ASYMPTOTIC BEHAVIOUR OF A NON-CLASSICAL AND NON-AUTONOMOUS DIFFUSION EQUATION CONTAINING SOME HEREDITARY CHARACTERISTIC

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Abstract. We show existence and uniqueness of solutions for a non-classical and non-autonomous diffusion equation with infinite delay terms, analyzing the asymptotic behaviour in the pullback sense and obtaining global exponential decay to stationary solutions.

1. **Introduction and theoretical framework.** In this work we study the following non-classical diffusion equation with infinite delays, written in an abstract functional formulation,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \gamma(t)\Delta \dfrac{\partial u}{\partial t} - \Delta u = g(u) + f(t, u_t) \text{ in } (\tau, +\infty) \times \Omega, \\[2mm] u = 0 \text{ on } (\tau, +\infty) \times \partial\Omega \\[2mm] u(t, x) = \phi(t - \tau, x), t \in (-\infty, \tau], x \in \Omega \end{cases} \qquad (1)$$

where $\tau \in \mathbb{R}$ is the initial time, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $\gamma : \mathbb{R} \to (0, +\infty)$ is a continuous bounded function with $0 < \gamma_0 \leq \gamma(t) \leq \gamma_1 < \infty$, and the non-linearity g is a function satisfying the following growth conditions:

$$g \in C^1(\mathbb{R}), \qquad \limsup_{|a| \to +\infty} \frac{g(a)}{a} \leq \frac{\lambda_1}{6} \qquad (2)$$

$$|g(a) - g(b)| \leq c|a - b|(1 + |a|^{\rho-1} + |b|^{\rho-1}), \qquad (3)$$

with $1 < \rho < \frac{n+2}{n-2}$ and λ_1 the first eigenvalue of the Laplacian with Dirichlet boundary conditions. The time-dependent delay term $f(t, u_t)$ represents, for instance, the influence of an external force with some kind of delay, memory or hereditary characteristics, although can also model some kind of feedback control. Here, u_t denotes a segment of the solution, that is, given a function $u : (-\infty, +\infty) \times \Omega \to \mathbb{R}$, for each $t \in \mathbb{R}$ we can define the mapping $u_t : (-\infty, 0] \times \Omega \to \mathbb{R}$ by

$$u_t(\theta, x) = u(t + \theta, x), \text{ for } \theta \in (-\infty, 0], x \in \Omega.$$

This abstract formulation allows to consider different kinds of delay terms like

$$F_1(u(t - \sigma(t))), \int_{-\infty}^{0} F_2(t, \theta, u(t + \theta)) \, d\theta, \tag{4}$$

where F_i $(i = 1, 2)$ are suitable functions, and $\sigma : \mathbb{R} \to [0, +\infty)$. Both can be described by the following corresponding f_i defined as

$$f_1(t, \psi) = F_1(\psi(-\sigma(t))), \ f_2(t, \psi) = \int_{-\infty}^{0} F_2(t, \theta, \psi(\theta)) \, d\theta, \tag{5}$$

where $\psi : (-\infty, 0] \to X$ (X denotes certain Banach or Hilbert space concerning the spatial variable). Then, when we replace ψ by u_t in (5), we obtain (4).

Nonclassical parabolic equations are used to model physical phenomena such as non-Newtonian flow, soil mechanics, heat conduction, etc (see [1, 15, 16, 2, 4, 18, 21, 23, 24] and references therein). The asymptotic behaviour of the model without the delay term and with constant coefficients is studied in [25]. It is shown there the well-posedness of the problem and the existence of the global attractor in $H_0^1(\Omega)$ and in $H^2(\Omega)$, depending on the regularity of the initial data. However, there are situations in which the model is better described if some terms containing delays are considered in the equations.

The introduction of a time dependence in coefficient $\gamma(t)$ represents the variability of viscosity in time due to, for example, external environment temperatures. This time dependence endows the system with a non-autonomous nature.

First of all we are going to introduce the framework for the study of the asymptotic behaviour of our non-autonomous system. Although there exist different kinds of framework like non-autonomous dynamical systems, skew-product semiflow or evolution processes, we are interested in the existence of the pullback attractor for (1), and to this end, we will first recall some theoretical results from the framework of evolution processes.

Given a metric space $(\mathcal{X}, d_{\mathcal{X}})$ and two subsets A and B of $\mathcal{X}$, the Hausdorff semidistance between A and B is defined as

$$\text{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} d_{\mathcal{X}}(a, b).$$

Definition 1.1. An *evolution process* in a metric space $(\mathcal{X}, d_{\mathcal{X}})$ is a family of continuous maps $\{S(t, \tau) : t \geq \tau\}$ from $\mathcal{X}$ into itself with the following properties

  i) $S(t, t) = I$, for all $t \in \mathbb{R}$,
 ii) $S(t, \tau) = S(t, s)S(s, \tau)$, for all $t \geq s \geq \tau$,
iii) $\{(t, \tau) \in \mathbb{R}^2 : t \geq \tau\} \times \mathcal{X} \ni (t, \tau, x) \mapsto S(t, \tau)x \in \mathcal{X}$ is continuous.

Let $\mathcal{P}(\mathcal{X})$ denote the family of all nonempty subsets of $\mathcal{X}$, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(\mathcal{X})$. Let $\mathcal{D}$ be a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 1.2. It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}-$absorbing for the process $\{S(t, \tau) : t \geq \tau\}$ on $\mathcal{X}$ if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \leq t$ such that

$$S(t, \tau)D(\tau) \subset D_0(t) \quad \text{for all } \tau \leq \tau_0(t, \widehat{D}).$$

Definition 1.3. The family $\mathcal{A}_\mathcal{D} = \{\mathcal{A}_\mathcal{D}(t) : t \in \mathbb{R}\}$ is the $\mathcal{D}$-pullback attractor for the process $\{S(t, \tau) : t \geq \tau\}$ in $\mathcal{X}$ if:

1. for any $t \in \mathbb{R}$, the set $\mathcal{A}_\mathcal{D}(t)$ is a nonempty compact subset of $\mathcal{X}$.
2. $\mathcal{A}_\mathcal{D}$ is pullback $\mathcal{D}-$attracting, i.e.,

$$\lim_{\tau \to -\infty} \text{dist}(S(t, \tau)D(\tau), \mathcal{A}_\mathcal{D}(t)) = 0$$

   for all $\widehat{D} \in \mathcal{D}$, for $t \in \mathbb{R}$,
3. $\mathcal{A}_\mathcal{D}$ is invariant, i.e.,

$$S(t, \tau)\mathcal{A}_\mathcal{D}(\tau) = \mathcal{A}_\mathcal{D}(t) \quad \text{for all } \tau \leq t.$$

The family $\mathcal{A}_\mathcal{D}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$,

$$\lim_{\tau \to -\infty} \text{dist}(S(t, \tau)D(\tau), C(t)) = 0,$$

then $\mathcal{A}_\mathcal{D}(t) \subset C(t)$.

Definition 1.4. Given a family parameterized in time, $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(\mathcal{X})$, it is said that a process $\{S(t, \tau) : t \geq \tau\}$ on $\mathcal{X}$ is pullback $\widehat{D}-$asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset \mathcal{X}$ bounded satisfying $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ for all n, the sequence $\{S(t, \tau_n)x_n\}$ is relatively compact in $\mathcal{X}$.

Definition 1.5. A process $\{S(t, \tau) : t \geq \tau\}$ on $\mathcal{X}$ is said to be pullback $\mathcal{D}-$asymptotically compact if it is $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

Theorem 1.6. *Consider a process $\{S(t, \tau) : t \geq \tau\}$ in $\mathcal{X}$, a universe $\mathcal{D}$ in $\mathcal{P}(\mathcal{X})$, a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(\mathcal{X})$ which is pullback $\mathcal{D}-$absorbing and assume also that the process is pullback $\widehat{D}_0-$asymptotically compact.*
Then, there exists the pullback attractor $\mathcal{A}_\mathcal{D} = \{\mathcal{A}_\mathcal{D}(t) : t \in \mathbb{R}\}$.

