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THE POSTPROCESSED MIXED FINITE-ELEMENT METHOD FOR THE NAVIER–STOKES EQUATIONS: REFINED ERROR BOUNDS*

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Abstract. A postprocessing technique for mixed finite-element methods for the incompressible Navier–Stokes equations is analyzed. The postprocess, which amounts to solving a (linear) Stokes problem, is shown to increase the order of convergence of the method to which it is applied by one unit (times the logarithm of the mesh diameter). In proving the error bounds, some superconvergence results are also obtained. Contrary to previous analysis of the postprocessing technique, in the present paper we take into account the loss of regularity suffered by the solutions of the Navier–Stokes equations at the initial time in the absence of nonlocal compatibility conditions of the data. As in [H. G. Heywood and R. Rannacher, *SIAM J. Numer. Anal.*, 25 (1988), pp. 489–512], where the same hypothesis is assumed, no better than fifth-order convergence is achieved.

Key words. Navier–Stokes equations, mixed finite-element methods, optimal regularity, error estimates

AMS subject classifications. 65M60, 65M20, 65M15, 65M12

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1. Introduction.

It is well known that, no matter how regular the data are, solutions of the Navier–Stokes equations cannot be assumed to have more than second-order spatial derivatives bounded in L^2 up to initial time $t = 0$, since this requires the data to satisfy nonlocal compatibility conditions unlikely to be fulfilled in practical situations [36], [37]. Therefore, error analysis of numerical methods for the Navier–Stokes equations is more meaningful if this fact is taken into account. This is the case of the present paper, where we analyze a postprocessing technique that improves the errors of mixed finite-element (MFE) methods for the Navier–Stokes equations from $O(h^r)$ to $O(h^{r+1}\,|\log(h)|)$, where h is the mesh size.

This postprocessing technique was first developed for spectral methods in [26], [27], in connection with approximate inertial manifolds [14], [15]. Later it was extended to methods based on Chebyshev and Legendre polynomials [16], spectral-element methods [18], and finite-element methods [28], [19], together with being applied in the study of nonlinear shell vibrations [43]. Also, it has been effectively applied to reduce the order of practical engineering problems modeled by nonlinear differential systems [46], [45].

More recently, the postprocessing technique has also been developed for MFE methods for the Navier–Stokes equations in [4], [6], but for solutions more regular up to $t = 0$ than what can be realistically assumed in practice, and this allows the

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above-mentioned improvement of the error for $r \geq 2$. In the present paper, however, due to the loss of regularity at $t = 0$, no better than $O(h^5 |\log(h)|)$ error bounds are proved, a limitation similar to that already found in [37] for MFE methods. On the other hand, as in [6], although for simplicity we concentrate on Hood–Taylor [39] elements, the postprocessing technique can be easily adapted to other kinds of mixed LBB-stable elements.

We analyze the postprocessing technique for the incompressible Navier–Stokes equations

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f, \qquad \mathrm{div}(u) = 0, \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a smooth boundary subject to homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. In (1.1), u is the velocity field, p the pressure, and f a given force field. We assume that the fluid density and viscosity have been normalized by an adequate change of scale in space and time.

The postprocessing technique to approximate up to time $T > 0$ the solution u and p corresponding to a given initial condition

$$u(\cdot, 0) = u_0 \tag{1.2}$$

is as follows. Compute first standard MFE approximations u_h and p_h to the velocity and pressure, respectively, by integrating in time the corresponding discretization of (1.1)–(1.2). Then compute MFE approximations $\tilde{u}_h$ and $\tilde{p}_h$ to the solution $\tilde{u}$ and $\tilde{p}$ of the following Stokes problem:

$$\left.\begin{aligned} -\Delta\tilde{u} + \nabla\tilde{p} &= f - \tfrac{d}{dt}u_h - (u_h \cdot \nabla)u_h \\ \mathrm{div}(\tilde{u}) &= 0 \end{aligned}\right\} \quad \text{in } \Omega, \qquad \tilde{u} = 0, \quad \text{on } \partial\Omega. \tag{1.3}$$

The MFE on this postprocessing step can be either the same-order Hood–Taylor element over a finer grid or a higher-order Hood–Taylor element over the same grid. Observe that, since the MFE approximations u_h and p_h do not depend on $\tilde{u}$ and $\tilde{p}$ or their approximations $\tilde{u}_h$ and $\tilde{p}_h$, these may be computed only when needed. If this happens only at the final time $t = T$, then an $O(h^{r+1} |\log(h)|)$ error is obtained at the cost of an $O(h^r)$ one, since only a single Stokes problem such as (1.3) is solved with the enhanced MFE method, and this is done when the time integration is completed.

The method studied in the present paper can be seen as a two-level method, where the postprocessed (or fine-mesh) approximations $\tilde{u}_h$ and $\tilde{p}_h$ are an improvement of the previously computed (coarse-mesh) approximations u_h and p_h. Recent research on two-level finite-element methods for the transient Navier–Stokes equations can be found in [30], [31], [32], [44] (see also [34], [33], [40], [42] for spectral discretizations), where the full nonlinear problem is dealt with on the coarse mesh, and a linear problem is solved on the fine mesh. Let us mention that, whereas in the present paper we are concerned with higher-order methods, in [30], [31], [32], [44] only low-order methods are dealt with. Also, in the present paper, computations on the finer grid are done only at the final time $t = T$ (or at those time levels where more accuracy is wanted), whereas in the previous works computations on the finer grids are carried out from $t = 0$ to $t = T$. It must be mentioned though, as opposed to [2], that the coarse-grid component computation in [30], [31], [32], [44] is fully decoupled from that on the fine grid.

Although in the present paper we concentrate on the spatial discretization, practical computations are affected by the errors induced by the time discretization. Numerical experiments in [6], [16], [19], [17], [26], [28] have repeatedly shown, for the different discretizations considered, that the increase in accuracy and convergence rate predicted by the theory is also seen in practice (provided errors arising from the time discretization are kept sufficiently small). Nevertheless, in the present paper we give an explanation of this fact; that is, the gain in (spatial) accuracy in the postprocessing step takes place independently of errors arising from the temporal discretization being present or not. Similar results are obtained in [47] for finite-element methods of degree 3 or larger for reaction-diffusion problems when integrated by the implicit Euler method.

To prove our error estimates, we need to prove first some superconvergence results. These are not with respect to the Stokes projection (as in [17], [6], [4]) but with respect to the solution of a certain linear evolution problem. In this process, we also obtain error bounds for the pressure that improve those originally proved in [37, Theorem 3.1] by a factor of $t^{1/2}$.

Finally, we remark that in recent works [20], [21], [22] the postprocessing technique has shown itself useful in obtaining efficient a posteriori error estimators for partial differential equations of evolution, a field which is remarkably less developed than that of steady problems. The application of the postprocessing technique to a posteriori error estimates for the Navier–Stokes equations using the results obtained in the present paper will be the subject of future research.

The rest of the paper is as follows. In section 2 we introduce the notation and some standard material. In section 3 we comment on how to approximate the Stokes problem of the postprocessing step. Section 4 is devoted to the superconvergence analysis of the MFE approximation. In section 5 error bounds of the postprocessed approximation are obtained. Finally, we make some remarks on postprocessing when time discretization is taken into account.

2. Preliminaries and notations. We will assume that Ω is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, not necessarily convex, of class $\mathcal{C}^m$, for $m \geq 3$, and we consider the Hilbert spaces

$$H = \{u \in \big(L^2(\Omega)\big)^d \mid \operatorname{div}(u) = 0,\ u \cdot n_{|\partial\Omega} = 0\},$$

$$V = \{u \in \big(H_0^1(\Omega)\big)^d \mid \operatorname{div}(u) = 0\},$$

endowed with the inner product of $L^2(\Omega)^d$ and $H_0^1(\Omega)^d$, respectively. For $l \geq 0$ integer and $1 \leq q \leq \infty$, we consider the standard Sobolev spaces $W^{l,q}(\Omega)^d$ of functions with derivatives up to order l in $L^q(\Omega)$, and $H^l(\Omega)^d = W^{l,2}(\Omega)^d$. We will denote by $\|\cdot\|_l$ the norm in $H^l(\Omega)^d$, and $\|\cdot\|_{-l}$ will represent the norm of its dual space. We consider also the quotient spaces $H^l(\Omega)/\mathbb{R}$ with norm $\|p\|_{H^l/\mathbb{R}} = \inf\{\|p + c\|_l \mid c \in \mathbb{R}\}$.

Let us recall the following Sobolev's imbeddings [1]: For $q \in [1, \infty)$, there exists a constant $C = C(\Omega, q)$ such that

$$\|v\|_{L^{q'}(\Omega)^d} \leq C\|v\|_{W^{s,q}(\Omega)^d}, \qquad \frac{1}{q'} \geq \frac{1}{q} - \frac{s}{d} > 0, \quad q < \infty, \quad v \in W^{s,q}(\Omega)^d. \tag{2.1}$$

For $q' = \infty$, (2.1) holds with $\frac{1}{q} < \frac{s}{d}$.

Let $\Pi : L^2(\Omega)^d \longrightarrow H$ be the $L^2(\Omega)^d$ projection onto H. We denote by A the Stokes operator on Ω:

$$A : \mathcal{D}(A) \subset H \longrightarrow H, \quad A = -\Pi\Delta, \quad \mathcal{D}(A) = H^2(\Omega)^d \cap V.$$

Applying Leray's projector to (1.1), the equations can be written in the form

$$u_t + Au + B(u,u) = \Pi f \qquad \text{in } \Omega,$$

where $B(u,v) = \Pi(u\cdot\nabla)v$ for u, v in $H_0^1(\Omega)^d$.

We shall use the trilinear form $b(\cdot,\cdot,\cdot)$ defined by

$$b(u,v,w) = (F(u,v),w) \quad \forall u,v,w \in H_0^1(\Omega)^d,$$

where

$$F(u,v) = (u\cdot\nabla)v + \frac{1}{2}(\nabla\cdot u)v \quad \forall u,v \in H_0^1(\Omega)^d.$$

It is straightforward to verify that b enjoys the skew-symmetry property

$$b(u,v,w) = -b(u,w,v) \quad \forall u,v,w \in H_0^1(\Omega)^d. \tag{2.2}$$

Let us observe that $B(u,v) = \Pi F(u,v)$ for $u \in V$, $v \in H_0^1(\Omega)^d$.

