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PAPER *Special Section on Nonlinear Theory and Its Applications*

On the Takens-Bogdanov Bifurcation in the Chua's Equation

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SUMMARY The analysis of the Takens-Bogdanov bifurcation of the equilibrium at the origin in the Chua's equation with a cubic nonlinearity is carried out. The local analysis provides, in first approximation, different bifurcation sets, where the presence of several dynamical behaviours (including periodic, homoclinic and heteroclinic orbits) is predicted. The local results are used as a guide to apply the adequate numerical methods to obtain a global understanding of the bifurcation sets. The study of the normal form of the Takens-Bogdanov bifurcation shows the presence of a degenerate (codimension-three) situation, which is analyzed in both homoclinic and heteroclinic cases.

key words: *bifurcations, oscillations, Chua circuit*

1. Introduction

The main objective of this work is to provide a deep understanding of some nontrivial dynamical behaviour related to the Takens-Bogdanov bifurcation (double-zero eigenvalue of the linearization matrix) in the Chua's equation with a cubic nonlinearity. This equation models an electronic circuit, whose most important features are its simplicity (only one nonlinearity, which we have taken as an odd cubic polynomial), and the complex behaviours that can exhibit. Some of these behaviours are analytically explained in the study we will perform.

The analysis of this circuit has been source of a large bibliography (see Matsumoto et al. [9]). The most widely considered case corresponds to a piecewise linear implementation of the nonlinear device (see, e.g., Madan [8] and references therein). Under this hypothesis, the theoretical analysis of the state equations cannot profit from many results of differentiable dynamics and, in particular, of bifurcation theory (see, for instance, [5]).

We consider here the Chua's equation with a cubic nonlinearity:

$$\begin{aligned} \dot{x} &= \alpha(y - ax^3 - cx), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y. \end{aligned} \tag{1}$$

We are interested in those bifurcation aspects related to the Takens-Bogdanov bifurcation that the equilibrium at the origin in the above equation exhibits. The importance of this bifurcation lies in the possibility of finding global effects (homoclinic and heteroclinic motions) from a local bifurcation study (see Matsumoto et al. [9]).

Chua's equation (1) has been analyzed, e.g., by Khibnik et al. [7], Huang et al. [6], Pivka et al. [10]. In the two last papers, a new linear term is included in the last equation of (1), in order to take into account small resistive effects in the inductor. In Khibnik et al. [7], the analysis is done by keeping fixed a, c, so that there is no possibility of a Takens-Bogdanov bifurcation. The authors carry out the analysis for the Hopf bifurcation, and establish its connection with a homoclinic bifurcation. This connection can be explained with the analysis we will perform here (compare Fig. 10 of [7] with Fig. 1 here).

In our analysis of the Chua's equation (1), we will consider the nontrivial cases $a \neq 0$, $\alpha \neq 0$.

Note the symmetry $(x, y, z) \to (-x, -y, -z)$ it exhibits. The origin is always an equilibrium point, and the linearization matrix at this point is:

$$\begin{pmatrix} -\alpha c & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}. \tag{2}$$

It is a straightforward computation to show that, taking

$$c = c_c = 0, \qquad \beta = \beta_c = \alpha,$$

the linearization matrix at the origin has a double zero eigenvalue and a third eigenvalue -1. Then, we have a bidimensional center manifold and a one-dimensional stable manifold. To analyze this linear codimension-two bifurcation, we take c and β as bifurcation parameters, and look for the bifurcation behaviours corresponding to parameter values close to the critical ones: c_c, β_c.

2. Normal Form

Firstly, we consider the Chua's equation with the parameter evaluated at their critical values. In order to put the equation in an appropriate form, we make the linear transformation:

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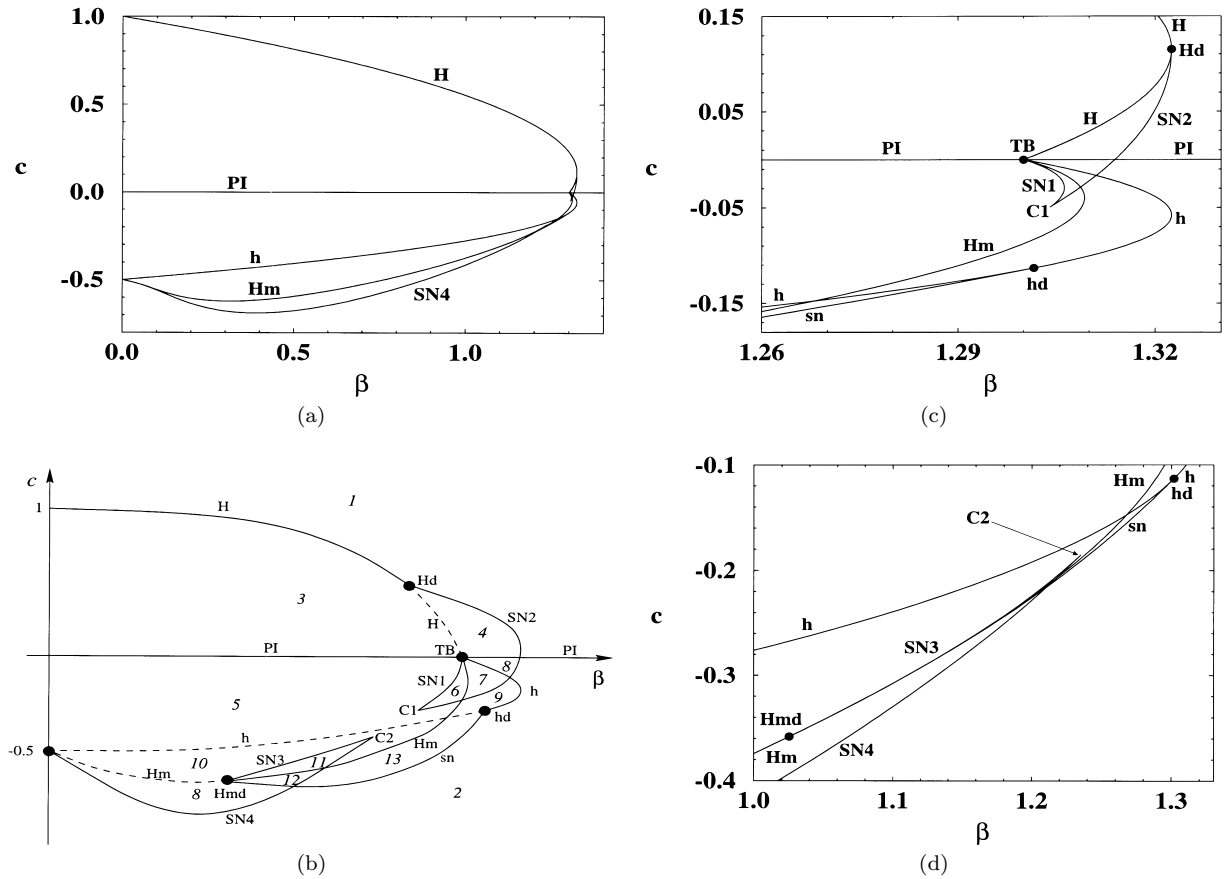

Fig. 1 Bifurcation set for the Chua's equation, for $a = 1$, $\alpha = 1.3$. (a) Numerical bifurcation set. (b) Qualitative bifurcation set. The configuration of equilibria and periodic orbits in each zone appears in Fig. 4. (c) Zoom of the neighbourhood of Takens-Bogdanov point. (d) Zoom of the neighbourhood of the degenerate homoclinic connection point.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha & 0 & -\alpha \\ 0 & 1 & 1 \\ -\alpha & 1 & \alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

bringing the linearization matrix (2) to Jordan form:

$$\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & -1 \end{array}\right).$$

In the new variables, a third-order center manifold (computed by using a recursive procedure developed in Freire et al. [2]) is given by

$$Z = \alpha^4 a(X^3 - 3X^2Y + 6XY^2 - 6Y^3).$$

This approximation enables us to obtain the fifth-order reduced system on the center manifold.