In [22] the existence of the pullback attractor and its continuity under non-autonomous perturbations without delay is showed, giving a concrete structure under some assumptions on the non-linearity. The finite delay case was first studied in [7], establishing the well-posedness of the problem when $\gamma(t) \equiv \gamma$ constant, showing the stability of the stationary solutions under some appropriate hypotheses on the delay term. In [8] we studied the asymptotic behaviour of solutions within the framework of pullback attractors' theory for the time dependent perturbation case. The infinite delay case started to be analyzed in [9], where we proved the existence and uniqueness of solutions as well as the continuous dependence on the initial values. In [14], Hu and Wang studied this equation with a specific variable delay term with bounded derivative, showing the existence of the pullback attractor in H_0^1 and H^2 without neither non-linearity nor variable coefficients.

The content of this paper is as follows. In Section 2 we prove the existence and uniqueness of local solutions for (1). Section 3 is devoted to the study of the global existence of solutions and the existence of a pullback absorbing family within the

universe of global bounded families. Then, in Section 4 we show the existence of a pullback attractor. Finally, the existence of stationary solutions of our problem and the asymptotic behaviour of such stationary solutions are treated in Section 5.

2. **Existence of solution.** We consider the following usual spaces $H = L^2(\Omega)$ with inner product $(\cdot, \cdot)$ and associate norm $|\cdot|$, and $V = H_0^1(\Omega)$ with scalar product $((u, v)) = (A^{1/2}u, A^{1/2}v)$, for $u, v \in V$, and associate norm $\|\cdot\|$, where $Au = -\Delta u$ for any $u \in D(A)$ with $D(A) = \{u \in V : Au \in H\} = H_0^1(\Omega) \cap H^2(\Omega)$.
One possibility to deal with infinite delays, and which we will use here (cf. [12, 13, 19]), is to consider, for any $\delta > 0$, the space :

$$C_\delta(V) = \left\{ \varphi \in C((-\infty, 0]; H_0^1(\Omega)) \ : \ \exists \lim_{s \to -\infty} e^{\delta s} \varphi(s) \in H_0^1(\Omega) \right\},$$

which is a Banach space with the norm

$$\|\varphi\|_\delta := \sup_{s \in (-\infty, 0]} e^{\delta s} \|\varphi(s)\|.$$

For the delay term, we assume that $f : \mathbb{R} \times C_\delta(V) \to V$ and satisfies:

f1) is continuous in t,

f2) is locally Lipschitz in $C_\delta(V)$ uniformly in time, that is, there exists a nondecreasing function $L_f : \mathbb{R} \to \mathbb{R}$, such that for all $R > 0$ if $\|\xi\|_\delta, \|\eta\|_\delta \leq R$, then

$$\|f(t, \xi) - f(t, \eta)\| \leq L_f(R)\|\xi - \eta\|_\delta,$$

for all $t \in \mathbb{R}$, and

f3) there exist a constant $C_f > 0$ and a nonnegative function $\psi \in L^1(\tau, T)$, for all $T > \tau$, such that, for any $\xi \in C_\delta(V)$,

$$\|f(t, \xi)\|^2 \leq C_f \|\xi\|_\delta^2 + \psi(t), \quad \text{for all} \quad t > \tau.$$

Finally, we suppose that $\phi \in C_\delta(V)$.

Proceeding as in [22], we can define operators $B(t) = (I + \gamma(t)A)^{-1}$ and $\tilde{A}(t) = AB(t)$, where $A = -\Delta$ with Dirichlet boundary conditions and the functions $\tilde{g}(t, u) = B(t)g(u)$ and $\tilde{f}(t, \phi) = B(t)f(t, \phi)$, $\forall t \in \mathbb{R}$, $\forall \phi \in C_\delta(V)$.

Then, the equation in (1) can be written as

$$\frac{du}{dt} = h(t, u_t), \tag{6}$$

with $h : \mathbb{R} \times C_\delta(V) \to V$ defined as $h(t, \psi) = \tilde{A}(t)\psi(0) + \tilde{g}(t, \psi(0)) + \tilde{f}(t, \psi)$, $\forall t \in \mathbb{R}$, $\forall \psi \in C_\delta(V)$, where operator $\tilde{A}(t)$ can be written as

$$\tilde{A}(t) = \frac{1}{\gamma(t)} \left[I - (1 + \gamma(t)A)^{-1} \right], \tag{7}$$

for any $t \in \mathbb{R}$, for any $\alpha > 0$ and $x \in D(A^\alpha)$, $A^\alpha \tilde{A}(t)x = \tilde{A}(t)A^\alpha x$. Moreover, we have that this operator is uniformly bounded and its domain does not depend on time.

Thanks to the continuity of the function $\mathbb{R} \ni t \mapsto B(t) \in \mathcal{L}(H_0^1(\Omega))$, we obtain the following estimate (see [22] for more details)

$$\|\tilde{A}(t) - \tilde{A}(s)\|_{\mathcal{L}(H_0^1(\Omega))} \leq C|\gamma(t) - \gamma(s)|,$$

for a constat $C \in \mathbb{R}$.

We can now state the existence of solution to our problem.

Theorem 2.1. *For each $\phi \in C_\delta(V)$ and under assumptions (2), (3) and (f1-f2), there exists $\epsilon > 0$ such that in the interval $(-\infty, \tau + \epsilon)$ there is a unique solution of problem (1). In other words, there exists a function $u \in C((-\infty, \tau + \epsilon); H_0^1(\Omega))$ with $u(t, \tau; \phi) = \phi(t - \tau)$ for all $t \in (-\infty, \tau]$ which satisfies*

$$u(t, \tau; \phi) = \phi(0) + \int_\tau^t h(r, u_r) dr,$$

for all $t \in [\tau, \tau + \epsilon)$.

Proof. First of all we define the metric space where we apply the contraction mapping theorem. Then, for the given initial datum $\phi \in C_\delta(V)$, and for a time $T > 0$, we define the following space

$$X_\phi^T = \big\{ u \in C\left((-\infty, T); H_0^1(\Omega)\right) : u(t) = \phi(t - \tau) \text{ for all } \quad t \in (-\infty, \tau], $$

$$\text{and } \|u\|_{X_\phi^T} \leq 2\|\phi\|_\delta \big\}, \tag{8}$$

where $\|u\|_{X_\phi^T} = \sup_{\sigma \in (-\infty, T)} \|u(\sigma)\|$.

This space X_ϕ^T is a complete metric space (since it is a closed subset of a Banach space).

Now we consider the operator $\Phi : X_\phi^T \to X_\phi^T$ given by

$$\Phi(u)(t) = \begin{cases} \phi(t - \tau), & t \in (-\infty, \tau] \\ \phi(0) + \int_\tau^t h(r, u_r) dr, & t \in (\tau, T). \end{cases}$$

In [9] the well-possedness of Φ is proved.