We shall assume that

$$\|u(t)\|_1 \le M_1, \quad \|u(t)\|_2 \le M_2, \qquad 0 \le t \le T,$$

and, for $k \ge 2$ integer,

$$\sup_{0\le t\le T} \left\|\partial_t^{\lfloor k/2\rfloor} f\right\|_{k-1-2\lfloor k/2\rfloor} + \sum_{j=0}^{\lfloor (k-2)/2\rfloor} \sup_{0\le t\le T} \left\|\partial_t^j f\right\|_{k-2j-2} < +\infty,$$

so that, according to Theorems 2.4 and 2.5 in [36], there exist positive constants M_k and K_k such that for $k \ge 2$

$$\|u(t)\|_k + \|u_t(t)\|_{k-2} + \|p(t)\|_{H^{k-1}/\mathbb{R}} \le M_k \tau(t)^{1-k/2} \tag{2.3}$$

and for $k \ge 3$

$$\int_0^t \sigma_{k-3}(s)\big(\|u(s)\|_k^2 + \|u_s(s)\|_{k-2}^2 + \|p(s)\|_{H^{k-1}/\mathbb{R}}^2 + \|p_s(s)\|_{H^{k-3}/\mathbb{R}}^2 \big)\, ds \le K_k^2, \tag{2.4}$$

where $\tau(t) = \min(t,1)$ and $\sigma_n = e^{-\alpha(t-s)}\tau^n(s)$ for some $\alpha > 0$. Observe that, for $t \le T < \infty$, we can take $\tau(t) = t$ and $\sigma_n(s) = s^n$. For simplicity, we will take these values of τ and σ_n. We note that no further than $k \le 6$ will be needed in the present paper.

Let $\mathcal{T}_h = (\tau_i^h, \phi_i^h)_{i\in I_h}$, $h > 0$, be a family of partitions of suitable domains Ω_h, where h is the maximum diameter of the elements $\tau_i^h \in \mathcal{T}_h$ and ϕ_i^h are the mappings of the reference simplex τ_0 onto τ_i^h. We restrict ourselves to quasi-uniform and regular meshes $\mathcal{T}_h$.

Let $r \ge 3$, and we consider the finite-element spaces

$$S_{h,r} = \big\{\chi_h \in \mathcal{C}(\overline{\Omega}_h) \,|\, \chi_h|_{\tau_i^h} \circ \phi_i^h \in P^{r-1}(\tau_0)\big\} \subset H^1(\Omega_h),$$

$$S_{h,r}^0 = \big\{\chi_h \in \mathcal{C}(\overline{\Omega}_h) \,|\, \chi_h|_{\tau_i^h} \circ \phi_i^h \in P^{r-1}(\tau_0),\ \chi_h(x) = 0\ \forall x \in \partial\Omega_h\big\} \subset H_0^1(\Omega_h),$$

where $P^{r-1}(\tau_0)$ denotes the space of polynomials of degree at most $r-1$ on τ_0. Since we are assuming that the meshes are quasi-uniform, the following inverse inequality holds for each $v_h \in (S^0_{h,r})^d$ (see, e.g., [9, Theorem 3.2.6])

$$\|v_h\|_{W^{m,q}(\tau)^d} \le C h^{l-m-d(\frac{1}{q'}-\frac{1}{q})} \|v_h\|_{W^{l,q'}(\tau)^d}, \tag{2.5}$$

where $0 \le l \le m \le 1$, $1 \le q' \le q \le \infty$, and τ is an element in the partition $\mathcal{T}_h$.

We shall denote by $(X_{h,r}, Q_{h,r-1})$ the so-called Hood–Taylor element, where

$$X_{h,r} = \left(S^0_{h,r}\right)^d, \quad Q_{h,r-1} = S_{h,r-1} \cap L^2(\Omega_h)/\mathbb{R}, \qquad r \ge 3.$$

For this mixed element a uniform inf-sup condition is satisfied (see [8]); that is, there exists a constant $\beta > 0$ independent of the mesh grid size h such that

$$\inf_{q_h \in Q_{h,r-1}} \sup_{v_h \in X_{h,r}} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\|_1 \|q_h\|_{L^2/\mathbb{R}}} \ge \beta. \tag{2.6}$$

The approximate velocity belongs to the discretely divergence-free space

$$V_{h,r} = X_{h,r} \cap \Big\{ \chi_h \in H^1_0(\Omega_h) \mid (q_h, \nabla \cdot \chi_h) = 0 \quad \forall q_h \in Q_{h,r-1} \Big\}.$$

We observe that, for the Hood–Taylor element, $V_{h,r}$ is not a subspace of V.

Let $\Pi_h : L^2(\Omega)^d \longrightarrow V_{h,r}$ be the discrete Leray's projection defined by

$$(\Pi_h u, \chi_h) = (u, \chi_h) \quad \forall \chi_h \in V_{h,r}.$$

We will use the following well known bounds

$$\|(I - \Pi_h)u\|_j \le C h^{l-j} \|u\|_l, \qquad 1 \le l \le r, \quad j = 0, 1, \tag{2.7}$$

and, also, since we are assuming that Ω is at least $\mathcal{C}^2$.

$$\left\|A^{-m/2}\Pi(I - \Pi_h)u\right\|_0 \le C h^{l+\min(m,r-2)} \|u\|_l, \qquad 1 \le l \le r, \quad m = 1, 2, \tag{2.8}$$

We will denote by $A_h : V_{h,r} \to V_{h,r}$ the discrete Stokes operator defined by

$$(\nabla v_h, \nabla \phi_h) = (A_h v_h, \phi_h) = \left(A_h^{1/2} v_h, A_h^{1/2} \phi_h\right) \qquad \forall v_h, \phi_h \in V_{h,r}.$$

Let $\mathcal{A}$ denote either $\mathcal{A} = A$ or $\mathcal{A} = A_h$. Notice that both are positive self-adjoint operators with compact resolvent in H and V_h, respectively. Let us consider then for $\alpha \in \mathbb{R}$ and $t > 0$ the operators $\mathcal{A}^\alpha$ and $e^{-t\mathcal{A}}$, which are easily defined by means of the spectral properties of $\mathcal{A}$ (see, e.g., [10, p. 33], [24]). An easy calculation shows that

$$\|\mathcal{A}^\alpha e^{-t\mathcal{A}}\|_0 \le (\alpha e^{-1})^\alpha t^{-\alpha}, \qquad \alpha \ge 0, \ t > 0, \tag{2.9}$$

where, here and in what follows, $\|\cdot\|_0$ when applied to an operator denotes the associated operator norm. Also, using the change of variables $\tau = s/t$, it is easy to show that

$$\int_0^t s^{-1/2} \left\|A_h^{1/2} e^{-(t-s)A_h}\right\|_0 ds \le \frac{1}{\sqrt{2e}} B\left(\frac{1}{2}, \frac{1}{2}\right), \tag{2.10}$$

where B is the Beta function (see, e.g., [12]).

Let $(u,p) \in (H^2(\Omega)^d \cap V) \times (H^1(\Omega)/\mathbb{R})$ be the solution of a Stokes problem with right-hand side g, and we will denote by $S_h(u) \in V_{h,r}$ the so-called Stokes projection defined by (see [37])

$$(\nabla S_h(u), \nabla \chi_h) = (\nabla u, \nabla \chi_h) - (p, \nabla \cdot \chi_h) = (g, \chi_h) \quad \forall \chi_h \in V_{h,r}.$$

We will also consider the above definition of $S_h(u)$ written in mixed form: Find $(s_h, q_h) \in (X_h, Q_h)$ such that

$$(\nabla s_h, \nabla \phi_h) + (\nabla q_h, \phi_h) = (g, \phi_h) \quad \forall \phi_h \in X_h, \tag{2.11}$$

$$(\nabla \cdot s_h, \psi_h) = 0 \quad \forall \psi_h \in Q_h. \tag{2.12}$$

Obviously, $s_h = S_h(u)$. The following bound holds for $2 \le l \le r$:

$$\|u - s_h\|_0 + h\|u - s_h\|_1 \le Ch^l \left(\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}\right). \tag{2.13}$$

The proof of (2.13) for $\Omega = \Omega_h$ can be found in [37]. For the general case superparametric approximation at the boundary is assumed [5]. Under the same conditions, the bound for the pressure is [29]

$$\|p - q_h\|_{L^2/\mathbb{R}} \le C_\beta h^{l-1} \left(\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}\right), \tag{2.14}$$

where the constant C_β depends on the constant β in the inf-sup condition (2.6).

We will assume that the domain Ω is of class $\mathcal{C}^r$, so that standard bounds for the Stokes problem [5], [25] imply that

$$\left\|A^{-1}\Pi g\right\|_{2+j} \le \|g\|_j, \qquad -1 \le j \le r-2. \tag{2.15}$$

Then, using standard duality arguments and (2.13), it is easy to show that

$$\|u - s_h\|_{-m} \le Ch^{l+\min(m,r-2)}(\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}), \quad m = 1, 2. \tag{2.16}$$

In what follows we will apply the above estimates to the particular case in which (u,p) is the solution of the Navier–Stokes problem (1.1)–(1.2). In that case s_h is the discrete velocity in problem (2.11)–(2.12) with $g = f - u_t - (u \cdot \nabla u)$. Note that the temporal variable t appears here merely as a parameter, and then, taking the time derivative, the error bounds (2.13), (2.16) can also be applied to the time derivative of s_h changing u, p by u_t, p_t.

Since we are assuming that Ω is of class $\mathcal{C}^r$ and $r \ge 3$, from (2.13) and standard bounds for the Stokes problem [5], [25], we deduce that

$$\|(A^{-1}\Pi - A_h^{-1}\Pi_h)f\|_j \le Ch^{2-j}\|f\|_0 \quad \forall f \in L^2(\Omega)^d, \quad j = 0, 1, \tag{2.17}$$

$$\|(A^{-1}\Pi - A_h^{-1}\Pi_h)f\|_1 \le Ch^2\|f\|_1 \quad \forall f \in H^1(\Omega)^d. \tag{2.18}$$

In our analysis we shall frequently use the following relations for $f \in L^2(\Omega)^d$:

$$\|A_h^{-s/2}\Pi_h f\|_0 \le Ch^s\|f\|_0 + \|A^{-s/2}\Pi f\|_0, \quad s = 1, 2, \tag{2.19}$$

$$\|A^{-s/2}\Pi f\|_0 \le Ch^s\|f\|_0 + \|A_h^{-s/2}\Pi_h f\|_0, \quad s = 1, 2, \tag{2.20}$$

which are a consequence of (2.17) and the fact that for $v \in V$ and $v_h \in V_{h,r}$ we have $\|\nabla v\|_0 = \|A^{1/2}v\|_0$ and $\|\nabla v_h\|_0 = \|A_h^{1/2}v_h\|_0$. Notice also that, since $(A_h^{-1/2}\Pi_h f, v_h) = (f, A_h^{-1/2}v_h)$, for all $v_h \in V_{h,r}$, it follows that

$$\|A_h^{-1/2}\Pi_h f\|_0 \le C\|f\|_{-1}. \tag{2.21}$$

Finally, we will use the following inequality whose proof can be obtained by applying [36, Lemma 4.4]:

$$\|v_h\|_\infty \le C\,\|A_h v_h\|_0 \quad \forall v_h \in V_{h,r}. \tag{2.22}$$

Remark 2.1. We remark that our analysis applies also to any pair of LBB-stable mixed finite elements satisfying (2.7), (2.13), and especially (2.16). This is the case, for example, of the Crouzeix–Raviart element when $r \ge 3$ [11], [29]. In the case of low-order LBB-stable elements, the analysis is much simpler; see Remark 4.2 below. However, since to simplify the analysis we make use of several results from [6] which are stated and proved for Hood–Taylor elements, we will restrict ourselves to these elements in the present paper.

