0^+ \times P \times X \rightarrow X$ which satisfies

- (i) $\psi(0, p, x) = x$ for all $(p, x) \in P \times X$;
- (ii) $\psi(t + \tau, p, x) = \psi(t, \theta_\tau(p), \psi(\tau, p, x))$ for all $t, \tau \in \mathbb{T}_0^+$ and $(p, x) \in P \times X$;
- (iii) the mapping $(t, p, x) \mapsto \psi(t, p, x)$ is continuous.

Definition 3.4. Let (X, d_X) and (P, d_P) be metric spaces. A driving dynamical system θ acting on parameter space P along with a cocycle mapping ψ acting on the state space X forms a *nonautonomous dynamical system*, denoted by (θ, ψ) .

For $\mathbb{T}_0^+ = \{t \in \mathbb{T} : t \geq 0\}$, define the mapping $\pi : \mathbb{T}_0^+ \times P \times X \rightarrow P \times X$ by

$$\pi(t, (p, x)) := (\theta_t(p), \psi(t, p, x)).$$

Then π forms an autonomous semi-dynamical system on the product space $\mathcal{X} = P \times X$, and is called the *skew product flow* associated with the nonautonomous dynamical system (θ, ψ) .

Remark 3.1. At this point it seems that the process formulation of a nonautonomous dynamical system is more intuitive and the skew product flow formulation would only serve in complicating the analysis. However, the skew product flow formulation contains more information than the process formulation regarding input-driven dynamical systems, since the input dynamics in the skew product flow formulation is explicitly stated by the driving dynamical system, and is only implicitly considered in the process formulation. Moreover, in the particular case where the parameter space P is compact, the topological methods used in the classical theory of skew product flows provide interesting results on the recurrence properties of solutions, etc. Furthermore, with its general form discussed above, the skew product flow formulation can be easily generalized to the formulation of random dynamical systems as we will see in Chapter 4.

The skew product flow formulation of nonautonomous dynamical systems does not only arise from triangular differential systems; it can also be generated by nonautonomous ODEs. In fact, given a continuous vector field f in the IVP (3.1), we define the *hull* of f as

$$\mathcal{H}(f) = \overline{\{f(t + \cdot, \cdot) : t \in \mathbb{T}\}}. \quad (3.9)$$

Then $\mathcal{H}(f)$ is a compact metric space provided f satisfies appropriate hypotheses (e.g., f is periodic or almost periodic in t , or it belongs to a class of functions more general than almost periodic ones, as introduced by Kloeden and Rodrigues in [46]) and the closure in (3.9) is taken in the uniform convergence topology (see e.g., [51]).

Define the shift mapping $\theta_t : \mathcal{H}(f) \mapsto \mathcal{H}(f)$ by

$$\theta_t h := h(\cdot + t, \cdot), \quad \forall t \in \mathbb{T}, \quad h \in \mathcal{H}(f).$$

It is easy to check that $\theta = \{\theta_t\}_{t \in \mathbb{T}}$ is a driving dynamical system on $\mathcal{H}(f)$. Now consider the IVP

$$\frac{dx}{dt} = F(\theta_t p, x), \quad x(0) = x_0 \in \mathbb{R}^d, \quad (3.10)$$

where $p \in \mathcal{H}(f)$, $F(p, x) := p(0, x)$. Let $x(t; p, x_0)$ denote the solution to IVP (3.10) with initial value x_0 at $t = 0$, then we can construct a cocycle mapping generated by (3.10) via

$$\psi(t, p, x_0) = x(t; p, x_0).$$

Notice that if we take $p = f \in \mathcal{H}(f)$, then $\psi(t, f)x_0 = x(t; f, x_0)$, and hence IVP (3.10) becomes

$$\frac{dx}{dt} = \theta_t f(0, x), \quad x(0) = x_0 \in \mathbb{R}^d,$$

or, equivalently,

$$\frac{dx}{dt} = f(t, x), \quad x(0) = x_0 \in \mathbb{R}^d.$$

In addition, we have

$$\psi(t, f, x_0) = \varphi(t, 0, x_0),$$

where $\varphi(t, \cdot, \cdot)$ denotes the process generated by IVP (3.1). Therefore, problem (3.10) generates a skew product flow with driving system $\theta = \{\theta_t\}_{t \in \mathbb{T}}$ on $\mathcal{H}(f)$ and associated cocycle $\{\psi(t, p, \cdot)\}_{p \in \mathcal{H}(f), t \in \mathbb{T}}$ on \mathbb{R}^d . Moreover the following relation holds:

$$\psi(t, \theta_\tau f, x_0) = \varphi(t + \tau, \tau, x_0).$$

Remark 3.2. When we consider the skew product flow based on the hull of the nonautonomous vector field, we are studying not only a single ODE but all those ODEs whose vector field is a translation of the original vector field as well as the limiting equations. This means that we would be able to obtain the most complete information on the dynamics of the underlying system. On the other hand, when we are interested in studying only a single equation, the process formalism is simpler to apply.

3.2 Nonautonomous Attractors

In this section we will introduce the concept of nonautonomous attractors, for both the process and skew product flow formulations of nonautonomous dynamical systems.

3.2.1 Nonautonomous attractors for processes

Denote by $\varphi(t, t_0, x_0)$ the nonautonomous dynamical system generated by the solution mapping of IVP (3.1), then φ generates a two-parameter semigroup that provides two-time description of the system evolution. Recall that, in the autonomous case, a one-parameter semigroup suffices to determine completely the evolution, and hence the system evolution is invariant with respect to the elapsed time, i.e., $\varphi(t, t_0, x_0) = \varphi(t - t_0, 0, x_0)$. This results in the same asymptotic behavior of the underlying system when $t_0 \rightarrow -\infty$ and t fixed and when t_0 fixed and $t \rightarrow \infty$. Without such translation invariance property, the limiting behavior when $t_0 \rightarrow -\infty$ and t fixed may be different from the one obtained in the forward sense with t_0 fixed and $t \rightarrow \infty$.

To illustrate the fundamental character of this distinction, consider the following simple nonautonomous initial value problem:

$$\frac{dx}{dt} = -ax + b \sin t, \quad x(t_0) = x_0, \quad t \geq t_0, \quad (3.11)$$

where a and b are positive constants. The solution mapping $\varphi(t, t_0, x_0)$ is given explicitly by

$$\varphi(t, t_0, x_0) = \frac{b(a \sin t - \cos t)}{a^2 + 1} + \left(x_0 - \frac{b(a \sin t_0 - \cos t_0)}{a^2 + 1} \right) e^{a(t_0 - t)}. \quad (3.12)$$

Clearly for any x_0 , the forward limit $\lim_{t \rightarrow \infty} \varphi(t, t_0, x_0)$ does not exist. But if we consider the difference between any two solutions corresponding to two different initial values x_0 and y_0 , it follows from an easy computation that

$$\varphi(t, t_0, x_0) - \varphi(t, t_0, y_0) = (x_0 - y_0) e^{a(t_0 - t)}.$$

Hence although individually each solution does not have a limit as t goes to $+\infty$, all solutions behave similarly in the sense that the difference between any two solutions goes to zero when t goes to $+\infty$, i.e., all solutions approach the same target as time goes on. The goal is then to find a particular or special solution such that the trajectory of this solution describes the path that any other solution should approach during their evolution. It is easy to check that the function

$$A(t) := \frac{b(a \sin t - \cos t)}{a^2 + 1} \quad (3.13)$$

is such a particular solution to the IVP (3.11), that provides the long term information on the future evolution of the system, in the sense that any other solution will eventually approach it, i.e.,

$$\lim_{t \rightarrow \infty} |\varphi(t, t_0, x_0) - A(t)| = 0, \quad \text{for all } t_0, x_0 \in \mathbb{R}.$$

However, if we take limit with t fixed and the initial time t_0 goes to $-\infty$ in the expression in (3.12), we obtain that

$$\lim_{t_0 \rightarrow -\infty} \varphi(t, t_0, x_0) = A(t), \quad \text{for all } t, x_0 \in \mathbb{R}. \quad (3.14)$$

The limit in (3.14) is said to be taken in the “pullback” sense, as we are solving a series of IVPs with the same initial value x_0 at retrospective initial times t_0 , and look at the solution at a current fixed time t . The concepts of the forward and pullback limit are illustrated in Figures 3.1 and 3.2 by solution trajectories of equation (3.11).

In some occasions we can obtain special solutions to an underlying ODE that attract any other solution in both the pullback and the forward sense such as in example (3.11). But it is also possible that the attraction happens in either the pullback

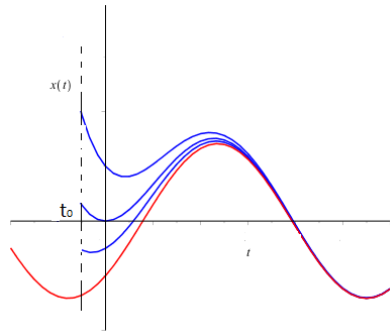


Fig. 3.1: Forward limit of solutions to equation (3.11) with t_0 fixed and $t \rightarrow \infty$.

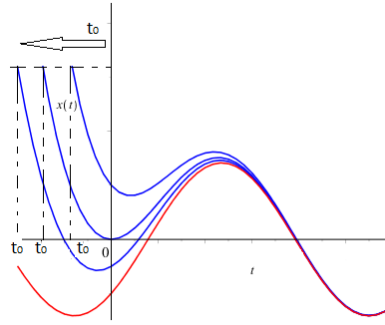


Fig. 3.2: Pullback limit of solutions to equation (3.11) with t fixed and $t_0 \rightarrow -\infty$.

or the forward sense, but not both. Taking limit in the pullback sense may result in obtaining special (particular) solutions of a nonautonomous ODE which are independent of the initial values x_0 . This fact is very important in the random cases, because such stationary solutions can be seen as a natural extension of the concept of equilibrium points in the random framework.

The set $A(t)$ defined in (3.13) can be further shown to be invariant under the dynamical system φ (or φ -invariant), i.e., $\varphi(t, t_0, A(t_0)) = A(t)$ for every $t \geq t_0$. We have therefore constructed a family of limiting objects $A(t)$, which in addition is a particular solution of the equation in (3.11). These objects exist in actual time t rather than asymptotically in the future ($t \rightarrow \infty$), and convey the effect of the dissipation due to the term $-ax$. Note that $A(t)$ is merely a time-dependent object that attracts solutions starting from all the initial data x_0 in the past. More generally, in the force-dissipative case, we can obtain, for all t , a collection of objects, $\mathcal{A} = \{A(t) : t \in \mathbb{R}\} = \{A(t)\}_{t \in \mathbb{R}}$ that depend on time t , by letting $t_0 \rightarrow -\infty$. This collection \mathcal{A} is the so-called *pullback attractor* for processes. Each $A(t)$ may be more complicated than a point, and attracts particular subsets of initial data taken in the asymptotic past. The rigorous definition of pullback attractors and forward attractors will be given below, following the definition of pullback and forward attraction.

Recall that in the basic definition of attractor (Definition 2.6), the attractor for a dynamical system on X attracts all bounded subsets of X . While for nonautonomous dynamical systems, the candidates for nonautonomous attractors are families of sets parameterized by time. Thus we will define the concept of attraction based on families of sets $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$, instead of a single set (which is the special case where $D(t) \equiv D$).

Definition 3.5. Let φ be a process on \mathbb{R}^d . A family of sets $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ is said to

(i) pullback attract another family $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$ if

$$\lim_{t_0 \rightarrow -\infty} \text{dist}(\varphi(t, t_0, D(t_0)), A(t)) = 0, \quad \text{for all } t \in \mathbb{R};$$

(ii) forward attract another family $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$ if

$$\lim_{t \rightarrow \infty} \text{dist}(\varphi(t, t_0, D(t_0)), A(t)) = 0 \quad \text{for all } t_0 \in \mathbb{R}.$$

The pullback attraction is said to be uniform if

$$\lim_{t_0 \rightarrow -\infty} \sup_{t \in \mathbb{R}} \text{dist}(\varphi(t, t_0, D(t_0)), A(t)) = 0, \quad (3.15)$$

and the forward attraction is said to be uniform if

$$\lim_{t \rightarrow \infty} \sup_{t_0 \in \mathbb{R}} \text{dist}(\varphi(t, t_0, D(t_0)), A(t)) = 0. \quad (3.16)$$

Remark 3.3. In general the pullback and forward attraction are independent concepts. However, when the attraction is uniform, then it is straightforward to prove that both concepts are equivalent.

Observe that in the motivating example (3.11), the convergence of solutions onto $A(t)$ defined in (3.13) is both pullback and forward. To illustrate the difference between pullback and forward attractions, consider the following IVPs:

$$\frac{du}{dt} = 2tu + 1, \quad u(t_0) = u_0, \quad (3.17)$$

$$\frac{dv}{dt} = -2tv + 1, \quad v(t_0) = v_0. \quad (3.18)$$

whose solutions are given explicitly by

$$u(t; t_0, u_0) = u_0 e^{t^2 - t_0^2} + e^{t^2} \int_{t_0}^t e^{-s^2} ds$$

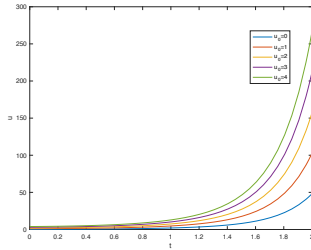
$$v(t; t_0, v_0) = v_0 e^{-(t^2 - t_0^2)} + e^{-t^2} \int_{t_0}^t e^{s^2} ds.$$

It is clear that

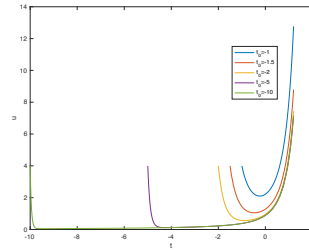
$$u(t; t_0, u_0) \longrightarrow +\infty \quad \text{as } t \rightarrow +\infty,$$

$$u(t; t_0, u_0) \longrightarrow e^{t^2} \int_{-\infty}^t e^{-s^2} ds \quad \text{as } t_0 \rightarrow -\infty,$$

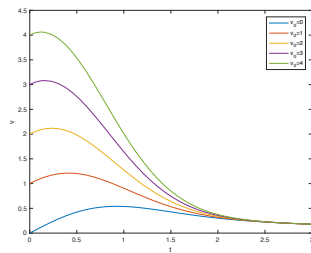
and hence $u(\cdot; \cdot, \cdot)$ possesses a limit in the pullback sense but not in the forward sense. On the other hand,



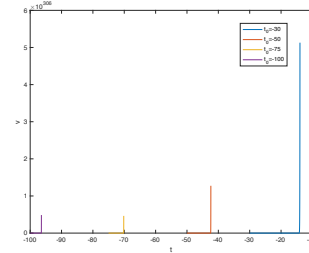
(a) Solution trajectories of Eq. (3.17) starting from $t_0 = 0$ and $u_0 = 0, 1, 2, 3, 4$.



(b) Solution trajectories of Eq. (3.17) starting from $t_0 = -1, -1.5, -2, -5, -10$ and $u_0 = 4$.



(c) Solution trajectories of Eq. (3.18) starting from $t_0 = 0$ and $u_0 = 0, 1, 2, 3, 4$.



(d) Solution trajectories of Eq. (3.18) starting from $t_0 = -30, -50, -75, -100$ and $v_0 = 4$.

Fig. 3.3: Forward and pullback trajectories of Eqs. (3.17) and (3.18).

$$\begin{aligned} v(t; t_0, v_0) &\longrightarrow 0 \quad \text{as } t \rightarrow +\infty, \\ v(t; t_0, v_0) &\longrightarrow +\infty \quad \text{as } t_0 \rightarrow -\infty. \end{aligned}$$

Hence v is convergent to 0 in the forward sense, but is not convergent in the pullback sense (see Fig. 3.3).

Next we will define the concept of pullback and forward attractors. Notice that the family of sets \mathcal{D} in Definition 3.5 is the key to define attractors; different characterizations of \mathcal{D} will give different attractors. In the general abstract framework for attractors, we should consider the attraction (by the attractor) of a *universe* (denoted by \mathcal{U}) of nonautonomous sets, i.e., a collection of families $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$ that satisfies some specific property. For example, \mathcal{U} can be the collection of all families \mathcal{D} that satisfies

$$\lim_{t \rightarrow +\infty} e^{-t} \sup_{d \in D(t)} |d| = 0.$$

However, all the applications in this chapter requires no more than the attraction (by the attractor) of families of sets which are formed by the same bounded set for all t , i.e., $\mathcal{D} = \{D(t) \equiv D_0\}_{t \in \mathbb{R}}$, where D_0 is a bounded set. Hence at this point we will simply restrict our definition of attractors based on families of nonempty bounded subsets of X ,

$$\mathcal{D} = \{D(t) : t \in \mathbb{R}\} \text{ where } D(t) \text{ is a bounded subset of } X.$$

Later in Chapter 4 we will generalize the definition of attractors by using the concept of universe.

Definition 3.6. A family of sets $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ is said to be a *(global) pullback attractor for the process φ* , if

- (i) $A(t)$ is compact for all $t \in \mathbb{R}$,
- (ii) $\varphi(t, t_0, A(t_0)) = A(t)$ for all $t \geq t_0$,
- (iii) \mathcal{A} pullback attracts all families of bounded subsets of X , i.e.,

$$\lim_{t_0 \rightarrow -\infty} \text{dist}(\varphi(t, t_0, D(t_0)), A(t)) = 0 \quad \text{for any fixed } t \in \mathbb{R}.$$

The pullback attractor is said to be uniform if the attraction property is uniform in time as established in (3.15).

Definition 3.7. A family of sets $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ is said to be a *(global) forward attractor for the process φ* , if

- (i) $A(t)$ is compact for all $t \in \mathbb{R}$,
- (ii) $\varphi(t, t_0, A(t_0)) = A(t)$ for all $t \geq t_0$,
- (iii) \mathcal{A} forward attracts all families of bounded subsets of X , i.e.,

$$\lim_{t \rightarrow \infty} \text{dist}(\varphi(t, t_0, D(t_0)), A(t)) = 0 \quad \text{for any fixed } t_0 \in \mathbb{R}.$$

The forward attractor is said to be uniform if the attraction property is uniform in time as established in (3.16).

Remark 3.4. When the attraction is uniform then pullback and forward attraction are equivalent. However in general the forward limit defining a nonautonomous forward attractor is different from the pullback limit. In fact, the forward limit is taken asymptotically in the future whereas the pullback limit is taken asymptotically in the past. Moreover, the limiting objects obtained forward in time do not have the same dynamical meaning at current time as the limiting objects obtained pullback in time. For a more complete discussion about differences between pullback and forward attractions, and in particular how to construct forward attractors the reader is referred to the papers [47, 50].

The concept of attractor is closely related to that of absorbing sets, which is defined below.

Definition 3.8. A family $\mathcal{B} = \{B(t)\}_{t \in \mathbb{R}}$ of nonempty subsets of X is said to be (*pullback*) *absorbing for the process* φ if for each $t \in \mathbb{R}$ and every family $\mathcal{D} = \{D(t) : t \in \mathbb{R}\}$ of nonempty bounded subsets of X , there exists $T_{\mathcal{D}}(t) > 0$ such that

$$\varphi(t, t_0, D(t_0)) \subset B(t) \quad \text{for all } t_0 \leq t - T_{\mathcal{D}}(t).$$

The absorption is uniform if $T_{\mathcal{D}}(t)$ does not depend on the time variable t .

The existence of compact absorbing sets for a nonautonomous dynamical system is the key to obtain the existence of pullback attractors, as stated in the following theorem due to Caraballo *et al.* (see [11] for more details).

Theorem 3.1. *Suppose that the process $\varphi(\cdot, \cdot, \cdot)$ has a family $\mathcal{B} = \{B(t)\}_{t \in \mathbb{R}}$ of nonempty compact subsets of X which is pullback absorbing for $\varphi(\cdot, \cdot, \cdot)$. Then $\varphi(\cdot, \cdot, \cdot)$ has a global pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$, whose component subsets are defined by*

$$A(t) = \Lambda(\mathcal{B}, t) := \bigcap_{s \leq t} \left(\overline{\bigcup_{\tau \leq s} \varphi(t, \tau, B(\tau))} \right), \quad t \in \mathbb{R}.$$

Moreover, \mathcal{A} is minimal in the sense that if $\tilde{\mathcal{A}} = \{\tilde{A}(t)\}_{t \in \mathbb{R}}$ is a family of closed sets such that $\lim_{\tau \rightarrow -\infty} \text{dist}(\varphi(t, \tau, B(\tau)), \tilde{A}(t)) = 0$, then $A(t) \subset \tilde{A}(t)$.

The pullback attractor given in Theorem 3.1 is minimal with respect to set inclusion (see [29]). However it may not be unique in general (see [10]). For example, consider the following nonautonomous ODE

$$\frac{dx(t)}{dt} = (-\alpha + \max\{t, 0\}) x(t), \quad (3.19)$$

which generates a process by setting $\varphi(t, t_0, x_0) = x(t; t_0, x_0)$, where $x(\cdot; t_0, x_0)$ denotes the unique solution to equation (3.19) satisfying $x(t_0; t_0, x_0) = x_0$. Let $a(t) \leq 0$ and $b(t) \geq 0$ be any entire solutions to equation (3.19) defined for all $t \in \mathbb{R}$, then the family

$\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ defined by $A(t) = [a(t), b(t)]$ satisfies all the conditions in Definition 3.6, and thus is a global pullback attractor for the process φ .

Similar to the autonomous case, in order to guarantee the uniqueness of global pullback attractors, we need to impose additional hypotheses on the attractor such as assuming the family $\{A(t)\}_{t \in \mathbb{R}}$ is uniformly bounded (i.e., there exists a bounded set $B \subset X$ such that $A(t) \subset B$ for all $t \in \mathbb{R}$). It is not difficult to check in example (3.19) that the unique uniformly bounded attractor is given by $A(t) = \{0\}$, for all $t \in \mathbb{R}$.

In fact, if we assume in addition in Theorem 3.1 that \mathcal{A} is uniformly bounded, i.e., $\cup_{t \in \mathbb{R}} A(t)$ is bounded, which means that there exists a bounded set $B \subset X$ such that $A(t) \subset B$ for all $t \in \mathbb{R}$, then \mathcal{A} is the unique pullback attractor with all properties defined in Definition 3.6. A sufficient condition to ensure the uniform boundedness of \mathcal{A} is that the family of compact absorbing sets in Theorem 3.1 is uniformly bounded. The existence and uniqueness of pullback attractors can be summarized in the following theorem.