Let us take $u, v \in X_\phi^T$. We have

$$\|\Phi(u)(t) - \Phi(v)(t)\| \leq \int_\tau^t \|h(r, u_r) - h(r, v_r)\| dr$$

$$\leq \int_\tau^t \|\tilde{A}(r)\|_{\mathcal{L}(H_0^1)} \|u(r) - v(r)\| dr$$

$$+ \int_\tau^t \|B(r)\left(g(u(r)) - g(v(r)) + f(r, u_r) - f(r, v_r)\right)\| dr.$$

Using the uniform bound in time for $\tilde{A}(t)$ and $B(t)$, that $B(t) \circ g$ is locally Lipschitz in $H_0^1(\Omega)$, and (f2), taking into account that $\|u(t)\|, \|v(t)\| \leq R$ for all $t \in [s, T]$ and that $\|u_r\|_\delta \leq 2\|\phi\|_\delta$, for all $\tau \leq r \leq t < T$, we obtain

$$\|\Phi(u)(t) - \Phi(v)(t)\| \leq K_1 \int_\tau^t \|u(r) - v(r)\| dr + K_2 \int_\tau^t \|f(r, u_r) - f(r, v_r)\| dr$$

$$\leq K_1 \int_\tau^t \|u(r) - v(r)\| dr$$

$$+ K(R) \int_\tau^t \sup_{\theta \in (-\infty, 0]} \|u(r + \theta) - v(r + \theta)\| dr.$$

Taking supremum in $[\tau, T)$ with $T = \tau + \epsilon$

$$\|\Phi(u) - \Phi(v)\|_{X_\phi^T} \leq K_1 \epsilon \|u - v\|_{X_\phi^T} + K(R)\epsilon \left(\sup_{r \in [\tau, T)} \sup_{\theta \in (-\infty, 0]} \|u(r + \theta) - v(r + \theta)\| \right),$$

but, if $u, v \in X_\phi^T$,

$$\sup_{r \in [\tau, T]} \sup_{\theta \in (-\infty, 0]} \|u(r + \theta) - v(r + \theta)\| = \sup_{r \in (-\infty, T)} \|u(r) - v(r)\| = \|u - v\|_{X_\phi^T}.$$

Therefore, for $\epsilon > 0$ small enough, Φ is well defined and is a contraction in X_ϕ.

Then, by the contraction mapping principle and the Banach fixed point theorem, there exists a unique fixed point for Φ, ensuring the existence of solution for (1). □

3. **Global solution and pullback absorbing family.** In this section we will prove that the local solution, whose existence has been proved in Theorem 2.1, is in fact a global one. The way to prove it will provide us also with the existence of pullback absorbing sets for the process generated by our model in the universe of the families with bounded union.

For any $\varphi \in H_0^1(\Omega)$, taking into account (2) and arguing as in [11], for each $\rho > 0$ there is a constant $K_\rho > 0$ such that

$$\begin{aligned}
\int_\Omega g(u)u &\leq \rho|u|^2 + K_\rho, \\
\int_\Omega G(u) &\leq \rho|u|^2 + K_\rho
\end{aligned} \tag{9}$$

for all $u \in L^2(\Omega)$, where $G(r) = \int_0^r g(\theta)d\theta$.

Let $L_b(\varphi)$ be the following energy functional

$$L_b(\varphi) = \frac{1}{2}\left(|\varphi|^2 + b\|\varphi\|^2\right) - b\int_\Omega G(\varphi), \tag{10}$$

with $b \geq 0$. It is easy to prove that for $\rho = \frac{\lambda_1}{6}$,

$$L_b(\varphi) \geq \frac{b}{3}\|\varphi\|^2 - bK_{\frac{\lambda_1}{6}} \tag{11}$$

and for any $\rho > 0$,

$$L_b(\varphi) \leq \frac{1 + b(\lambda_1 + 2\rho)}{2\lambda_1}\|\varphi\|^2 + bK_\rho, \tag{12}$$

with λ_1 the first eingenvalue of A.

Taking a solution $u(t, \tau; \phi)$ of (1) and for $b > 0$,

$$\begin{aligned}
\frac{d}{dt}L_b(u) &\leq -\left(1 - \frac{\gamma_1\varepsilon_1}{2} - \frac{2\rho + \varepsilon_2}{2\lambda_1}\right)\|u\|^2 + \frac{\varepsilon_2 + 1}{2\varepsilon_2}|f(t, u_t)|^2 \\
&\quad + \gamma(t)\left(\frac{1}{2\varepsilon_1} - b\right)\|\frac{du}{dt}\|^2 + K_\delta,
\end{aligned}$$

for $\varepsilon_1, \varepsilon_2, \rho > 0$. Taking $\varepsilon_1 = \frac{1}{4\gamma_1}$, $\varepsilon_2 = \frac{\lambda_1}{4}$, $\rho = \frac{\lambda_1}{8}$ and $b \geq \frac{1}{2\varepsilon_1} = 2\gamma_1$, we obtain

$$\begin{aligned}
\frac{d}{dt}L_b(u) &\leq -\frac{1}{2}\|u\|^2 + \left(\frac{\lambda_1 + 4}{2\lambda_1}\right)|f(t, u_t)|^2 + K_{\frac{\lambda_1}{8}} \\
&\leq -\left(\frac{\lambda_1}{1 + b(\lambda_1 + 2\tilde{\rho})}\right)L_b(u) + \left(\frac{\lambda_1 + 4}{2\lambda_1}\right)|f(t, u_t)|^2 \\
&\quad + \left(\frac{\lambda_1}{1 + b(\lambda_1 + 2\tilde{\rho})}\right)K_{\tilde{\rho}} + K_{\frac{\lambda_1}{8}},
\end{aligned}$$

where $\tilde{\rho}$ is a fixed positive constant.

Denoting $C_b = \left(\frac{\lambda_1}{1+b(\lambda_1+2\tilde{\rho})}\right)$, $C_{\lambda_1} = \left(\frac{\lambda_1+4}{2\lambda_1}\right)$ and $\tilde{K}_b = C_b K_{\tilde{\rho}} + K_{\frac{\lambda_1}{8}}$, we have that

$$\frac{d}{dt}\left(e^{C_b t} L_b(u)\right) \leq e^{C_b t}(C_{\lambda_1}|f(t,u_t)|^2 + \tilde{K}_b).$$

Integrating between τ and t, $t \geq \tau$, and using the hypothesis f3),

$$e^{C_b t} L_b(u(t)) \leq e^{C_b \tau} L_b(\phi(0)) + \lambda_1^{-1} C_{\lambda_1} C_f \int_\tau^t e^{C_b r} \|u_r\|_\delta^2 dr + \lambda_1^{-1} C_{\lambda_1} \int_\tau^t e^{C_b r} \psi(r) dr$$

$$+ \frac{\tilde{K}_b}{C_b}\left(e^{C_b t} - e^{C_b \tau}\right).$$

Taking into account (11) and (12), we obtain

$$\frac{b}{3} e^{C_b t} \|u(t)\|^2 \leq e^{C_b \tau}(\tilde{C}_b \|\phi(0)\|^2 + bK_{\rho_2}) + \lambda_1^{-1} C_{\lambda_1} C_f \int_\tau^t e^{C_b r} \|u_r\|_\delta^2 dr$$

$$+ \lambda_1^{-1} C_{\lambda_1} \int_\tau^t e^{C_b r} \psi(r) dr + e^{C_b t}\left(\frac{\tilde{K}_b}{C_b} + bK_{\frac{\lambda_1}{6}}\right),$$

where $\rho_2 > 0$ is chosen and

$$\tilde{C}_b = \frac{1 + b(\lambda_1 + 2\rho_2)}{2\lambda_1}.$$

Consequently, if $t \geq \tau$, we have

$$e^{C_b t}\|u(t)\|_\delta^2 \leq \max\left\{ \sup_{\theta \in (-\infty, \tau-t]} e^{C_b t} e^{2\delta\theta} \|\phi(t+\theta-\tau)\|^2, \right.$$

$$\sup_{\theta \in [\tau-t, 0]} \frac{3}{b} e^{C_b \tau} e^{(2\delta - C_b)\theta}(\tilde{C}_b \|\phi(0)\|^2 + bK_{\rho_2})$$