3. The postprocessed method. The postprocessing technique can be seen as a two-level method. In the first level, the mixed finite-element approximation to (1.1)–(1.2) for a given partition $\mathcal{T}_h$ of Ω_h is computed. That is, given $u_h(0) = \Pi_h(u_0)$, we compute $u_h(t) \in X_{h,r}$ and $p_h(t) \in Q_{h,r-1}$, $t \in (0,T]$, satisfying

$$(\dot{u}_h, \phi_h) + (\nabla u_h, \nabla\phi_h) + b(u_h, u_h, \phi_h) + (\nabla p_h, \phi_h) = (f, \phi_h) \quad \forall\, \phi_h \in X_{h,r}, \tag{3.1}$$

$$(\nabla\cdot u_h, \psi_h) = 0 \qquad \forall\, \psi_h \in Q_{h,r-1}. \tag{3.2}$$

In the second level, the discrete velocity and pressure $(u_h(t), p_h(t))$ are postprocessed by solving the following discrete Stokes problem: Find $(\tilde{u}_h(t), \tilde{p}_h(t)) \in (\widetilde{X}, \widetilde{Q})$ satisfying

$$\big(\nabla\tilde{u}_h(t), \nabla\tilde{\phi}\big) + \big(\nabla\tilde{p}_h(t), \tilde{\phi}\big) = (f, \tilde{\phi}) - b(u_h(t), u_h(t), \tilde{\phi}) - (\dot{u}_h(t), \tilde{\phi}) \quad \forall\, \tilde{\phi} \in \widetilde{X}, \tag{3.3}$$

$$\big(\nabla\cdot\tilde{u}_h(t), \tilde{\psi}\big) = 0 \quad \forall\, \tilde{\psi} \in \widetilde{Q}, \tag{3.4}$$

where $(\widetilde{X}, \widetilde{Q})$, is either

(a) the same-order Hood–Taylor element over a finer grid (that is, for $h' < h$, we choose $(\widetilde{X}, \widetilde{Q}) = (X_{h',r}, Q_{h',r-1})$), or
(b) a higher-order Hood–Taylor element over the same grid. In this case we choose $(\widetilde{X}, \widetilde{Q}) = (X_{h,r+1}, Q_{h,r})$.

In both cases, we will denote by $\widetilde{V}$ the corresponding discretely divergence-free space that can be either $\widetilde{V} = V_{h',r}$ or $\widetilde{V} = V_{h,r+1}$ depending on the selection of the postprocessing space. The discrete orthogonal projection into $\widetilde{V}$ will be denoted by $\widetilde{\Pi}_h$, and we will represent by $\widetilde{A}_h$ the discrete Stokes operator acting on functions in $\widetilde{V}$.

We remark that in (3.3)–(3.4) the time variable appears merely as a parameter. Thus, in practice, $(\tilde{u}_h, \tilde{p}_h)$ may be computed only for those $t \in (0,T]$ where improved accuracy is desired, which are usually a small set of selected times. Nevertheless, here we obtain error estimates for $t \in (0,T]$ since this adds no further difficulty.

Section 5 is devoted to the analysis of convergence of the postprocessed MFE method. In Theorems 5.2 and 5.3 the rate of convergence of the postprocessed method is obtained. We notice that the order of approximation is optimal with respect to the pair $(\widetilde{X}, \widetilde{Q})$ in which the standard MFE approximation is postprocessed. Comparing with the (optimal) rate of convergence of the MFE method (see Corollaries 4.8, 4.16, and 4.19), we deduce that in case b an increase in the rate of convergence of one unit is achieved. Similar conclusions can be reached in case a by taking h' small enough.

The analysis of the postprocessed method will be done in two steps. We will first obtain superconvergence bounds for the MFE approximation in Theorems 4.7, 4.15, and 4.18 with respect to an auxiliary approximation to be introduced in section 4.1. In Theorems 4.7 and 4.15 we deal with the velocity, for the quadratic and cubic approximations, respectively, and with the pressure in Theorem 4.18. In the second step, the errors bounds of the postprocessed approximation are given in Theorem 5.2 for the velocity and in Theorem 5.3 for the pressure.

4. Analysis of the MFE approximation.

4.1. An auxiliary approximation. For a u and p solution of (1.1)–(1.2) let us consider the approximations $v_h : [0,T] \longrightarrow X_{h,r}$ and $g_h : [0,T] \longrightarrow Q_{h,r-1}$, respectively, solutions of

$$(\dot v_h, \phi_h) + (\nabla v_h, \nabla \phi_h) + (\nabla g_h, \phi_h) = (f, \phi_h) - b(u,u,\phi_h) \qquad \forall \phi_h \in X_{h,r}, \tag{4.1}$$

$$(\nabla \cdot v_h, \psi_h) = 0 \qquad \forall\, \psi_h \in Q_{h,r-1}, \tag{4.2}$$

with initial condition $v_h(0) = \Pi_h u_0$.

Let us observe that the MFE approximation u_h and the recently defined v_h satisfy

$$\dot u_h + A_h u_h + B_h(u_h, u_h) = \Pi_h f, \qquad u_h(0) = \Pi_h u_0, \tag{4.3}$$

$$\dot v_h + A_h v_h + B_h(u, u) = \Pi_h f, \qquad v_h(0) = \Pi_h u_0, \tag{4.4}$$

respectively, where $B_h(u, v) = \Pi_h F(u, v)$. Then $e_h = v_h - u_h$ satisfies

$$\dot e_h + A_h e_h = B_h(u_h, u_h) - B_h(u, u), \qquad e_h(0). \tag{4.5}$$

We will also use the following notation:

$$z_h = \Pi_h u - v_h, \qquad \theta_h = \Pi_h u - s_h.$$

It is easy to show then that

$$\dot z_h + A_h z_h = A_h \theta_h, \quad z_h(0) = 0. \tag{4.6}$$

Some technical lemmas are stated in this section. For the convenience of the reader, we will reproduce here the following two lemmas from [6].

LEMMA 4.1. *Let $v \in (H^2(\Omega))^d \cap V$. Then there exists a constant $K = K(\|v\|_2)$ such that for all $w \in H^1_0(\Omega)^d$ we have that*

$$\big\|A^{-1}\Pi[F(v,v) - F(w,w)]\big\|_0 \le K\big(\|v-w\|_{-1} + \|v-w\|_1 \|v-w\|_0\big). \tag{4.7}$$

LEMMA 4.2. *For any $f \in C([0,T]; L^2(\Omega)^d)$, the following estimate holds for all $t \in [0,T]$:*

$$\int_0^t \big\|A_h e^{-(t-s)A_h}\Pi_h f(s)\big\|_0 ds \le C|\log(h)| \max_{0\le t\le T} \|f(t)\|_0.$$

LEMMA 4.3 (estimates for z_h). *Let (u, p) be the solution of (1.1)–(1.2). Then there exists a positive constant C such that the error $z_h = \Pi_h u - v_h$ of the discrete velocity v_h defined by (4.4) satisfies the following bounds:*

$$\|A_h^{-j/2} z_h\|_0 \le C h^{2+j}, \qquad j = 0, 1, 2, \qquad r \ge 3, \tag{4.8}$$

$$\|A_h^{(-1+j)/2} z_h\|_0 \le \frac{C}{t^{1/2}} h^{4-j}, \qquad j = -1, 0, 1, \qquad r \ge 4, \tag{4.9}$$

$$\|A_h^{(-1+j)/2} z_h\|_0 \le \frac{C}{t^{1/2}} h^{4-j}, \qquad j = 0, 1, 2, \qquad r \ge 3, \tag{4.10}$$

$$\|A_h^{(-1+j)/2} z_h\|_0 \le \frac{C}{t} h^{5-j}, \qquad j = 0, 1, 2, \qquad r \ge 4, \tag{4.11}$$

$$\int_0^t \|z_h(s)\|_j^2 \, ds \le C h^{6-2j}, \qquad j = 0, 1, \qquad r \ge 3. \tag{4.12}$$

Proof. From (4.6) it follows that

$$z_h(t) = \int_0^t e^{-A_h(t-s)} A_h \theta_h(s) \, ds. \tag{4.13}$$

If we multiply both sides of this equation by A_h^{-1}, then using (2.10), (2.8), and (2.16) we have

$$\|A_h^{-1} z_h\|_0 \le CB\left(\frac{1}{2}, \frac{1}{2}\right) \max_{0 \le s \le t} s^{1/2} \|A_h^{-1/2} \theta_h(s)\|_0 \le C h^4 \max_{0 \le s \le t} s^{1/2} \|u(s)\|_3,$$

which, applying estimates (2.3), proves (4.8) for the case $j = 2$. Since $\|A_h^{-1/2} z_h\|_0 \le C h^{-1} \|A_h^{-1} z_h\|_0$ and $\|z_h\|_0 \le C h^{-2} \|A_h^{-1} z_h\|_0$, the cases $j = 1$ and $j = 0$ are readily deduced, and then (4.8) is concluded.

We will now prove (4.9). Multiplying (4.6) by $t^{1/2} A_h^{-1/2}$, it is easy to see that $y_h = t^{1/2} A_h^{-1/2} z_h$ satisfies

$$\dot{y}_h + A_h y_h = t^{1/2} A_h^{1/2} \theta_h + (2t)^{-1/2} A_h^{-1/2} z_h,$$

so that we have

$$y_h(t) = \int_0^t e^{-A_h(t-s)} s^{1/2} A_h^{1/2} \theta_h(s) \, ds + \frac{1}{2} \int_0^t e^{-A_h(t-s)} s^{-1/2} A_h^{-1/2} z_h \, ds. \tag{4.14}$$

Then, using (2.10), (2.8), and (2.16), it follows that

$$\begin{aligned} t^{1/2} \|A_h^{-1} z_h\|_0 &\le CB\left(\frac{1}{2}, \frac{1}{2}\right) \left(\max_{0 \le s \le t} s \|A_h^{-1/2} \theta_h\|_0 + \max_{0 \le s \le t} \|A_h^{-3/2} z_h\|_0 \right) \\ &\le C \left(M_4 h^5 + \max_{0 \le s \le t} \|A_h^{-3/2} z_h\|_0 \right). \end{aligned}$$

In order to estimate the last term on the right-hand side above, we multiple both sides of (4.13) by $A_h^{-3/2}$, so that using (2.10), (2.8), and (2.16) we obtain

$$\|A_h^{-3/2} z_h\|_0 \le CB\left(\frac{1}{2}, \frac{1}{2}\right) \max_{0 \le s \le t} s^{1/2} \|A_h^{-1} \theta_h(s)\|_0 \le C h^5 M_3,$$

and then (4.9) is proved in the case $j = -1$, from which, using the same arguments as with (4.8), the cases $j = 0, 1$ are inferred.