Theorem 3.2. *Suppose that the process $\varphi(\cdot, \cdot, \cdot)$ possesses a uniformly bounded family $\mathcal{B} = \{B(t)\}_{t \in \mathbb{R}}$ of nonempty compact subsets of X which is pullback absorbing for $\varphi(\cdot, \cdot, \cdot)$. Then $\varphi(\cdot, \cdot, \cdot)$ has a unique global pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$, whose component subsets are defined by*

$$A(t) = \Lambda(\mathcal{B}, t) := \bigcap_{s \leq t} \left(\overline{\bigcup_{\tau \leq s} \varphi(t, \tau, B(\tau))} \right), \quad t \in \mathbb{R}.$$

Remark 3.5. There are other conditions to ensure the uniqueness of the pullback attractor, such as requiring the attractor to belong to a certain class of set valued functions which are attracted by the attractor (see [23]).

Remark 3.6. If the family \mathcal{B} in Theorem 3.1 and 3.2 is in addition φ -positively invariant, then the components of the pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ are determined by

$$A(t) = \bigcap_{t_0 \leq t} \varphi(t, t_0, B(t_0)) \quad \text{for each } t \in \mathbb{R}.$$

A pullback attractor consists of *entire solutions*, i.e., functions $\xi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\xi(t) = \varphi(t, t_0, \xi(t_0))$ for all $(t, t_0) \in \mathbb{R}_{\geq}^2$. In special cases it consists of a single entire solution. We will next state a theorem to obtain a pullback attractor that consists of a single entire solution in the finite dimensional space \mathbb{R}^d . Before that we first define the following property that is required in the theorem.

Definition 3.9. A nonautonomous dynamical system φ is said to satisfy a *uniform strictly contracting property* if for each $R > 0$, there exist positive constants K and α such that

$$|\varphi(t, t_0, x_0) - \varphi(t, t_0, y_0)|^2 \leq K e^{-\alpha(t-t_0)} \cdot |x_0 - y_0|^2 \quad (3.20)$$

for all $(t, t_0) \in \mathbb{R}_{\geq}^2$ and $x_0, y_0 \in \overline{\mathbb{B}}(0; R)$, the closed ball in \mathbb{R}^d centered at the origin with radius $R > 0$.

This uniform strictly contracting property along with the existence of a pullback absorbing set ensure the existence of a global attractor that consists of singleton sets (i.e., a single entire solution) in both the forward and pullback senses, as stated in the following theorem.

Theorem 3.3. *Suppose that a process φ on \mathbb{R}^d is uniform strictly contracting on a φ -positively invariant pullback absorbing family $\mathcal{B} = \{B(t) : t \in \mathbb{R}\}$ of nonempty compact subsets of \mathbb{R}^d . Then the process φ has a unique global forward and pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ with component sets consisting of singleton sets, i.e., $A(t) = \{\xi^*(t)\}$ for each $t \in \mathbb{R}$, where ξ^* is an entire solution of the process.*

The proof of Theorem 3.3 is done by constructing an appropriate Cauchy sequence which converges to a unique limit. The reader is referred to [48, 49] for more details about the proof.

3.2.2 Nonautonomous attractors for skew product flows

Let (θ, ψ) be a nonautonomous dynamical system over the metric spaces P (a base space or a parameter space) and X (the phase space) with corresponding metrics d_P and d_X , respectively. Consider its associated skew product semiflow $\pi : \mathbb{R}_0^+ \times P \times X \rightarrow P \times X$ defined by

$$\pi(t, (p, x)) := (\theta_t(p), \psi(t, p, x)).$$

Then as it was shown in Subsection 3.1.2, π is an autonomous semi-dynamical system on the extended product space $\mathcal{X} = P \times X$. Consequently the theory of global attractors for autonomous semi-dynamical systems in Chapter 2 can be directly applied here. In fact, a global attractor for π is a nonempty compact subset \mathcal{A} of \mathcal{X} that is π -invariant, i.e., $\pi(t, \mathcal{A}) = \mathcal{A}$ for all $t \geq 0$ and

$$\lim_{t \rightarrow +\infty} \text{dist}_{\mathcal{X}}(\pi(t, D), \mathcal{A}) = 0,$$

for each bounded subset D of \mathcal{X} .

Comparing to the global attractor for processes, the global attractor for skew product flows is valuable only if it provides us with some more specific information on the dynamics of an underlying system on the phase space X , as this is the space where the evolution of any dynamical system takes place. In fact, when P is compact and θ -invariant, the global attractor \mathcal{A} of the autonomous semi-dynamical systems π can be rewritten in the form

$$\mathcal{A} = \bigcup_{p \in P} \{p\} \times A(p), \quad (3.21)$$

where $A(p)$ is a compact subset of X for each $p \in P$. More precisely, $A(p) = \{x \in X : (p, x) \in \mathcal{A}\}$, i.e., $A(p)$ is the projection of the global attractor \mathcal{A} on its second com-

ponent. Therefore, this family of subsets $\{A(p)\}_{p \in P}$ of the phase space X serves as a good candidate to provide information on the dynamics of an underlying system.

In addition, observe that the invariance property of the global attractor \mathcal{A} implies that $\pi(t, \mathcal{A}) = \mathcal{A}$ for all $t \geq 0$. In fact, it follows directly from (3.21) that

$$\mathcal{A} = \bigcup_{\tilde{p} \in P} \{\tilde{p}\} \times A(\tilde{p}) = \bigcup_{p \in P} \{\theta_t(p)\} \times \psi(t, p, A(p)) = \pi(t, \mathcal{A}),$$

which implies that $\psi(t, p, A(p)) = A(\theta_t(p))$ for all $t \geq 0$ and $p \in P$. This confirms that the family of sets $\{A(p)\}_{p \in P}$ of the phase space X can better describe the dynamics of a system than the global attractor on the extended space \mathcal{X} , since \mathcal{X} also includes the base (or parameter) space P as a component, which usually does not have a physical meaning similar to the state space component. For these reasons, in addition to the global attractor for skew product flows, we also want to define other types of attractors consisting of families of compact subsets of the phase space X , that are similar to the forward and pullback attractors for processes.

Definition 3.10. Let $\pi = (\theta, \psi)$ be a skew product flow on the metric space $P \times X$. A *pullback attractor* for π is a nonautonomous set $\mathcal{A} = \{A(p) \subset X\}_{p \in P}$ such that

- (i) $A(p)$ is a nonempty, compact set for every $p \in P$;
- (ii) \mathcal{A} is invariant for the cocycle, i.e., $\psi(t, p, A(p)) = A(\theta_t(p))$ for all $t \geq 0$ and $p \in P$;
- (iii) pullback attracts the bounded subsets of X , i.e.,

$$\lim_{t \rightarrow \infty} \text{dist}_X(\psi(t, \theta_{-t}(p), D), A(p)) = 0 \quad (3.22)$$

holds for every nonempty bounded subset D of X and $p \in P$.

Definition 3.11. Let $\pi = (\theta, \psi)$ be a skew product flow on the metric space $P \times X$. A *forward attractor* for π is a nonautonomous set $\mathcal{A} = \{A(p) \subset X\}_{p \in P}$ such that

- (i) $A(p)$ is a nonempty, compact set for every $p \in P$;
- (ii) \mathcal{A} is invariant for the cocycle, i.e., $\psi(t, p, A(p)) = A(\theta_t(p))$ for all $t \geq 0$ and $p \in P$;
- (iii) forward attracts the bounded subsets of X , i.e.,

$$\lim_{t \rightarrow +\infty} \text{dist}_X(\psi(t, p, D), A(\theta_t(p))) = 0 \quad (3.23)$$

holds for every nonempty bounded subset D of X and $p \in P$.

Remark 3.7. As in the case of attractors for processes, there is no clear relationship between the existence of forward and pullback attractors for skew product flows; one can exist while the other does not. See, for instance, Kloeden and Lorenz [47, 50] for a more detailed discussion.

If we assume further that the limits taken in (3.22) and (3.23) are uniform, in the sense that

$$\begin{aligned}\lim_{t \rightarrow +\infty} \sup_{p \in P} \text{dist}_X(\psi(t, \theta_{-t}(p), D), A(p)) &= 0 \quad (\text{pullback}), \\ \lim_{t \rightarrow +\infty} \sup_{p \in P} \text{dist}_X(\psi(t, p, D), A(\theta_t(p))) &= 0 \quad (\text{forward}),\end{aligned}$$

then we obtain another type of attractor, called *uniform pullback attractor* and *uniform forward attractor*, respectively. Similar to the idea of equivalence between uniform pullback and forward attractions for processes, when an attractor for the skew product flow is uniform (either pullback or forward), it will be both uniform pullback and uniform forward, so one can simply refer to it as a *uniform attractor*. For more details regarding relationships among different types of nonautonomous attractors, the reader is referred to [24] and [51].

We next illustrate how to obtain attractors for the skew product flow. Consider again the system (3.11) which can be written as

$$\frac{dx}{dt} = -ax + p_0(t), \quad x(t_0) = x_0, \quad (t, t_0) \in \mathbb{R}_+^2, \quad (3.24)$$

where $p_0(t) := b \sin t$. In order to set the problem in the skew product framework, we need to define (1) the base (parameter) space P , (2) the driving dynamical system θ_t , and (3) the cocycle ψ . To this end, we define P by

$$P = \{p_0(t + \cdot) : 0 \leq t \leq 2\pi\},$$

define $\theta_t : P \rightarrow P$ by

$$\theta_t(p)(\cdot) = p(t + \cdot) \quad \text{for all } p \in P,$$

and define $\psi(t, p, x_0)$ by the solution to the IVP (3.24), which can be calculated explicitly to be

$$\psi(t, p, x_0) = x_0 e^{-a(t-t_0)} + e^{-at} \int_{t_0}^t e^{as} p(s) ds. \quad (3.25)$$

At this point to estimate the pullback limit, we need to obtain first the expression for $\psi(t, \theta_{-t}(p), x_0)$ and then evaluate the expression at the limit of $t \rightarrow \infty$. By (3.25) we obtain that

$$\begin{aligned}\psi(t, \theta_{-t}(p), x_0) &= x_0 e^{-a(t-t_0)} + e^{-at} \int_{t_0}^t e^{as} \theta_{-t}(p)(s) ds \\ &= x_0 e^{-a(t-t_0)} + e^{-at} \int_{t_0}^t e^{as} p(s-t) ds \\ &= x_0 e^{-a(t-t_0)} + \int_{t_0-t}^0 e^{a\tau} p(\tau) d\tau.\end{aligned} \quad (3.26)$$

Taking the limit of (3.26) at $t \rightarrow \infty$ yields

$$\lim_{t \rightarrow +\infty} \psi(t, \theta_{-t}(p), x_0) = \int_{-\infty}^0 e^{as} p(s) ds.$$

Denote by $A(p) = \int_{-\infty}^0 e^{as} p(s) ds$, then fibers of the pullback attractor for the skew product flow are given by $A(p)$ for all $p \in P$. As a result, the component subsets of the pullback attractor for system (3.24) is given by $A(\theta_t(p_0))$ which can be calculated as follows.

$$A(\theta_t p_0) = \int_{-\infty}^0 e^{as} \theta_t(p_0)(s) ds = b \int_{-\infty}^0 e^{as} \sin(t+s) ds = \frac{b(a \sin t - \cos t)}{a^2 + 1}.$$

Observe that the above pullback convergence is actually uniform. Hence the pullback attractor is uniform and consequently is also a uniform forward attractor, and a uniform attractor. In addition, the skew product flow possesses a global attractor given by

$$\mathcal{A} = \bigcup_{p \in P} \{p\} \times A(p).$$

Remark 3.8. As it can be seen in examples (3.11) and (3.24), the skew product formulation appears to be more complicated and less straightforward (from calculation point of view) than the process formulation, although it provides more information on the dynamics in the phase space. It is because of this reason that we will adopt the process formulation instead of the skew formulation of nonautonomous dynamical systems for the applications later in this chapter.

Despite the difference between the process formulation and the skew product formulation of nonautonomous dynamical systems, the existence of a global pullback attractor for both formulations depends on some absorbing property. The absorbing property for processes was established in Definition 3.8. The corresponding absorbing property for skew product flows will be defined next.

Definition 3.12. Let (θ, ψ) be a skew product flow on the metric space $P \times X$. A nonempty compact subset B of X is said to be *pullback absorbing* if for each $p \in P$ and every bounded subset D of X , there exists a $T = T(p, D) > 0$ such that $\psi(t, \theta_{-t}(p), D) \subset B$ for all $t \geq T$.

The following theorem states the existence of pullback attractors for skew product flows.

Theorem 3.4. *Let (θ, ψ) be a skew product flow on a metric space $P \times X$. Assume that there exists a compact pullback absorbing set B for ψ such that*

$$\psi(t, p, B) \subset B, \text{ for all } t \geq 0, p \in P. \quad (3.27)$$

Then there exists a unique pullback attractor \mathcal{A} with fibers in B determined by

$$A(p) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \psi(t, \theta_{-t}(p), B)}, \text{ for all } p \in P.$$

Moreover, if P is a compact metric space then

$$\lim_{t \rightarrow +\infty} \sup_{p \in P} \text{dist} \left(\psi(t, p, D), \overline{\bigcup_{p \in P} A(p)} \right) = 0$$

for any bounded subset D of X .

When we have a skew product flow generated by a system of nonautonomous differential equations whose vector field depends explicitly on the parameter of the base space, a certain type of Lipschitz condition can allow us to prove that the fibers of the pullback attractor consist only of a singleton which, in addition, forms an entire solution of the system. This result is established as follows.

Consider the following equation in \mathbb{R}^d

$$\frac{dx}{dt} = f(p, x), \quad (3.28)$$

with a driving system θ on a compact metric space P . Assume that (3.28) generates a skew product flow (θ, ψ) .

Theorem 3.5. *If the vector field f is uniformly dissipative, i. e., for any $p \in P$ and $x \in \mathbb{R}^d$ there exist positive constants K and L such that*

$$\langle x, f(p, x) \rangle \leq K - L|x|^2,$$

then the skew product flow generated by (3.28) possesses a pullback attractor. In addition, if the vector field f satisfies the following uniform one-sided Lipschitz condition: there exists a positive constant L such that:

$$\langle x_1 - x_2, f(p, x_1) - f(p, x_2) \rangle \leq -L|x_1 - x_2|^2, \text{ for all } x_1, x_2 \in \mathbb{R}^d, p \in P,$$

then the skew product flow generated by (3.28) possesses a unique pullback attractor \mathcal{A} that consists of singleton fibers, i. e., $A(p) = \{a(p)\}$ for each $p \in P$. Moreover, the mapping $t \mapsto a(\theta_t p)$ is an entire solution to (3.28) for each $p \in P$.

We have already shown by several simple examples for the process formulation that a pullback attractor does not need to be a forward attractor, and vice versa. Now we will discuss the relationship between pullback and forward attractors for the skew product flow generated by nonautonomous differential equations, based on the concept of global attractors for the associated autonomous semi-dynamical system. In particular, when the base (parameter) space P is compact, we have the following result from [51] (Propositions 3.30 and 3.31) and [19] (Theorem 3.4).

Theorem 3.6. *Let (θ, ψ) be a skew product flow over the metric spaces P and X , where P is compact. Let $\{\pi(t) : t \geq 0\}$ be the associated skew product semiflow on $P \times X$ with a global attractor \mathbb{A} . Then $\mathcal{A} = \{A(p)\}_{p \in P}$ with $A(p) = \{x \in X : (x, p) \in \mathbb{A}\}$ is the pullback attractor for (θ, ψ) .*

The following result offers a converse statement, but requires additionally some uniformity condition.

Theorem 3.7. *Suppose that $\{A(p)\}_{p \in P}$ is the pullback attractor of the skew product flow (θ, ψ) , and denote by $\{\pi(t) : t \geq 0\}$ the associated skew product semiflow. Assume that $\{A(p)\}_{p \in P}$ is uniformly attracting, i.e., there exists a compact subset $K \subset X$ such that, for all $B \subset X$ bounded,*

$$\lim_{t \rightarrow +\infty} \sup_{p \in P} \text{dist}(\psi(t, \theta_{-t}(p), B), K) = 0,$$

and that $\bigcup_{p \in P} A(p)$ is precompact in X . Then the set \mathbb{A} associated with $\{A(p)\}_{p \in P}$, given by

$$\mathbb{A} = \bigcup_{p \in P} \{p\} \times A(p), \quad (3.29)$$

is the global attractor for the semigroup $\{\pi(t) : t \geq 0\}$.

It has been mentioned several times in this chapter that the pullback attraction does not necessarily imply forward attraction, and the forward attraction does not necessarily imply pullback attraction. As a result, the pullback attractor and forward attractor do not necessarily imply each other (see [47, 50] for more details).

3.3 Applications

In this section we will study the nonautonomous counterparts of the chemostat, SIR and Lorenz–84 models presented in Chapter 2.

3.3.1 Nonautonomous chemostat model

In the simple chemostat model (2.10) – (2.11), the availability of the nutrient and its supply rate are assumed to be fixed. However, the availability of a nutrient in a natural system usually depends on the nutrient consumption rate and input nutrient concentration, which may lead to a nonautonomous dynamical system. Another assumption in the chemostat model (2.10) – (2.11) is that the flow rate is assumed to be fast enough that it does not allow growth on the cell walls. Yet wall growth does occur when the washout rate is not fast enough and may cause problems in bio-reactors. In this subsection we study the chemostat models with a variable nutrient supplying rate or a variable input nutrient concentration, with or without wall growth.

Denote by $x(t)$ and $y(t)$ the concentrations of the nutrient and the microorganism at any specific time t . When I and D are both constant, equations (2.10) – (2.11) describe the limited resource-consumer dynamics. Often, the microorganisms grow not only in the growth medium, but also along the walls of the container. This is either due to the ability of the microorganisms to stick to the walls of the container or

the flow rate is not fast enough to wash these organisms out of the system. Naturally, we can regard the consumer population $y(t)$ as an aggregate of two categories of populations, one in the growth medium, denoted by $y_1(t)$, and the other on the walls of the container, denoted by $y_2(t)$. These individuals may switch their categories at any time, i.e., the microorganisms on the walls may join those in the growth medium or the biomass in the medium may prefer walls. Let r_1 and r_2 represent the rates at which the species stick on to and shear off from the walls, respectively, then $r_1 y_1(t)$ and $r_2 y_2(t)$ represent the corresponding terms of species changing the categories. Assume that the nutrient is equally available to both of the categories, therefore it is assumed that both categories consume the same amount of nutrient and at the same rate.

When the flow rate is low, the organisms may die naturally before being washed out and thus washout is no longer the only prime factor of death. Denote by $\nu(> 0)$ the collective death rate coefficient of $y(t)$ representing all the aforementioned factors such as diseases, aging, etc. On the other hand, when the flow rate is small, the dead biomass is not sent out of the system immediately and is subject to bacterial decomposition which in turn leads to regeneration of the nutrient. Expecting not 100% recycling of the dead material but only a fraction, we let constant $b \in (0, 1)$ describe the fraction of dead biomass that is recycled. Note that only $y_1(t)$ contributes to the material recycling of the dead biomass in the medium. Moreover, since the microorganisms on the wall are not washed out of the system, the term $-Dy_2(t)$ is not included in the equation representing the growth of $y_2(t)$.

All the parameters are same as those of system (2.10) - (2.11), but $0 < c \leq a$ replaces a as the growth rate coefficient of the consumer species. When I and D vary in time, and there are no time delays in the system, the following model describes the dynamics of chemostats with variable inputs and wall growth.

$$\frac{dx}{dt} = D(t)(I(t) - x(t)) - aU(x(t))(y_1(t) + y_2(t)) + b\nu y_1(t), \quad (3.30)$$

$$\frac{dy_1}{dt} = -(\nu + D(t))y_1(t) + cU(x(t))y_1(t) - r_1 y_1(t) + r_2 y_2(t), \quad (3.31)$$

$$\frac{dy_2}{dt} = -\nu y_2(t) + cU(x(t))y_2(t) + r_1 y_1(t) - r_2 y_2(t). \quad (3.32)$$

We assume here that the consumption function follows the Michaelis-Menten or Holling type-II form:

$$U(x) = \frac{x}{\lambda + x}, \quad (3.33)$$

where $\lambda > 0$ is the half-saturation constant [76]. We will provide some results for two scenarios: a) when the nutrient supply rate D is time dependent but not I ; b) when I depends on t but not D . The reader is referred to [15, 17, 18] for more details and the complete proofs of the results stated in this section.

Variable nutrient supply rate

First we present the existence and uniqueness of a nonnegative and bounded solution, given a nonnegative initial data.

Lemma 3.1. *Assume that $D : \mathbb{R} \rightarrow [d_m, d_M]$, where $0 < d_m < d_M < \infty$, is continuous and $I(t) = I$ for all $t \in \mathbb{R}$. Suppose also that $(x_0, y_{1,0}, y_{2,0}) \in \mathbb{R}_+^3 := \{(x, y_1, y_2) \in \mathbb{R}^3 : x \geq 0, y_1 \geq 0, y_2 \geq 0\}$. Then all solutions to system (3.30)–(3.32) corresponding to initial data in \mathbb{R}_+^3 are*

- (i) *nonnegative for all $t > t_0$;*
- (ii) *uniformly bounded in \mathbb{R}_+^3 .*

Moreover, the nonautonomous dynamical system on \mathbb{R}_+^3 generated by the system of ODEs (3.30)–(3.32) has a pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ in \mathbb{R}_+^3 .

Proof. (i) By continuity each solution has to take value 0 before it reaches a negative value. With $x = 0$ and $y_1 \geq 0, y_2 \geq 0$, the ODE for $x(t)$ reduces to

$$x' = D(t)I + bv y_1,$$

and thus $x(t)$ is strictly increasing at $x = 0$. With $y_1 = 0$ and $x \geq 0, y_2 \geq 0$, the reduced ODE for $y_1(t)$ is

$$y_1' = r_2 y_2 \geq 0,$$

thus $y_1(t)$ is non-decreasing at $y_1 = 0$. Similarly, y_2 is non-decreasing at $y_2 = 0$. Therefore, $(x(t), y_1(t), y_2(t)) \in \mathbb{R}_+^3$ for any t .