Next, we will put the reduced system in an appropriate form. For that, we use near-identity transformations leading to normal form. Using the algorithm developed in Gamero et al. [3], we obtain the following fifth-order normal form for the reduced system on the center manifold:

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= a_3X^3 + b_3X^2Y + a_5X^5 + b_5X^4Y, \end{aligned} \tag{3}$$

where

$$\begin{aligned} a_3 &= -a\alpha^4, & b_3 &= 3a\alpha^3(\alpha - 1), \\ a_5 &= 3a^2\alpha^8, & b_5 &= -3a^2\alpha^7(8\alpha - 5). \end{aligned}$$

We observe that the coefficient a_3 is always nonzero (although it may be positive or negative). However, the coefficient b_3 vanishes for the value $\alpha = 1$, where a degenerate Takens-Bogdanov bifurcation takes place.

To know how the parameters c, β affect to the normal form (3), we suspend the system (adding the trivial equations $\dot{c} = 0$, $\dot{\beta} = 0$) and compute the center manifold for the suspended system. After some computations, and neglecting the higher-order terms in the parameters, we put in correspondence the Chua's equation (1) –taking $c \approx c_c$ and $\beta \approx \beta_c$– with:

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= \epsilon_1 X + \epsilon_2 Y + a_3X^3 \\ &\quad + b_3X^2Y + a_5X^5 + b_5X^4Y, \\ \dot{Z} &= -Z, \end{aligned}$$

where

$$\epsilon_1 = -\alpha^2 c,$$
$$\epsilon_2 = -(\beta - \alpha) + \alpha(\alpha - 1)c.$$

It is a straightforward computation to show that these unfolding parameters satisfy the transversality condition:

$$\left.\frac{\partial(\epsilon_1, \epsilon_2)}{\partial(c, \beta)}\right|_{c=0, \beta=\alpha} = \alpha^2 \neq 0,$$

and then, the change of parameters c, β by ϵ_1, ϵ_2 is a local diffeomorphism.

In next sections we will address the study of both, nondegenerate and degenerate cases.

3. Nondegenerate Cases

When both a_3 and b_3 are nonzero, we are dealing with a nondegenerate Takens-Bogdanov bifurcation [5]. Its classification depends on the sign of a_3, and the coefficients a_5, b_5 do not play any role in the subsequent analysis. So, in the study of the nondegenerate cases it is enough to compute the third-order normal form, i.e., the normal form (3) truncated up to third order (anyway, the whole fifth-order normal form will be required later in the analysis of the degenerate case).

Depending on the sign of a_3 (which is determined by the sign of the parameter a), two different situations arise:

3.1 Homoclinic Case

This case applies when the coefficient a_3 is negative, that is, for $a > 0$. We can consider $b_3 < 0$, whose analysis can be found in Guckenheimer and Holmes [5] (the study for $b_3 > 0$ can be reduced to the above one by changing the sign of t, X, ϵ_2).

In the bifurcation set around the origin, the following codimension-one bifurcations are present:

- A pitchfork bifurcation of the origin **PI**.
- A Hopf bifurcation of the origin **H**.
- A Hopf bifurcation of the nontrivial equilibria **h**.
- A homoclinic connection of the origin **Hm** (in fact, a pair of homoclinic connections, due to the symmetry).
- A saddle-node bifurcation of periodic orbits **SN**1.

The local approximations of these bifurcations appear in the quoted book, and they are not included here for the sake of brevity. It is important to notice that, as we have computed ϵ_1, ϵ_2 to first order as a function of parameters c, β, we have a useful starting point for the numerical analysis we will present later.

3.2 Heteroclinic Case

This situation corresponds to $a_3 > 0$. It occurs in Chua's equation (1) when $a < 0$. We can restrict to the case $b_3 < 0$ (the case $b_3 > 0$ can be handled by changing the sign of t, X, ϵ_2). The following bifurcations are present (see Guckenheimer and Holmes [5]):

- A pitchfork bifurcation of the origin **PI**.
- A Hopf bifurcation of the origin **H**.
- A heteroclinic connection between nontrivial equilibria **Ht**.

As above, we have not included the local expressions for these bifurcations.

4. Degenerate Cases

The complexity of bifurcation behaviours grows in the degenerate case corresponding to the vanishing of b_3, which occurs at the critical value $\alpha_c = 1$. In this case, the fifth-order terms in (3) are necessary to determine the local bifurcation behaviour.

For the critical value of α, we find $b_5 = -9a^2\alpha^7 \neq 0$. This degenerate case is a codimension-three situation, and a third unfolding parameter is required. We will take $\epsilon_3 = \alpha - \alpha_c = \alpha - 1$, together with ϵ_1 and ϵ_2, to describe this bifurcation.