$$+ \frac{3}{b}\lambda_1^{-1} C_{\lambda_1} C_f e^{(2\delta - C_b)\theta} \int_\tau^{t+\theta} e^{C_b r} \|u_r\|_\delta^2 dr$$

$$+ \frac{3}{b}\lambda_1^{-1} C_{\lambda_1} e^{(2\delta - C_b)\theta} \int_\tau^{t+\theta} e^{C_b r} \psi(r) dr$$

$$\left. + \frac{3}{b} e^{(2\delta - C_b)\theta} e^{C_b t}\left(\frac{\tilde{K}_b}{C_b} + 2K_{\frac{\lambda_1}{6}}\right)\right\},$$

and, taking $2\delta > C_b$, we deduce

$$\frac{b}{3} e^{C_b t} \|u_t\|_\delta^2 \leq e^{C_b \tau}\left(\tilde{C}_b \|\phi\|_\delta^2 + bK_{\rho_2}\right) + \lambda_1^{-1} C_{\lambda_1} \int_\tau^t e^{C_b r} \psi(r) dr$$

$$+ e^{C_b t}\left(\frac{\tilde{K}_b}{C_b} + bK_{\frac{\lambda_1}{6}}\right) + \lambda_1^{-1} C_{\lambda_1} C_f \int_\tau^t e^{C_b r} \|u_r\|_\delta^2 dr.$$

Assuming that

$$\frac{3}{b}\lambda_1^{-1} C_{\lambda_1} C_f < C_b \tag{13}$$

and calling $\beta = \frac{3}{b}\lambda_1^{-1} C_{\lambda_1} C_f$ (it means $\beta < C_b$) and

$$\alpha(t) = \frac{3}{b} e^{C_b \tau}\left(\tilde{C}_b \|\phi\|_\delta^2 + bK_{\rho_2}\right) + \frac{3}{b}\lambda_1^{-1} C_{\lambda_1} \int_\tau^t e^{C_b r} \psi(r) dr + \frac{3}{b} e^{C_b t}\left(\frac{\tilde{K}_b}{C_b} + bK_{\frac{\lambda_1}{6}}\right),$$

by the Gronwall Lemma we obtain that

$$e^{C_b t}\|u_t\|_\delta^2 \leq \alpha(t) + \beta \int_\tau^t \alpha(r) e^{\beta(t-r)} dr.$$

Now,

$$\beta \int_\tau^t \alpha(r) e^{\beta(t-r)} dr \leq \frac{3}{b} e^{C_b \tau} e^{\beta(t-\tau)} \left(\tilde{C}_b \|\phi\|_\delta^2 + bK_{\rho_2} \right)$$

$$+ \frac{3}{b} \frac{\beta}{C_b - \beta} \left(\frac{\tilde{K}_b}{C_b} + bK_{\frac{\lambda_1}{6}} \right) e^{C_b t}$$

$$+ \frac{3}{b} \beta \lambda_1^{-1} C_{\lambda_1} e^{\beta t} \int_\tau^t e^{(C_b - \beta) r} \psi(r) dr.$$

Then,

$$\|u_t\|_\delta^2 \leq e^{-C_b t} \alpha(t) + \frac{3}{b} e^{(C_b - \beta)(\tau - t)} \left(\tilde{C}_b \|\phi\|_\delta^2 + bK_{\rho_2} \right) + \frac{3}{b} \frac{\beta}{C_b - \beta} \left(\frac{\tilde{K}_b}{C_b} + bK_{\frac{\lambda_1}{6}} \right)$$

$$+ \frac{3}{b} \beta \lambda_1^{-1} C_{\lambda_1} e^{-(C_b - \beta) t} \int_\tau^t e^{(C_b - \beta) r} \psi(r) dr.$$

Assuming that there exists a $\eta_0 \geq 0$ such that for any $\eta \in [0, \eta_0]$,

$$\int_{-\infty}^t e^{\eta r} \psi(r) dr < +\infty, \tag{14}$$

we have

$$\|u_t\|_\delta^2 \xrightarrow{\tau \to -\infty} l(t), \tag{15}$$

where

$$l(t) = \frac{3}{b} \left(\frac{\tilde{K}_b}{C_b} + bK_{\frac{\lambda_1}{6}} \right) \left(1 + \frac{\beta}{C_b - \beta} \right)$$

$$+ \frac{3}{b} \lambda_1^{-1} C_{\lambda_1} \left(\int_{-\infty}^t e^{C_b r} \psi(r) dr + \beta e^{-(C_b - \beta) t} \int_{-\infty}^t e^{(C_b - \beta) r} \psi(r) dr \right).$$

Then, we have the global existence of any solution $u(t, \tau; \phi)$ of (1), i.e. for each $\phi \in C_\delta(V)$, $u(\cdot, \tau; \phi) \in C((-\infty, +\infty), H_0^1(\Omega))$ in Theorem 2.1, and, once we justify that the solutions of our problem generate a non-autonomous dynamical system, this also ensures the existence of a family of closed subsets $\{\overline{B}_{C_\delta(V)}(0, l^{1/2}(t)) : t \in \mathbb{R}\}$ which pullback attracts bounded subsets of $C_\delta(V)$.
For a more detailed proof of this result the reader is referred to [9].

We also need a result on the continuous dependence on the initial data.

Proposition 1. *Under assumptions of Theorem 2.1, any solution $u(t, \tau; \phi)$ of (1) is continuous with respect to the initial condition $\phi \in C_\delta(V)$. More precisely, if u^i, for $i = 1, 2$, are the corresponding solutions to the initial data $\phi^i \in C_\delta(V)$, $i = 1, 2$, the following estimate holds:*

$$\max_{r \in [\tau, t]} \|u^1(r) - u^2(r)\| \leq \left(\|\phi^1(0) - \phi^2(0)\| + \frac{\tilde{L}(R)}{\delta} \|\phi^1 - \phi^2\|_\delta \right) e^{\tilde{L}(R)(t-\tau)}, \tag{16}$$

for all $t \in [\tau, T]$, where $\tilde{L}(R) = \sup_{t \in \mathbb{R}} \|\tilde{A}(t)\|_{\mathcal{L}(H_0^1)} + L_g(R) + b_0 L_f(R)$ and $R \geq 0$ is given by

$$R = \max(2\|\phi^1\|_\delta, 2\|\phi^2\|_\delta).$$

The detailed proof of this result can be found in [9].

4. **Existence of the pullback attractor.** In this section we will prove the existence of the pullback attractor in the universe $\mathcal{D}_b$ of all families with bounded union, that is, the family $\{D(t) : t \in \mathbb{R}\}$ is in $\mathcal{D}_b$ if and only if $\bigcup\{D(t) : t \in \mathbb{R}\}$ is bounded in $C_\delta(V)$.

Assuming that $f(t,0) = 0$ for all $t \in \mathbb{R}^1$, by estimates in Section 3, there exists a pullback $\mathcal{D}_b-$absorbing family $\widehat{B}_0 = \{B_0(t) : t \in \mathbb{R}\}$ in $\mathcal{D}_b$.

By the previous results, will be able to construct a process $S : C_\delta(V) \to C_\delta(V)$ associated to (1), and can prove the existence of a pullback attractor for such process $S(t,\tau)$ in $C_\delta(V)$ defined as

$$S(t,\tau)\phi = u_t(\cdot;\tau,\phi) \quad \forall t \geq \tau,$$

where $\phi \in C_\delta(V)$ and $\tau \in \mathbb{R}$.