(ii) Define $\|X(t)\|_1 := x(t) + y_1(t) + y_2(t)$ for $X(t) = (x(t), y_1(t), y_2(t)) \in \mathbb{R}_+^3$. Then $\|X(t)\|_1 \leq S(t) \leq \frac{a}{c} \|X(t)\|_1$, where

$$S(t) = x(t) + \frac{a}{c}(y_1(t) + y_2(t)).$$

The time derivative of $S(t)$ along solutions to (3.30)–(3.32) satisfies

$$\begin{aligned} \frac{dS(t)}{dt} &= D(t)[I - x(t)] - \left[\frac{a}{c}(v + D(t)) - bv \right] y_1(t) - \frac{a}{c} v y_2(t) \\ &\leq d_M I - d_m x(t) - \left[\frac{a}{c}(v + d_m) - bv \right] y_1(t) - \frac{a}{c} v y_2(t) \end{aligned}$$

Note that $\frac{a}{c}(v + d_m) - bv > \frac{a}{c} d_m$ since $a \geq c$ and $0 < b < 1$. Let $\mu := \min\{d_m, v\}$, then

$$\frac{dS(t)}{dt} \leq d_M I - \mu S(t).$$

If $S(t_0) < \frac{d_M I}{\mu}$, then $S(t) \leq \frac{d_M I}{\mu}$ for all $t \geq t_0$. On the other hand, if $S(t_0) \geq \frac{d_M I}{\mu}$, then $S(t)$ will be non-increasing for all $t \geq t_0$ and thus $S(t) \leq S(t_0)$. These imply that $\|X(t)\|_1$ is bounded above, i.e.,

$$\|X(t)\|_1 \leq \max \left\{ \frac{d_M I}{\mu}, x(t_0) + \frac{a}{c}(y_1(t_0) + y_2(t_0)) \right\},$$

for all $t \geq t_0$.

It follows that for every $\varepsilon > 0$ the nonempty compact set

$$B_\varepsilon := \left\{ (x, y_1, y_2) \in \mathbb{R}_+^3 : x + \frac{a}{c}(y_1 + y_2) \leq \frac{d_M I}{\mu} + \varepsilon \right\}$$

is positively invariant and absorbing in \mathbb{R}_+^3 . The nonautonomous dynamical system on \mathbb{R}_+^3 generated by the ODE system (3.30)–(3.32) thus has a pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$, consisting of nonempty compact subsets of \mathbb{R}_+^3 that are contained in B_ε . \square

Now we will obtain more information about the internal structure of the pullback attractor of the nonautonomous dynamical system generated by the ODE system (3.30)–(3.32). First we make the following change of variables:

$$\alpha(t) = \frac{y_1(t)}{y_1(t) + y_2(t)}, \quad z(t) = y_1(t) + y_2(t). \quad (3.34)$$

System (3.30)–(3.32) then becomes

$$\frac{dx(t)}{dt} = D(t)(I - x(t)) - \frac{ax(t)}{\lambda + x(t)}z(t) + bv\alpha(t)z(t), \quad (3.35)$$

$$\frac{dz(t)}{dt} = -vz(t) - D(t)\alpha(t)z(t) + \frac{cx(t)}{\lambda + x(t)}z(t), \quad (3.36)$$

$$\frac{d\alpha(t)}{dt} = -D(t)\alpha(t)(1 - \alpha(t)) - r_1\alpha(t) + r_2(1 - \alpha(t)). \quad (3.37)$$

Note that the steady state solution $(I, 0, 0)$ of system (3.30)–(3.32) has no counterpart for system (3.35)–(3.37), since α is not defined for it. On the other hand, $(I, 0)$ is a steady state solution for the subsystem (3.35)–(3.36).

As the dynamics of $\alpha(t) = \alpha(t; t_0, \alpha_0)$ is uncoupled from $x(t)$ and $z(t)$ and satisfies the Riccati equation (3.37), we can study the behavior of $\alpha(\cdot)$ first, and use this information later on when we analyze the asymptotic behavior for the other components of the solution. For any positive y_1 and y_2 we have $0 < \alpha(t) < 1$ for all t . Note that $\alpha'|_{\alpha=0} = r_2 > 0$ and $\alpha'|_{\alpha=1} = -r_1 < 0$, and thus the interval $(0, 1)$ is positively invariant. This is the biologically relevant region.

When D is a constant, equation (3.37) has a unique asymptotically stable steady state. We investigate now the case in which D varies in time (e.g., periodically or almost periodically) in a bounded positive interval $D(t) \in [d_m, d_M]$ for all $t \in \mathbb{R}$. In such a situation we need to consider the pullback attractor $\mathcal{A}_\alpha = \{A_\alpha(t) : t \in \mathbb{R}\}$ in the interval $(0, 1)$. Such an attractor exists since the unit interval $(0, 1)$ is positively invariant (see e.g., [51]), so its component subsets are given by

$$A_\alpha(t) = \bigcap_{t_0 < t} \alpha(t, t_0, [0, 1]), \quad \forall t \in \mathbb{R}.$$

In addition, these component subsets have the form

$$A_\alpha = [\alpha_l^*(t), \alpha_u^*(t)],$$

where $\alpha_l^*(t)$ and $\alpha_u^*(t)$ are entire bounded solutions of the Riccati equation. All other bounded entire solutions of the Riccati equation (3.32) lie between $\alpha_l^*(t)$ and $\alpha_u^*(t)$.

We can use differential inequalities to obtain bounds on these entire solutions. Indeed, denoting

$$\beta^* = \frac{r_2}{r_1 + r_2}, \text{ and } \gamma^* = \frac{r_2}{r_1 + r_2 + d_M},$$

it is not difficult to prove that

$$\mathcal{A}(t) = [\alpha_l^*(t), \alpha_u^*(t)] \subset [\gamma^*, \beta^*].$$

To investigate the case where the pullback attractor consists of a single entire solution, we need to find conditions under which

$$\alpha_l^*(t) \equiv \alpha_u^*(t), \quad t \in \mathbb{R}.$$

Suppose that they are not equal and consider their difference $\Delta_\alpha(t) = \alpha_u^*(t) - \alpha_l^*(t)$. Then

$$\begin{aligned} \frac{d\Delta_\alpha(t)}{dt} &= D(t)(\alpha_u^*(t) + \alpha_l^*(t))\Delta_\alpha(t) - (D(t) + r_1 + r_2)\Delta_\alpha(t) \\ &\leq d_M \cdot 2\alpha_u^*(t)\Delta_\alpha(t) - (d_m + r_1 + r_2)\Delta_\alpha(t) \\ &\leq \left(\frac{2d_M r_2}{r_1 + r_2} - d_m - r_1 - r_2 \right) \Delta_\alpha(t). \end{aligned}$$

Thus

$$0 \leq \Delta_\alpha(t) \leq e^{\left(\frac{2d_M r_2}{r_1 + r_2} - d_m - r_1 - r_2 \right)(t-t_0)} \Delta_\alpha(t_0) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

(as well as when $t_0 \rightarrow -\infty$) provided

$$\frac{2d_M r_2}{r_1 + r_2} - d_m - r_1 - r_2 < 0, \quad (3.38)$$

which is equivalent to $2d_M r_2 < d_m(r_1 + r_2) + (r_1 + r_2)^2$. To conclude, the pullback attractor for the Riccati ODE (3.37) consists of a singleton entire solution, $\alpha^*(t)$, provided (3.38) holds. Moreover, $\alpha^*(t)$ is also asymptotically stable in the forward sense.

Remark 3.9. Since $d_m < d_M$, this holds, e.g., if $d_M(r_2 - r_1) < (r_1 + r_2)^2$. Inequality (3.38) essentially puts a restriction on the width of the interval in which $D(t)$ can take its values, unless $r_1 > r_2$.

Now we can analyze the global dynamics of $x(t)$ and $z(t)$.

Suppose that $\alpha^*(t)$ is the unique entire solution in the pullback attractor of the Riccati ODE (3.37). Then $\alpha^*(t) \in [\gamma^*, \beta^*] \subset (0, 1)$ for all $t \in \mathbb{R}$. Moreover, for t sufficiently large, $x(t)$ and $z(t)$ components of system (3.35)–(3.37) satisfy

$$\frac{dx(t)}{dt} = D(t)(I - x(t)) - \frac{ax(t)}{\lambda + x(t)}z(t) + bv\alpha^*(t)z(t), \quad (3.39)$$

$$\frac{dz(t)}{dt} = -vz(t) - D(t)\alpha^*(t)z(t) + \frac{cx(t)}{\lambda + x(t)}z(t). \quad (3.40)$$

System (3.39)–(3.40) has a steady state equilibrium $(I, 0)$. Hence $(I, 0, \alpha^*(t))$ is a nonautonomous “equilibrium” solution of the system (3.35)–(3.37).

Theorem 3.8. *Assume that $D : \mathbb{R} \rightarrow [d_m, d_M]$, with $0 < d_m < d_M < \infty$, is continuous, $a \geq c$, $b \in (0, 1)$ and $v > 0$. Then, system (3.39) - (3.40) possesses a pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ inside the nonnegative quadrant. Moreover,*

(i) *When*

$$v + d_m\gamma^* > c,$$

the axial steady state solution $(I, 0)$ is asymptotically stable in the nonnegative quadrant and the pullback attractor \mathcal{A} has a singleton component subset $A(t) = \{(I, 0)\}$ for all $t \in \mathbb{R}$.

(ii) *When*

$$v + d_M\beta^* < \frac{cd_m I}{\lambda(a - c + v + d_M - bv\beta^*) + d_m I},$$

the pullback attractor \mathcal{A} also contains points strictly inside the positive quadrant in addition to the point $\{(I, 0)\}$.

Variable nutrition input rate

Now we assume that the nutrition input value I can vary continuously with time while the consumption rate D is a constant. Similarly we assume that I is bounded with positive values, in particular, $I(t) \in [i_m, i_M]$ for all $t \in \mathbb{R}$, where $0 < i_m \leq i_M < \infty$.

When D is a constant, I varies in time and there are no delays in time, the system (3.30) - (3.32) with U taking the form (3.33) becomes

$$\frac{dx(t)}{dt} = D(I(t) - x(t)) - \frac{ax(t)}{\lambda + x(t)}(y_1 + y_2) + bvy_1(t), \quad (3.41)$$

$$\frac{dy_1(t)}{dt} = -(v + D)y_1(t) + \frac{cx(t)}{\lambda + x(t)}y_1(t) - r_1y_1(t) + r_2y_2(t), \quad (3.42)$$

$$\frac{dy_2(t)}{dt} = -vy_2(t) + \frac{cx(t)}{\lambda + x(t)}y_2(t) + r_1y_1(t) - r_2y_2(t). \quad (3.43)$$

Lemma 3.2. *Suppose that $(x_0, y_{1,0}, y_{2,0}) \in \mathbb{R}_+^3$. Then, all solutions to system (3.41)–(3.43) with initial value $(x(t_0), y_1(t_0), y_2(t_0)) = (x_0, y_{1,0}, y_{2,0})$ are*

- (i) *nonnegative for all $t > t_0$;*
- (ii) *uniformly bounded in \mathbb{R}_+^3 .*

Moreover, the nonautonomous dynamical system on \mathbb{R}_+^3 generated by the system of ODES (3.41)–(3.43) has a pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ in \mathbb{R}_+^3 .

Proof. Similar to the proof of Lemma 3.1. □

Using the new variables $z(t)$ and $\alpha(t)$ defined as in (3.34), equations (3.41)–(3.43) become

$$\frac{dx(t)}{dt} = D(I(t) - x(t)) - \frac{ax(t)}{\lambda + x(t)}z(t) + bv\alpha(t)z(t), \quad (3.44)$$

$$\frac{dz(t)}{dt} = -vz(t) - D\alpha(t)z(t) + \frac{cx(t)}{\lambda + x(t)}z(t), \quad (3.45)$$

$$\frac{d\alpha(t)}{dt} = -D\alpha(t)(1 - \alpha(t)) - r_1\alpha(t) + r_2(1 - \alpha(t)). \quad (3.46)$$

Equation (3.46) has a unique steady state solution

$$\alpha^* = \frac{D + r_1 + r_2 - \sqrt{(D + r_1 + r_2)^2 - 4Dr_2}}{2D}$$

which is asymptotically stable on $(0, 1)$. Hence, when $t \rightarrow \infty$, replacing $\alpha(t)$ by α^* in equations (3.44) and (3.45), we have

$$\frac{dx(t)}{dt} = D(I(t) - x(t)) - \frac{ax(t)}{\lambda + x(t)}z(t) + bv\alpha^*z(t), \quad (3.47)$$

$$\frac{dz(t)}{dt} = -vz(t) - D\alpha^*z(t) + \frac{cx(t)}{\lambda + x(t)}z(t). \quad (3.48)$$

For more details of the long term dynamics of the solutions to (3.47) - (3.48) we establish the following theorem.

Theorem 3.9. *Assume that $I : \mathbb{R} \rightarrow [i_m, i_M]$, with $0 < i_m < i_M < \infty$, is continuous, $a \geq c$, $b \in (0, 1)$ and $v > 0$. Then system (3.47) - (3.48) has a pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ inside the nonnegative quadrant. Moreover,*

- (i) *when $v + D\alpha^* > c$, the entire solution $(w^*(t), 0)$ is asymptotically stable in \mathbb{R}_+^2 where*

$$w^*(t) = De^{-Dt} \int_{-\infty}^t I(s)e^{Ds} ds,$$

and the pullback attractor \mathcal{A} has a singleton component subset $A(t) = \{(w^(t), 0)\}$ for all $t \in \mathbb{R}$,*

(ii) when

$$\nu + D\alpha^* < \frac{cDi_M}{\lambda(a - c + \nu - b\nu\alpha^* + D) + Di_M}$$

the pullback attractor \mathcal{A} also contains points strictly inside the positive quadrant in addition to the set $\{(w^*(t), 0)\}$.

Over-yielding in nonautonomous chemostats

An interesting feature related to nonautonomous chemostats and also other nonautonomous population models in general, is the so called over-yielding effect produced by temporal variation of one or more parameters in the model. For a given amount of nutrient that is fed in a chemostat during a given period of time T , one can compare the biomass production over the time period taking into account the way the amount of nutrient is distributed over the time period. It is said that there exists a biomass *over-yielding* when a time varying input produces more biomass than a constant input. To illustrate the effect of over-yielding in nonautonomous chemostats, we consider the chemostat model with wall growth and variable inputs (3.30)-(3.32), rewritten also with the new variables as (3.35)-(3.37).

When $D(t) = D$ is constant and $I(\cdot)$ is a non-constant T -periodic function with

$$\frac{1}{T} \int_t^{t+T} I(s) ds = \bar{I},$$

a periodic solution of system (3.47) - (3.48) has to fulfill the equations

$$0 = D(\bar{I} - \bar{x}) - a \frac{1}{T} \int_t^{t+T} U(x(s))z(s) ds + b\gamma\alpha^*\bar{z}, \quad (3.49)$$

$$0 = -(\gamma + D\alpha^*)\bar{z} + c \frac{1}{T} \int_t^{t+T} U(x(s))z(s) ds, \quad (3.50)$$

where \bar{x} , \bar{z} denote the average values of the variables $x(\cdot)$, $z(\cdot)$ over the period T . Combining equations (3.49) and (3.50), one obtains the relation

$$D(\bar{I} - \bar{x}) = \left[\frac{a(\gamma + D\alpha^*)}{c} - b\gamma\alpha^* \right] \bar{z}. \quad (3.51)$$

One can also write from equation (3.48)

$$0 = \frac{1}{T} \int_t^{t+T} \frac{z'(s)}{z(s)} ds = -(\gamma + D\alpha^*) + c \frac{1}{T} \int_t^{t+T} U(x(s)) ds.$$

As the function $U(\cdot)$ as defined in (3.33) is concave and increasing, one deduces the inequality $\bar{x} > x^*$, where x^* stands for the steady state of the variable $x(\cdot)$ with the constant input $I(t) = \bar{I}$. Similarly, x^* satisfies the equality $cU(x^*) = \gamma + D\alpha^*$. One

can then compare the corresponding biomass variables, by using equation (3.51), to obtain

$$\left[b\gamma\alpha^* - \frac{a(\gamma + D\alpha^*)}{c} \right] (\bar{z} - z^*) > 0.$$

We conclude that an over-yielding occurs when the condition

$$bc\gamma\alpha^* > a(\gamma + D\alpha^*) \quad (3.52)$$

is fulfilled. One can see that the nutrient recycling of the dead biomass ($b\gamma \neq 0$) is essential to obtain an over-yielding.

In the special case where the wall growth is neglected, our system (3.30) - (3.32) reduces to the system of two ODEs

$$\frac{dx(t)}{dt} = D(t)[I(t) - x(t)] - \frac{ax(t)}{\lambda + x(t)}y(t), \quad (3.53)$$

$$\frac{dy(t)}{dt} = -D(t)y(t) + \frac{ax(t)}{\lambda + x(t)}y(t). \quad (3.54)$$

Assume now that $I(\cdot) = I$ is constant and $D(\cdot)$ is a non-constant T -periodic function with

$$\frac{1}{T} \int_t^{t+T} D(s)ds = \bar{D}.$$

From equations (3.53)-(3.54) a periodic solution has to fulfill

$$I = x(t) + y(t) \quad (3.55)$$

$$0 = \frac{1}{T} \int_t^{t+T} \frac{y'(s)}{y(s)} ds = -\bar{D} + a \frac{1}{T} \int_t^{t+T} U(x(s)) ds. \quad (3.56)$$

From equation (3.56), one obtains, as before, the inequality $\bar{x} > x^*$ and thus $\bar{y} < y^*$. Consequently over-yielding never occurs.

Remark 3.10. For the chemostat model with wall growth and periodic $D(\cdot)$, it is still unknown whether an over-yielding is possible or not, although numerical simulations tend to show that it is not. For more general time varying inputs (i.e. not necessarily periodic), one can also study the influence of the variations of the inputs on the characteristics of the pullback attractor as can be observed in our previous theorems 3.8 and 3.9.

3.3.2 Nonautonomous SIR model

Here we release the assumption that the total population is a constant, i.e., $N(t) = S(t) + I(t) + R(t)$ can vary (either deterministically or randomly) with time. This can be achieved in various ways, but here we consider the simplest situation, where

the reproduction spam is modeled by a time-dependent function, $\Lambda(t)$. To this end, consider a temporal forcing term given by a continuous function $\Lambda(t) : \mathbb{R} \rightarrow \mathbb{R}$ with positive bounded values

$$\Lambda(t) \in [\Lambda_1, \Lambda_2], \quad \forall t \in \mathbb{R}, \quad 0 < \Lambda_1 \leq \Lambda_2.$$

With this time-varying force, system (2.12) – (2.14) becomes

$$\frac{dS}{dt} = \Lambda(t) - \frac{\beta}{N}SI - \nu S, \quad (3.57)$$

$$\frac{dI}{dt} = \frac{\beta}{N}SI - \nu I - \gamma I, \quad (3.58)$$

$$\frac{dR}{dt} = \gamma I - \nu R. \quad (3.59)$$

For simplicity write $u(t) := (S(t), I(t), R(t))$ and $u_0 = (S_0, I_0, R_0)$.

Lemma 3.3. *Suppose that $u_0 = (S_0, I_0, R_0) \in \mathbb{R}_+^3$. Then, the solution to system (3.57)–(3.59) with initial value $u(t_0) = (S(t_0), I(t_0), R(t_0)) = u_0$ is defined globally in time and is nonnegative for all $t \geq t_0$.*

Proof. The proof follows a similar argument to the proof of Lemma 3.1. \square

Thanks to Lemma 3.3 we can define a process on \mathbb{R}_+^3 by

$$\varphi(t, t_0, u_0) = u(t; t_0, u_0), \quad \text{for } u_0 \in \mathbb{R}_+^3 \text{ and } t \geq t_0, \quad (3.60)$$

where $u(t; t_0, u_0)$ is the solution to (3.57)–(3.59) corresponding to the initial value u_0 . Next we prove that this process possesses a pullback attractor with respect to bounded subsets.

Lemma 3.4. *Assume $2\nu > \max\{\gamma, 1\}$ and $\gamma > \beta$. Then, for any initial value $u_0 \in \mathbb{R}_+^3$, the corresponding solution to (3.57)–(3.59) satisfies*

$$|u(t; t_0, u_0)|^2 \leq e^{-\kappa(t-t_0)}|u_0|^2 + e^{-\kappa t} \int_{-\infty}^t e^{\kappa s} \Lambda^2(s) ds$$

for all $t \geq t_0$, provided

$$\kappa = \min\{2\nu - \gamma, 2\nu - 1, 2\nu + \gamma - \beta\} > 0.$$

Moreover, the family $\mathcal{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ given by $D_0(t) = \overline{\mathbb{B}}(0, \rho_\kappa(t)) \cap \mathbb{R}_+^3$, where $\rho_\kappa(t)$ is the nonnegative constant

$$\rho_\kappa^2(t) = 1 + \nu e^{-\kappa t} \int_{-\infty}^t e^{\kappa s} \Lambda^2(s) ds,$$

is a pullback absorbing family for the process φ .