In the degenerate cases, the knowledge of the whole fifth-order normal form (3) is required. One of the fifth-order terms in the normal form (3) —the coefficient a_5— can be eliminated by rescaling the time in terms of the state variables. This operation do not alter the values of the remaining coefficients in the fifth-order normal form.

As in the nondegenerate cases, two different situations are possible. Each one will be considered in next subsections.

4.1 Homoclinic Case

This is the case when $a_3 < 0$. We can assume $b_5 > 0$ (the case $b_5 < 0$ can be reduced to the above by changing the sign of $t, Y, \epsilon_2, \epsilon_3$). In Rousseau and Li [14] and Rodríguez-Luis et al. [12], it is shown that, besides the bifurcations present in the nondegenerate case (see Sect. 3.1), the following ones appear:

- A degenerate Hopf bifurcation of the origin **Hd**. From here, a saddle-node bifurcation of periodic orbits **SN**2 emerges.
- A degenerate Hopf bifurcation of the nontrivial equilibria **hd**. From here, a saddle-node bifurcation of periodic orbits **sn** emerges.
- A degenerate homoclinic connection **Hmd**. From here, two saddle-node bifurcations of periodic orbits **sn** and **SN**3 emerge.
- A cusp of saddle-node bifurcations of periodic orbits **C**1, where **SN**1 and **SN**2 collapse.

4.2 Heteroclinic Case

This corresponds to the case $a_3 > 0$. We can reduce our study to the case $b_5 > 0$, by changing the sign of $t, Y, \epsilon_2, \epsilon_3$ if necessary. Beyond the bifurcations present in the nondegenerate case (see Sect. 3.2), the following bifurcations appear (see Rousseau [13]):

- A degenerate Hopf bifurcation **Hd**. From here, a saddle-node bifurcation of periodic orbits **SN** emerges.
- A degenerate heteroclinic connection **Htd**, where the heteroclinic connection changes its stability. Here, the above saddle-node bifurcation of periodic orbits **SN** ends.

5. Numerical Study

Now, we will look for this rich bifurcation behaviour predicted by the theory, around this degenerate Takens-Bogdanov bifurcation, in the Chua's equation. Firstly, we have selected $a = 1$ in order to present bifurcation sets corresponding to the homoclinic case. We will focus on the degenerate case that occurs at $c_c = 0$, $\beta_c = 1$, $\alpha_c = 1$.

The numerical results are presented in Figs. 1 and 3, corresponding to fixed values of $\alpha = 1.3$ and 0.8, respectively. They are located on both sides of the critical value $\alpha_c = 1$ where the degeneracy takes place.

The local results achieved in previous sections have been essentials as a guide in the use of the adequate numerical continuation methods (see Doedel et al. [1], Rodríguez-Luis et al. [11]), in order to extend globally the local information.

In Fig. 1 we have taken $\alpha = 1.3$. This is the richest situation from the point of view of different bifurcation behaviours. In the bifurcation set drawn in (a) we have considered the range $\beta \in (0, 1.4)$, $c \in (-0.8, 1)$. The numerical results of the quoted picture have been obtained with AUTO [1]. Several curves are so close that are almost indistinguishable. We refer to the qualitative curves (Fig. 1(b)) and the two zooms ((c)–(d)) to clarify the bifurcation set. All the bifurcations predicted by the analysis of Sect. 3.1 are present. Moreover, a cusp of saddle-node bifurcations of periodic orbits **C**2, where **SN**3 and **SN**4 collapse, also appears. The presence of this cusp may be related to a higher degeneracy in the homoclinic **Hm**. Further study will be needed to understand it.

For the moment, we show numerically that, increasing α, the two cusps **C**1 and **C**2 meets in a beak-to-beak singularity. This fact is presented in the bifurcation sets of Fig. 2, where we only draw the four saddle-node bifurcations of periodic orbits. Notice that, for $\alpha = 1.45$, the saddle-node bifurcations $\mathbf{SN}_1$ and $\mathbf{SN}_2$ collapse at **C**1, and $\mathbf{SN}_3$ and $\mathbf{SN}_4$ at **C**2. Later, for $\alpha = 1.5$, a saddle-node curve ($\mathbf{SN}_1$–$\mathbf{SN}_3$) joins **TB** and **Hmd**, whereas $\mathbf{SN}_2$–$\mathbf{SN}_4$ are not the same curve.