It is not difficult to prove that $S(\cdot,\cdot)$ is a process and we can also write

$$\begin{aligned}
(S(t,\tau)\phi)(\theta) &= u_t(\theta;\tau,\phi) \\
&= u(t+\theta;\tau,\phi) \\
&= T(t+\theta,\tau)\phi(0) + \int_\tau^{t+\theta} T(t+\theta,s)\tilde{f}(s,u_s)ds \\
&\quad + \int_\tau^{t+\theta} T(t+\theta,s)\tilde{g}(s,u)ds,
\end{aligned} \tag{17}$$

for all $t \geq \tau$ and $\theta \in (-\infty,0]$, where $T(t,\tau)$ is the evolution process associated to (1) with $f = 0$ and $g = 0$.

The following result gives a characterization of asymptotically compact processes, useful in order to prove the existence of the pullback attractor.

Theorem 4.1. *Let $\{S(t,\tau) : t \geq \tau\}$ be a process such that $S(t,\tau) = T(t,\tau) + U(t,\tau)$, where $U(t,\tau)$ is compact and there exists a non-increasing function*

$$k : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$$

with $k(\sigma,r) \to 0$ when $\sigma \to \infty$, and for all $\tau \leq t$ and $x \in C_H$ with $\|x\|_{C_H} \leq r$, $\|T(t,\tau)x\|_{C_H} \leq k(t-\tau,r)$. Then, $\{S(t,\tau) : t \geq \tau\}$ is $\mathcal{D}_b$-pullback asymptotically compact.

Proof. Using the fact that any family $\widehat{D}$ of $\mathcal{D}_b$ has bounded union, the result follows from Theorem 2.8 in [6]. $\qquad\square$

Since

$$T(t+\theta,\tau)\phi(0) + \int_\tau^{t+\theta} T(t+\theta,s)\tilde{f}(s,u_s)ds$$

tends to zero exponentially in $C_\delta(V)$ (this fact easily follows by arguing as in Section 3, taking into account that $f(t,0) = 0$ for all $t \geq \tau$), we only need to prove that $U(t,\tau)B$ defined as

$$(U(t,\tau)\phi)(\theta) = \int_\tau^{t+\theta} T(t+\theta,s)\tilde{g}(s,u(s,\tau;\phi))ds,$$

is relatively compact for any bounded subset $B \subset C_\delta(V)$,.

---

[1]This is not a real restriction as we can subtract such term to $f(t,\cdot)$ and add it to the term g.

To this end, for any bounded subset $D \subset C_\delta(V)$, $U(t, \tau)D$ is pre-compact in $C_\delta(V)$ for any $t \geq \tau$, and to prove this we will apply the Azcoli-Arzelà theorem, (see [8] for more details)

i) $U(t, \tau)D$ is bounded, $\forall t \geq \tau$.

ii) For each $\theta \in [-h, 0]$, $\overline{\bigcup_{\phi \in D} (U(t, \tau)\phi)(\theta)}$ is a compact subset of $H_0^1(\Omega)$.

iii) The set $U(t, \tau)D$ is equicontinuous (i.e., from all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|\theta_1 - \theta_2| \leq \delta$, then $\|(U(t, \tau)\phi)(\theta_1) - (U(t, \tau)\phi)(\theta_2)\| \leq \varepsilon$, for all $t \geq \tau$, and for all $\phi \in D$),

Assertion i) follows from the same estimates obtained in the proof of the existence of the absorbing family and ii) is a consequence of the same analysis carried out in [22], just using the fact that $\rho < \frac{n+2}{n-2}$ and, for any $\eta \in \left(\frac{1}{2}, 1\right)$, we have the following chain of inclusions:

$$H_0^1 \hookrightarrow L^{\frac{2n}{n-2}} \xrightarrow{g} L^{\frac{2n}{n+2\eta}} \hookrightarrow H^{-\eta} \subset\subset H^{-1} \xrightarrow{B(t)} H_0^1.$$

Finally, to prove iii) we need to estimate

$$\left| \int_\tau^{t+\theta_1} T(t+\theta_1, s)\tilde{g}(s, u(s, \tau; \phi))ds - \int_\tau^{t+\theta_2} T(t+\theta_2, s)\tilde{g}(s, u(s, \tau; \phi))ds \right|.$$

Taking into account (3), the exponencial decay of $\|T(t, \tau)\|_{\mathcal{L}(C_H)}$, the fact that any solution of (1) is in $C((-\infty, T), H_0^1(\Omega))$, for all $T > \tau$ and the uniform bound of operator $\tilde{A}(t)$, following the ideas of [8] we obtain that

$$\int_\tau^{t+\theta_1} |(T(t+\theta_1, s) - T(t+\theta_2, s))\tilde{g}(s, u(s, \tau; \phi))| \, ds \leq C |\theta_1 - \theta_2|,$$

for a certain positive constant $C \in \mathbb{R}$.

Therefore, by Theorem 4.1 and Theorem 1.6 there exists the $\mathcal{D}_b$-pullback attractor for our evolution process $S(t, \tau)$.

5. **Stationary solutions and their stability.** In this section we will prove that, under additional assumptions, there exists a unique stationary solution of problem (1) which is globally asymptotically exponentially stable.

From now on we assume that $f : \mathbb{R} \times C_\delta(V) \to V$ satisfies f1)–f3) with $\psi(t) = |\psi| \geq 0$ for all $t \geq \tau$, a constant function.

We also suppose that f is autonomous, in the sense that there exists a function $f_0 : V \to V$ such that

f4) $f(t, w) = f_0(w)$ for all $(t, w) \in [\tau, \infty) \times V$,

where, with a slight abuse of notation, we identify every element $w \in V$ with the constant function in $C_\delta(V)$ which is equal to w for any time $t \in (-\infty, \tau]$.
Moreover, we assume function g is globally Lipschitz in $\mathbb{R}$, with C_g the Lipschitz constant.

We consider the following equation,

$$\frac{d}{dt}(u) + \gamma(t)\frac{d}{dt}(Au) + Au = g(u) + f_0(u_t) \quad t > \tau. \tag{18}$$

A stationary solution to (18) will be an element $u^* \in V$ such that

$$((u^*, v)) = (g(u^*), v) + (f_0(u^*), v) \quad \forall v \in V. \tag{19}$$

Theorem 5.1.

a) *Under the above assumptions and notation, the problem (18) admits at least one stationary solution u^* (which indeed belongs to $D(A)$) if $\lambda_1 > C_g + C_f^{1/2}$. Moreover, any such stationary solution satisfies the estimate*

$$(\lambda_1 - C_g - C_f^{1/2})\|u^*\| \le \|g(0)\| + |\psi|^{1/2}. \tag{20}$$

b) *If we have also*

$$\lambda_1 > C_g + L_f(R) \tag{21}$$

where $R = \dfrac{\|g(0)\| + |\psi|^{1/2}}{\lambda_1 - C_g - C_f^{1/2}}$, then the stationary solution is unique.

Proof. First, we will obtain the estimate (20). If u^* is a stationary solution, it must verify

$$((u^*, v)) = (g(u^*), v) + (f_0(u^*), v) \quad \forall v \in V,$$

and, therefore, taking into account f3),

$$
\begin{aligned}
\|u^*\|^2 &\le \ |g(u^*) - g(0)||u^*| + |g(0)||u^*| + |f_0(u^*)||u^*| \\
&\le \ \lambda_1^{-1/2}C_g\|u^*\||u^*| + |g(0)||u^*| + \lambda_1^{-1/2}(C_f^{1/2}\|u^*\|_\delta + |\psi|^{1/2})|u^*| \\
&\le \ \lambda_1^{-1}C_g\|u^*\|^2 + \lambda_1^{-1}\|g(0)\|\|u^*\| + \lambda_1^{-1/2}(C_f^{1/2}\|u^*\| + |\psi|^{1/2})|u^*| \\
&\le \ \lambda_1^{-1}C_g\|u^*\|^2 + \lambda_1^{-1}\|g(0)\|\|u^*\| + \lambda_1^{-1}C_f^{1/2}\|u^*\|^2 + \lambda_1^{-1}|\psi|^{1/2}\|u^*\|
\end{aligned}
$$

Now, it is easy to deduce (20).