Proof. First, multiplying (3.57) by S , (3.58) by I , (3.59) by R and summing them up we obtain

$$\frac{d}{dt}|u(t)|^2 = 2\Lambda(t)S - 2\nu|u(t)|^2 + 2\gamma IR - 2\gamma I^2 - 2\beta\left(\frac{S^2 I}{N} - \frac{S I^2}{N}\right).$$

Since

$$2\Lambda(t)S \leq \Lambda^2(t) + S^2, \quad 2\gamma IR \leq \gamma I^2 + \gamma R^2,$$

on account that $0 \leq \frac{S}{N}, \frac{I}{N} \leq 1$, we deduce

$$\frac{d}{dt}|u(t)|^2 + \kappa|u(t)|^2 \leq \Lambda^2(t) \leq \Lambda_2^2.$$

Multiplying by $e^{\kappa t}$ and integrating between t_0 and t

$$\begin{aligned} e^{\kappa t}|u(t)|^2 &\leq e^{\kappa t_0}|u_0|^2 + \int_{t_0}^t e^{\kappa s}\Lambda^2(s)ds \\ &\leq e^{\kappa t_0}|u_0|^2 + \int_{-\infty}^t e^{\kappa s}\Lambda^2(s)ds. \end{aligned}$$

Let $D \subset \mathbb{R}_+^3$ be bounded. Then there exists $d > 0$ such that $|u_0| \leq d$ for all $u_0 \in D$. By Lemma 3.4 we deduce for $t_0 \leq t$ and $u_0 \in D$,

$$\begin{aligned} |\varphi(t, t_0, u_0)|^2 &\leq e^{-\kappa t} e^{\kappa t_0} |u_0|^2 + e^{-\kappa t} \int_{-\infty}^t e^{\kappa s} \Lambda^2(s) ds \\ &\leq e^{-\kappa t} e^{\kappa t_0} d^2 + e^{-\kappa t} \int_{-\infty}^t e^{\kappa s} \Lambda^2(s) ds. \end{aligned}$$

Denoting $T(t, D) = \kappa^{-1} \log(e^{\kappa t} d^{-2})$, then

$$|\varphi(t, t_0, u_0)|^2 \leq 1 + e^{-\kappa t} \int_{-\infty}^t e^{\kappa s} \Lambda^2(s) ds,$$

for all $t_0 \leq T(t, D)$ and all $u_0 \in D$.

Consequently, the family \mathcal{D}_0 is pullback absorbing for the process φ . \square

These results enable us to conclude with the existence of the pullback attractor for our nonautonomous SIR model.

Theorem 3.10. *Under the assumptions in Lemma 3.4 the process φ previously defined in \mathbb{R}_+^3 possesses a pullback attractor \mathcal{A} .*

3.3.3 Nonautonomous Lorenz-84 model

In the simple Lorenz-84 model (2.17) – (2.19), the forcing terms F and G are assumed to be constant. To make the model more realistic, we allow the forcing terms to vary over time, resulting in the following nonautonomous Lorenz model:

$$\frac{dx}{dt} = -ax - y^2 - z^2 + aF(t), \quad (3.61)$$

$$\frac{dy}{dt} = -y + xy - bxz + G(t), \quad (3.62)$$

$$\frac{dz}{dt} = -z + bxy + xz, \quad (3.63)$$

The nonautonomous terms $F(\cdot)$ and $G(\cdot)$ can be periodic, almost periodic or even belong to a type of functions more general than almost periodic, as in [46]. To illustrate the theory of attractors for the process formalism, we will consider very general nonautonomous terms. Indeed, we will assume that the mappings $F, G : \mathbb{R} \mapsto \mathbb{R}$ are continuously differentiable and satisfy

$$\int_{-\infty}^t e^{ls} F^2(s) ds < +\infty, \quad \int_{-\infty}^t e^{ls} G^2(s) ds < +\infty, \quad \forall t \in \mathbb{R},$$

where $l := \min\{a, 1\}$.

First we need to construct the process generated by (3.61)-(3.63). It is easy to check that the right hand side of system (3.61)-(3.63) defines a vector field which is continuously differentiable. Thus, the standard theory of existence and uniqueness of local solutions for systems of ordinary differential equations guarantees the existence of local solutions for the initial value problems associated to (3.61)-(3.63). The next theorem will ensure that the local solutions can be extended to a global one defined for all $t \in \mathbb{R}$. For simplicity, denote by $u(t) := (x(t), y(t), z(t))$ and $u_0 = (x_0, y_0, z_0)$

Lemma 3.5. *Assume that $a > 0$ and $b \in \mathbb{R}$. Then for any initial condition $u_0 \in \mathbb{R}^3$ and any initial time $t_0 \in \mathbb{R}$, the solution $u(\cdot) := u(\cdot; t_0, u_0)$ of the IVP associated to (3.61)-(3.63) satisfies*

$$|u(t; t_0, u_0)|^2 \leq e^{-l(t-t_0)} |u_0|^2 + ae^{-lt} \int_{-\infty}^t e^{ls} F^2(s) ds + e^{-lt} \int_{-\infty}^t e^{ls} G^2(s) ds, \quad (3.64)$$

for all $t \geq t_0$, where $l := \min\{a, 1\}$.

Proof. We immediately deduce that

$$\frac{d}{dt} |u(t)|^2 = -2(ax^2 + y^2 + z^2) + 2axF(t) + 2yG(t),$$

and, consequently,

$$\frac{d}{dt} |u(t)|^2 + l|u(t)|^2 \leq aF^2(t) + G^2(t). \quad (3.65)$$

Multiplying (3.65) by e^{lt} , we obtain that

$$\frac{d}{dt} \left(e^{lt} |u(t)|^2 \right) \leq a e^{lt} F^2(t) + e^{lt} G^2(t).$$

Integrating between t_0 and t

$$\begin{aligned} e^{lt} |u(t)|^2 &\leq e^{lt_0} |u_0|^2 + a \int_{t_0}^t e^{ls} F^2(s) ds + \int_{t_0}^t e^{ls} G^2(s) ds \\ &\leq e^{lt_0} |u_0|^2 + a \int_{-\infty}^t e^{ls} F^2(s) ds + \int_{-\infty}^t e^{ls} G^2(s) ds, \end{aligned}$$

whence (3.64) follows. \square

Thanks to Lemma 3.5 we can define a process on \mathbb{R}^3 by

$$\varphi(t, t_0, u_0) = u(t; t_0, u_0) \quad \text{for } u_0 \in \mathbb{R}^3 \quad \text{and } t \geq t_0. \quad (3.66)$$

Considering now the universe of fixed nonempty bounded subsets of \mathbb{R}^3 , we will be able to prove that there exists a pullback absorbing family for the process $\varphi(\cdot, \cdot, \cdot)$.

Theorem 3.11. *Assume that $a > 0$ and $b \in \mathbb{R}$. Let F, G satisfy (4.69). Then, the family $\mathcal{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{\mathbb{B}}(0, \rho_l(t))$, where $\rho_l(t)$ is given by*

$$\rho_l^2(t) = 1 + a e^{-lt} \int_{-\infty}^t e^{ls} F^2(s) ds + e^{-lt} \int_{-\infty}^t e^{ls} G^2(s) ds, \quad \forall t \in \mathbb{R},$$

is a pullback absorbing family for the process $\varphi(\cdot, \cdot, \cdot)$. Furthermore, there exists the pullback attractor for the process φ .

Proof. Let $D \subset \mathbb{R}^3$ be bounded. Then, there exists $d > 0$ such that $|u_0| \leq d$ for all $u_0 \in D$. Thanks to Lemma 4.2, we deduce that for every $t_0 \leq t$ and any $u_0 \in D$,

$$\begin{aligned} |\varphi(t, t_0, u_0)|^2 &\leq e^{-lt} e^{lt_0} |u_0|^2 + a e^{-lt} \int_{-\infty}^t e^{ls} F^2(s) ds + e^{-lt} \int_{-\infty}^t e^{ls} G^2(s) ds \\ &\leq e^{-lt} e^{lt_0} d^2 + a e^{-lt} \int_{-\infty}^t e^{ls} F^2(s) ds + e^{-lt} \int_{-\infty}^t e^{ls} G^2(s) ds. \end{aligned}$$

If we set $T(t, D) := l^{-1} \log(e^{lt} d^{-2})$, we have

$$|\varphi(t, t_0, u_0)|^2 \leq 1 + a e^{-lt} \int_{-\infty}^t e^{ls} F^2(s) ds + e^{-lt} \int_{-\infty}^t e^{ls} G^2(s) ds,$$

for all $t_0 \leq T(t, D)$ and for all $u_0 \in D$.

Consequently the family $\mathcal{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{\mathbb{B}}(0, \rho_l(t))$ is pullback absorbing for the process φ , and thanks to Theorem 3.1 the process defined in (3.66) possesses a pullback attractor. \square

Remark 3.11. A more detailed analysis concerning the structure of the pullback attractor for the nonautonomous Lorenz-84 model can be found in [1]. More precisely, one can see how the properties of the nonautonomous terms (periodicity, almost periodicity or a property called more general than almost periodicity by Kloeden and Rodrigues in [46]) in the model are inherited by the pullback attractor. Also an estimate of the Hausdorff dimension of the pullback attractor is constructed in [1], for which the skew product flow formalism has to be used. However, we do not include more details in this work due to the page limit. The reader is referred to the paper [1] for more details.

Chapter 4

Random dynamical systems

In this chapter we will introduce methods and techniques to analyze models with stochasticity or randomness. In particular we will establish the framework of random dynamical systems and introduce the concept of random attractors. To make the content more accessible to readers from the applied sciences and engineering, we start from the motivation of generalizing the concept of nonautonomous pullback attractors to the stochastic context.

4.1 Noise is present almost everywhere

It has been well understood that noise, also referred to as randomness or stochasticity, presents in most of the real world phenomena. Therefore modeling such phenomena by systems of deterministic differential equations is merely a simplistic approximation. One question then arises naturally, does the approximating deterministic system behave similarly to or differently from the real system? Furthermore, if we take into account the noisy disturbance and model the phenomena by systems of stochastic differential equations or random differential equations, then what are the effects induced by noise in comparison with the corresponding deterministic approximations?

Another question of great interest and importance is the type of stochastic processes that best describe the noise. There are many different choices of the noise suitable for different applications, such as Brownian motion, fractional Brownian motion, Poisson noise, Lévy noise, etc. On the other hand, there are different ways to include the noise term, e.g., as an additional term to a deterministic system, in the Itô or the Stratonovich sense, or as random parameters or functions of stochastic processes. It is worth mentioning that different types of noise or different ways of including the noise may yield different results while analyzing the long-time behavior of the system. In this work we will not justify which is the most suitable type of noise to model a certain phenomenon. Instead, we will illustrate, by using simple examples, different results produced by the inclusion of different types of noise.

Due to the limitation of the length of this book, it is impossible even just to analyze existence, uniqueness and stability of solutions for all possible inclusion of different noises. Here we will develop a set of theories that require a minimum amount of knowledge on stochastic differential equations, and present the most basic and necessary results only. The best candidate for this purpose, no doubt, is the theory of random attractors which is a natural though nontrivial generalization of skew product flows introduced in Chapter 3.

It is worth mentioning that the theories of pullback attractors for nonautonomous and random dynamical systems were originally developed in the opposite way of what is organized in this work. The first studies on pullback attractors were done in the stochastic context at the beginning of the 90's, where the attractor was termed as "random" [28, 29, 72]. But it was only around the end of that decade when the first papers containing the concept of pullback attractors appeared published ([22, 45]). Nevertheless, with both concepts available today, introducing random attractors after pullback attractors is more logical and more straightforward for the reader.

In the rest of this Chapter we will first introduce the basic tools for the analysis of random dynamical systems and then apply them to random counterparts of the systems discussed in Chapters 2 and 3. We will also discuss briefly an interesting topic on the effects produced by the noise in stabilizing or destabilizing deterministic systems.

4.2 Formulation of Random Dynamical System and Random Attractor

When the time-dependent forcing is random, the pullback attractor introduced in Chapter 3 then becomes a random pullback attractor or random attractor for short. The framework introduced in Section 3.2 allows us to show that where the parameter space is a probability space $P = \Omega$, for almost all realizations $\omega \in \Omega$, the evolution in the state space X of a stochastic system from time $t_0 < t$ to time t can be described by a two-parameter family of transformations $\varphi(t, t_0, \omega)$. It is then natural to adopt the pullback approach described in Section 3.2 in an ω -parametrized version, to obtain the analog of a pullback attractor in the stochastic context. Such a generalization, however, needs additional characterizations, as the resulting object $\bigcup_{t \in \mathbb{R}} A(t, \omega)$ does not exhibit any relation between different realizations ω . For example, from an experimentalist point of view, for an experiment to be repeatable, there has to be a reasonable description for its random aspects. These random aspects may change in time, and thus the noise has to be modeled as a time-dependent stochastic process with certain known properties.

Representing mathematically such a stochastic process starts with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where \mathcal{F} is a σ -algebra of measurable subsets of Ω , called "events", and \mathbb{P} is the probability measure. To track the noise effect by time, we need to connect each realization, i.e., each state ω of the random environment at time $t = 0$

with its state after a time of t has elapsed. Denoting this connection by $\theta_t\omega$ and setting $\theta_0\omega = \omega$ establishes a mapping $\theta_t : \Omega \rightarrow \Omega$ for all times t . As a result, instead of describing a random ordinary differential equation (RODE) as

$$\frac{df}{dt} = f(t, \omega, x), \quad (4.1)$$

it is more accurate to consider

$$\frac{df}{dt} = f(\theta_t\omega, x). \quad (4.2)$$

In fact, equation (4.1) is a simple differential equation depending on only one random parameter ω at all time, while (4.2) takes into account the time evolution of the random parameter as well. Moreover, as will be shown later in this chapter, RODEs of the form (4.2) also arise from a suitable change of variables in particular stochastic differential equations.

In this chapter we will consider two types of noise that have been widely used in engineering and the applied sciences: the *white noise* and the *real noise*. When the random aspect has no memory of the past, is not correlated with itself, and at each time the probability of obtaining a specific value $x \in \mathbb{R}$ is the same as obtaining $-x$, it can be described by the white noise, commonly denoted as W_t . White noise has been widely used in engineering applications, where the notation refers to a signal (or process), named in analogy to white light, with equal energy over all frequency bands. When the random aspect is investigated in a pathwise sense instead of statistically, it can be described by the real noise, commonly expressed as a function of the flow $\theta_t\omega$, e.g., $D(\theta_t\omega)$. The use of real noise is well justified for many applications in the applied sciences such as biology and ecology, when population (must be non-negative) dynamics are studied.

The consideration of the white noise yields systems of stochastic differential equations (SDEs), while the consideration of the real noise yields systems of random differential equations (RDEs). In some cases where the white noise has a special structure, SDEs could be transformed into RDEs by suitable change of variables. In this chapter we will focus on the RDEs and SDEs whose solutions generate a random dynamical system (RDS). Roughly speaking, a RDS is essentially a nonautonomous dynamical system formulated as a skew product flow with the base space Ω being a probability space rather than a metric space. The driving system, $\theta = (\theta_t)_{t \in \mathbb{R}}$ on Ω is now a dynamical system satisfying some measurability instead of continuity conditions in the base space variables.

In practice, the mapping $(t, \omega) \mapsto \theta_t\omega$ is required to be measurable, and to satisfy the one-parameter group property $\theta_{t+\tau} = \theta_t \circ \theta_\tau$ for any t and τ , along with $\theta_0 = Id_\Omega$. Such requirements lead to a time-dependent family $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ of invertible transformations of Ω that keeps track of the noise. Furthermore, a stationary condition required such that the statistics of the external noise are invariant under θ_t . Mathematically, this means that the probability measure \mathbb{P} is preserved by θ_t , i.e., $\theta_t\mathbb{P} = \mathbb{P}$.

In summary these properties of θ_t result in a driving dynamical system defined as follows.

Definition 4.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta = (\theta_t)_{t \in \mathbb{R}}$ be a flow on Ω which is defined as a mapping $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$ satisfying

- (i) $\theta_0 \omega = \omega$ for all $\omega \in \Omega$;
- (ii) $\theta_{t+\tau} \omega = \theta_t \circ \theta_\tau \omega$ for all $t, \tau \in \mathbb{T}$.
- (iii) the mapping $(t, \omega) \mapsto \theta_t \omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$ -measurable and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Then the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a driving dynamical system.

With the definition of driving dynamical system, we are ready to introduce the definition of RDS.

Definition 4.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X = \mathbb{R}^d$. A random dynamical system (θ, ψ) on X consists of a driving dynamical system $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ acting on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a cocycle mapping $\psi : \mathbb{R}_0^+ \times \Omega \times X \rightarrow X$ satisfying

- (i) initial condition: $\psi(0, \omega, x) = x$ for all $\omega \in \Omega$ and $x \in X$,
- (ii) cocycle property: $\psi(t+s, \omega, x) = \psi(t, \theta_s \omega, \psi(s, \omega, x))$ for all $t, s \in \mathbb{R}_0^+, \omega \in \Omega$ and $x \in X$.
- (iii) measurability: $(t, \omega, x) \mapsto \psi(t, \omega, x)$ is measurable,
- (iv) continuity: $x \mapsto \psi(t, \omega, x)$ is continuous for all $(t, \omega) \in \mathbb{R} \times \Omega$.

Remark 4.1. In the above definition we assume $X = \mathbb{R}^d$ because we are mainly interested in RDS on finite dimensional spaces in this book, but in general X could be any metric space.

To make latter presentation more accessible for a wider audience, we first consider the canonical representation of the driving system θ on the probability space Ω that is based on Wiener process. Recall that an m -dimensional two-sided Wiener process is a stochastic process, i.e., a mapping $W(\cdot) : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^m$ satisfying the following properties:

- (i) $W_0(\omega) = 0$ for all $\omega \in \Omega$.
- (ii) $W_t - W_s$ is a Gaussian random variable with zero mean value and variance $t - s$, for all $t \geq s$,
- (iii) W possesses independent increments, i.e., if $s_1 < t_1 \leq s_2 < t_2 \cdots s_k < t_k$ then $W_{t_1} - W_{s_1}, W_{t_2} - W_{s_2}, \dots, W_{t_k} - W_{s_k}$ are independent random variables.
- (iv) W has continuous paths with probability one, i.e. for almost all $\omega \in \Omega$, the mapping $t \in \mathbb{R} \mapsto W_t(\omega)$ is continuous.

Alternatively, we can define the two-sided Wiener process as two independent Wiener processes pasted together W_t and W_{-t} for $t \geq 0$.

When a RDS is generated by an Itô stochastic differential equation driven by an m -dimensional two-sided Wiener process W_t defined for $t \in \mathbb{R}$, then the probability space can be identified with the canonical space of continuous mappings

$\Omega = C_0(\mathbb{R}, \mathbb{R}^m)$, i.e., every event $\omega \in \Omega$ is a continuous functions $\omega : \mathbb{R} \rightarrow \mathbb{R}^m$ such that $\omega(0) = 0$ and θ is defined by the “Wiener shift”

$$\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t).$$

Moreover, we can identify $W_t(\omega) = \omega(t)$ for every $\omega \in \Omega$. Some other examples of driving systems and RDS can be found in Arnold [3].

The definition of attractor for an RDS requires the definition of random sets.

Definition 4.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random set C on $X = \mathbb{R}^d$ is a measurable subset of $X \times \Omega$ with respect to the product σ -algebra of the Borel σ -algebra of X and \mathcal{F} .

A random set can be regarded as a family of sets parametrized by the random parameter ω and satisfies some measurability property. More precisely, a random set C can be identified by the family of its ω -fibers $C(\omega)$, defined by

$$C(\omega) = \{x \in X : (x, \omega) \in C\}, \quad \omega \in \Omega.$$

When a random set $C \subset X \times \Omega$ has closed fibers, it is said to be a *closed random set* if and only if for every $x \in X$ the mapping

$$\omega \in \Omega \rightarrow d(x, C(\omega)) \in [0, +\infty)$$

is measurable (see Castaing and Valadier [21, Chapter 2]). Similarly, when the fibers of C are compact, C is said to be a *compact random set*. On the other hand, an *open random set* is a set $U \subset X \times \Omega$ such that its complementary set $U^c = (X \times \Omega) \setminus U$ is a closed random set.

Definition 4.4. A bounded random set $K(\omega) \subset X$ is said to be tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} e^{-\beta t} \sup_{x \in K(\theta_{-t}\omega)} \|x\|_X = 0, \quad \text{for all } \beta > 0;$$

a random variable $\omega \mapsto r(\omega) \in \mathbb{R}$ is said to be tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} e^{-\beta t} \sup_{t \in \mathbb{R}} |r(\theta_{-t}\omega)| = 0, \quad \text{for all } \beta > 0.$$

Now we can define the forward and pullback random attractors based on the concept similar to skew product flows. Only the basic contents needed for the applications later this chapter are presented here. For more details the reader is referred to the recent survey by Crauel and Kloeden [31].

Definition 4.5. Let (θ, ψ) be an RDS on X and let \mathcal{B} be a family of random sets. A compact random set $\mathcal{A} \subset X \times \Omega$ is said to be a random pullback attractor for \mathcal{B} if

(i) it is strictly invariant, i.e., $\psi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t \omega)$ for every $t \geq 0$, \mathbb{P} -a.s.

(ii) it attracts all the sets of the universe \mathcal{B} in the pullback sense (i.e. \mathcal{A} is a pullback attracting set for \mathcal{B}):

$$\lim_{t \rightarrow +\infty} \text{dist}(\psi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0, \quad \mathbb{P} - \text{a.s.}, \quad \text{for every } B \in \mathcal{B}.$$

The pullback invariance of the random attractor is illustrated in Figure 4.1.

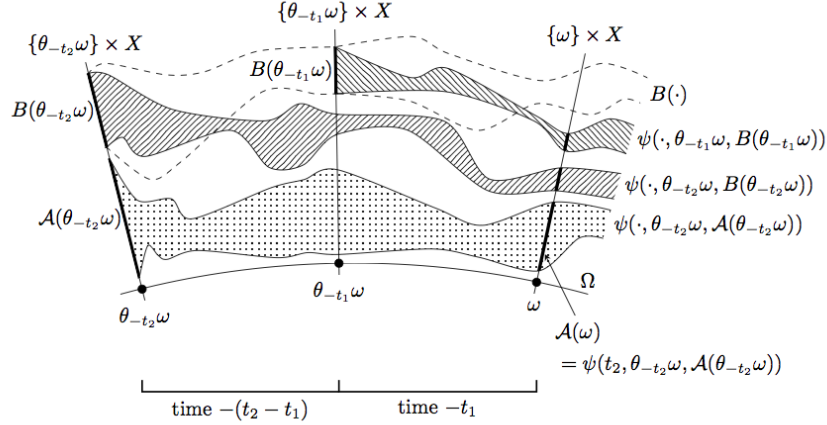


Fig. 4.1: Pullback invariance of the random attractor and pullback attraction of random sets.