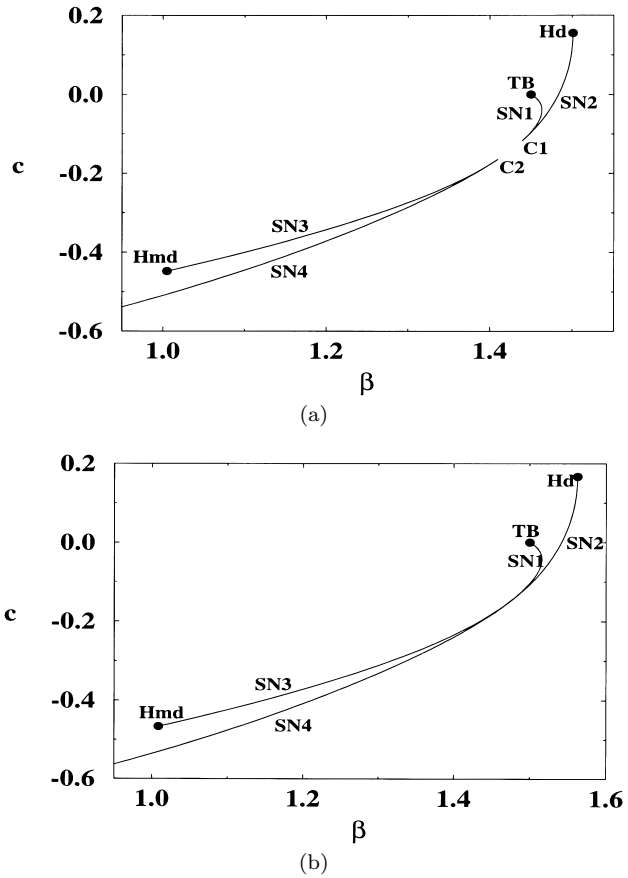

Fig. 2 Beak-to-beak singularity for $a = 1$. Partial bifurcation set including only the saddle-node bifurcation of periodic orbits curves. (a) Situation at $\alpha = 1.45$, before the collision of the cusps. (b) Situation at $\alpha = 1.5$, after the collision.

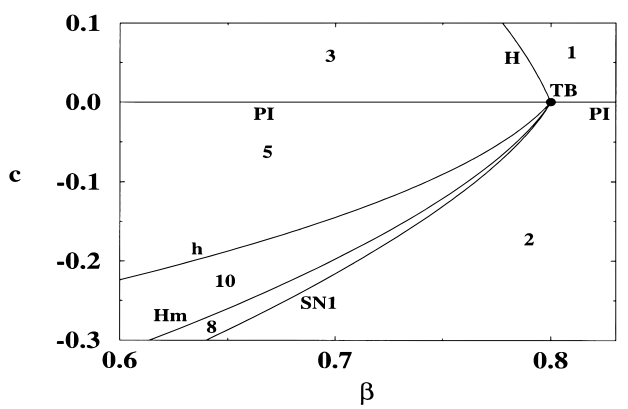

Fig. 3 Numerical bifurcation set for the Chua's equation for $a = 1$, $\alpha = 0.8$. The configuration of equilibria and periodic orbits in each zone appears in Fig. 4.

In Fig. 3, the situation is simpler, and only the bifurcations predicted by the local analysis appear. This bifurcation set also illustrates the homoclinic nondegenerate case, considered in Sect. 3.1.