As for the existence, let us consider $\{v_j\} \subset V$, the orthonormal basis of H formed by all the eigenfunctions of the operator A. For each integer $m \ge 1$, let us denote again $V_m = \text{span}[v_1, \ldots, v_m]$, with the inner product $((\cdot, \cdot))$ and norm $\|\cdot\|$. Define the operators $R_m : V_m \to V_m$, $m \ge 1$, by

$$((R_m u, v)) = ((u, v)) - (g(u), v) - (f_0(u), v), \quad \forall u, v \in V_m. \tag{22}$$

Since the right hand side is a continuous linear map from V_m to $\mathbb{R}$, by the Riesz theorem, each $R_m u \in V_m$ is well defined. We check now that R_m is continuous.

$$
\begin{aligned}
((R_m u - R_m \widetilde{u}, v)) &= \ ((u - \widetilde{u}, v)) - (g(u) - g(\widetilde{u}), v) - (f_0(u) - f_0(\widetilde{u}), v) \\
&\le \ \|u - \widetilde{u}\|\,\|v\| + \lambda_1^{-1}C_g\,\|u - \widetilde{u}\|\,\|v\| + \lambda_1^{-1/2}L_f(R)\|u - \widetilde{u}\|_\delta|v| \\
&\le \ (1 + \lambda_1^{-1}C_g + \lambda_1^{-1}L_f(R))\,\|u - \widetilde{u}\|\,\|v\|, \tag{23}
\end{aligned}
$$

for all $u, \widetilde{u}, v \in V_m$, where $R = max\{\|u\|, \|\widetilde{u}\|\}$.

Therefore,

$$\|R_m u - R_m \widetilde{u}\| \le \left(1 + \lambda_1^{-1}C_g + \lambda_1^{-1}L_f(R)\right)\|u - \widetilde{u}\|,$$

for all $u, \widetilde{u}$.

On the other hand, for all $u \in V_m$,

$$
\begin{aligned}
((R_m u, u)) &= \ ((u, u)) - (g(u), u) - (f_0(u), u) \\
&\ge \ \|u\|^2 - \lambda_1^{-1}C_g\|u\|^2 - \lambda_1^{-1}\|g(0)\|\|u\| - \lambda_1^{-1}|\psi|^{1/2}\|u\| - \lambda_1^{-1}C_f^{1/2}\|u\|^2.
\end{aligned}
$$

Thus, if we take

$$\beta = \frac{\lambda_1^{-1}\|g(0)\| + \lambda_1^{-1}|\psi|^{1/2}}{1 - \lambda_1^{-1}C_g - \lambda_1^{-1}C_f^{1/2}} = \frac{\|g(0)\| + |\psi|^{1/2}}{\lambda_1 - C_g - C_f^{1/2}},$$

we obtain $((R_m u, u)) \geq 0$ for all $u \in V_m$ such that $\|u\| = \beta$.

Consequently, by a corollary of the Brouwer fixed point theorem (see [17, p. 53]), for each $m \geq 1$ there exists $u_m \in V_m$ such that $R_m(u_m) = 0$, with $\|u_m\| \leq \beta$.

Observe moreover that $Au_m \in V_m$, and therefore

$$
\begin{aligned}
|Au_m|^2 &= (g(u_m), Au_m) + (f_0(u_m), Au_m) \qquad (24) \\
&\leq \frac{1}{2}|Au_m|^2 + 2\lambda_1^{-1}C_g^2\|u_m\|^2 + 2|g(0)|^2 + \lambda_1^{-1}(C_f\|u_m\|^2 + |\psi|).
\end{aligned}
$$

From (24), for all u_m such that $\|u_m\| \leq \beta$, we deduce that the sequence $\{u_m\}$ is bounded in $D(A)$, and consequently, by the compact injection of $D(A)$ in V, we can extract a subsequence $\{u_{m'}\} \subset \{u_m\}$, which converges, weakly in $D(A)$ and strongly in V, to an element $u^* \in D(A)$. It is now standard to take limits in (22) and to obtain that u^* is a stationary solution.

Uniqueness

Let us suppose that u^* and $\widetilde{u}^*$ are two stationary solutions of (18). Then,

$$((u^* - \widetilde{u}^*, v)) = (g(u^*) - g(\widetilde{u}^*), v) + (f_0(u^*) - f_0(\widetilde{u}^*), v), \quad \forall v \in V, \ t > 0. \quad (25)$$

Taking $v = u^* - \widetilde{u}^*$ and proceeding as in (23) we obtain from (25)

$$\|u^* - \widetilde{u}^*\|^2 \leq (\lambda_1^{-1}C_g + \lambda_1^{-1}L_f(R))\|u^* - \widetilde{u}^*\|^2,$$

where $R = \dfrac{\|g(0)\| + |\psi|^{1/2}}{\lambda_1 - C_g - C_f^{1/2}}$.

Then, it is obvious that $u^* = \widetilde{u}^*$ if condition (21) is satisfied.

$\square$

Theorem 5.2. *Assume that f1)$-$f4) hold with ψ time-independent and that we have $\lambda_1 > C_g + C_f^{1/2}$, and (21) is fulfilled. Then, there exists a value $0 < \lambda < 2\delta$ such that for the solution $u(\cdot, \tau, \phi)$ of (1) and $\phi \in C_\delta(V)$, the following estimates hold for all $t \geq \tau$:*

a) *If function f is globally lipschitz, i.e., $L_f(R) = L_f$, then*

$$|u(t,\tau,\phi) - u^*|^2 \leq e^{-\lambda t}\left(|\phi(0) - u^*|^2 + \tilde{\gamma}\|\phi(0) - u^*\|^2 + \frac{e^{2\delta\tau}\lambda_1^{-1}L_f}{2\delta - \lambda}\|\phi - u^*\|_\delta^2\right),$$
$$(26)$$

$$
\begin{aligned}
&\|u_t(\cdot,\tau,\phi) - u^*\|_\delta^2 \\
&\leq \max\Big\{e^{-2\delta(t-\tau)}\|\phi - u^*\|_\delta^2, \\
&\quad \frac{e^{-\lambda t}}{\tilde{\gamma}}\left(|\phi(0) - u^*|^2 + \tilde{\gamma}\|\phi(0) - u^*\|^2 + \frac{e^{2\delta\tau}\lambda_1^{-1}L_f}{2\delta - \lambda}\|\phi - u^*\|_\delta^2\right)\Big\}, \ (27)
\end{aligned}
$$

b) *Assume that $L_f(R)$ is a continuous function of R, and there exists $0 < \mu < \frac{2\lambda_1 - 2C_g}{\lambda_1\tilde{\gamma} + 1}$ such that $\mu\lambda_1\tilde{\gamma}(2\lambda_1 - \mu\tilde{\gamma}\lambda_1 - \mu - 2C_g) > 2C_f$, and*

$$\lambda_1 > L_f(\widetilde{R}), \qquad (28)$$

where $\widetilde{R}$ is the positive number given by

$$
\begin{aligned}
\widetilde{R}^2 &= \max\Big\{2\lambda_1^{-1}\tilde{\gamma}^{-1}(2\lambda_1 - \mu\tilde{\gamma}\lambda_1 - \mu - 2C_g)^{-1} \times \\
&\quad \times(\mu - (2\lambda_1^{-1}\tilde{\gamma}^{-1}C_f(2\lambda_1 - \mu\tilde{\gamma}\lambda_1 - \mu - 2C_g)^{-1})^{-1}(\|g(0)\|^2 + |\psi|), R^2\Big\},
\end{aligned}
$$