We will now prove (4.10) and (4.11) for the case $j = 0$. As before, the cases $j = 1$ and $j = 2$ follow immediately. For $y_h(t) = tA_h^{-1/2} z_h(t)$, it is easy to see that

$$y_h(t) = \int_0^t e^{-A_h(t-s)} s A_h^{1/2} \theta_h(s)\, ds + \int_0^t e^{-A_h(t-s)} A_h^{-1/2} z_h(s)\, ds,$$

so that, integrating by parts, we have

$$y_h(t) = tA_h^{-1/2}\theta_h(t) - \int_0^t e^{-A_h(t-s)} A_h^{-1/2}\big(\theta_h(s) + s\dot\theta_h(s)\big)\, ds + \int_0^t e^{-A_h(t-s)} A_h^{-1/2} z_h(s)\, ds. \tag{4.15}$$

We deal first with (4.10). For the last term on the right-hand side above, by writing $e^{-A_h(t-s)}A_h^{-1/2} z_h(s) = A_h^{1/2} e^{-A_h(t-s)} A_h^{-1} z_h(s)$, then, thanks to (2.9) and (4.8), it follows that

$$\left\| \int_0^t e^{-A_h(t-s)} A_h^{-1/2} z_h(s)\, ds \right\|_0 \le Ct^{1/2} \max_{0\le s\le t} \|A_h^{-1} z_h(s)\|_0 \le Ct^{1/2} h^4. \tag{4.16}$$

For the second term on the right-hand side of (4.15), we start by writing $e^{-A_h(t-s)} = e^{-A_h(t-s)} s^{-1/2} s^{1/2}$, and then, since, trivially, $\|s^{-1/2} e^{-A_h(t-s)}\|_0 \le s^{-1/2}$, we have

$$\begin{aligned} &\left\| \int_0^t e^{-A_h(t-s)} A_h^{-1/2} (\theta_h(s) + s\dot\theta_h(s))\, ds \right\|_0 \\ &\quad \le 2t^{1/2} \max_{0\le s\le t} s^{1/2} \big(\|A_h^{-1/2}(\theta_h(s) + s\dot\theta_h(s))\|_0 \big) \\ &\quad \le Ct^{1/2} h^4 \max_{0\le s\le t} s^{1/2} \big(\|u(s)\|_3 + s\, \|u_s(s)\|_3 \big) \le Ct^{1/2} h^4 (M_3 + M_5), \end{aligned} \tag{4.17}$$

where in the last inequality we have applied (2.8) and (2.16). The same argument shows that the first term on the right-hand side of (4.15) can be bounded by $Ct^{1/2}h^4 M_3$, which together with (4.16) and (4.17) finishes the proof of (4.10).

In order to prove (4.11), we go back to (4.15). Using (2.10) to estimate both integrals on the right-hand side of (4.15), we may write

$$\begin{aligned} \|y_h(t)\|_0 &\le t \big\| A_h^{-1/2} \theta_h(t) \big\|_0 \\ &\quad + CB\left(\frac{1}{2}, \frac{1}{2}\right) \Big(\max_{0\le s\le t} s^{1/2} \big\| A_h^{-1} (\theta_h(s) + s\dot\theta_h(s)) \big\|_0 + \max_{0\le s\le t} s^{1/2} \|A_h^{-1} z_h(s)\|_0 \Big), \end{aligned}$$

so that, thanks to (2.8), (2.16), and (4.9), it follows that (4.11) holds for $j = 0$.

It only remains to prove (4.12) in the case $j = 0$ (the case $j = 1$ is deduced by applying inverse inequality (2.5)). From (4.6) we get

$$(A_h^{-1} \dot z_h, z_h) + (z_h, z_h) = (\theta_h, z_h).$$

Then

$$\frac{d}{dt}\|A_h^{-1/2} z_h\|_0^2 + \|z_h\|_0^2 \le \|\theta_h\|_0^2.$$

Finally, by integrating with respect to t and applying (2.7) and (2.13) we obtain

$$\int_0^t \|z_h(s)\|_0^2 \, ds \le \int_0^t \|\theta_h(s)\|_0^2 \, ds \le h^6 \int_0^t \|u(s)\|_3^2 \, ds \le K_3^2 h^6,$$

which finishes the proof. ∎

4.2. Superconvergence for the velocity: $r = 3$. We need several auxiliary lemmas before proving Theorem 4.7. We start with the continuity of the nonlinear term in several different norms.

LEMMA 4.4. *For each $\alpha > 0$ there exists a constant $K > 0$ depending on α and M_2 such that, for every $w_h^1(\cdot), w_h^2(\cdot) \in V_{h,r}$ satisfying the threshold condition*

$$\left\|u(t) - w_h^1(t)\right\|_j \le \alpha h^{2-j}, \quad \left\|u(t) - w_h^2(t)\right\|_j \le \alpha h^{2-j}, \quad j = 0, 1, \ t \in [0, T], \tag{4.18}$$

the following bounds hold:

$$\left\|F(w_h^1, w_h^1) - F(w_h^2, w_h^2)\right\|_0 \le K \left\|w_h^1 - w_h^2\right\|_1, \tag{4.19}$$

$$\left\|F(w_h^1, w_h^1) - F(w_h^2, w_h^2))\right\|_{-1} \le K \left\|w_h^1 - w_h^2\right\|_0, \tag{4.20}$$

$$\left\|B_h(w_h^1, w_h^1) - B_h(w_h^2, w_h^2)\right\|_0 \le K \left\|w_h^1 - w_h^2\right\|_1, \tag{4.21}$$

$$\left\|A_h^{-1/2}(B_h(w_h^1, w_h^1) - B_h(w_h^2, w_h^2))\right\|_0 \le K \left\|w_h^1 - w_h^2\right\|_0, \tag{4.22}$$

$$\left\|A_h^{-1}(B_h(w_h^1, w_h^1) - B_h(w_h^2, w_h^2))\right\|_0 \le K\left\|A_h^{-1/2}(w_h^1 - w_h^2)\right\|_0. \tag{4.23}$$

Proof. The bounds (4.19) and (4.20) are obtained by arguing exactly as in the proof of [6, Lemma 3.1]. Using the stability of Π_h and (2.21), the proofs of (4.21) and (4.22) are easily deduced from (4.19) and (4.20).

Finally, let us prove (4.23). Let us write $w_h = w_h^1 - w_h^2$. Taking into account that

$$h^2\|w_h\|_1 \le C\|A_h^{-1/2} w_h\|_0 \tag{4.24}$$

and using (2.19) and (4.21), it follows that

$$\begin{aligned}\left\|A_h^{-1}(B_h(w_h^1, w_h^1) - B_h(w_h^2, w_h^2))\right\|_0 \le& \left\|A^{-1}\Pi(F(w_h^1, w_h^1) - F(w_h^2, w_h^2))\right\|_0 \\ &+ C\left\|A_h^{-1/2} w_h\right\|_0.\end{aligned} \tag{4.25}$$

We now estimate the first term on the right-hand side above by duality, since

$$\left\|A^{-1}\Pi(F(w_h^1, w_h^1) - F(w_h^2, w_h^2))\right\|_0 \le \max_{\phi \in H, \phi \ne 0} \frac{\left|(F(w_h^1, w_h^1) - F(w_h^2, w_h^2), A^{-1}\phi)\right|}{\|\phi\|_0}.$$

For this purpose, using the skew-symmetry property of b, we write

$$\begin{aligned}(F(w_h^1, w_h^1) - F(w_h^2, w_h^2), A^{-1}\phi) &= b(w_h, w_h^2, A^{-1}\phi) + b(w_h^1, w_h, A^{-1}\phi) \\ &= b(w_h, u(t), A^{-1}\phi) - b(u(t), A^{-1}\phi, w_h) \\ &\quad + b(w_h, A^{-1}\phi, u(t) - w_h^2) + b(u(t) - w_h^1, A^{-1}\phi, w_h).\end{aligned} \tag{4.26}$$

We now notice that $|b(w_h, A^{-1}\phi, u(t)-w_h^2)| \le C\|w_h\|_1 \|\phi\|_0 \|u(t)-w_h^2\|_0$, so that, thanks to (4.24) and (4.18), we have

$$\left|b(w_h, A^{-1}\phi, u(t)-w_h^2)\right| \le C\|w_h\|_1 h^2 \|\phi\|_0 \le C\left\|A_h^{-1/2}w_h\right\|_0 \|\phi\|_0 .$$

Similarly, $|b(u(t)-w_h^1, A^{-1}\phi, w_h)| \le C\|w_h\|_0 \|\phi\|_0 \|u(t)-w_h^1\|_1$. Since $h\|w_h\|_0 \le C\|A_h^{-1/2}w_h\|_0$, it follows that

$$\left|b(u(t)-w_h^1, A^{-1}\phi, w_h)\right| \le Ch\|w_h\|_0 \|\phi\|_0 \le C\left\|A_h^{-1/2}w_h\right\|_0 \|\phi\|_0 ,$$

so that from (4.26) we get

$$\begin{aligned}\left|(F(w_h^1,w_h^1)-F(w_h^2,w_h^2), A^{-1}\phi)\right| \le& \left|b(w_h, u(t), A^{-1}\phi)\right| + \left|b(u(t), A^{-1}\phi, w_h)\right| \\ &+ C\left\|A_h^{-1/2}w_h\right\|_0 \|\phi\|_0 .\end{aligned} \tag{4.27}$$

Finally, since by applying the divergence theorem we have

$$b(w_h, u(t), A^{-1}\phi) = (w_h, \nabla u(t)\cdot A^{-1}\phi) - \frac{1}{2}(w_h, \nabla(u(t)\cdot A^{-1}\phi)),$$

it follows that

$$\left|b(w_h, u(t), A^{-1}\phi)\right| \le C\|w_h\|_{-1}\left(\left\|\nabla u(t)\cdot A^{-1}\phi\right\|_1 + \left\|\nabla(A^{-1}\phi\cdot u(t))\right\|_1\right), \tag{4.28}$$

$$\left|b(u(t), A^{-1}\phi, w_h)\right| \le C\|w_h\|_{-1}\left(\left\|(u(t)\cdot\nabla)A^{-1}\phi\right\|_1 + \left\|(\nabla\cdot u(t))A^{-1}\phi\right\|_1\right). \tag{4.29}$$

Since

$$\begin{aligned}\left\|\nabla u\cdot A^{-1}\phi\right\|_1 &\le C\big(\|u\|_2\left\|A^{-1}\phi\right\|_{L^\infty} + \|u\|_{W^{1,4}}\left\|A^{-1}\phi\right\|_{W^{1,4}}\big)\\ &\le C\|u\|_2\left\|A^{-1}\phi\right\|_2 \le C\|u\|_2\|\phi\|_0,\end{aligned}$$

arguing similarly with the rest of the terms on the right-hand side of (4.28)–(4.29), from (4.25) and (4.27) the proof is easily finished. ∎

Remark 4.1. In what follows we will apply Lemma 4.4 to functions $w_h^1 = v_h$, satisfying (4.4), and $w_h^2 = u_h$, satisfying (4.3). Let us now check that the threshold condition (4.18) is satisfied. We first observe that $\|v_h - u\|_0 \le \|z_h\|_0 + \|u - \Pi_h u\|_0$ so that by applying (4.8) and (2.7) we get $\|v_h - u\|_0 \le Ch^2$, and then v_h satisfies (4.18). For the bound $\|u-u_h\|_0 \le Ch^2$, we refer the reader to [36, Theorem 3.1].