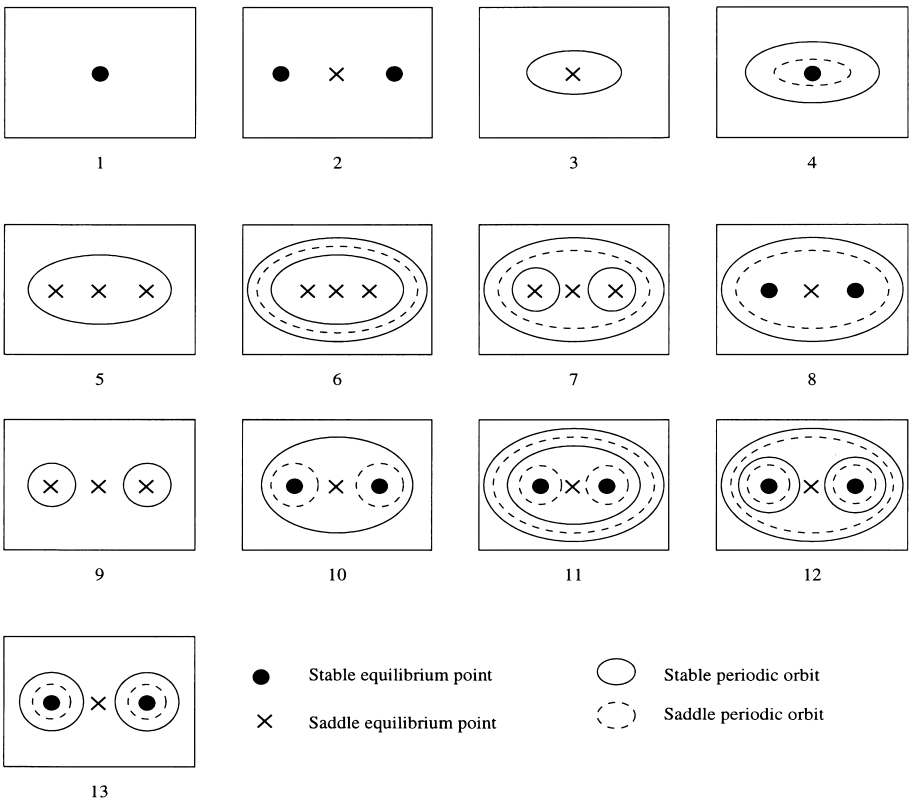

Fig. 4 Configuration of equilibria and periodic orbits in the homoclinic case.

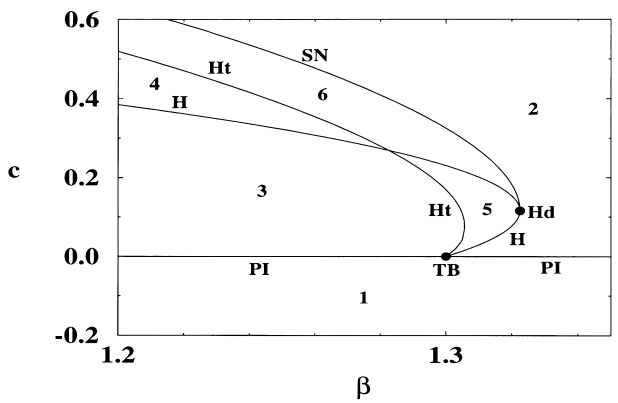

Fig. 5 Numerical bifurcation set for the Chua's equation for $a = -1$, $\alpha = 1.3$. The configuration of equilibria and periodic orbits in each zone appears in Fig. 7.

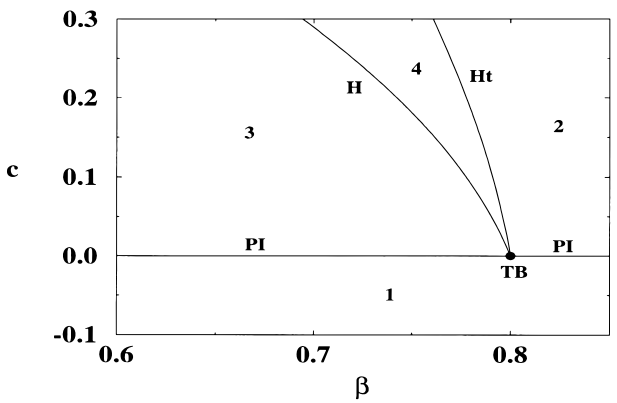

Fig. 6 Numerical bifurcation set for the Chua's equation for $a = -1$, $\alpha = 0.8$. The configuration of equilibria and periodic orbits in each zone appears in Fig. 7.

The configurations of equilibria and periodic orbits present in each zone of the bifurcation sets, for this homoclinic case, are depicted schematically in Fig. 4.