It is said that the random set $\mathcal{A} \subset X \times \Omega$ is a random forward attractor for \mathcal{B} if the convergence in (ii) is forward, i.e.,

$$\lim_{t \rightarrow +\infty} \text{dist}(\psi(t, \omega, B(\omega)), \mathcal{A}(\theta_t\omega)) = 0, \quad \mathbb{P} - \text{a.s.}, \quad \forall B \in \mathcal{B}.$$

Similar to deterministic nonautonomous systems, both forward and pullback random attractors can be used to study the long term behavior of a random dynamical system. But there are surprising differences between them and technical complications in studying the differences, as randomness always allows the possibility of exceptional null sets. However, attractors of RDS have much stronger uniqueness properties than those of general nonautonomous systems. For more details on RDS that are not covered in this book, the reader is referred to the monograph by Arnold [3].

Remark 4.2. As in the deterministic nonautonomous framework, there are examples of pullback attractors of RDS which are not forward attractors and vice versa. However, the construction of such an example needs more sophisticated calculations and techniques than in the deterministic case.

Similar to the theory of nonautonomous pullback attractors, the existence of a random attractor relies on the existence of a pullback absorbing set, which can be defined following the same manner as the definition of pullback absorbing sets for a cocycle (see Definition 3.8).

Definition 4.6. A random set $K \in \mathcal{B}$ is said to be pullback absorbing for the universe \mathcal{B} if for every $B \in \mathcal{B}$ and $\omega \in \Omega$, there exists $T(B, \omega) > 0$ such that

$$\psi(t, \theta_{-t}(\omega), B(\theta_{-t}(\omega))) \subset K(\omega), \quad \text{for all } t \geq T(B, \omega).$$

In a general phase space, an absorbing compact set is clearly an attracting compact set while the opposite is not necessarily true. However, for the special space $X = \mathbb{R}^d$ considered in all of the applications in this book, a compact attracting set is also a compact absorbing set. In fact, the closure of an ε -neighborhood of a compact attracting set, is absorbing and compact.

Now we are ready to establish a necessary and sufficient condition ensuring the existence of random pullback attractor.

Theorem 4.1. (*Existence of random attractor: necessary & sufficient condition*) Let (θ, ψ) be an RDS on X and let \mathcal{B} be a family of random sets. Then, there exists a random pullback attractor \mathcal{A} for \mathcal{B} if and only if there exists a compact random set K which is attracting for \mathcal{B} . Furthermore, there exists a unique minimal random pullback attractor $\mathcal{A}_{\mathcal{B}}$ for \mathcal{B} .

The above theorem presents an elegant result, however from the application point of view, it is more effective to have a sufficient condition which ensures the existence of random attractors, stated as follows.

Theorem 4.2. (*Existence of random attractor: sufficient condition*) Let (θ, ψ) be an RDS on X and let \mathcal{B} be a family of random sets. Assume that there exists a compact random set K which is absorbing for \mathcal{B} . Then, there exists a random pullback attractor A for \mathcal{B} . Moreover, this is the unique minimal random pullback attractor $A_{\mathcal{B}}$ for \mathcal{B} .

Analogous results ensuring the existence of a random forward attractor are still not available up to date. However, it is possible to develop a different theory in which a certain type of pullback attractor is also forward attractor, and vice versa. Thanks to the θ_t -invariance of the probability measure \mathbb{P} , for any $\varepsilon > 0$

$$\begin{aligned} & \mathbb{P}\{\omega \in \Omega : \text{dist}((\psi(t, \theta_{-t}(\omega), B(\theta_{-t}(\omega))), A(\omega)) \geq \varepsilon\} \\ &= \mathbb{P}\{\omega \in \Omega : \text{dist}((\psi(t, \omega, B(\omega)), A(\theta_t \omega)) \geq \varepsilon\}. \end{aligned}$$

Since \mathbb{P} -almost sure convergence implies convergence in probability, then a random pullback attractor also converges forward, but only in the weaker sense of convergence in probability. The same argument states that a random forward attractor is also a pullback attractor, but only in probability. This gives rise to a new type of random attractors which are also forward attractors and are termed as “attractors in

probability” or “weak attractors”. The reader is referred to [5] for more details on this topic.

The random attractor, while exists, is closely related to the universe \mathcal{B} of random sets to be attracted. For this reason, one RDS can have different pullback random attractors, each of which associated to a different universe. It is then interesting to discover if there is any relationship among these attractors. One interesting result along this line can be found in [26], stated as follows.

Theorem 4.3. *Let (θ, ψ) be an RDS on X and let \mathcal{B} be the family of all compact deterministic subsets of X (i.e., if $D \in \mathcal{B}$ then $D(\omega) = D_0$, which is a compact set, for all $\omega \in \Omega$). Assume that there exists a compact deterministic set K which is attracting for \mathcal{B} . Then, there exists a random pullback attractor \mathcal{A} for \mathcal{B} , and this attractor is unique in the sense that if $\tilde{\mathcal{A}}$ is a random pullback attractor for every compact deterministic set, then $\mathcal{A} = \tilde{\mathcal{A}}$, \mathbb{P} -a.s. Furthermore, every random compact, invariant set is a subset of \mathcal{A} , \mathbb{P} -almost surely.*

As an immediate consequence we deduce that if \mathcal{B} is an arbitrary universe of random sets for which an RDS (θ, ψ) possesses a random attractor $\mathcal{A}_{\mathcal{B}}$, then as this random set is invariant, we have that $\mathcal{A}_{\mathcal{B}} \subset \mathcal{A}$, \mathbb{P} -a.s. Furthermore, if the universe \mathcal{B} contains every compact deterministic set, then $\mathcal{A}_{\mathcal{B}} = \mathcal{A}$, \mathbb{P} -a.s.

An interesting point to be highlighted here is that the large class of random attractors which can exist associated to different universes can be reduced to just one unique attractor. In summary, if an RDS possesses an attractor for the universe of compact deterministic sets, then for any other universe \mathcal{B} of random sets there are only three possibilities:

- (a) There exists no random pullback attractor for the universe \mathcal{B} .
- (b) There exists a random pullback attractor $\mathcal{A}_{\mathcal{B}}$ for the universe \mathcal{B} and is unique, then $\mathcal{A}_{\mathcal{B}} = \mathcal{A}$, \mathbb{P} -a.s. (when \mathcal{B} contains the compact deterministic sets)
- (c) There exists a random pullback attractor $\mathcal{A}_{\mathcal{B}}$ for the universe \mathcal{B} but is not unique, then $\mathcal{A}_{\mathcal{B}} \subset \mathcal{A}$, \mathbb{P} -a.s.

4.2.1 Some properties of the random attractor

In this section we will discuss some important properties of the random pullback attractor for an RDS (θ, ψ) on the phase space X . First of all, the random attractor associated to the universe of compact deterministic subsets of X will be called the *global random attractor* in what follows and this is the largest random pullback attractor that an RDS may have, at light of the arguments stated in the previous subsection.

Recall first the definition of ω -limit set of a random set B , analogous to the ω -limit set defined in the nonautonomous framework, is given by

$$\Gamma(B, \omega) = \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \psi(t, \theta_{-t}(\omega), B(\theta_{-t}(\omega)))}. \quad (4.3)$$

The set defined by (4.3) is also characterized as the set of points $x \in X$ such that there exists sequences $t_n \rightarrow +\infty$, $b_n \in B(\theta_{-t_n}(\omega))$, $n \in \mathbb{N}$ such that

$$x = \lim_{n \rightarrow \infty} \psi(t_n, \theta_{-t_n}(\omega), b_n).$$

The ω -limit set $\Gamma(B, \omega)$ define in (4.3) is always forward invariant, but not strictly invariant in general. A particular case that $\Gamma(B, \omega)$ is also strict invariant happens when B is attracted by some compact random set. We next elaborate some properties of the structure of the global random attractor. Unfortunately, it is impossible to make the following contents less abstract. But to provide necessary information to the readers who are keen to get a deeper understanding of the global random attractor, we still include the most typical results whose proofs can be found in Crauel [26, 27]. Note skipping the following results will not affect the understanding of the applications to be studied later.

Theorem 4.4. *Let (θ, ψ) be an RDS on the phase space X , and assume that there exists the global random attractor $\omega \mapsto \mathcal{A}(\omega)$. Then,*

- (i) \mathcal{A} is measurable with respect to the past of ψ , i.e., for every $x \in X$, the mapping $\omega \mapsto \text{dist}(x, \mathcal{A}(\omega))$ is measurable with respect to the σ -algebra

$$\mathcal{F}^- = \sigma\{\psi(s, \theta_{-t}(\omega), x) : x \in X, 0 \leq s \leq t\} \subset \mathcal{F}.$$

- (ii) $\Gamma(B, \omega) \subset \mathcal{A}(\omega)$, \mathbb{P} -a.s., for every random set B attracted by \mathcal{A} , in particular, for every compact deterministic set B .
- (iii) $\mathcal{A}(\omega) = \overline{\cup \Gamma(K, \omega)}$, where the union is taken over all compact deterministic subsets $K \subset X$.
- (v) the global random attractor is connected \mathbb{P} -a.s.
- (vi) if θ is ergodic (see e.g. Arnold [3]) then $\mathcal{A}(\omega) = \Gamma(K, \omega)$ for every compact deterministic $K \subset X$ satisfying that $\mathbb{P}\{A(\omega) \subset K\} > 0$.

In order to characterize a random attractor for a universe \mathcal{B} , the following result was established in Crauel [27].

Theorem 4.5. *Let (θ, ψ) be an RDS on the phase space X , and assume that there exists a random pullback attractor $\mathcal{A}(\omega)$ for the universe \mathcal{B} . Then*

- (i) the omega limit set $\Gamma(B, \omega)$ of any $B \in \mathcal{B}$ is a subset of the attractor, i.e. $\Gamma(B, \omega) \subset \mathcal{A}(\omega)$.
- (ii) the minimal attractor for \mathcal{B} is given by

$$\mathcal{A}_{\mathcal{B}}(\omega) = \overline{\bigcup_{B \in \mathcal{B}} \Gamma(B, \omega)}.$$

Furthermore, when the RDS satisfies a strictly uniformly contracting property, i.e., there exists $K > 0$ such that

$$\|\psi(t, \omega, x_0) - \psi(t, \omega, y_0)\|_X \leq e^{-Kt} \|x_0 - y_0\|_X$$

for all $t \geq 0$, $\omega \in \Omega$ and $x_0, y_0 \in X$, then the random attractor consists of singleton subsets $\mathcal{A}(\omega) = \{A(\omega)\}$.

4.2.2 Generation of random dynamical systems

The theory of random dynamical systems allows us to analyze the asymptotic behavior of stochastic models based on every realization of the system rather than statistically in the mean or mean square sense. Hence the main concern for applying the theory of RDS to a stochastic model is to know when the stochastic model generates an RDS. To make the presentation less abstract, we will develop the theory in $X = \mathbb{R}^d$, which is sufficient for the study of stochastic models formed by systems of differential equations, either random differential equations or stochastic differential equations. The underlying theory, indeed, could be carried out in a more general metric space if necessary.

Two types of stochastic models will be discussed. In the first type of models, the noise appears as random parameters expressed as continuous, temporally varying, and bounded functions of the flow $\theta_t \omega$. The resulting systems are random differential equations in the form of (4.2), that have been used in biological and ecological applications. In the second type of models, the noise is included as a random perturbation to the deterministic models, expressed by terms involving white noise. The resulting systems are then Itô or Stratonovich stochastic differential equations which have been widely used in engineering applications. In this book we mainly focus on stochastic differential equations with special structured white noise, e.g., additive or multiplicative white noise, that can be transformed into random differential equations with random parameters in the form of (4.2). More precisely, we will study stochastic models that can be described directly or indirectly by the formulation

$$\frac{dx(t)}{dt} = f(\theta_t \omega, x), \quad (4.4)$$

where $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ is a measurable and measure-preserving dynamical system acting on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The generation of an RDS by the solutions to an IVP associated to (4.4) can be carried out as follows. Denote by $x(t; t_0, \omega, x_0)$ the global solution to the IVP

$$\frac{dx(t)}{dt} = f(\theta_t \omega, x), \quad x(t_0) = x_0$$

for $t \geq t_0, x_0 \in X$, and define

$$\psi(t, \omega, x_0) := x(t; 0, \omega, x_0). \quad (4.5)$$

Then, it is straightforward to check that

$$\psi(t - t_0, \theta_{t_0} \omega, x_0) = x(t - t_0; 0, \theta_{t_0} \omega, x_0) = x(t; t_0, \omega, x_0),$$

and thus

$$\psi(t, \theta_{t_0} \omega, x_0) = x(t + t_0; t_0, \omega, x_0).$$

Observe that in the cocycle expression $\psi(t, \theta_{t_0} \omega, x_0)$ the variable t represents the elapsed time since the initial time t_0 . Assuming that the vector field f satisfies suitable conditions to ensure the existence and uniqueness of a global solution, the definition for ψ in (4.5) allows us to verify that all conditions in Definition 4.2 are fulfilled, and therefore an RDS has been successfully generated.

The next goal is to show how a stochastic differential equation can be transformed into a random differential equation of the type (4.2). Moreover, we want to show that the RDS generated by the solution to the random differential equation after transformation is homeomorphic to the RDS generated by the solution to the stochastic equation before transformation. To this end, a brief review of the required knowledge on stochastic differential equations is provided below.

4.2.3 A brief introduction to stochastic differential equations

Let $(W_t)_{t \in \mathbb{R}}$ be a standard Brownian motion or one dimensional Wiener process defined on the (canonical, if necessary) probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A traditional way to model random perturbation to a deterministic system

$$\frac{dx(t)}{dt} = f(t, x), \quad t, x \in \mathbb{R}, \quad (4.6)$$

is to include a white noise term in the system and consider the stochastic system

$$\frac{dx}{dt} = f(t, x) + \sigma(t, x) \frac{dW_t}{dt}, \quad t, x \in \mathbb{R}. \quad (4.7)$$

Note that the derivative of the Brownian motion, often referred to as the white noise, is only a formal notation (as W_t is nowhere differentiable) although it is commonly used in engineering applications. Equation (4.7) can be presented more rigorously as

$$dx(t) = f(t, x)dt + \sigma(t, x)dW_t, \quad t, x \in \mathbb{R}.$$

A more general inclusion of the noise, e.g.,

$$\frac{dx}{dt} = f\left(t, x, \frac{dW_t}{dt}\right), \quad t, x \in \mathbb{R},$$

can be approximated by a system in the form (4.7) by using the Taylor formula.

The differential equation (4.7) is only meaningful while being written as the following integral equation

$$x(t) = x(0) + \int_0^t f(s, x(s))ds + \int_0^t \sigma(s, x(s))dW_s, \quad t \geq 0. \quad (4.8)$$

The first integral in (4.8) is a regular deterministic integral and exists, for instance, when f is continuous. But the second integral in (4.8), which is a stochastic integral with respect to the Wiener process, is completely different. Recall that for every fixed $\omega \in \Omega$, the path $t \in \mathbb{R} \mapsto W_t(\omega)$ is continuous, but is not of bounded variation on any time interval $[t_1, t_2]$. Thus although every sample path of W_t is continuous, it cannot be used as an integrator function to define any Riemann-Stieltjes integral with respect to the Wiener process. In fact a stochastic integral has to be defined as a limit in $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$. Detailed construction of stochastic integrals can be found in many references in probability, e.g., [2, 64]; in what follows we only discuss the difficulties in defining the stochastic integrals. These difficulties make the random dynamical system approach, while applicable, a powerful alternative to the probabilistic approaches which rely heavily on stochastic integrals.

For simplicity of exposition, consider the integral $\int_0^T W_s dW_s$ and suppose that it can be calculated by using Riemann-Stieltjes sums. If this integral made sense, and could be defined pathwise, then, at least formally its value should be

$$\int_0^T W_s dW_s = \frac{1}{2} (W_T^2 - W_0^2) = \frac{1}{2} W_T^2. \quad (4.9)$$

Choose a sequence of partitions (Δ_n) of $[0, T]$,

$$\Delta_n = \{0 = t_0^n < t_1^n < \dots < t_n^n = T\},$$

such that

$$\delta_n := \max_{0 \leq k \leq n-1} (t_{k+1}^n - t_k^n) \xrightarrow{n \rightarrow \infty} 0.$$

Let $a \in [0, 1]$, denote by $\tau_k^n := at_k^n + (1-a)t_{k-1}^n$, and consider the corresponding Riemann-Stieltjes sums

$$S_n = \sum_{k=1}^n W_{\tau_k^n} (W_{t_k^n} - W_{t_{k-1}^n}). \quad (4.10)$$

Then, by using the decomposition

$$\begin{aligned} S_n &= \frac{W_T^2}{2} - \frac{1}{2} \sum_{k=1}^n (W_{t_k^n} - W_{t_{k-1}^n})^2 + \sum_{k=1}^n (W_{\tau_k^n} - W_{t_{k-1}^n})^2 \\ &\quad + \sum_{k=1}^n (W_{t_k^n} - W_{\tau_k^n})(W_{\tau_k^n} - W_{t_{k-1}^n}), \end{aligned}$$

it is not difficult to check that

$$\lim_{n \rightarrow \infty} S_n = \frac{W_T^2}{2} - \frac{(1-2a)T}{2} \quad \text{in } L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}),$$

or equivalently

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left| S_n - \left(\frac{W_T^2}{2} - \frac{(1-2a)T}{2} \right) \right|^2 d\mathbb{P}(\omega) = 0.$$

Therefore defining $\int_0^T W_s dW_s$ as a limit in mean square is dependent on the choice of the intermediate point τ_k^n , and, consequently, dependent on the value of a .

For each $a \in [0, 1]$, denote by $\mathcal{I}_a(t)$ the stochastic integral induced by the ‘‘Riemann-Stieltjes sums’’ (4.10) corresponding to $\tau_k^n := at_k^n + (1-a)t_{k-1}^n$. Then we have

$$\mathcal{I}_a(t) = \frac{W_t^2}{2} - \frac{(1-2a)t}{2},$$

which achieves different values for different choice of a . This property also holds true for a general integrand $f(t, W_t)$ instead of just W_t . When $a = 1/2$, i.e., the middle point in each interval of the partition is chosen, the resulting integral is called a *Stratonovich* stochastic integral (Stratonovich integral in short), with the integrator usually denoted by $\circ dW_t$. It is worth mentioning that the differentiation properties of Stratonovich integrals follow the classical rules in calculus. When $a = 0$, i.e., the left point of each interval of partition is chosen, the resulting integral is called *Itô* stochastic integral (Itô integral in short), with the integrator denoted simply by dW_t . The classical rules of calculus do not hold any more for Itô integrals; the differentiation of Itô integrals requires Itô’s Calculus. But on the other hand, Itô integrals possess nice properties concerning measurability, such as the martingale property, and the expectation of any Itô integral is always 0.

Note that general Itô and Stratonovich integrals over finite time can be converted between each other according to the formula

$$\int_{t_0}^T f(t, X_t) \circ dW_t = \int_{t_0}^T f(t, X_t) dW_t + \frac{1}{2} \int_{t_0}^T \frac{\partial f}{\partial X}(t, X_t) f(t, X_t) dt,$$

which guarantees that both types of integrals provide solutions with equivalent behaviors within finite period of time. However, this is not necessarily true for long time behavior, i.e., when $T \rightarrow \infty$ or $t_0 \rightarrow -\infty$. For example, one SDE with Stratonovich integrals can be stable while an otherwise identical SDE with Itô integrals is not, and vice versa. This will be discussed further in Section 4.4. The readers who are interested in a rigorous construction of the Itô integral are referred to the books by Arnold [2] or Oksendal [64]. We next introduce the Itô formula, the stochastic calculus counterpart of the chain rule in classical deterministic calculus, that allows the conversion between an Itô SDE and a Stratonovich SDE.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and let $\{W_t\}_{t \in \mathbb{R}}$ be a one dimensional two-sided Wiener process. Denote by $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ the family of σ -algebras given by the filtration $\mathcal{F}_t = \sigma\{W_s : s \leq t\}$. Denote by $\mathcal{L}_{\mathcal{F}_t}^p(0, T)$, $p = 1, 2$ the linear space of measurable processes $\phi : [0, T] \times \Omega \rightarrow \mathbb{R}$ such that for each $t \in [0, T]$ the mapping $\phi(t, \cdot)$ is \mathcal{F}_t -measurable (i.e., \mathcal{F}_t -adapted), and $\int_0^T |\phi(s)|^p ds < \infty$ \mathbb{P} -a.s. Then we have the following result.

Theorem 4.6. (*Itô's formula*) Let X_0 be a random variable which is \mathcal{F}_0 -measurable, $b \in \mathcal{L}_{\mathcal{F}_t}^1(0, T)$ and $\sigma \in \mathcal{L}_{\mathcal{F}_t}^2(0, T)$. Denote by X_t the stochastic process $\{\mathcal{F}_t\}$ -measurable for any $t \in [0, T]$ defined by

$$X_t = X_0 + \int_0^t b(s) ds + \int_0^t \sigma(s) dW_s \quad \text{for all } t \in [0, T], \quad (4.11)$$

where the integral of b is in the pathwise sense. Let $Y(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$ be a real-valued function which is differentiable in t and twice differentiable in x . Then

$$\begin{aligned} Y(t, X_t) &= Y(0, X_0) + \int_0^t Y_s(s, X_s) ds + \int_0^t Y_x(s, X_s) b(s) ds \\ &+ \int_0^t Y_x(s, X_s) \sigma(s) dW_s + \frac{1}{2} \int_0^t Y_{xx}(s, X_s) \sigma^2(s) ds \quad \text{for all } t \in [0, T]. \end{aligned} \quad (4.12)$$

Remark 4.3. Note that the expression for the Itô formula (4.12) contains an additional term $\frac{1}{2} \int_0^t Y_{xx}(s, X_s) \sigma^2(s) ds$ comparing to the stochastic integral otherwise calculated by the deterministic chain rule of differentiation. This additional term, however, does not appear if the stochastic equation (4.11) is in the Stratonovich sense, i.e.,

$$X_t = X_0 + \int_0^t b(s) ds + \int_0^t \sigma(s) \circ dW_s \quad \text{for all } t \in [0, T].$$

We next recall the definitions of stochastic differential equations and their solutions. For $T > 0$, assume that the mappings $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable, and the random variable ξ is \mathcal{F}_0 -measurable with values in \mathbb{R} .