LEMMA 4.5. *Let (u,p) be the solution of* (1.1)–(1.2). *Then there exists a positive constant C such that the discrete velocity v_h defined by* (4.4) *satisfies*

$$\|A_h^{-1}(B_h(v_h(t),v_h(t)) - B_h(u(t),u(t)))\|_0 \le \frac{C}{t^{(r-2)/2}}h^{r+1}, \quad t\in(0,T], \quad r=3,4.$$

Proof. Let us write $\rho_h = B_h(v_h,v_h) - B_h(u,u)$. By applying (2.19) we have

$$\|A_h^{-1}\rho_h\|_0 \le Ch^2\|F(v_h,v_h)-F(u,u)\|_0 + \|A^{-1}\Pi(F(v_h,v_h)-F(u,u))\|_0.$$

Using then (4.19) from Lemmas 4.4 and 4.1 we get

$$\|A_h^{-1}\rho_h\|_0 \le Ch^2\|v_h-u\|_1 + C(\|v_h-u\|_1\|v_h-u\|_0 + C\|v_h-u\|_{-1}).$$

Applying now the standard bounds for Π_h (see (2.7), (2.8)) together with the estimates (4.8), (4.10) for z_h in Lemma 4.3 when $r = 3$ and (4.11), (4.8) when $r = 4$, the conclusion is reached. ∎

LEMMA 4.6. *Let* (u, p) *be the solution of* (1.1)–(1.2). *Then there exists a positive constant* C *such that the discrete velocity* v_h *defined by* (4.4) *and the Hood–Taylor element approximation to* u, u_h, *satisfy the following bound:*

$$\|A_h^{-1/2}(v_h(t) - u_h(t))\|_0 \le Ch^4, \quad r \ge 3, \quad t \in (0,T].$$

Proof. Let us consider $y_h(t) = A_h^{-1/2} e_h(t)$ and $\rho_h = B_h(v_h, v_h) - B_h(u, u)$. From (4.5) it follows that

$$y_h(t) = \int_0^t e^{-(t-s)A_h} A_h^{-1/2} \big(B_h(u_h, u_h) - B_h(v_h, v_h)\big)\, ds + \int_0^t e^{-(t-s)A_h} A_h^{-1/2} \rho_h(s)\, ds.$$

Applying (2.9) and (4.23) we have that

$$\|y_h(t)\|_0 \le \int_0^t \frac{C}{\sqrt{t-s}} \|y_h(s)\|_0 \, ds + \left\| \int_0^t e^{-(t-s)A_h} A_h^{-1/2} \rho_h(s)\, ds \right\|_0,$$

so that a generalized Gronwall lemma [35, pp. 188–189] allows us to write

$$\max_{0 \le t \le T} \|y_h(t)\|_0 \le C \max_{0 \le t \le T} \left\| \int_0^t e^{-(t-s)A_h} A_h^{-1/2} \rho_h(s)\, ds \right\|_0.$$

Using (2.10) we obtain

$$\max_{0 \le t \le T} \|y_h(t)\|_0 \le CB\left(\frac{1}{2}, \frac{1}{2}\right) \max_{0 \le s \le T} s^{1/2} \big\| A_h^{-1} \big(B_h(v_h(s), v_h(s)) - B_h(u(s), u(s))\big)\big\|_0,$$

where, by applying Lemma 4.5 with $r = 3$, the proof is finished. ∎

THEOREM 4.7 (superconvergence for the velocity). *Let* (u, p) *be the solution of* (1.1)–(1.2). *There exists a positive constant* C *such that the discrete velocity* v_h *defined by* (4.4) *and the Hood–Taylor element approximation to* u, u_h, *satisfy the following bound:*

$$\|v_h(t) - u_h(t)\|_j \le \frac{C}{t^{1/2}} |\log(h)| h^{4-j}, \quad t \in (0,T], \quad j = 0, 1, \quad r \ge 3. \tag{4.30}$$

Proof. We prove the error bound for the L^2 norm, from which the bound in the H^1 norm is readily obtained by applying the inverse inequality (2.5). Let us consider $y_h(t) = t^{1/2} e_h(t)$. An easy calculation shows that

$$\dot{y}_h + A_h y_h + t^{1/2}(B_h(v_h, v_h) - B_h(u_h, u_h)) = t^{1/2} \rho_h + \frac{1}{2t^{1/2}} e_h,$$

where $\rho_h = B_h(v_h, v_h) - B_h(u, u)$, and, thus,

$$y_h(t) = \int_0^t e^{-A_h(t-s)} s^{1/2} (B_h(u_h, u_h) - B_h(v_h, v_h))\, ds$$

$$+ \int_0^t e^{-A_h(t-s)} \left(s^{1/2} \rho_h + \frac{1}{2s^{1/2}} e_h \right) ds.$$

For the first term on the right-hand side above, using (2.9) and (4.22) we have

$$\left\| \int_0^t e^{-A_h(t-s)} s^{1/2} (B_h(u_h,u_h) - B_h(v_h,v_h))\, ds \right\|_0 \le C \int_0^t \frac{\|y_h\|_0}{\sqrt{t-s}}\, ds,$$

so that by applying a generalized Gronwall lemma [35, pp. 188–189], it follows that

$$\max_{0\le t\le T} \|y_h(t)\|_0$$
$$\le C\left(\max_{0\le t\le T} \left\| \int_0^t e^{-A_h(t-s)} s^{1/2} \rho_h\, ds \right\|_0 + \max_{0\le t\le T} \left\| \int_0^t e^{-A_h(t-s)} \frac{e_h}{s^{1/2}}\, ds \right\|_0 \right).$$

Applying now Lemma 4.2 and (2.10) we have

$$\max_{0\le t\le T} \|y_h(t)\|_0$$
$$\le C|\log(h)| \max_{0\le s\le T} \left\| s^{1/2} A_h^{-1} \rho_h(s) \right\|_0 + CB\left(\frac{1}{2},\frac{1}{2}\right) \max_{0\le s\le T} \left\| A_h^{-1/2} e_h(s) \right\|_0,$$

where Lemmas 4.5 and 4.6 finish the proof. □

COROLLARY 4.8. *Let* (u,p) *be the solution of* (1.1)–(1.2). *Then, for* $r\ge 3$, *there exists a positive constant* C *such that the Hood–Taylor element approximation to* u, u_h, *satisfies the following bound for* $j=0,1$:

$$\|u_h(t) - u(t)\|_j \le \frac{C}{t^{1/2}} h^{3-j}, \quad t\in(0,T].$$

Proof. By rewriting $u - u_h = (u - \Pi_h u) + (\Pi_h u - v_h) + (v_h - u_h)$, rewriting $u - u_h = (u - \Pi_h u) + (\Pi_h u - v_h) + (v_h - u_h)$, and applying (2.7), Lemma 4.3, and Theorem 4.7, the corollary is proved. □

Remark 4.2. Notice that by applying Lemma 4.2 to (4.13), for the z_h solution of (4.6), we obtain

$$\|z_h(t)\|_0 \le C M_2 h^2 \left|\log(h)\right|. \tag{4.31}$$

With this estimate and a much simpler analysis than the previous one, it is possible to prove superconvergence results in the H^1 norm for low-order LBB-stable pairs of mixed finite-element methods, such as the so-called mini element [3]. More precisely, for the e_h solution of (4.5), the following bound holds:

$$\max_{0\le t\le T} \|e_h(t)\|_1 \le C \left|\log(h)\right|^2 h^2,$$

with C depending only on M_2.

4.3. Superconvergence for the velocity: $r = 4$. As before, several auxiliary lemmas are needed before Theorem 4.15. We start with a generalized Gronwall lemma.

LEMMA 4.9. *If* u *is a nonnegative function, continuous in* $[0,T]$ *and satisfying*

$$u(t) \le u_0 + \beta \int_0^t \left(\frac{1}{(t-s)^{1/2}} + \frac{1}{s^{1/2}} \right) u(s)\, ds, \qquad 0\le t<T,$$

for some $u_0>0$ *and* $\beta>0$, *then*

$$u(t) \le E_{1/2}(2\beta(\pi t)^{1/2}) u_0, \qquad 0\le t<T,$$

where $E_\alpha(z)$ denotes the Mittag–Leffler function (see 4.36 below).

Proof. Let us consider the operator

$$Eu(t) = \beta \int_0^t \left(\frac{1}{(t-s)^{1/2}} + \frac{1}{s^{1/2}} \right) u(s)\, ds, \qquad 0 \le t < T.$$

We notice that

$$u(t) \le u_0 + Eu(t) \le u_0 + Eu_0 + E^2 u(t) \le (1 + E + \cdots + E^n) u_0 + E^{n+1} u(t). \tag{4.32}$$

We have that

$$Eu_0 = 4\beta t^{1/2} u_0 = (2 + B(1/2, 1))(\beta t^{1/2}) u_0,$$

where $B(x, y)$ denotes the Beta function (see, e.g., [12]). We also notice that

$$\begin{aligned} E^2 u_0 &= 4\beta^2 u_0 \int_0^t \left(\frac{1}{(t-s)^{1/2}} + \frac{1}{s^{1/2}} \right) s^{1/2}\, ds \\ &= 4\beta^2 u_0 \left(t + \int_0^t \frac{s^{1/2}}{(t-s)^{1/2}}\, ds \right). \end{aligned}$$

Now taking into account that, by means of the change of variables, $s = tx$ we have

$$\int_0^t \frac{s^{1/2}}{(t-s)^{1/2}}\, ds = t \int_0^1 \frac{x^{1/2}}{(1-x)^{1/2}}\, dx = tB(1/2, 3/2),$$

and, thus,

$$E^2 u_0 = 4(1 + B(1/2, 3/2)) \left(\beta t^{1/2}\right)^2 u_0 = \left(\prod_{l=1}^{2} \left(\frac{2}{l} + B\left(\frac{1}{2}, \frac{l+1}{2} \right) \right) \right) \left(\beta t^{1/2}\right)^2 u_0.$$

Assume that for $j = 2, \dots, n$

$$E^j u_0 = \left(\prod_{l=1}^{j} \left(\frac{2}{l} + B\left(\frac{1}{2}, \frac{l+1}{2} \right) \right) \right) (\beta t^{1/2})^j, \tag{4.33}$$

Let us compute $E^{n+1} u_0$. Since $E^{n+1} u_0 = E E^n u_0$, we have

$$E^{n+1} u_0 = \beta^{n+1} u_0 \left(\prod_{l=1}^{n} \left(\frac{2}{l} + B\left(\frac{1}{2}, \frac{l+1}{2} \right) \right) \right) \int_0^t \left(\frac{1}{(t-s)^{1/2}} + \frac{1}{s^{1/2}} \right) s^{n/2}\, ds.$$

But

$$\int_0^t \left(\frac{s^{n/2}}{(t-s)^{1/2}} + s^{(n-1)/2} \right) ds = \frac{2}{n+1} t^{(n+1)/2} + \int_0^t \frac{s^{n/2}}{(t-s)^{1/2}}\, ds,$$

and, using again the change of variables $s = tx$,

$$\int_0^t \frac{s^{n/2}}{(t-s)^{1/2}}\, ds = t^{(n+1)/2} \int_0^1 \frac{x^{n/2}}{(1-x)^{1/2}}\, ds = t^{(n+1)/2} B\left(\frac{1}{2}, \frac{n}{2} + 1 \right),$$

and since $\frac{n}{2} + 1 = \frac{(n+1)+1}{2}$, it follows that (4.33) also holds for $j = n + 1$.