with R defined by (21). Then, there exists a value $0 < \lambda < 2\delta$ such that for each $\phi \in C_\delta(V)$, there exists $T_\phi > \tau$ such that:

$$|u(t, \tau, \phi) - u^*|^2 \le e^{-\lambda t} \left(|\phi(0) - u^*|^2 + \tilde{\gamma}\|\phi(0) - u^*\|^2 + \frac{e^{2\delta\tau}\lambda_1^{-1}L_f(\widetilde{R})}{2\delta - \lambda}\|\phi - u^*\|_\delta^2 \right),$$
(29)

$$\|u_t(\cdot, \tau, \phi) - u^*\|_\delta^2$$
$$\le \max\left\{ e^{-2\delta(t-\tau)}\|\phi - u^*\|_\delta^2, \right.$$
$$\left. \frac{e^{-\lambda t}}{\tilde{\gamma}} \left(|\phi(0) - u^*|^2 + \tilde{\gamma}\|\phi(0) - u^*\|^2 + \frac{e^{2\delta\tau}\lambda_1^{-1}L_f(\widetilde{R})}{2\delta - \lambda}\|\phi - u^*\|_\delta^2 \right) \right\}$$
(30)

for all $t \ge T_\phi$, where u^* is the unique stationary solution of (18) given by Theorem 5.1.

Proof. For short we denote $u(t) = u(\cdot, \tau, \phi)$. Let us also denote $w(t) = u(t) - u^*$. Considering equations (18) for $u(t)$ and (19) for u^*, one has

$$\frac{d}{dt}(w(t), v) + \gamma(t)\frac{d}{dt}((w(t), v)) + ((w(t), v)) = (g(u(t)) - g(u^*), v) + (f(t, u_t) - f_0(u^*), v),$$

for $t > \tau$, for any $v \in V$.

Then, taking into account that $0 < \gamma_0 < \gamma(t) < \gamma_1 < \infty$ for all t, we have that

$$\frac{d}{dt}|w(t)|^2 + \tilde{\gamma}\frac{d}{dt}\|w(t)\|^2 + 2\|w(t)\|^2$$
$$\le \frac{d}{dt}|w(t)|^2 + \gamma(t)\frac{d}{dt}\|w(t)\|^2 + 2\|w(t)\|^2$$
$$\le 2|g(u(t) - g(u^*)||w(t)| + 2|f(t, u_t) - f_0(u^*)||w(t)|,$$

for $t > \tau$, where $\tilde{\gamma} = \gamma_0$ or $\tilde{\gamma} = \gamma_1$ depending on if $\frac{d}{dt}\|w(t)\|^2$ is positive o negative. Therefore,

$$\frac{d}{dt}(|w(t)|^2 + \tilde{\gamma}\|w(t)\|^2) \le -2\|w(t)\|^2 + 2|g(u(t) - g(u^*)||w(t)| + 2|f(t, u_t) - f_0(u^*)||w(t)|,$$

for $t > \tau$.

Case a): We assume that f is globally Lipschitz.
From energy equality and the Lipschitz condition on g and f, and introducing an exponential term $e^{\lambda t}$ with a positive value λ to be fixed later on, we obtain

$$\frac{d}{dt}(e^{\lambda t}(|w(t)|^2 + \tilde{\gamma}\|w(t)\|^2)) \le e^{\lambda t}\left(\lambda(|w(t)|^2 + \tilde{\gamma}\|w(t)\|^2) - 2\|w(t)\|^2 \right.$$
$$+ 2|g(u(t) - g(u^*)||w(t)| + 2|f(t, u_t) - f_0(u^*)||w(t)|)$$
$$\le e^{\lambda t}\left(\lambda(|w(t)|^2 + \tilde{\gamma}\|w(t)\|^2) - 2\|w(t)\|^2 \right.$$
$$\left. + 2\lambda_1^{-1}C_g\|w(t)\|^2 + 2\lambda_1^{-1/2}L_f\|w_t\|_\delta|w(t)| \right),$$

for $t > \tau$.

Hence, using the Young inequality with $\epsilon > 0$ to be fixed later on, we conclude that

$$\frac{d}{dt}(e^{\lambda t}(|w(t)|^2 + \tilde{\gamma}\|w(t)\|^2)) \leq e^{\lambda t}(\lambda\lambda_1^{-1} + \lambda\tilde{\gamma} - 2 + 2C_g\lambda_1^{-1} + \epsilon\lambda_1^{-1}L_f)\|w(t)\|^2$$
$$+ \frac{\lambda_1^{-1}L_f}{\epsilon}e^{\lambda t}\|w_t\|_\delta^2.$$

Therefore, integrating from τ to t, we have

$$e^{\lambda t}(|w(t)|^2 + \tilde{\gamma}\|w(t)\|^2) \leq |w(\tau)|^2 + \tilde{\gamma}\|w(\tau)\|^2 + \frac{\lambda_1^{-1}L_f}{\epsilon}\int_\tau^t e^{\lambda s}\|w_s\|_\delta^2 ds$$

$$+ (\lambda\lambda_1^{-1} + \lambda\tilde{\gamma} - 2 + 2C_g\lambda_1^{-1} + \epsilon\lambda_1^{-1}L_f)\int_\tau^t e^{\lambda s}\|w(s)\|^2 ds. \tag{31}$$

In order to control the term $\int_\tau^t e^{\lambda s}\|w_s\|_\delta^2 ds$, we proceed as follows.

$$\int_\tau^t e^{\lambda s}\sup_{\theta\leq 0}e^{2\delta\theta}\|w(s+\theta)\|^2 ds$$

$$= \int_\tau^t e^{\lambda s}\max\{\sup_{\theta\leq\tau-s}e^{2\delta\theta}\|w(s+\theta)\|^2, \sup_{\theta\in[\tau-s,0]}e^{2\delta\theta}\|w(s+\theta)\|^2\}ds$$

$$= \int_\tau^t \max\{e^{2\delta\tau}e^{-(2\delta-\lambda)s}\|\phi-u^*\|_\delta^2, \sup_{\theta\in[\tau-s,0]}e^{(2\delta-\lambda)\theta}e^{\lambda(s+\theta)}\|w(s+\theta)\|^2\}ds.$$

So, if $0 < \lambda < 2\delta$, using the above equality in (31), we obtain

$$e^{\lambda t}(|w(t)|^2 + \tilde{\gamma}\|w(t)\|^2)$$

$$\leq |w(\tau)|^2 + \tilde{\gamma}\|w(\tau)\|^2 + \frac{e^{2\delta\tau}\lambda_1^{-1}L_f}{\epsilon}\|\phi-u^*\|_\delta^2\int_\tau^t e^{(\lambda-2\delta)s}ds$$

$$+ (\lambda\lambda_1^{-1} + \lambda\tilde{\gamma} - 2 + 2C_g\lambda_1^{-1} + \epsilon\lambda_1^{-1}L_f + \frac{\lambda_1^{-1}L_f}{\epsilon})\int_\tau^t \max_{r\in[\tau,s]}e^{\lambda r}\|w(r)\|^2 ds.$$

Observe that the (optimal) choice of $\epsilon = 1$ makes $\epsilon\lambda_1^{-1}L_f + L_f(\lambda_1\epsilon)^{-1}$ be minimal and the coefficient of the last integral is negative with a suitable choice of $\lambda \in (0, 2\delta)$ by (21). So, we can omit this term and deduce that

$$e^{\lambda t}(|w(t)|^2 + \tilde{\gamma}\|w(t)\|^2) \leq |w(\tau)|^2 + \tilde{\gamma}\|w(\tau)\|^2$$

$$+ \frac{e^{2\delta\tau}\lambda_1^{-1}L_f}{2\delta-\lambda}(1 - e^{(\lambda-2\delta)t})\|\phi-u^*\|_\delta^2 \tag{32}$$

whence (29) follows.