A (Itô) stochastic differential equations is properly interpreted as

$$\begin{cases} dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW_t, & \text{in } [0, T], \\ X(0) = \xi. \end{cases} \quad (4.13)$$

Definition 4.7. A stochastic process $\{X(t)\}_{t \in [0, T]}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a solution to (4.13) if it is a continuous and $\{\mathcal{F}_t\}$ -adapted process such that

- (i) $\{b(t, X(t))\}_{t \in [0, T]} \in \mathcal{L}_{\mathcal{F}_t}^1(0, T)$ and $\{\sigma(t, X(t))\}_{t \in [0, T]} \in \mathcal{L}_{\mathcal{F}_t}^2(0, T)$,
- (ii) it holds

$$X(t) = \xi + \int_0^t b(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dW_s \quad \text{for all } t \in [0, T], \mathbb{P} - a.s.$$

A Stratonovich stochastic differential equations is properly interpreted as

$$\begin{cases} dX(t) = b(t, X(t))dt + \sigma(t, X(t)) \circ dW_t, & \text{in } [0, T], \\ X(0) = \xi, \end{cases} \quad (4.14)$$

with an equivalent integral formulation

$$X(t) = \xi + \int_0^t b(s, X(s)) ds + \int_0^t \sigma(s, X(s)) \circ dW_s \quad \text{for all } t \in [0, T]. \mathbb{P} - a.s. \quad (4.15)$$

If $\sigma(t, X)$ is differentiable with respect its second variable X , then the Stratonovich SDE (4.15) is equivalent to the following Itô SDE (see, e.g., [78])

$$X(t) = \xi + \int_0^t b(s, X(s)) ds + \frac{1}{2} \int_0^t \sigma_X(s, X(s)) \sigma(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dW_s. \quad (4.16)$$

The contents presented above cover SDEs in general forms and thus required some background in probability theory. The SDE applications considered in this book, however, are much simpler. As the emphasis of this chapter is the theory of random dynamical systems, we only focus on the SDEs whose solutions generate RDS. In particular, we will consider SDEs with additive white noise such as

$$dx = f(x)dt + a dW_t, \quad a \in \mathbb{R},$$

or SDEs with linear multiplicative noise such as

$$dx = f(x)dt + \sigma x dW_t, \quad \sigma \in \mathbb{R}. \quad (4.17)$$

By applying the formula (4.16) we can easily derive that the Itô SDE (4.17) is equivalent to the following Stratonovich SDE

$$dx = \left(f(x) - \frac{\sigma^2}{2} x \right) dt + \sigma x \circ dW_t. \quad (4.18)$$

Thus, when studying SDEs with linear multiplicative noise, either Itô's or Stratonovich's formulation could be adopted.

4.2.4 Global asymptotic behavior of SDEs: conjugation of RDS

In this section we discuss the global asymptotic behavior of random dynamical systems generated by stochastic differential equations. The local asymptotic behavior in the neighborhood of equilibria has been extensively studied in the literature (see e.g., [57, 60, 61]), but will not be commented here. We are interested in analyzing the global behavior of SDEs by the theory of random dynamical systems.

The "prerequisite" for applying the theory of RDS to study SDEs is to show that solutions to an underlying SDE generate an RDS. The main technique is to transform the SDE into an equivalent random differential equation by a homeomorphism, which is referred to as the *conjugation* of random dynamical systems. The challenges of using this transformation technique lie in the fact that an explicit expres-

sion for such a homeomorphism is needed, but difficult or impossible to construct explicitly for specific SDEs system from real world applications. In other words, although we can prove that such a homeomorphism exists, an explicit expression which allows us to work directly with the solutions of both equations (random and stochastic) may not be available. For more details about the abstract aspects of this statement, the reader is referred to the paper by Imkeller and Schmalfuß [40].

Remark 4.4. The conjugation of RDS may work for more general SDEs than just the SDEs with additive or linear multiplicative noise. But for stochastic partial differential equations (SPDEs) the existence of conjugation is unknown for most cases except the special SPDEs with additive or linear multiplicative noise.

We next introduce one special type of stochastic process, the Ornstein-Uhlenbeck process, that has been widely used in the literature to obtain conjugation of RDS. Consider the one-dimensional stochastic differential equation

$$dz(t) = -\lambda z(t)dt + dW_t \quad (4.19)$$

for some $\lambda > 0$. Equation (4.19) has a special stationary solution known as the stationary Ornstein-Uhlenbeck (OU) process. The OU process has several nice properties proved by Caraballo *et al.* in [20], which have been extensively used in latter studies of conjugation of RDS. The properties are stated in the following Proposition.

Proposition 4.1. *Let λ be a positive number. There exists a $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant subset $\overline{\Omega} \in \mathcal{F}$ of $\Omega = C_0(\mathbb{R}, \mathbb{R})$ (set of continuous functions which vanish at zero) of full measure such that for $\omega \in \overline{\Omega}$,*

(i)

$$\lim_{t \rightarrow \pm\infty} \frac{|\omega(t)|}{t} = 0;$$

(ii) the random variable given by

$$z^*(\omega) := -\lambda \int_{-\infty}^0 e^{\lambda\tau} \omega(\tau) d\tau$$

is well defined;

(iii) the mapping

$$\begin{aligned} (t, \omega) \mapsto z^*(\theta_t \omega) &= -\lambda \int_{-\infty}^0 e^{\lambda\tau} \theta_t \omega(\tau) d\tau \\ &= -\lambda \int_{-\infty}^0 e^{\lambda\tau} \omega(t + \tau) d\tau + \omega(t) \end{aligned} \quad (4.20)$$

is a stationary solution of (4.19) with continuous trajectories satisfying

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} \frac{|z^*(\theta_t \omega)|}{|t|} &= 0, & \lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t z^*(\theta_\tau \omega) d\tau &= 0, \\ \lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t |z^*(\theta_\tau \omega)| d\tau &= \mathbb{E}|z^*| < \infty, \end{aligned} \quad (4.21)$$

where \mathbb{E} denotes the expectation.

Remark 4.5. While the OU process is used for transformation, the initial probability space Ω will be changed to the probability space $\bar{\Omega}$ given in Proposition 4.1, but still denoted by Ω if the context is clear.

The proposition below stating the conjugation of two random dynamical systems was also proved by Caraballo *et al.* in [20].

Proposition 4.2. *Let ψ be a random dynamical system on a phase space X . Suppose that the mapping $T : \Omega \times X \rightarrow X$ possesses the following properties:*

- (i) *for fixed $\omega \in \Omega$, the mapping $T(\omega, \cdot)$ is a homeomorphism on X ;*
- (ii) *for fixed $x \in X$, the mappings $T(\cdot, x)$, $T^{-1}(\cdot, x)$ are measurable.*

Then the mapping

$$(t, \omega, x) \rightarrow \phi(t, \omega, x) := T^{-1}(\theta_t \omega, \psi(t, \omega, T(\omega, x))) \quad (4.22)$$

defines a (conjugated) random dynamical system.

We next illustrate how to prove an SDE with additive noise or linear multiplicative noise generates an RDS. To show how the technique work for both Itô SDE and Stratonovich SDE, we will present one example of Itô SDE with additive noise and one example for Stratonovich SDE with linear multiplicative noise.

SDE with additive noise

Consider a stochastic differential equation with additive noise of the form

$$dx(t, \omega) = f(x)dt + dW_t, \quad x(0, \omega) = x_0 \in \mathbb{R}. \quad (4.23)$$

Assume that f is locally Lipschitz or continuously differentiable, such that solutions to (4.23) exists. We will perform the change of variable

$$y(t, \omega) = x(t, \omega) - z^*(\theta_t \omega),$$

where $z^*(\theta_t \omega)$ is defined as in (4.20). First observe that at the initial time $t = 0$ it holds

$$y(0, \omega) = x(0, \omega) - z^*(\omega) = x_0 - z^*(\omega).$$

Then noticing that $W_t(\omega) = \omega(t)$ we obtain

$$\begin{aligned}
dy(t, \omega) &= dx(t, \omega) - dz^*(\theta_t \omega) \\
&= f(x(t, \omega)) + dW_t(\omega) - (-\lambda z^*(\theta_t \omega) dt + dW_t(\omega)) \\
&= (f(x(t, \omega)) + \lambda z^*(\theta_t \omega)) dt \\
&= (f(y(t, \omega) + z^*(\theta_t \omega)) + \lambda z^*(\theta_t \omega)) dt,
\end{aligned}$$

i.e.,

$$\frac{dy(t, \omega)}{dt} = f(y(t, \omega) + z^*(\theta_t \omega)) + \lambda z^*(\theta_t \omega). \quad (4.24)$$

Equation (4.24) is indeed a random differential equation without white noise to which the theory of RDS can be applied directly. If the assumptions on the function f ensure that equation (4.24) generates an RDS ψ , then we can apply Proposition 4.2 to obtain a conjugated RDS for the original SDE (4.23). In fact, the mapping $T : \Omega \times X \rightarrow X$ defined by

$$T(\omega, x) = x - z^*(\omega)$$

clearly satisfies assumptions in Proposition 4.2. Hence by Proposition 4.2

$$\begin{aligned}
\phi(t, \omega, x_0) &:= T^{-1}(\theta_t \omega, \psi(t, \omega, T(\omega, x_0))) \\
&= T^{-1}(\theta_t \omega, y(t; \omega, x_0 - z^*(\omega))) \\
&= y(t; \omega, x_0 - z^*(\omega)) + z^*(\theta_t \omega) \\
&= x(t; \omega, x_0)
\end{aligned}$$

is an RDS for the original SDE (4.23).

SDE with linear multiplicative noise

Consider a Stratonovich SDE with linear multiplicative noise in the form

$$dx = f(x) dt + \sigma x \circ dW_t, \quad x(0; \omega) = x_0. \quad (4.25)$$

We will perform the change of variable

$$y(t, \omega) = e^{-\sigma z^*(\theta_t \omega)} x(t, \omega), \quad (4.26)$$

where $z^*(\theta_t \omega)$ is defined as in (4.20). First notice that $y(0; \omega) = x_0 e^{-\sigma z^*(\omega)}$. Then due to classical chain rules (which can be applied directly to Stratonovich integrals)

$$\begin{aligned}
dy(t, \omega) &= e^{-\sigma z^*(\theta_t \omega)} dx(t, \omega) - \sigma e^{-\sigma z^*(\theta_t \omega)} x(t, \omega) \circ dz^*(\theta_t \omega) \\
&= e^{-\sigma z^*(\theta_t \omega)} (f(x(t, \omega)) dt + \sigma x(t, \omega) \circ dW_t(\omega)) \\
&\quad - \sigma e^{-\sigma z^*(\theta_t \omega)} x(t, \omega) \circ (-\lambda z^*(\theta_t \omega) dt + dW_t(\omega)) \\
&= e^{-\sigma z^*(\theta_t \omega)} (f(x(t, \omega)) + \lambda \sigma z^*(\theta_t \omega) x(t, \omega)) dt, \\
&= \left(e^{-\sigma z^*(\theta_t \omega)} f(e^{\sigma z^*(\theta_t \omega)} y(t, \omega)) + \lambda \sigma z^*(\theta_t \omega) y(t, \omega) \right) dt,
\end{aligned}$$

i.e.,

$$\frac{dy(t, \omega)}{dt} = e^{-\sigma z^*(\theta_t \omega)} f(e^{\sigma z^*(\theta_t \omega)} y(t, \omega)) + \lambda \sigma z^*(\theta_t \omega) y(t, \omega). \quad (4.27)$$

Assume that solutions to the random differential equation (4.27) generate an RDS ψ , then we can construct a conjugated RDS for the original equation (4.25). In fact, define the homeomorphism $T : \Omega \times X \rightarrow X$ by

$$T(\omega, x) = e^{-\sigma z^*(\omega)} x, \quad (4.28)$$

whose inverse is given by

$$T^{-1}(\omega, x) = e^{\sigma z^*(\omega)} x.$$

Then T fulfills assumptions in Lemma 4.2 and, as a result, the mapping

$$\begin{aligned} \phi(t, \omega, x_0) &:= T^{-1}(\theta_t \omega, \psi(t, \omega, T(\omega, x_0))) \\ &= T^{-1}(\theta_t \omega, y(t, \omega, x_0 e^{-\sigma z^*(\omega)})) \\ &= e^{\sigma z^*(\theta_t \omega)} y(t, \omega, x_0 e^{-\sigma z^*(\omega)}) \\ &= x(t, \omega, x_0), \end{aligned}$$

defines an RDS for the original equation (4.25).

For the applications to be considered in the next section, whenever an underlying SDE system contains additive or linear multiplicative noise terms, the same procedure as introduced above will be used. More precisely, we will first transform the SDE into an RDE without white noise by performing a suitable change of variable and analyze the resulting RDE to confirm its solutions generate an RDS. Then we will use Proposition 4.2 to obtain that the solutions to the original equation generate a conjugated RDS. Thereby the theory of RDS can be adopted to analyze the long term behavior of both RDSs.

Remark 4.6. For simplicity of exposition, the transformation technique was illustrated by the simplest one-dimensional SDE with a one dimensional Brownian motion, in the space $X = \mathbb{R}$. However, similar analysis can also be carried out for a system of stochastic differential equations and with an m -dimensional Wiener process (see, e.g. [2, 64]). The state space can also be a general metric space instead of \mathbb{R} , including infinite-dimensional spaces where stochastic partial differential equations are included.

Remark 4.7. We have discussed only the situations of additive and multiplicative linear noises because we can obtain explicitly the conjugated random dynamical systems after the transformation. However, it is possible to consider a more general multiplicative noise such as a Lipschitz function multiplied by the Wiener process. In this case, it can be proved that a conjugated RDS can be constructed by using the solution of an ordinary differential equation whose existence is known but the explicit expression can be hardly obtained (see Imkeller and Schmalfuß [40] for more details).

4.3 Applications

In this section we will study random/stochastic versions of the chemostat, SIR and Lorenz model introduced in the previous chapters.

4.3.1 Random chemostat

As mentioned in Chapter 3, chemostat models in reality are neither autonomous nor deterministic. They are affected by physical or chemical inputs with noise, caused by environmental perturbations, internal variability, randomly fluctuating parameters, measurement errors, etc. This motivates the study of chemostat models with randomly fluctuating input parameters, including the nutrient supplying rate D and the nutrient supplying concentration I . From a biological point of view, for an experiment to be repeatable, one has to have a reasonable description of its random aspects. These aspects may change in time and thus the noise can be modeled as a time-dependent stochastic process with certain known properties.

Recall that representing mathematically such a stochastic process starts with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and to connect the state $\omega \in \Omega$ of the random environment with its state after a time t has elapsed, we need a family of time-dependent maps $\{\theta_t : \Omega \rightarrow \Omega\}$ that keeps track of the noise. Here we formulate the two input parameters D and I as $D(\theta_t\omega)$ and $I(\theta_t\omega)$, respectively. In addition, the random parameters $D(\theta_t\omega)$ and $I(\theta_t\omega)$ are assumed to be continuous and bounded. This is a natural formalism to model the realistic stochastic fluctuations of a biological system caused by its interaction with the external environment, because the parameters in dynamical systems of biological interest are inherently positive and bounded (see [16] for more details).

Bounded noise can be modeled in various ways. For example in [4], given a stochastic process Z_t such as an Ornstein-Uhlenbeck process, the stochastic process

$$\zeta(Z_t) := \zeta_0 \left(1 - 2\varepsilon \frac{Z_t}{1 + Z_t^2} \right),$$

where ζ_0 and ε are positive constants with $\varepsilon \in (0, 1)$, takes values in the interval $\zeta_0[1 - \varepsilon, 1 + \varepsilon]$ and tends to peak around $\zeta_0(1 \pm \varepsilon)$. It is thus suitable for a noisy switching scenario. In another example, the stochastic process

$$\eta(Z_t) := \eta_0 \left(1 - \frac{2\varepsilon}{\pi} \arctan Z_t \right),$$

where η_0 and ε are positive constants with $\varepsilon \in (0, 1)$, takes values in the interval $\eta_0[1 - \varepsilon, 1 + \varepsilon]$, and is centered on η_0 . In the theory of random dynamical systems the driving noise process $Z_t(\omega)$ is replaced by a canonical driving system $\theta_t\omega$. This simplification allows a better understanding of the path-wise approach to model

noise: a system influenced by stochastic processes for each single realization ω can be interpreted as wandering along a path $\theta_t\omega$ in Ω and thus may provide additional statistical/geological information to the modeler.

Our objective is to study the evolution of concentrations of the nutrient and microorganism when the input parameters are random and wall growth is taken into account, which can be described by the following random system:

$$x'(t) = D(\theta_t\omega)(I(\theta_t\omega) - x(t)) - a \frac{x(t)}{m + x(t)}(y_1(t) + y_2(t)) + bv y_1(t), \quad (4.29)$$

$$y_1'(t) = -(v + D(\theta_t\omega))y_1(t) + c \frac{x(t)}{m + x(t)}y_1(t) - r_1 y_1(t) + r_2 y_2(t), \quad (4.30)$$

$$y_2'(t) = -v y_2(t) + c \frac{x(t)}{m + x(t)}y_2(t) + r_1 y_1(t) - r_2 y_2(t), \quad (4.31)$$

where $0 < c \leq a$ is the growth rate coefficient of the consumer species. In particular, we assume that the inputs are perturbed by real noise, i.e., $D(\theta_t\omega)$ and $I(\theta_t\omega)$ are continuous and essentially bounded:

$$D(\theta_t\omega) \in d \cdot [1 - \varepsilon, 1 + \varepsilon], \quad I(\theta_t\omega) \in i \cdot [1 - \varepsilon, 1 + \varepsilon], \quad d > 0, \quad i > 0, \quad \varepsilon < 1.$$

We first state that equations (4.29)-(4.31) generates a random dynamical system, and the random dynamical system has a random attractor. Letting

$$\mathbb{R}_+^3 = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}, \quad u(t) = (x(t), y_1(t), y_2(t))$$

we have the following lemma which can be proved by standard techniques.

Theorem 4.7. *For any $\omega \in \Omega$, any $t_0 \in \mathbb{R}$, and any initial data $u_0 := (x(t_0), y_1(t_0), y_2(t_0)) \in \mathbb{R}_+^3$, system (4.29)-(4.31) admits a unique bounded solution $u(\cdot; t_0, \omega, u_0) \in C([t_0, \infty), \mathbb{R}_+^3)$ with $u(t_0; t_0, \omega, u_0) = u_0$. Moreover the solution generates a random dynamical system $\psi(t, \omega, \cdot)$ defined as*

$$\psi(t, \omega, u_0) = u(t; 0, \omega, u_0), \quad \forall t \geq 0, \quad u_0 \in \mathbb{R}_+^3, \quad \omega \in \Omega.$$

Moreover, the RDS $\psi(t, \omega, \cdot)$ possesses a global random attractor.

Proof. Denote by $u(t; \omega, u_0) = \psi(t, \omega, u_0)$ the solution of system (4.29)-(4.31) satisfying $u(0; \omega, u_0) = u_0$. Then for $u_0 := u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, ($B \in \mathcal{D}(\mathbb{R}_+^3)$), the universe of tempered random sets)

$$\|\psi(t, \theta_{-t}\omega u_0)\|_1 = \|u(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|_1 \leq s(t; \theta_{-t}\omega, s_0(\theta_{-t}\omega)),$$

where $s(t) := x(t) + \frac{a}{c}(y_1(t) + y_2(t))$. Since $a \geq c$ and $0 < b < 1$,

$$\frac{a}{c}(v + d(1 - \varepsilon) - bv) = \frac{a}{c}d(1 - \varepsilon) + \left(\frac{a}{c} - b\right)v > \frac{a}{c}d(1 - \varepsilon).$$

Therefore by letting $\lambda := \min\{d(1 - \varepsilon), \nu\}$ we obtain

$$\frac{ds(t)}{dt} \leq di(1 + \varepsilon)^2 - \lambda s(t). \quad (4.32)$$

According to inequality (4.32),

$$s(t, \omega) \leq s_0 e^{-\lambda t} + di(1 + \varepsilon)^2 / \lambda. \quad (4.33)$$

Substituting ω by $\theta_{-t}\omega$ in (4.33), we obtain

$$s(t; \theta_{-t}\omega, s_0(\theta_{-t}\omega)) \leq e^{-\lambda t} \sup_{(x, y_1, y_2) \in B(\theta_{-t}\omega)} \left(x + \frac{a}{c}(y_1 + y_2) \right) + \frac{di(1 + \varepsilon)^2}{\lambda}.$$

Therefore for any $\epsilon > 0$, there exists $T_B(\omega)$ such that when $t > T_B$,

$$\begin{aligned} \|u(t; \theta_{-t}\omega, u_0)\|_1 &= x(t; \theta_{-t}\omega, u_0) + y_1(t; \theta_{-t}\omega, u_0) + y_2(t; \theta_{-t}\omega, u_0) \\ &\leq di(1 + \varepsilon)^2 / \lambda + \epsilon, \end{aligned}$$

for all $u_0 \in B(\theta_{-t}\omega)$. Define

$$K_\epsilon(\omega) = \{(x, y_1, y_2) \in \mathbb{R}_+^3 : x + y_1 + y_2 \leq di(1 + \varepsilon)^2 / \lambda + \epsilon\},$$

then $K_\epsilon(\omega)$ is positively invariant, and is absorbing in \mathbb{R}_+^3 .

It follows from the discussion above that the random dynamical system generated by system (4.29)-(4.31) possesses a random attractor $\mathcal{A} = \{A(\omega) : \omega \in \Omega\}$, consisting of nonempty compact random subsets of \mathbb{R}_+^3 contained in $K_\epsilon(\omega)$. Next we will discuss the geometric structure of the random attractor of the RDS generated by (4.29)-(4.31). \square

To obtain more detailed information on the internal structure of the pullback attractor, we make the following change of variables:

$$y(t) = y_1(t) + y_2(t); \quad \gamma(t) = \frac{y_1(t)}{y(t)}. \quad (4.34)$$

System (4.29)-(4.31) then becomes

$$\frac{dx(t)}{dt} = D(\theta_t\omega)(I(\theta_t\omega) - x(t)) - \frac{ax(t)}{m + x(t)}y(t) + b\nu\gamma(t)y(t), \quad (4.35)$$

$$\frac{dy(t)}{dt} = -\nu y(t) - D(\theta_t\omega)\gamma(t)y(t) + \frac{cx(t)}{m + x(t)}y(t), \quad (4.36)$$

$$\frac{d\gamma(t)}{dt} = -D(\theta_t\omega)\gamma(t)(1 - \gamma(t)) - r_1\gamma(t) + r_2(1 - \gamma(t)). \quad (4.37)$$

By definition, $\gamma(t)$ represents the portion of microorganism that attaches to the wall. Noting that the dynamics of $\gamma(t) = \gamma(t; \omega, \gamma_0)$ is uncoupled with $x(t)$ and $y(t)$, we can thus study the dynamics of $\gamma(t)$ independently.