Since, on the one hand

$$B\left(\frac{1}{2},\frac{n+1}{2}\right)=\Gamma\left(\frac{1}{2}\right)\frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+1)}, \tag{4.34}$$

where $\Gamma(z)$ denotes the Gamma function, and on the other hand (see, e.g., [12, 6.1.47])

$$1=\lim_{z\to\infty} z^{1/2}\frac{\Gamma(z)}{\Gamma(z+1/2)},$$

we have that

$$E^{n+1}u(t)\le E^{n+1}\left\|u\right\|_{L^\infty(0,T)}\to 0 \qquad \text{as } n\to\infty.$$

Thus, from (4.32) and (4.33) it follows that

$$u(t)\le u_0\sum_{n=0}^{\infty} c_n\big(\beta t^{1/2}\big)^n, \qquad 0\le t\le T, \tag{4.35}$$

where

$$c_0=1, \qquad \frac{c_n}{c_{n-1}}=\frac{2}{n}+B\left(\frac{1}{2},\frac{n+1}{2}\right), \quad n=1,3,\dots .$$

Furthermore, on the one hand, for $n=1$ we have $2/n=B(\frac{1}{2},\frac{n+1}{2})=2$, and, on the other hand, since for $x\in(0,1)$ we have $1/\sqrt{x(1-x)}\ge 2$, it follows that, for $n\ge 2$,

$$B\left(\frac{1}{2},\frac{n+1}{2}\right)=\int_0^1\frac{x^{(n-1)/2}}{\sqrt{1-x}}\,dx=\int_0^1\frac{x^{n/2}}{\sqrt{x(1-x)}}\,dx\ge 2\int_0^1 x^{n/2}\,dx=\frac{4}{n+2}\ge\frac{2}{n},$$

so that by $c_n/c_{n-1}\le 2B(\frac{1}{2},\frac{n+1}{2})$ for $n=1,2,\dots$. Thus, if we set $k_0=2\Gamma(1/2)=2\sqrt{\pi}$, then in view of (4.34) we have $c_n\le k_0\Gamma(\frac{n}{2}+\frac{1}{2})\Gamma(\frac{n}{2}+1)^{-1}c_{n-1}$, that is,

$$c_n\le k_0\frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+1)}c_{n-1}\le k_0^2\frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+1)}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+\frac{1}{2})}c_{n-2}=k_0^2\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+1)}c_{n-2},$$

so that by iterating this process we have

$$c_n\le\frac{k_0^n}{\Gamma(\frac{n}{2}+1)}c_0, \qquad n=1,2,\dots .$$

Then from (4.35) it follows that

$$u(t)\le u_0\sum_{n=0}^{\infty}\frac{\big(\beta k_0 t^{1/2}\big)^n}{\Gamma(\frac{n}{2}+1)}=E_{1/2}(\beta k_0 t^{1/2})u_0,$$

where $E_\alpha(z)$ is the Mittag–Leffler function

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)}. \qquad \square \tag{4.36}$$

Lemma 4.10. *Let $v,w\in H_0^1(\Omega)\cap H^3(\Omega)$, and let $g\in H_0^1(\Omega)$, satisfying*

$$(\nabla\cdot\ g,\chi_h)=0, \qquad \forall\chi_h\in Q_{h,r-1}, \quad r\ge 4. \tag{4.37}$$

Then there exists a positive constant C such that the following bounds hold:

$$|b(v,w,g)| \le C\big(\left\|A^{-1}\Pi g\right\|_0 + h^2 \left\|g\right\|_0\big)\big(\|(v\cdot\nabla)w\|_2 + \|(\nabla\cdot v)w\|_2\big), \tag{4.38}$$

$$|b(g,w,v)| \le C\big(\left\|A^{-1}\Pi g\right\|_0 + h^2 \left\|g\right\|_0\big)\left\|\nabla w\cdot v\right\|_2 + Ch^2\left\|g\right\|_0\left\|\nabla v\cdot w\right\|_2. \tag{4.39}$$

Proof. We will first prove the following result whose proof is similar to that of Lemma 3.11 in [6]. Let $v \in H_0^1(\Omega)^d$, satisfying $(\nabla\cdot v, \chi_h) = 0$ for all $\chi_h \in Q_{h,r-1}$. Then for $f \in H_0^1(\Omega)^d \cap H^s(\Omega)^d$ the following bound holds:

$$|(v,f)| \le \big(\left\|A^{-s/2}\Pi v\right\|_0 + Ch^s\left\|v\right\|_0\big)\left\|f\right\|_s, \qquad s=1,2. \tag{4.40}$$

To prove (4.40) we decompose

$$f = \Pi f + (I-\Pi)f = \Pi f + \nabla q$$

for some $q \in H^{s+1}$. Since Leray's projector Π is continuous from $H_0^1(\Omega)\cap H^s(\Omega)$ onto H^s, we have

$$\left\|\Pi f\right\|_s \le C\left\|f\right\|_s, \qquad \left\|q\right\|_{H^{s+1}/\mathbb{R}} \le C\left\|f\right\|_s \tag{4.41}$$

for some positive constant C. Thus,

$$(v,f) = (v,\Pi f) + (v,\nabla q) = (A^{-s/2}\Pi v, A^{s/2}\Pi f) + (v,\nabla q). \tag{4.42}$$

Since, for any $\chi_h \in Q_{h,r-1}$, $|(v,\nabla q)| = |(v,\nabla(q-\chi_h))|$, by taking χ_h as the interpolant of q in $Q_{h,r-1}$, from (4.41) and (4.42) it follows that

$$|(v,f)| \le Ch^s\left\|v\right\|\left\|f\right\|_s + \big|(A^{-s/2}\Pi v, A^{s/2}\Pi f)\big|.$$

Finally, since $\|A^{s/2}\Pi f\|_0 \le C\left\|\Pi f\right\|_s \le C'\left\|f\right\|_s$, (4.40) follows.

Notice now that (4.38) is a direct consequence of (4.40). For the second bound, the same argument shows that $((g\cdot\nabla)w, v)) = (g, \nabla w\cdot v)$ can be bounded by the first term on the right-hand side of (4.39). Now by using the divergence theorem and (4.37) we have

$$\big((\nabla\cdot g)w, v\big) = -\big(g, \nabla(w\cdot v)\big) = -\big(g,\nabla(w\cdot v - \chi_h)\big) \qquad \forall \chi_h \in Q_{h,r-1}.$$

Taking χ_h as the interpolant of $(w{\cdot}v)$ in $Q_{h,r-1}$ and taking into account that $\nabla(w{\cdot}v) = \nabla w\cdot v + \nabla v\cdot w$, the proof is easily concluded. ∎

LEMMA 4.11. *There exists a positive constant C such that for any $\phi_h \in V_{h,r}$ the following bound holds for $\psi_h(t) = e^{-tA_h}A_h^{-1}\phi_h$ and $\Psi(t) = e^{-tA}A^{-1}\Pi\phi_h$:*

$$\|\Psi(t) - \psi_h(t)\|_1 \le \frac{Ch^2}{t^{1/2}}\|\phi_h\|_0, \quad t>0. \tag{4.43}$$

Proof. Let us first observe that, since $\Psi_t = -A\Psi$, then, thanks to (2.9) and (2.15), for $t>0$ we have

$$\left\|\Psi(t)\right\|_{2+j} \le C\left\|\Psi_t(t)\right\|_j \le Ct^{-j/2}\left\|\phi_h\right\|_0, \qquad j=0,1, \tag{4.44}$$

$$\left\|\Psi_{tt}(t)\right\|_0 \le Ct^{-1}\left\|\phi_h\right\|_0. \tag{4.45}$$

We set $\varphi_h(t) = e^{-tA_h}\Pi_h A^{-1}\Pi\phi_h$. We decompose $\psi_h - \Psi = (\psi_h - \varphi_h) + (\varphi_h - \Psi)$, and we will bound the two terms on the right-hand side separately. For the first one, taking into account that $\psi_h(t) - \varphi_h(t) = e^{-tA_h}\Pi_h(A_h^{-1}\Pi_h - A^{-1}\Pi)\phi_h$, and using (2.9) and (2.17), we have

$$\|\psi_h(t) - \varphi_h(t)\|_1 \le \frac{C}{t^{1/2}} \left\|(A_h^{-1}\Pi_h - A^{-1}\Pi)\phi_h\right\|_0 \le \frac{Ch^2}{t^{1/2}}\|\phi_h\|_0.$$

In order to bound $\varphi_h - \Psi$, we write $\varphi_h - \Psi = \varphi_h - \Pi_h\Psi + (I - \Pi_h)\Psi$, so that, in view of (2.7) and (4.44) we only have to estimate $\varphi_h - \Pi_h\Psi$. For this purpose, on the one hand, we have $\dot\varphi_h + A_h\varphi_h = 0$. On other hand, since $\Psi_t = -A\Psi$, we have

$$\Pi_h\Psi_t + A_h\Pi_h\Psi = \Pi_h\Psi_t - A_h\Pi_h A^{-1}\Psi_t = A_h(A_h^{-1}\Pi_h - \Pi_h A^{-1})\Psi_t,$$

so that for $\epsilon_h = \varphi_h - \Pi_h\Psi$ and $\rho_h = (A_h^{-1}\Pi_h - \Pi_h A^{-1})\Psi_t$ we have

$$\dot\epsilon_h + A_h\epsilon_h = A_h\rho_h,$$

an equation which is similar to (4.6). Thus, arguing as in the proof of (4.8) we have

$$\|\epsilon_h(t)\|_0 \le C \max_{0\le s\le t} s^{1/2}\left\|A_h^{1/2}\rho_h(s)\right\|_0 \le Ch^2 \|\phi_h\|_0\,, \tag{4.46}$$

where, in the last inequality, we have applied (2.7), (2.18), and the case $j = 1$ in (4.44). Furthermore, since for $y_h(t) = A_h^{1/2}t^{1/2}\epsilon_h(t)$ it is easy to see that

$$y_h(t) = \int_0^t e^{-A_h(t-s)}s^{1/2}A_h^{3/2}\rho_h(s)\,ds + \frac{1}{2}\int_0^t e^{-A_h(t-s)}s^{-1/2}A_h^{1/2}\epsilon_h(s)\,ds,$$

so that by integrating by parts in the first integral above and using (2.10) we have

$$\|y_h(t)\|_0 \le t^{1/2}\|\rho_h(t)\|_1 + CB\left(\frac{1}{2},\frac{1}{2}\right)\left(\max_{0\le s\le t}\left\|\frac{1}{2}\rho_h(s) + s\dot\rho_h(s)\right\|_0 + \max_{0\le s\le t}\|\epsilon_h(s)\|_0\right).$$