Finally, (30) can be deduced in the following way:

$$\|w_t\|_\delta^2 = \sup_{\theta\leq 0}e^{2\delta\theta}\|w(t+\theta)\|^2$$

$$= \max\{\sup_{\theta\in(-\infty,\tau-t]}e^{2\delta\theta}\|\phi(t+\theta-\tau)-u^*\|^2, \max_{\theta\in[\tau-t,0]}e^{2\delta\theta}\|w(t+\theta)\|^2\}$$

$$= \max\{e^{-2\delta(t-\tau)}\|\phi-u^*\|_\gamma^2, \max_{\theta\in[\tau-t,0]}e^{2\delta\theta}\|w(t+\theta)\|^2\},$$

and the second term can be estimated using (32) and that $e^{(2\delta-\lambda)\theta} \leq 1$.

Case b): Now, proceeding in the same way

$$\frac{d}{dt}(e^{\mu t}[|u(t)|^2 + \tilde{\gamma}\|u(t)\|^2])$$

$$\leq \mu e^{\mu t}(|u(t)|^2 + \tilde{\gamma}\|u(t)\|^2)$$
$$+ e^{\mu t}(-2\|u(t)\|^2 + 2|g(u)||u(t)| + 2|f(t,u_t)||u(t)|)$$
$$\leq e^{\mu t}(\mu\lambda_1^{-1} + \mu\tilde{\gamma} - 2 + 2C_g\lambda_1^{-1})\|u(t)\|^2 + 2e^{\mu t}(|g(0)| + |f(t,u_t)|)|u(t)|$$
$$\leq 2e^{\mu t}\lambda_1^{-1}\left(2 - \mu\lambda_1^{-1} - \mu\tilde{\gamma} - 2C_g\lambda_1^{-1}\right)^{-1}(|g(0)|^2 + |f(t,u_t)|^2),$$

a.e. $t > \tau$, and therefore, by f3),

$$\frac{d}{dt}(e^{\mu t}[|u(t)|^2 + \tilde{\gamma}\|u(t)\|^2]) \leq 2\lambda_1^{-1}C_f\left(2\lambda_1 - \mu\tilde{\gamma}\lambda_1 - \mu - 2C_g\right)^{-1}e^{\mu t}\|u_t\|_\delta^2$$
$$+ 2\lambda_1^{-1}\left(2\lambda_1 - \mu\tilde{\gamma}\lambda_1 - \mu - 2C_g\right)^{-1}e^{\mu t}(\|g(0)\|^2 + |\psi|) \text{ a.e. } t > \tau.$$

Integrating this last inequality, we obtain

$$e^{\mu t}[|u(t)|^2 + \tilde{\gamma}\|u(t)\|^2] \leq e^{\mu\tau}[|\phi(0)|^2 + \tilde{\gamma}\|\phi(0)\|^2]$$

$$+ 2\lambda_1^{-1}\left(2\lambda_1 - \mu\tilde{\gamma}\lambda_1 - \mu - 2C_g\right)^{-1}(\|g(0)\|^2 + |\psi|)\int_\tau^t e^{\mu s}\,ds$$

$$+ 2\lambda_1^{-1}C_f\left(2\lambda_1 - \mu\tilde{\gamma}\lambda_1 - \mu - 2C_g\right)^{-1}\int_\tau^t e^{\mu s}\|u_s\|_\delta^2\,ds$$

for all $t \geq \tau$.

Now,

$$
\begin{aligned}
e^{\mu t}\|u_t\|_\delta^2 &= e^{\mu t}\sup_{\theta \leq 0}e^{2\delta\theta}\|u(t+\theta)\|^2 \\
&= e^{\mu t}\max\{\sup_{\theta \in (-\infty,\tau-t]}e^{2\delta\theta}\|\phi(t+\theta-\tau)\|^2, \sup_{\theta \in [\tau-t,0]}e^{2\delta\theta}\|u(t+\theta)\|^2\} \\
&= \max\{e^{2\delta\tau}e^{-(2\delta-\mu)t}\|\phi\|_\delta^2, \sup_{r \in [\tau,t]}e^{(2\delta-\mu)(r-t)}e^{\mu r}\|u(r)\|^2\},
\end{aligned}
$$

Thus, taking $0 < \mu < 2\delta$, it is easy to deduce that

$$
\begin{aligned}
e^{\mu t}\|u_t\|_\delta^2 &\leq \tilde{\gamma}^{-1}e^{2\delta\tau}[|\phi(0)|^2 + \tilde{\gamma}\|\phi\|_\delta^2] \\
&+ 2\lambda_1^{-1}\tilde{\gamma}^{-1}\left(2\lambda_1 - \mu\tilde{\gamma}\lambda_1 - \mu - 2C_g\right)^{-1}(\|g(0)\|^2 + |\psi|)\int_\tau^t e^{\mu s}\,ds \\
&+ 2\lambda_1^{-1}\tilde{\gamma}^{-1}C_f\left(2\lambda_1 - \mu\tilde{\gamma}\lambda_1 - \mu - 2C_g\right)^{-1}\int_\tau^t e^{\mu s}\|u_s\|_\delta^2\,ds
\end{aligned}
$$

for all $t \geq \tau$, and therefore, thanks to the Gronwall lemma and calling

$$\xi = +2\lambda_1^{-1}\tilde{\gamma}^{-1}\left(2\lambda_1 - \mu\tilde{\gamma}\lambda_1 - \mu - 2C_g\right)^{-1},$$

we deduce

$$
\begin{aligned}
\|u_t\|_\delta^2 &\leq (\tilde{\gamma}^{-1}e^{2\delta\tau}[|\phi(0)|^2 + \tilde{\gamma}\|\phi\|_\delta^2] \\
&+ \xi(\|g(0)\|^2 + |\psi|))e^{\xi t} + \xi(\mu - \xi)^{-1}(\|g(0)\|^2 + |\psi|),
\end{aligned}
\tag{33}
$$

for all $t \geq \tau$.

By (28) and the continuity of L_f, there exists an $\varepsilon > 0$ such that

$$\lambda_1 > L_f(\widetilde{R} + \varepsilon),\tag{34}$$

and a time T_ϕ big enough such that

$$\|u_t\|_\delta \leq \widetilde{R} + \varepsilon \quad \forall\, t \geq T_\phi.$$

Then, reasoning as in case a), we can prove the convergence in this case b).   □

6. **Some remarks for future research in a set-valued framework.** The analysis we have carried out in the paper strongly relies on hypotheses $(f1)-(f3)$, where $(f2)$ is a locally Lipschitz assumption responsible for the uniqueness of solutions of the initial value problem associated to our non-autonomous model. However, there are many interesting situations in applications in which the function f can only be guaranteed to be continuous and satisfying some growth condition (like condition $(f3)$), or even can be a set-valued function (and therefore the differential equation in (1) becomes a differential inclusion). Then, in these situations, it is not possible to ensure uniqueness of solutions of our problem (1), and consequently, we cannot define a non-autonomous dynamical system according to Definition 1.1. However, these situations can also be analyzed by exploiting the tools and technique of the set-valued analysis. More precisely, there is a recently developed theory of set-valued or multi-valued dynamical systems (in both the autonomous and non-autonomous/random frameworks, see, e.g., [3, 20, 10, 5]) which has proven to be very useful in these cases of non-uniqueness of solutions as well as those concerning differential inclusions. The main feature is that, in many of these cases, one can construct a set-valued or multi-valued semigroup or process generated by taking into account all the possible solutions that the problem may have associated to every initial value.

It is worth mentioning that the extension of the results in this paper to this set-valued setup is nontrivial and requires of a much more sophisticate analysis with techniques from set-valued analysis. It is our intention to analyze this case in a future work.

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