It can be proved that equation (4.37) has a pullback attractor $\mathcal{A}_\gamma = \{A_\gamma(\omega)\}_{\omega \in \Omega}$ with its component subsets given by

$$A_\gamma(\omega) = \bigcap_{t \geq 0} \overline{\gamma(t; \theta_{-t}\omega, [0, 1])}.$$

These component subsets have the form

$$A_\gamma(\theta_t\omega) = [\gamma_l(\theta_t\omega), \gamma_u(\theta_t\omega)],$$

where $\gamma_l(\theta_t\omega)$ and $\gamma_u(\theta_t\omega)$ are entire bounded solutions of equation (4.37). All other bounded entire solutions of (4.37) lie between these two. We next estimate bounds of these entire solutions by using differential inequalities.

On the one hand, since $\gamma(t) \leq 1$ and $D(\theta_t\omega) > 0$, we have

$$\begin{aligned} \gamma'(t) &= D(\theta_t\omega)(\gamma^2(t) - \gamma(t)) - (r_1 + r_2)\gamma(t) + r_2 \\ &\leq -(r_1 + r_2)\gamma(t) + r_2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \gamma'(t) &= D(\theta_t\omega)\gamma^2(t) - (D(\theta_t\omega) + r_1 + r_2)\gamma(t) + r_2 \\ &\geq -(d(1 + \varepsilon) + r_1 + r_2)\gamma(t) + r_2. \end{aligned}$$

Let $\alpha(t)$ and $\beta(t)$ satisfy

$$\alpha'(t) = -(d(1 + \varepsilon) + r_1 + r_2)\alpha(t) + r_2, \quad \alpha(0) = \gamma(0), \quad (4.38)$$

$$\beta'(t) = -(r_1 + r_2)\beta(t) + r_2, \quad \beta(0) = \gamma(0). \quad (4.39)$$

Then $\alpha(t) \leq \gamma(t) \leq \beta(t)$ and $[\gamma_l(\theta_t\omega), \gamma_u(\theta_t\omega)] \subseteq [\alpha^*, \beta^*]$, where

$$\alpha^* = \frac{r_2}{r_1 + r_2 + d(1 + \varepsilon)}, \quad \beta^* = \frac{r_2}{r_1 + r_2} \quad (4.40)$$

are asymptotically stable steady states for (4.38) and (4.39) respectively. In summary,

$$A_\gamma(\omega) = [\gamma_l(\omega), \gamma_u(\omega)] \subseteq [\alpha^*, \beta^*].$$

We provide the sufficient condition for \mathcal{A} to consist of only a single entire solution in the next theorem.

Theorem 4.8. *The pullback attractor \mathcal{A} associated to the random dynamical system $\gamma(t, \omega, \cdot)$ generated by (4.37) consists of a single entire solution, denoted by $\gamma^*(\theta_t\omega)$, provided*

$$2r_2d(1 + \varepsilon) < (r_1 + r_2 + d(1 - \varepsilon))(r_1 + r_2).$$

Note that this sufficient condition is equivalent to

$$d\varepsilon < \frac{(r_1 + r_2)^2 + d(r_1 + r_2)}{r_1 + 3r_2},$$

which essentially represents the restriction on the magnitude of noise on D . The long term dynamics of $x(t)$ and $y(t)$ are given in the following theorem.

Theorem 4.9. *Given $a \geq c$, $0 < b < 1$, $v > 0$, assume that $D(\theta_t\omega)$ and $I(\theta_t\omega)$ are continuous and essentially bounded, with $d(1 - \varepsilon) \leq D(\theta_t\omega) \leq d(1 + \varepsilon)$ and $i(1 - \varepsilon) \leq I(\theta_t\omega) \leq i(1 + \varepsilon)$. Then, system (4.35)-(4.36) has a pullback attractor $\mathcal{A} = \{A(\omega) : \omega \in \Omega\}$ inside the nonnegative quadrant. Moreover, letting*

$$x^*(\omega) = \int_{-\infty}^0 D(\theta_s\omega)I(\theta_s\omega)e^{-\int_s^0 D(\theta_\tau\omega)d\tau}ds, \quad (4.41)$$

(i) *the pullback attractor \mathcal{A} has a singleton component subset $A(\omega) = \{(x^*(\omega), 0)\}$ provided*

$$v + d(1 - \varepsilon)a^* \geq c,$$

(ii) *the pullback attractor \mathcal{A} also contains points strictly inside the positive quadrant in addition to the singleton solution $\{(x^*(\omega), 0)\}$ provided*

$$ac^2di(1 - \varepsilon)^2 > (mc(av + ad(1 + \varepsilon) - cbv\beta^*) + acdi(1 - \varepsilon)^2) \cdot (v + d(1 + \varepsilon)\beta^*). \quad (4.42)$$

4.3.2 Random and stochastic SIR

In this section, a random and a stochastic version of the SIR model analyzed in the previous chapters will be considered. In particular, we will first consider an SIR model described by a system of random differential equations whose coefficients contain real noise perturbation, i.e., bounded and continuous functions of $\theta_t\omega$. Then we consider an SIR model described by a system of stochastic differential equations obtained from perturbing the deterministic SIR model by a white noise. Differences between the SDE model and the RDE model will be commented from modeling and analysis point of view. The techniques on construction of conjugated random dynamical systems introduced in Subsection 4.2.4 will be illustrated by the SDE SIR system. For a more detailed analysis the reader is referred to [12].

Random SIR with real noise

The total population $S + I + R$ was assumed to be a constant in the autonomous SIR model (2.12) – (2.14). This is due to the same constant birth rate and death rate for all the groups, which is not realistic. Here we will release this assumption and assume a total random reproduction spam, modeled by $\Lambda(\theta_t\omega)$ with

$$\Lambda(\theta_t\omega) \in \Lambda[1 - \varepsilon, 1 + \varepsilon], \quad \Lambda > 0, \quad \varepsilon \in (0, 1). \quad (4.43)$$

This assumption results in a system of random differential equations:

$$\frac{dS}{dt}(t, \omega) = \Lambda(\theta_t \omega) - \nu S(t) - \beta \frac{S(t) I(t)}{N(t)}, \quad (4.44)$$

$$\frac{dI}{dt}(t, \omega) = -(\nu + \gamma)I(t) + \beta \frac{S(t) I(t)}{N(t)}, \quad (4.45)$$

$$\frac{dR}{dt}(t, \omega) = \gamma I(t) - \nu R(t), \quad (4.46)$$

where ν, β, γ are positive constants which represent the death rate, the infection rate and the removal rate, respectively, and $N(t) = S(t) + I(t) + R(t)$ is the total population.

Remark 4.8. A more general model could be considered, in which more parameters are modeled to be random. But for simplicity of exposition, we only discuss the simplest RDE system (4.44) – (4.46). The analysis to be adopted, however, can be applied to a general RDE system with more random parameters.

First of all we prove that solutions corresponding to nonnegative initial conditions remain nonnegative, stated as below.

Lemma 4.1. *The set $\mathbb{R}_+^3 = \{(S, I, R) \in \mathbb{R}^3 : S \geq 0, I \geq 0, R \geq 0\}$ is positively invariant for the system (4.44)–(4.46), for each fixed $\omega \in \Omega$.*

Proof. It is not difficult to check that the vector field, at the boundary of \mathbb{R}_+^3 , points toward the inside of \mathbb{R}_+^3 . In fact, on the plane $S = 0$ we have $S' > 0$, the plane $I = 0$ is invariant since on it we have $I' = 0$ while on $R = 0$ we have $R' \geq 0$. The positive S -semi axis is invariant, and in addition we have:

$$\frac{dS(t, \omega)}{dt} = \Lambda(\theta_t \omega) - \nu S(t),$$

which can be solved to obtain explicitly

$$S(t, \omega) = S_0 e^{-\nu(t-t_0)} + e^{-\nu t} \int_{t_0}^t \Lambda(\theta_s \omega) e^{\nu s} ds. \quad (4.47)$$

Notice that last term in (4.47) is bounded,

$$\Lambda(1 - \varepsilon) \nu e^{-\nu t} \int_{t_0}^t e^{\nu s} ds \leq e^{-\nu t} \int_{t_0}^t \Lambda(\theta_s \omega) e^{\nu s} ds \leq \Lambda(1 + \varepsilon) \nu e^{-\nu t} \int_{t_0}^t e^{\nu s} ds,$$

i.e.,

$$\Lambda(1 - \varepsilon)(1 - e^{-\nu(t-t_0)}) \leq e^{-\nu t} \int_{t_0}^t \Lambda(\theta_s \omega) e^{\nu s} ds \leq \Lambda(1 + \varepsilon)(1 - e^{-\nu(t-t_0)}).$$

Thus if the system starts on the positive R -semi axis, the solution will enter the plane $I = 0$, while on the positive I -semi axis $S' > 0$ and $R' > 0$. \square

Observe that replacing ω by $\theta_{-t} \omega$ in (4.47) and setting $t_0 = 0$ gives

$$S(t; \omega, S_0) = S_0 e^{-\nu t} + \int_{-t}^0 \Lambda(\theta_p \omega) e^{\nu p} dp, \quad (4.48)$$

which, for $t \rightarrow \infty$, pullback converges to

$$S^*(\omega) := \int_{-\infty}^0 \Lambda(\theta_p \omega) e^{\nu p} dp. \quad (4.49)$$

Note that $S^*(\omega)$ is positive and bounded. In fact, for any $\omega \in \Omega$, we have:

$$\Lambda(1 - \varepsilon) \leq S^*(\omega) \leq \Lambda(1 + \varepsilon). \quad (4.50)$$

The generation of an RDS by the solutions to (4.44) – (4.46) is stated as follows.

Theorem 4.10. *For any $\omega \in \Omega$, any $t_0 \in \mathbb{R}$ and any initial data $u_0 = (S(t_0), I(t_0), R(t_0)) \in \mathbb{R}_+^3$, system (4.44) – (4.46) admits a unique nonnegative and bounded solution $u(\cdot; t_0, \omega, u_0) \in C([t_0, +\infty), \mathbb{R}_+^3)$ with $u(t_0; t_0, \omega, u_0) = u_0$, provided that (4.43) is satisfied. Moreover the solution generates a random dynamical system $\psi(t, \omega, \cdot)$ defined as*

$$\psi(t, \omega, u_0) = u(t; 0, \omega, u_0), \quad \forall t \geq 0, u_0 \in \mathbb{R}_+^3, \omega \in \Omega.$$

Proof. The system can be rewritten as

$$\frac{du(t)}{dt} = f(u, \theta_t \omega),$$

where $f(u, \theta_t \omega)$ is vector-valued function composed of the right hand sides of (4.44)-(4.46). Since $\Lambda(\theta_t \omega)$ is continuous with respect to t , the function $f(\cdot, \theta_t \omega) \in C(\mathbb{R}_+^3 \times [t_0, +\infty), \mathbb{R}_+^3)$ and is continuously differentiable with respect to u . Then, by classical results of ODEs system (4.44)-(4.46) possesses a unique local solution.

Summing the equations of the system we arrive at

$$\frac{dN}{dt}(t, \omega) = \Lambda(\theta_t \omega) - \nu N(t),$$

whose solution satisfying $N(t_0) = N_0$ is given by

$$N(t; t_0, \omega, N_0) = N_0 e^{-\nu(t-t_0)} + e^{-\nu t} \int_{t_0}^t \Lambda(\theta_s \omega) e^{\nu s} ds. \quad (4.51)$$

As in the proof of the previous lemma we have

$$\Lambda(1 - \varepsilon) + [N_0 - \Lambda(1 - \varepsilon)] e^{-\nu(t-t_0)} \leq N(t; t_0, \omega, N_0) \leq \Lambda(1 + \varepsilon) + [N_0 - \Lambda(1 + \varepsilon)] e^{-\nu(t-t_0)}, \quad (4.52)$$

which implies that the solutions are bounded. Furthermore, the forward and backward limits of $N(t; t_0, \omega, N_0)$ satisfy, respectively,

$$\lim_{t \rightarrow +\infty} N(t; t_0, \omega, N_0) \in [\Lambda(1 - \varepsilon), \Lambda(1 + \varepsilon)], \quad \forall t_0 \in \mathbb{R},$$

and

$$\lim_{t_0 \rightarrow -\infty} N(t; t_0, \omega, N_0) \in [\Lambda(1 - \varepsilon), \Lambda(1 + \varepsilon)], \quad \forall t \in \mathbb{R}.$$

Then the local solution can be extended to a global solution $u(\cdot; t_0, \omega, u_0) \in C^1([t_0, \infty), \mathbb{R}_+^3)$. It is clear that

$$u(t + t_0; t_0, \omega, u_0) = u(t; 0, \theta_{t_0} \omega, u_0),$$

for all $t_0 \in \mathbb{R}$, $t \geq 0$, $\omega \in \Omega$, $u_0 \in \mathbb{R}_+^3$. Then we can define a map $\psi(t, \omega, \cdot)$ that is a random dynamical system by

$$\psi(t, \omega, u_0) = u(t; 0, \omega, u_0), \quad \forall t \geq 0, u_0 \in \mathbb{R}_+^3, \omega \in \Omega.$$

□

Theorem 4.11. *For each $\omega \in \Omega$ there exists a tempered bounded closed random absorbing set $K(\omega) \in \mathcal{D}(\mathbb{R}_+^3)$ of the random dynamical system $\{\psi(t, \omega, \cdot)\}_{t \geq 0, \omega \in \Omega}$ such that for any $B \in \mathcal{D}(\mathbb{R}_+^3)$ and each $\omega \in \Omega$ there exists a $T_B(\omega) > 0$ such that*

$$\psi(t, \theta_{-t} \omega, B(\theta_{-t} \omega)) \subset K(\omega), \quad \forall t \geq T_B(\omega).$$

Moreover, for any $0 < \eta < \Lambda(1 - \varepsilon)$, the set $K(\omega)$ can be chosen as the deterministic set

$$K_\eta := \{(S, I, R) \in \mathbb{R}_+^3 : \Lambda(1 - \varepsilon) - \eta \leq S + I + R \leq \Lambda(1 + \varepsilon) + \eta\},$$

for all $\omega \in \Omega$.

Proof. Using (4.51) and (4.52) we deduce that $N'(t, \omega) \leq 0$ on $N = \Lambda(1 + \varepsilon) + \eta$, and $N'(t, \omega) \geq 0$ on $N = \Lambda(1 - \varepsilon) - \eta$ for all $\eta \in [0, \Lambda(1 - \varepsilon)]$. Then K_η is positively invariant for $\eta \in [0, \Lambda(1 - \varepsilon)]$. Suppose that $N_0 \geq \Lambda(1 + \varepsilon) + \eta$ (the other case is similar), then

$$N(t; t_0, \omega, N_0) \leq N_0(\omega) e^{-\nu(t-t_0)} + \Lambda(1 + \varepsilon)[1 - e^{-\nu(t-t_0)}].$$

Replacing ω by $\theta_{-t} \omega$ gives

$$N(t; \theta_{-t} \omega, N_0(\theta_{-t} \omega)) \leq \sup_{N_0 \in B(\theta_{-t} \omega)} N_0 e^{-\nu(t-t_0)} + \Lambda(1 + \varepsilon)[1 - e^{-\nu(t-t_0)}]. \quad (4.53)$$

Hence there exists a time $T_B(\omega)$ such that for $t > T_B(\omega)$, $\psi(t, \theta_{-t} \omega, u_0) \in K_\eta$ for all $u_0 \in B(\theta_{-t} \omega)$. That is, the set K_η is compact absorbing for all $\eta \in (0, \Lambda(1 - \varepsilon))$, and absorbs all tempered random sets of \mathbb{R}_+^3 and in particular its bounded sets. □

As a result of Theorem 4.10 and Theorem 4.53 we obtain

Theorem 4.12. *The random dynamical system generated by system (4.44)–(4.46) possesses a global random attractor.*

The reader is referred to Caraballo and Colucci [12] for additional analysis on the structure of the random attractor and individual dynamics of the three components.

Stochastic SIR with white noise

Similar to the random system (4.44) – (4.46), we assume that the production spam parameter, Λ , is perturbed by noise. Instead of real noise, we now consider the effect of white noise. There are different ways to introduce the white noise into an ODE system, such as perturbing one or more parameters in an ODE system by an additive noise. Such types of perturbation, though widely used in the literature, are not always legitimate. For example assuming that the parameter Λ is perturbed by some additive white noise $\sigma\dot{W}(t)$, then the system (4.44) – (4.46) would become

$$dS = \left[\Lambda - \nu S(t) - \beta \frac{S(t) I(t)}{N(t)} \right] dt + \sigma dW_t, \quad (4.54)$$

$$dI = \left[-(\nu + \gamma)I(t) + \beta \frac{S(t) I(t)}{N(t)} \right] dt, \quad (4.55)$$

$$dR = [\gamma I(t) - \nu R(t)] dt. \quad (4.56)$$

and it is straightforward check that equation (4.54) does not preserve the positive-ness of the variable S and, as a consequence, the population of S may become negative and hence the model is not realistic.

A proper choice of white noise perturbation should at least ensures the non-negativeness of solutions while population models are considered. Though not fully justified, to illustrate the conjugation of RDS, we introduce multiplicative white noise of Stratonovich type for equations (4.54) – (4.56) to obtain

$$dS = \left[\Lambda - \nu S(t) - \beta \frac{S(t) I(t)}{N(t)} \right] dt + \sigma S \circ dW(t), \quad (4.57)$$

$$dI = \left[-(\nu + \gamma)I(t) + \beta \frac{S(t) I(t)}{N(t)} \right] dt + \sigma I \circ dW(t), \quad (4.58)$$

$$dR = [\gamma I(t) - \nu R(t)] dt + \sigma R \circ dW(t), \quad (4.59)$$

For simplicity, we assume that the noise intensity is the same in all the equations but it does not make a substantial difference if we consider a different noise intensity in each equation. The same technique still apply, only with more complicated calculations.

Following similar arguments used for the random SIR model, solutions corresponding to nonnegative initial data will remain nonnegative, ensured by the multiplicative type of noise structure. We will next transform the stochastic equations (4.57) – (4.59) into random differential equations random coefficients but without white noise. To this end, we consider the Ornstein-Uhlenbeck process

$$z^*(\theta_t \omega) = - \int_{-\infty}^0 e^s \theta_t \omega(s) ds, \quad t \in \mathbb{R}, \quad \omega \in \Omega_0, \quad (4.60)$$

which is the stationary solution to (4.19) for $\lambda = 1$.

According to the techniques introduced in the subsection 4.2.4, perform the following change of variables

$$\tilde{S}(t) = S(t)e^{-\sigma z^*(\theta_t \omega)}, \quad \tilde{I}(t) = I(t)e^{-\sigma z^*(\theta_t \omega)}, \quad \tilde{R}(t) = R(t)e^{-\sigma z^*(\theta_t \omega)}.$$

Then system (4.57) – (4.59) become

$$\frac{d\tilde{S}}{dt}(t, \omega) = \Lambda e^{-\sigma z^*} - \nu \tilde{S}(t) - \beta \frac{\tilde{S}(t) \tilde{I}(t)}{\tilde{N}(t)} + \sigma \tilde{S} z^*, \quad (4.61)$$

$$\frac{d\tilde{I}}{dt}(t, \omega) = -(\nu + \gamma) \tilde{I}(t) + \beta \frac{\tilde{S}(t) \tilde{I}(t)}{\tilde{N}(t)} + \sigma \tilde{I} z^*, \quad (4.62)$$

$$\frac{d\tilde{R}}{dt}(t, \omega) = \gamma \tilde{I}(t) - \nu \tilde{R}(t) + \sigma \tilde{R} z^*. \quad (4.63)$$

Note that adding equations (4.61) – (4.63) results in

$$\frac{d\tilde{N}}{dt}(t, \omega) = \Lambda e^{-\sigma z^*} - \nu \tilde{N}(t) + \sigma \tilde{N} z^* = \Lambda e^{-\sigma z^*} - (\nu - \sigma z^*) \tilde{N}. \quad (4.64)$$

The equation (4.64) has a nontrivial random solution that is both forward and pullback attracting. In fact, for any initial datum N_0 we have:

$$N(t; \omega, N_0) = N_0 e^{-\int_0^t [\nu - \sigma z^*(\theta_s \omega)] ds} + \int_0^t \Lambda e^{-\sigma z^*(\theta_s \omega)} e^{\int_s^t [\nu - \sigma z^*(\theta_\tau \omega)] d\tau} ds,$$

and replacing ω by $\theta_{-t} \omega$ we obtain

$$N(t; \theta_{-t} \omega, N_0) = N_0 e^{-\int_{-t}^0 [\nu - \sigma z^*(\theta_p \omega)] dp} + \int_{-t}^0 \Lambda e^{-\sigma z^*(\theta_p \omega)} e^{-\int_p^0 [\nu - \sigma z^*(\theta_q \omega)] dq} dp,$$

which pullback converges to

$$N^*(\omega) = \Lambda \int_{-\infty}^0 e^{-\sigma z^*(\theta_p \omega)} e^{-\int_p^0 [\nu - \sigma z^*(\theta_q \omega)] dq} dp \quad (4.65)$$

as $t \rightarrow +\infty$.

Observe that $N^*(\omega)$ is well defined thanks to Proposition 4.1 because the integrand behaves like $e^{\nu p}$ for $p \rightarrow -\infty$ and

$$\frac{z^*(\theta_p \omega)}{p}, \quad \frac{1}{p} \int_p^0 z^*(\theta_q \omega) dq \rightarrow 0, \quad \text{for } p \rightarrow -\infty.$$

Some comments are made below on differences between the random SIR model and the stochastic SIR model.