The first and the last terms on the right-hand side above are already bounded in (4.46), whereas the second term, applying (2.7), (2.17), and (4.44)–(4.45), can also be bounded by $Ch^2\|\phi_h\|_0$, which completes the proof. ∎

LEMMA 4.12. *Let (u, p) be the solution of* (1.1)–(1.2). *There exists a constant $C > 0$ such that for any $g \in H_0^1(\Omega)^d$ and $\phi_h \in V_{h,r}$ the following bounds hold for $\psi_h(t) = e^{-tA_h}A_h^{-1}\phi_h$, $\Psi(t) = e^{-tA}A^{-1}\Pi\phi_h$, and $0 < s < t \le T$:*

$$|b(g, u(s), \psi_h(t-s))| \le |b(g, u(s), \Psi(t-s))| + C\frac{M_2}{\sqrt{t-s}}h^2\|g\|_0\|\phi_h\|_0\,, \tag{4.47}$$

$$|b(u(s), \psi_h(t-s), g)| \le |b(u(s), \Psi(t-s), g)| + C\frac{M_2}{\sqrt{t-s}}h^2\|g\|_0\|\phi_h\|_0\,. \tag{4.48}$$

Proof. We will prove only (4.47) since (4.48) can be proved by reasoning similarly. Also, we will drop the dependence on s and $t - s$. We first observe that

$$b(g, u, \psi_h) = b(g, u, \Psi) + b(g, u, \psi_h - \Psi).$$

On the one hand, we have

$$|((g\cdot\nabla)u, \psi_h - \Psi)| \le C\|g\|_0\|\nabla u\|_{L^4(\Omega)}\|\psi_h - \Psi\|_{L^4(\Omega)}\,,$$

so that by using (2.1) and (4.43) it follows that

$$|((g\cdot\nabla)u,\psi_h-\Psi)|\le CM_2\|g\|_0\|\psi_h-\Psi\|_1\le C\frac{M_2}{\sqrt{t-s}}h^2\|g\|_0\|\phi_h\|_0. \tag{4.49}$$

On the other hand, since $((\nabla\cdot g)u,\psi_h-\Psi)=-(g,\nabla(u\cdot(\psi_h-\Psi)))$, we have

$$\|((\nabla\cdot g)u,(\psi_h-\Psi))\|_0\le C\|g\|_0\big(\|\nabla u\|_{L^4}\|\psi_h-\Psi\|_{L^4}+\|u\|_{L^\infty}\|\nabla(\psi_h-\Psi)\|_0\big).$$

Applying again (2.1) and (4.43) we get

$$\|((\nabla\cdot g)u,(\psi_h-\Psi))\|_0\le CM_2\|g\|_0\|\psi_h-\Psi\|_1\le C\frac{M_2}{\sqrt{t-s}}h^2\|g\|_0\|\phi_h\|_0,$$

which together with (4.49) allow us to conclude (4.47). □

LEMMA 4.13. *Let (u,p) be the solution of (1.1)–(1.2). There exists a positive constant C such that, for $r\ge 4$ and $0<s<t\le T$, the discrete velocity v_h defined by (4.4) and the Hood–Taylor element approximation to u, u_h, satisfy the following bound:*

$$\begin{aligned}&\big\|e^{-A_h(t-s)}A_h^{-1}\big(B_h(u_h(s),u_h(s))-B_h(v_h(s),v_h(s))\big)\big\|_0\\ &\quad\le C\left(\frac{1}{\sqrt{t-s}}+\frac{1}{\sqrt{s}}\right)\big\|A_h^{-1}\big(u_h(s)-v_h(s)\big)\big\|_0.\end{aligned} \tag{4.50}$$

Proof. We write $\rho_h=B_h(u_h(s),u_h(s))-B_h(v_h(s),v_h(s))$. We will bound the norm of $e^{-A_h(t-s)}A_h^{-1}\rho_h$ by duality. As usual, we will omit the dependence on s whenever this does not cause confusion. For $\phi_h\in V_{h,r}$ we set $\psi_h=e^{-A_h(t-s)}A_h^{-1}\phi_h$, so that

$$\big\|e^{-A_h(t-s)}A_h^{-1}\rho_h\big\|_0=\max_{\phi_h\in V_{h,r}\phi_h\neq 0}\frac{|(\rho_h,\psi_h)|}{\|\phi_h\|_0}.$$

We have

$$(\rho_h,\psi_h)=-b(e_h,\psi_h,u_h)-b(v_h,\psi_h,e_h), \tag{4.51}$$

where we recall that $e_h=v_h-u_h$. For the second term on the right-hand side of (4.51) we have

$$|b(v_h,\psi_h,e_h)|\le|b(u,\psi_h,e_h)|+|b(v_h-u,\psi_h,e_h)|$$

and

$$|b(v_h-u,\psi_h,e_h)|\le\|v_h-u\|_0\|\nabla\psi_h\|_\infty\|e_h\|_0+\|\nabla\cdot(v_h-u)\|_0\|\psi_h\|_\infty\|e_h\|_0.$$

Applying (2.5) and (2.22) we have $\|\nabla\psi_h\|_\infty\le Ch^{-1}\|\psi_h\|_\infty\le Ch^{-1}\|\phi_h\|_0$. Also, since $(v_h-u)=z_h+(\Pi_h u-u)$, by taking into account (2.7) and Lemma 4.3 we get

$$|b(v_h,\psi_h,e_h)|\le|b(u,\psi_h,e_h)|+\frac{CM_4}{s^{1/2}}h^2\|e_h\|_0\|\phi_h\|_0. \tag{4.52}$$

Similarly,

$$|b(e_h,\psi_h,u_h)|\le|b(e_h,\psi_h,u)|+|b(e_h,\psi_h,u_h-u)|,$$

and

$$|b(e_h, \psi_h, u_h - u)| \le \left(\|e_h\|_0 \|u_h - u\|_1 + \|e_h\|_1 \|u_h - u\|_0 \right) \|\psi_h\|_\infty .$$

Since by virtue of (2.5) we have $\|e_h\|_1 \le Ch^{-1} \|e_h\|_0$, by applying Corollary 4.8 we get

$$|b(e_h, \psi_h, u_h)| \le |b(e_h, \psi_h, u)| + \frac{C}{s^{1/2}} h^2 \|e_h\|_0 \|\phi_h\|_0 . \tag{4.53}$$

Thus, from (4.51), (4.52), and (4.53) and, since

$$h^2 \|e_h\|_0 \le C \|A_h^{-1} e_h\|_0, \tag{4.54}$$

it follows that

$$|(\rho_h, \psi_h)| \le |b(e_h, \psi_h, u)| + |b(u, \psi_h, e_h)| + \frac{C}{\sqrt{s}} \left\|A_h^{-1} e_h\right\|_0 \|\phi_h\|_0 . \tag{4.55}$$

Next, we apply Lemma 4.12 to get

$$\begin{aligned} |(\rho_h, \psi_h)| &\le |b(e_h, \Psi(t-s), u)| + |b(u, \Psi(t-s), e_h)| \\ &\quad + \left(\frac{C}{\sqrt{t-s}} + \frac{C}{\sqrt{s}} \right) \left\|A_h^{-1} e_h\right\|_0 \|\phi\|_0 . \end{aligned} \tag{4.56}$$

Now we apply Lemma 4.10 to the first two terms on the right-hand side of (4.56), with g replaced by e_h and v by u. Let us first observe that, using (2.20) and (4.54), we have $\|A^{-1}\Pi e_h\|_0 + h^2 \|e_h\|_0 \le C\|A_h^{-1} e_h\|_0$, so that in order to finish the proof we only have to bound $\|(u \cdot \nabla)\Psi\|_2$, $\|\nabla \Psi \cdot u\|_2$, and $\|\nabla u \cdot \Psi\|_2$ (recall that $(\nabla \cdot u) = 0$).

Since

$$\|(u \cdot \nabla)\Psi\|_2 \le \|u\|_2 \|\nabla \Psi\|_\infty + \|u\|_{W^{1,4}} \|\Psi\|_{W^{2,4}} + \|u\|_\infty \|\Psi\|_3 ,$$

taking into account that $\|\Psi\|_3 \le Ct^{-1/2}\|\phi_h\|_0$ and applying standard Sobolev inequalities (2.1) we obtain

$$\|(u(s) \cdot \nabla)\Psi(t-s)\|_2 \le \frac{CM_2}{\sqrt{t-s}} \|\phi_h\|_0 . \tag{4.57}$$

Arguing similarly we get

$$\|\nabla \Psi(t-s) \cdot u(s)\|_2 \le \frac{CM_2}{\sqrt{t-s}} \|\phi_h\|_0 . \tag{4.58}$$

Finally, since

$$\|\nabla u \cdot \Psi\|_2 \le \|\nabla u\|_{L^4} \|\Psi\|_{W^{2,4}} + \|\nabla u\|_1 \|\Psi\|_{W^{1,\infty}} + \|\nabla u\|_2 \|\Psi\|_\infty ,$$

taking into account that $\|\Psi\|_2 \le C\|\phi_h\|_0$ and using (2.1) again, we deduce

$$\begin{aligned} \|\nabla u(s) \cdot \Psi(t-s)\|_2 &\le C \left(\frac{M_2}{\sqrt{t-s}} + \|u(s)\|_3 \right) \|\phi_h\|_0 \\ &\le C \left(\frac{M_2}{\sqrt{t-s}} + \frac{M_3}{\sqrt{s}} \right) \|\phi_h\|_0 . \end{aligned} \tag{4.59}$$

Thus, inequalities (4.57), (4.58), and (4.59), together with (4.56) and Lemma 4.10, allow us to conclude the proof. ∎

LEMMA 4.14. *Let (u,p) be the solution of* (1.1)–(1.2). *There exists a positive constant C such that the discrete velocity v_h defined by* (4.4) *and the Hood–Taylor element approximation to u, u_h, satisfy the following bound for $r \geq 4$:*

$$\|A_h^{-1}(v_h(t) - u_h(t))\|_0 \leq Ch^5, \quad t \in (0,T].$$

Proof. Let us consider $y_h(t) = A_h^{-1}e_h(t)$ and $\rho_h = B_h(v_h,v_h) - B_h(u,u)$. From (4.5) we get

$$y_h(t) = \int_0^t e^{-(t-s)A_h} A_h^{-1}\big(B_h(u_h,u_h) - B_h(v_h,v_h)\big)\,ds + \int_0^t e^{-(t-s)A_h} A_h^{-1}\rho_h(s)\,ds.$$

Thanks to Lemma 4.13 we obtain

$$\|y_h(t)\|_0 \leq C\int_0^t \left(\frac{1}{\sqrt{t-s}} + \frac{1}{\sqrt{s}}\right)\|y_h(t)\|_0 + \left\|\int_0^t e^{-(t-s)A_h} A_h^{-1}\rho_h(s)\,ds\right\|_0,$$

so that by applying Lemma 4.9, we only have to show that

$$\max_{0\leq t\leq T}\left\|\int_0^t e^{-A_h(t-s)} A_h^{-1}\rho_h(s)\,ds\right\|_0 \leq Ch^5. \tag{4.60}$$