Now, we select $a = -1$, to deal with the heteroclinic case. As above, we have taken $\alpha = 1.3$ and $\alpha = 0.8$. Each value corresponds to a side of the critical value $\alpha_c = 1$ where the degeneracy takes place. The numerical results are presented in Figs. 5 and 6. The configurations of equilibria and periodic orbits are sketched in Fig. 7. All the curves and configurations are explained by the previous theoretical analysis. In particular, the configuration in the vicinity of a heteroclinic nondegenerate Takens-Bogdanov bifurcation coincides with the situation of Fig. 6.

The analysis carried out is a straight way to detect global dynamics from a local analysis. For instance, in Fig. 8 we show two global connections. Namely, a homoclinic orbit located on the curve **Hm** of Fig. 1. It corresponds to the parameter values $a = 1$, $\alpha = 1.3$, $\beta = 0.892$ and $c \approx -0.435$. We have represented its projection onto the xy plane. Also, a heteroclinic orbit located on the curve **Ht** of Fig. 5, corresponding to the

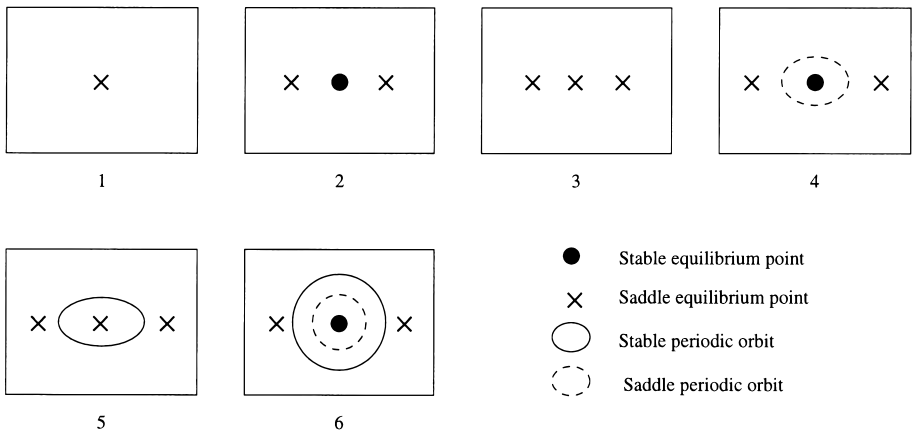

Fig. 7 Configuration of equilibria and periodic orbits in the heteroclinic case.

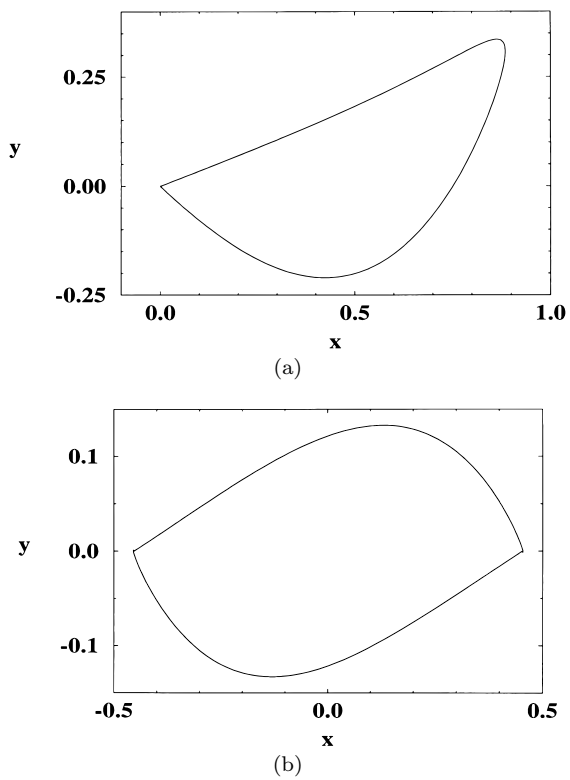

Fig. 8 (a) Projection on the xy plane of the homoclinic connection that appears, for $a = 1$, $\alpha = 1.3$, at $\beta = 0.892$ and $c \approx -0.435$. (b) Projection on the xy plane of a pair of heteroclinic connections that appears, for $a = -1$, $\alpha = 1.3$, at $\beta = 1.295$ and $c \approx 0.2052$.