- The stochastic differential system with white noise (4.57) – (4.59) can be transformed to an equivalent random differential system with unbounded random coefficients, while the random system with real noise (4.54) – (4.56) has bounded random coefficients.
- The differential equation describing the behavior of the total population $N(t) = S(t) + I(t) + R(t)$ is similar in both cases and provides us with a stationary process that pullback and forward attracting any other solution. However, an important difference is that in the random system (4.54) – (4.56), this stationary process is bounded, what allows for a posterior control of this process to validate the underlying model. While in the stochastic system (4.57) – (4.59) we are not able to obtain such posterior estimates and the corresponding additional information.
- The system with real noise, (4.54) – (4.56), preserves the form and structure of the underlying model. In other words, if we replace the random parameter $\Lambda(\theta_t\omega)$ by another random parameter with a special structure, the model is not a priori affected. While in the stochastic system (4.54) – (4.56) if there is a change in the noise structure, the underlying model may need to be redeveloped, even just to preserve the positiveness of solutions.

4.3.3 Stochastic Lorenz models

In the simplest autonomous Lorenz-84 model (2.17) – (2.19), the forcing terms F and G are assumed to be constant. To make the model more realistic, we allowed the forcing terms to vary with respect to time, resulting in a nonautonomous Lorenz model studied in Chapter 3. Now we will study the model in which the forcing terms may vary randomly in time, described by the following system of RDEs with real noise:

$$\frac{dx}{dt} = -ax - y^2 - z^2 + aF(\theta_t\omega), \quad (4.66)$$

$$\frac{dy}{dt} = -y + xy - bxz + G(\theta_t\omega), \quad (4.67)$$

$$\frac{dz}{dt} = -z + bxy + xz. \quad (4.68)$$

In the system (4.66) – (4.68), the time dependence of the forcing terms is defined by a metric dynamical system $(\theta_t)_{t \in \mathbb{R}}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By analysis analog to those performed in the nonautonomous Lorenz model (2.17) – (2.19), the existence of a random attractor for the RDS generated by the system (4.66) – (4.68) can also be obtained. To this end we assume that the mappings $F, G : \Omega \mapsto \mathbb{R}$ are measurable and for every fixed $\omega \in \Omega$, the mappings $t \in \mathbb{R} \mapsto F(\theta_t\omega)$ and $t \in \mathbb{R} \mapsto G(\theta_t\omega)$ are continuous, and satisfy

$$\int_{-\infty}^t e^{ls} F^2(\theta_s\omega) ds < +\infty, \quad \int_{-\infty}^t e^{ls} G^2(\theta_s\omega) ds < +\infty \quad \forall t \in \mathbb{R}, \omega \in \Omega \quad (4.69)$$

where $l := \min\{a, 1\}$.

The first step is to construct the cocycle generated by (4.66)-(4.68). It is straightforward to check that the right hand sides of equations (4.66)-(4.68) define a vector field which is continuous with respect to t and locally Lipschitz with respect to the other variables. Thus, the existence and uniqueness of maximal solutions to the initial value problems associated to (4.66)-(4.68) is guaranteed by the results in Chapter 1. The next theorem ensures that the maximal solutions are also defined globally in time. For simplicity, denote by $u(t) := (x(t), y(t), z(t))$ and $u_0 := (x_0, y_0, z_0)$.

Lemma 4.2. *Assume that $a > 0$ and $b \in \mathbb{R}$. Then for any initial condition $u_0 \in \mathbb{R}^3$ and any initial time $t_0 \in \mathbb{R}$, the solution $u(\cdot) := u(\cdot, t_0, \omega, u_0)$ of the IVP associated to (4.66)-(4.68) satisfies*

$$|u(t, t_0, \omega, u_0)|^2 \leq e^{-l(t-t_0)} |u_0|^2 + ae^{-lt} \int_{-\infty}^t e^{ls} F^2(\theta_s \omega) ds + e^{-lt} \int_{-\infty}^t e^{ls} G^2(\theta_s \omega) ds, \quad (4.70)$$

for all $t \geq t_0$, where $l := \min\{a, 1\}$.

Proof. We immediately deduce that

$$\frac{d}{dt} |u(t)|^2 = -2(ax^2 + y^2 + z^2) + 2axF(\theta_t \omega) + 2yG(\theta_t \omega),$$

and, consequently,

$$\frac{d}{dt} |u(t)|^2 + l|u(t)|^2 \leq aF^2(\theta_t \omega) + G^2(\theta_t \omega). \quad (4.71)$$

Multiplying (4.71) by e^{lt} , we obtain that

$$\frac{d}{dt} (e^{lt} |u(t)|^2) \leq ae^{lt} F^2(\theta_t \omega) + e^{lt} G^2(\theta_t \omega).$$

Integrating between t_0 and t

$$\begin{aligned} e^{lt} |u(t)|^2 &\leq e^{lt_0} |u_0|^2 + a \int_{t_0}^t e^{ls} F^2(\theta_s \omega) ds + \int_{t_0}^t e^{ls} G^2(\theta_s \omega) ds \\ &\leq e^{lt_0} |u_0|^2 + a \int_{-\infty}^t e^{ls} F^2(\theta_s \omega) ds + \int_{-\infty}^t e^{ls} G^2(\theta_s \omega) ds, \end{aligned} \quad (4.72)$$

whence (4.70) follows. \square

Thanks to Lemma 4.2 we can define the cocycle on \mathbb{R}^3 by

$$\psi(t, \omega, u_0) = u(t; 0, \omega, u_0) \quad \text{for } u_0 \in \mathbb{R}^3, \quad t \geq 0.$$

Considering now the universe of tempered random subsets of \mathbb{R}^3 , we will be able to prove that there exists a pullback absorbing family for the cocycle $\psi(\cdot, \cdot, \cdot)$.

Theorem 4.13. *Assume $a > 0$ and $b \in \mathbb{R}$. Let F, G satisfy (4.69). Then, the family $\mathcal{D}_0 = \{D_0(\omega) : \omega \in \Omega\}$ defined by $D_0(\omega) = \overline{B}(0, \rho_l(\omega))$, where $\rho_l(\omega)$ is given by*

$$\rho_l^2(\omega) = 1 + a \int_{-\infty}^0 e^{ls} F^2(\theta_s \omega) ds + \int_{-\infty}^0 e^{ls} G^2(\theta_s \omega) ds, \quad \forall t \in \mathbb{R}, \quad (4.73)$$

is a pullback absorbing family for the cocycle $\psi(\cdot, \cdot, \cdot)$. Furthermore, there exists a pullback random attractor for ψ .

Proof. Let $\mathcal{D} = \{D(\omega) : \omega \in \Omega\}$ be a tempered random set. Then,

$$\lim_{t \rightarrow \infty} e^{-\beta t} \sup_{u_0 \in D(\theta_{-t} \omega)} |u_0| = 0, \quad \text{for all } \beta > 0;$$

Then, pick $u_0 \in D(\omega)$. Thanks to Lemma 4.2, we can deduce that

$$\begin{aligned} |\psi(t, \omega, u_0)|^2 &\leq e^{-lt} |u_0(\omega)|^2 + ae^{-lt} \int_{-\infty}^t e^{ls} F^2(\theta_s \omega) ds + e^{-lt} \int_{-\infty}^t e^{ls} G^2(\theta_s \omega) ds \\ &\leq e^{-lt} |u_0(\omega)|^2 + ae^{-lt} \int_{-\infty}^t e^{ls} F^2(\theta_s \omega) ds + e^{-lt} \int_{-\infty}^t e^{ls} G^2(\theta_s \omega) ds. \end{aligned}$$

Replacing ω by $\theta_{-t} \omega$, and performing a suitable change of variable in the integrals results in

$$\begin{aligned} |\psi(t, \theta_{-t} \omega, u_0(\theta_{-t} \omega))|^2 &\leq e^{-lt} |u_0(\theta_{-t} \omega)|^2 + ae^{-lt} \int_{-\infty}^t e^{ls} F^2(\theta_{s-t} \omega) ds \\ &\quad + e^{-lt} \int_{-\infty}^t e^{ls} G^2(\theta_{s-t} \omega) ds \\ &\leq e^{-lt} |u_0(\theta_{-t} \omega)|^2 + a \int_{-\infty}^0 e^{ls} F^2(\theta_s \omega) ds + \int_{-\infty}^0 e^{ls} G^2(\theta_s \omega) ds \end{aligned}$$

Taking limit when t goes to $+\infty$ and on account of the temperedness of the family \mathcal{D} , we easily obtain the pullback absorption of the family \mathcal{D}_0 and therefore the existence of the pullback random attractor is proved. \square

Remark 4.9. Although we have considered a random version for the Lorenz-84 model so that we have coherent autonomous, nonautonomous and random variations of the same system, we would like to mention that in the paper [73], Schmalfuß considered a different stochastic Lorenz model. More precisely, a stochastic version with multiplicative linear Stratonovich noise is analyzed. The technique made use of the transformation from a stochastic differential equation into a random differential system by mean of the Ornstein-Uhlenbeck process as we have done in the previous stochastic SIR model. Moreover, the study of the finite dimensionality of the random attractor was carried out in the paper [73], which went beyond the objectives of the present work.

4.4 Stabilization of dynamical systems

It is clear that the analyzing a stochastic system is more challenging than analyzing its deterministic counterparts, and may require more techniques than the classical theory of ODE. Hence, very often the study of a stochastic model is done by analyzing a deterministic approximation of the model. One question then arises naturally, is there any difference between the long time behavior of solutions to the stochastic model and its deterministic counterpart? If there is a difference, how to quantify it? Along this line of research one interesting and important topic is the stabilization effects induced by the noise, which is closely related to controllability problems and many other applications in applied sciences and engineering. Many studies have been done regarding the stabilization effects of noise by different methods, but here we only discuss those related to the theory of random dynamical systems. In particular, we illustrate the stabilization effects of noise by using simple examples, which can serve as motivation for more complicated situations.

In the following examples, we will show that when a deterministic ODE is perturbed by a linear multiplicative white noise, the random dynamical system generated by the resulting SDE may have a “smaller” random attractor than the global attractor associated to the original deterministic equation. Such behavior can be regarded as a “stabilization” effect produced by the noise, that can happen not only to equilibrium points but also to attractors. The existence of such a “stabilization” effect, moreover, may depend on the type of stochastic integral considered. More precisely, the global attractor of an ODE can be different from the random attractor of its corresponding Itô SDE, but very similar to the random attractor of its corresponding Stratonovich SDE. Similarly, when a deterministic ODE is perturbed by an additive white noise, the random attractor for the resulting SDE may be a random singleton set while the global attractor for the original ODE is a compact interval.

Consider the following simple one-dimensional differential equation

$$\frac{dx}{dt} = x(1 - x^2), \quad (4.74)$$

which has three equilibrium points, $\{-1, 0, 1\}$, among which 0 unstable and the others are asymptotically stable. In addition, the global attractor exists for this equation and is given by the compact interval $K = [-1, 1]$. We next investigate the stability of SDEs obtained from perturbing (4.74) by different types of noise.

Itô's perturbation

Including a multiplicative noise in (4.74) results in the Itô SDE

$$dx = x(1 - x^2)dt + \sigma x dW_t, \quad (4.75)$$

where $\sigma \in \mathbb{R}$ is a constant and W_t is the standard and canonical two-sided Brownian motion (Wiener process). We will prove that the Itô SDE (4.75) generates an RDS which possesses a global random attractor \mathcal{A} given by $A(\omega) = \{0\}$ for every $\omega \in \Omega$, provided the intensity of the noise is large enough. In other words, the random attractor is just one fixed equilibrium (also called a random fixed point). This implies that the equilibrium 0 has been stabilized and the attractor has also been stabilized in some sense.

To analyze equation (4.75) by using the theory of RDS, we first consider its equivalent Stratonovich formulation

$$dx = \left(\left(1 - \frac{\sigma^2}{2} \right) x - x^3 \right) dt + \sigma x \circ dW_t. \quad (4.76)$$

Performing the change of variables (4.26) (corresponding to the conjugation (4.28)), by using the OU process z^* in (4.20) with $\lambda = 1$, we obtain $y(t) = x(t)e^{-\sigma z^*(\theta_t(\omega))}$ and the associated random differential equation

$$\frac{dy}{dt} = \left(1 - \frac{\sigma^2}{2} + \sigma z^*(\theta_t \omega) \right) y - e^{2\sigma z^*(\theta_t \omega)} y^3. \quad (4.77)$$

Equation (4.77) generates a random dynamical system given by

$$\psi(t, \omega, y_0) = y(t; 0, \omega, y_0), \quad (4.78)$$

where, as usual, $y(\cdot; s, \omega, y_0)$ denotes the solution to (4.77) such that at the initial time $t = 0$ takes the value y_0 .

Observe that equation (4.77) is of Benouilli type, hence can be solved by standard integration methods to obtain

$$\begin{aligned} \frac{1}{y^2(t; 0, \omega, y_0)} &= \frac{1}{y_0^2} e^{-(2-\sigma^2)t - 2\sigma \int_0^t z^*(\theta_s \omega) ds} \\ &\quad + 2 \int_0^t e^{-2(t-s) + 2\sigma z^*(\theta_s \omega) - 2\sigma \int_s^t z^*(\theta_r \omega) dr} ds, \end{aligned} \quad (4.79)$$

which gives an explicit expression of the cocycle ψ .

Choose σ large enough so that $\sigma^2 > 2$. Then, it follows from (4.79) that

$$\frac{1}{y^2(t; 0, \omega, y_0)} \geq \frac{1}{y_0^2} e^{-(2-\sigma^2)t - 2\sigma \int_0^t z^*(\theta_s \omega) ds}, \quad (4.80)$$

and replacing ω by $\theta_{-t}\omega$ with a suitable change of variable in the integral in the exponent,

$$\frac{1}{y^2(t; 0, \theta_{-t}\omega, y_0)} \geq \frac{1}{y_0^2} e^{-(2-\sigma^2)t - 2\sigma \int_{-t}^0 z^*(\theta_s \omega) ds}. \quad (4.81)$$

Consequently, taking limits as $t \rightarrow +\infty$ in both equations (4.80) and (4.81), at light of the properties of the OU process in Proposition 4.1, we deduce that the right hand sides in both expressions go to $+\infty$, and therefore

$$\lim_{t \rightarrow \infty} y^2(t; 0, \omega, y_0) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} y^2(t; 0, \theta_{-t}\omega, y_0) = 0.$$

Therefore for the corresponding RDS we have

$$\lim_{t \rightarrow \infty} \psi(t, \omega, y_0) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \psi(t, \theta_{-t}\omega, y_0) = 0.$$

It is then straightforward to see that the RDS ψ has a random pullback attractor \mathcal{A} with each component set $A(\omega) = 0$. Moreover this set also attracts any solution in the forward sense.

Stratonovich perturbation

Now consider the perturbation of (4.74) by multiplicative white noise in the Stratonovich sense

$$dx = x(1 - x^2)dt + \sigma x \circ dW_t. \quad (4.82)$$

The expression for $y(t; 0, \omega, y_0)$ is given explicitly by

$$\begin{aligned} \frac{1}{y^2(t; 0, \omega, y_0)} &= \frac{1}{y_0^2} e^{-2t - 2\sigma \int_0^t z^*(\theta_s \omega) ds} \\ &+ 2 \int_0^t e^{-2(t-s) + 2\sigma z^*(\theta_s \omega) - 2\sigma \int_s^t z^*(\theta_r \omega) dr} ds, \end{aligned} \quad (4.83)$$

whenever $y_0 \neq 0$ and $y(t; 0, \omega, 0) = 0$. Thanks to the properties of the OU process z^* , the first term on the right hand side of (4.83) goes to zero when t goes to ∞ no matter how large or small σ is. Then, replacing ω by $\theta_{-t}\omega$ in (4.83), and performing a suitable change of variables in each integral, we obtain

$$\begin{aligned} \frac{1}{y^2(t; 0, \theta_{-t}\omega, y_0)} &= \frac{1}{y_0^2} e^{-2t - 2\sigma \int_{-t}^0 z^*(\theta_s \omega) ds} \\ &+ 2 \int_{-t}^0 e^{2s + 2\sigma z^*(\theta_s \omega) - 2\sigma \int_s^0 z^*(\theta_r \omega) dr} ds. \end{aligned} \quad (4.84)$$

Consequently, taking limit when t goes to ∞ in (4.84) gives

$$\lim_{t \rightarrow +\infty} \frac{1}{y^2(t; 0, \theta_{-t}\omega, y_0)} = 2 \int_{-\infty}^0 e^{2s + 2\sigma z^*(\theta_s \omega) - 2\sigma \int_s^0 z^*(\theta_r \omega) dr} ds.$$

Denote by

$$\frac{1}{a(\omega)} := 2 \int_{-\infty}^0 e^{2s + 2\sigma z^*(\theta_s \omega) - 2\sigma \int_s^0 z^*(\theta_r \omega) dr} ds.$$

Then it is clear that the random pullback attractor \mathcal{A} for this system is given by $A(\omega) = [-a(\omega)^{1/2}, a(\omega)^{1/2}]$, which possesses the same geometrical structure of the global attractor for the deterministic ODE (4.74). Notice also that, by standard computations, it is not difficult to show that zero is still an unstable equilibrium of the stochastic equation, so the instability of the null solution persists under such a Stratonovich linear multiplicative white noise perturbation.

Additive noise

Last consider the following SDE obtained from perturbing the ODE (4.74) by an additive white noise:

$$dx = x(1 - x^2)dt + dW_t. \quad (4.85)$$

System (4.85) has been studied in details in [30] where it was proved that the random attractor was just a single fixed point. More precisely, Crauel and Flandoli proved that the SDE (4.85) possesses a random attractor $\mathcal{A}(\omega)$ whose component sets are singleton sets, i.e., $\mathcal{A}(\omega) = \{\alpha(\omega)\}$, where $\alpha(\cdot)$ is a random variable. In this situation the unstable fixed point 0 has been eliminated by the noise, and the global attractor of the deterministic equation (4.74), namely the interval $[-1, 1]$, has “collapsed” to a random attractor formed by a random fixed point. In other words, one could say that additive noise in (4.85) has produced a stabilization effect on the attractor of its deterministic counterpart (4.74).

It is worth mentioning that it is also possible to show an effect of the noise in the sense that, while a deterministic ODE does not have a global attractor, after including a noise term, the resulting SDE can have a random attractor. As an illustrative and simple example, consider the initial value problem

$$\frac{dx(t)}{dt} = x(t) + 1, \quad x(0) = x_0 \in \mathbb{R} \quad (4.86)$$

The solution to (4.86) is

$$x(t; 0, x_0) = -1 + (x_0 + 1)e^t, \quad (4.87)$$

and the dynamical system generated by (4.86) does not possess a global attractor.

Adding a linear Itô noise to the equation (4.86) results in

$$dx(t) = (x(t) + 1)dt + \sigma x(t)dW_t, \quad x(t_0) = x_0, \quad (4.88)$$

where $\sigma \in \mathbb{R}$. Then it is straightforward to check that the equation (4.88) generates an RDS. In fact, we can first transform (4.88) into an equivalent Stratonovich stochastic equation

$$dx(t) = \left(1 - \frac{\sigma^2}{2}\right)x(t) + 1 \Big) dt + \sigma x(t) \circ dW_t, \quad x(t_0) = x_0. \quad (4.89)$$

A change of variables by using the standard OU process could still be done, and will transform (4.89) to a random differential equation, to which the theory of RDS can be applied. But here we present a different technique, which at first sight may seem much easier and straightforward, but as a trade-off does not provide a conjugated random dynamical system. To this end, perform the change of variables

$$y(t) = e^{-\sigma W_t(\omega)} x(t), \quad (4.90)$$

to obtain the random equation

$$\frac{dy(t)}{dt} = \left(1 - \frac{\sigma^2}{2}\right) y(t) + e^{-\sigma W_t(\omega)}, \quad y(t_0) = y_0 = e^{-\sigma W_{t_0}(\omega)} x_0, \quad (4.91)$$

whose solution is explicitly given by

$$y(t; t_0, \omega, y_0) = e^{(1-\frac{\sigma^2}{2})(t-t_0)} e^{-\sigma W_{t_0}(\omega)} x_0 + e^{(1-\frac{\sigma^2}{2})t} \int_{t_0}^t e^{-(1-\frac{\sigma^2}{2})r} e^{-\sigma W_r(\omega)} dr. \quad (4.92)$$

Therefore, the RDS $\psi(t, \omega, \cdot)$ generated by the random equation (4.91) can be defined by

$$\psi(t, \omega, x_0) = e^{\sigma W_t(\omega)} y(t; 0, \omega, x_0).$$

Choose σ so that $1 - \frac{\sigma^2}{2} < 0$. Then taking limits $t_0 \rightarrow -\infty$ in (4.92) results in

$$\lim_{t_0 \rightarrow -\infty} y(t; t_0, \omega, y_0) = e^{(1-\frac{\sigma^2}{2})t} \int_{-\infty}^t e^{-(1-\frac{\sigma^2}{2})r} e^{-\sigma W_r(\omega)} dr,$$

and consequently

$$\lim_{t_0 \rightarrow -\infty} e^{\sigma W_t(\omega)} y(t; t_0, \omega, y_0) = e^{(1-\frac{\sigma^2}{2})t} e^{\sigma W_t(\omega)} \int_{-\infty}^t e^{-(1-\frac{\sigma^2}{2})r} e^{-\sigma W_r(\omega)} dr.$$

Denote by

$$\mathcal{A}(\omega) = \int_{-\infty}^0 e^{-(1-\frac{\sigma^2}{2})r} e^{-\sigma W_r(\omega)} dr.$$

Then it is straightforward to check that $x(t, \omega) := \mathcal{A}(\theta_t \omega)$ is a stationary solution of (4.89) and

$$\mathcal{A}(\theta_t \omega) = e^{(1-\frac{\sigma^2}{2})t} e^{\sigma W_t(\omega)} \int_{-\infty}^t e^{-(1-\frac{\sigma^2}{2})r} e^{-\sigma W_r(\omega)} dr.$$

Note that although the transformation (4.90) does not provide a conjugated random dynamical system, since every solution to the stochastic equation (4.89) is also a solution to the random equation (4.91), all solutions to the stochastic system (4.89) will approach in the pullback sense to the random attractor possessed by the random system (4.91). Therefore, $\mathcal{A}(\omega)$ is also a non-trivial random attractor for (4.88) which proves that Itô's noise may also cause stabilization to a random attractor.

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