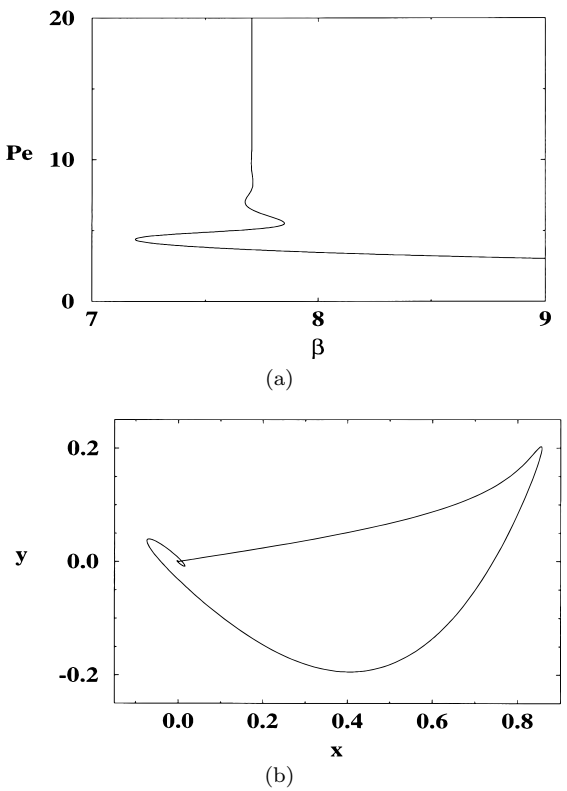

Fig. 9 (a) Bifurcation diagram (period versus β) for $a = 1$, $\alpha = 10$, $c = -0.5$. (b) Projection on the xy plane of the homoclinic connection that appears, for $a = 1$, $\alpha = 10$, $c = -0.5$, $\beta \approx 0.7705$.

values $a = -1$, $\alpha = 1.3$, $\beta = 1.295$ and $c \approx 0.2052$.

These global connections, that near the Takens-Bogdanov bifurcation are planar phenomena, develop a tridimensional structure by moving, for example, the parameter α. In the planar situation, the periodic orbits have a monotonically increasing period when approaching homoclinicity. On the other hand, when the homoclinic connection enters in the Shil'nikov region, a wiggling behaviour appears, with a sequence of saddle-node and period-doubling bifurcations of periodic orbits (see Glendinning and Sparrow [4]). This behaviour is presented in Fig. 9, where we have included a bifurcation diagram, plotting the period against β. Also, a phase portrait of the Shil'nikov homoclinic connection is drawn. Notice the saddle-focus character of the equilibrium at the origin.

6. Conclusions

The study of the Takens-Bogdanov bifurcation is a pow-

erful method that provides valuable information about periodic behaviour and global dynamics.

Near the Takens-Bogdanov bifurcation, the phenomena are planar, but moving the parameters far away, we can expect that they develop a tridimensional structure, and then they can provide a route to chaotic dynamics.

In this paper, we have carry out the analysis of the Takens-Bogdanov bifurcation of the equilibrium at the origin in the Chua's equation with a cubic nonlinearity. Deriving the corresponding normal form, we put in evidence the presence of degenerate cases. Then, we obtain theoretically local and global bifurcations, that provide information about periodic behaviours and homoclinic and heteroclinic motions. The completion of the bifurcation set requires numerical methods. These allow us to detect the presence of a cusp of saddle-node bifurcation of periodic orbits, and also of a beak-to-beak singularity. Moreover, our analysis explains the presence of several codimension-two bifurcations detected numerically in Khibnik et al. [7]. Namely, a degenerate Hopf bifurcation of the origin, a degenerate homoclinic and a cusp of saddle-node of periodic orbits (see Fig. 10 of [7]).

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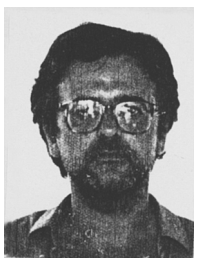

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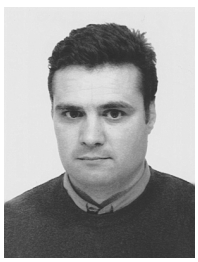

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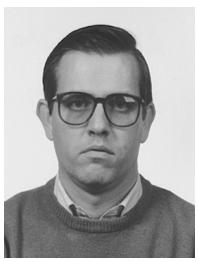

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