For this purpose, we will first estimate $e^{-A_h(t-s)}A_h^{-1}\rho_h(s)$ by duality. Let us take $\phi_h \in V_{h,r}$ and set $\psi_h = e^{-A_h(t-s)}A_h^{-1}\phi_h$ and $z = u - v_h$, so that we have

$$(\rho_h,\psi_h) = b(z,u,\psi_h) + b(u,z,\psi_h) - b(z,z,\psi_h). \tag{4.61}$$

For the third term on the right-hand side above, by applying (2.22), we have

$$\big|b(z,z,\psi)\big| \leq C\,\|z\|_0\,\|z\|_1\,\|\psi_h\|_{L^\infty} \leq C\,\|z\|_0\,\|z\|_1\,\|\phi_h\|_0\,.$$

For the other two terms on the right-hand side of (4.61), Lemma 4.12 shows that

$$|b(z,u,\psi_h) + b(u,z,\psi_h)| \leq |b(z,\Psi(t-s),u)| + |b(u,\Psi(t-s),z)|$$
$$+\frac{CM_2}{\sqrt{t-s}}h^2\,\|z\|_0\,\|\phi\|_0\,,$$

where, as in Lemma 4.12, $\Psi(t) = e^{-At}A^{-1}\Pi\phi_h$. Furthermore, by applying Lemma 4.10 to the first two terms on the right-hand side above, we have

$$|b(z,\Psi(t-s),u)| + |b(u,\Psi(t-s),z)| \leq Ch^2\,\|z\|_0\,\|\nabla u\cdot\Psi\|_2$$
$$+C\big(\big\|A^{-1}\Pi z\big\|_0 + h^2\,\|z\|_0\big)\big(\|(u\cdot\nabla)\Psi\|_2 + \|\nabla\Psi\cdot u\|_2\big).$$

Recalling the bounds of $\|(u\cdot\nabla)\Psi\|_2$, $\|\nabla\Psi\cdot u\|_2$ and $\|\nabla u\cdot\Psi\|_2$ in (4.57)–(4.59) we reach

$$\left\|e^{-(t-s)A_h}\rho_h(s)\right\|_0 \leq C\big(\|z\|_1 + h^2\,\|u\|_3\big)\,\|z\|_0 + \frac{CM_2}{\sqrt{t-s}}\big(h^2\,\|z\|_0 + \big\|A^{-1}\Pi z\big\|_0\big). \tag{4.62}$$

By applying Hölder's inequality, writing $z = (u - \Pi_h u) + z_h$, and using (2.7) and (4.12) we get

$$\int_0^t \left(\|z(s)\|_1 + h^2 \|u(s)\|_3\right) \|z(s)\|_0 \, ds \le Ch^5. \tag{4.63}$$

Using again (2.7) together with (4.9) we obtain $s^{1/2}\|z(s)\|_0 \le Ch^3$, and then

$$\int_0^t \frac{M_2}{\sqrt{t-s}} h^2 \|z(s)\|_0 \, ds \le CB\left(\frac{1}{2}, \frac{1}{2}\right) h^5. \tag{4.64}$$

Finally, as a consequence of (2.8) and (2.20) we have

$$\left\|A^{-1}\Pi z\right\|_0 \le \left\|A^{-1}\Pi z_h\right\|_0 + \left\|A^{-1}\Pi(I - \Pi_h)u\right\|_0 \le \left\|A_h^{-1} z_h\right\|_0 + C\left(h^2 \|z_h\|_0 + \|u\|_3 h^5\right),$$

so that using (4.9) again we get $s^{1/2}\|A^{-1}\Pi z(s)\|_0 \le Ch^5$, and then

$$\int_0^t \frac{M_2}{\sqrt{t-s}} \|A^{-1}\Pi z(s)\|_0 \, ds \le CB\left(\frac{1}{2}, \frac{1}{2}\right) h^5. \tag{4.65}$$

Now (4.60) follows from (4.62), (4.63), (4.64), and (4.65). ∎

THEOREM 4.15. *Let (u, p) be the solution of (1.1)–(1.2). There exists a positive constant C such that the discrete velocity v_h defined by (4.4) and the Hood–Taylor element approximation to u, u_h, satisfy the following bound:*

$$\|v_h(t) - u_h(t)\|_j \le \frac{C}{t} |\log(h)| h^{5-j}, \quad t \in (0, T], \quad j = 0, 1, \quad r \ge 4. \tag{4.66}$$

Proof. We prove the error bound for the L^2 norm from which the H^1 norm bound is obtained by applying inverse inequality (2.5). We consider $y_h(t) = te_h(t)$. By arguing similarly as in Theorem 4.7 we have

$$\max_{0 \le t \le T} \|y_h(t)\|_0 \le C\left(\max_{0 \le s \le T} \left\|sA_h^{-1}\rho_h(s)\right\|_0 |\log(h)| + \max_{0 \le s \le T} \left\|A_h^{-1} e_h(s)\right\|_0 |\log(h)|\right).$$

To conclude the estimate we apply Lemmas 4.5 and 4.14 to the first and second terms on the right-hand side above, respectively. ∎

COROLLARY 4.16. *Let (u, p) be the solution of (1.1)–(1.2), and let u_h be the Hood–Taylor element approximation to u. Then, for $r \ge 4$, there exists a positive constant C such that the following bound holds for $j = 0, 1$:*

$$\|u_h(t) - u(t)\|_j \le \frac{C}{t} h^{4-j}, \quad t \in (0, T].$$

Proof. The proof is obtained by reasoning as in Corollary 4.8 using Theorem 4.15 instead of Theorem 4.7. ∎

4.4. Superconvergence for the pressure. We begin with some error estimates for the time derivative of $v_h - u_h$.

LEMMA 4.17. *Let (u, p) be the solution of (1.1)–(1.2). Then there exists a positive constant C such that the discrete velocity v_h defined by (4.4) and the Hood–Taylor* *element approximation to* u, u_h, *satisfy the following bounds:*

$$\|\dot v_h(t) - \dot u_h(t)\|_0 \le \frac{C}{t^{s/2}} |\log(h)| h^{1+s}, \quad t \in (0, T], \tag{4.67}$$

$$\|\dot v_h(t) - \dot u_h(t)\|_{-1} \le \frac{C}{t^{s/2}} |\log(h)| h^{2+s}, \quad t \in (0, T], \tag{4.68}$$

$$\|A^{-1}\Pi(\dot v_h(t) - \dot u_h(t))\|_0 \le \frac{C}{t^{s/2}} |\log(h)| h^{3+s}, \quad t \in (0, T], \tag{4.69}$$

where $s = 1$ *in the case* $r = 3$ *and* $s = 2$ *in the case* $r = 4$.

Proof. Let us first observe that

$$\dot e_h = -A_h e_h + B_h(u_h, u_h) - B_h(u, u),$$

where we recall that $e_h = v_h - u_h$ Then by using (4.21) we get

$$\|\dot e_h\|_0 \le C h^{-1} \|e_h\|_1 + \|u_h - u\|_1.$$

Applying now Theorems 4.7 and 4.15 and Corollaries 4.8 and 4.16 to bound the first and second terms on the right-hand side above, we conclude (4.67).

In order to bound $\|\dot e_h\|_{-1} \le C\|A_h^{-1/2}\dot e_h\|_0$ we observe that

$$\|A_h^{-1/2}\dot e_h\|_0 \le \|A_h^{1/2} e_h\|_0 + \|A_h^{-1/2}(B_h(u_h, u_h) - B_h(u, u))\|_0,$$

so that by applying (4.22) from Lemma 4.4 we get

$$\|A_h^{-1/2}\dot e_h\|_0 \le C\|e_h\|_1 + C\|u - u_h\|_0,$$

and (4.68) is reached by applying again Theorems 4.7 and 4.15 and Corollaries 4.8 and (4.16).

We now prove (4.69). By using (2.20) and Lemma 4.1 we obtain

$$\begin{aligned}\|A^{-1}\Pi\dot e_h\|_0 &\le \|A^{-1}\Pi A_h e_h\|_0 + \|A^{-1}\Pi(B_h(u_h, u_h) - B_h(u, u))\|_0 \\ &\le C(h^2\|A_h e_h\|_0 + \|A_h^{-1} A_h e_h\|_0) + K(\|u - u_h\|_{-1} + \|u_h - u\|_0 \|u_h - u\|_1) \\ &\le C\|e_h\|_0 + K(\|e_h\|_{-1} + \|v_h - u\|_{-1} + \|u_h - u\|_0 \|u_h - u\|_1).\end{aligned}$$

To finish the proof we notice that $\|e_h\|_{-1} \le C\|e_h\|_0$, and then we apply Theorems 4.7 and 4.15 and Corollaries 4.8 and 4.16 together with Lemma 4.3. ☐

THEOREM 4.18 (superconvergence for the pressure). *Let* (u, p) *be the solution of* (1.1)–(1.2). *Then, for* $r = 3, 4$, *there exists a positive constant* C *such that the discrete pressure* g_h *defined by* (4.1) *and the Hood–Taylor element approximation to* p, p_h, *satisfy the following bound:*

$$\|p_h(t) - g_h(t)\|_{L^2/\mathbb{R}} \le \frac{C}{t^{(r-2)/2}} h^r |\log(h)|, \qquad t \in (0, T].$$

Proof. By subtracting (3.1) from (4.1), we obtain for the difference $g_h - p_h$

$$(g_h - p_h, \nabla\cdot\phi_h) = (\nabla e_h, \nabla\phi_h) + (F(u, u) - F(u_h, u_h), \phi_h) + (\dot e_h, \phi_h)$$

for all $\phi_h \in X_{h,r}$. Using the inf-sup condition (2.6),

$$\beta\|g_h - p_h\|_{L^2/\mathbb{R}} \le \|e_h\|_1 + \|F(u,u) - F(u_h,u_h)\|_{-1} + \|\dot{e}_h\|_{-1}.$$

By applying Theorems 4.7 and 4.15, (4.20), and (4.68) we obtain

$$\beta\|g_h - p_h\|_{L^2/\mathbb{R}} \le \frac{C}{t^{(r-2)/2}} h^r|\log(h)| + C\|u-u_h\|_0.$$

Finally, by using Corollaries 4.8 and 4.16, the conclusion is reached. □

COROLLARY 4.19. *Let (u,p) be the solution of* (1.1)–(1.2)*. Then, for $r = 3,4$, there exists a positive constant C such that the Hood–Taylor element approximation to p, p_h, satisfies the following bound:*

$$\|p_h(t) - p(t)\|_{L^2/\mathbb{R}} \le \frac{C}{t^{(r-2)/2}} h^{r-1}, \qquad t \in (0,T]. \tag{4.70}$$

Proof. Let g_h be the discrete pressure defined by (4.1). We decompose

$$\|p_h - p\|_{L^2/\mathbb{R}} \le \|p_h - g_h\|_{L^2/\mathbb{R}} + \|g_h - p\|_{L^2/\mathbb{R}}$$

and apply Theorem 4.18 to bound the first term on the right-hand side above. To bound the second one we decompose $\|g_h - p\|_{L^2/\mathbb{R}} \le \|g_h - q_h\|_{L^2/\mathbb{R}} + \|q_h - p\|_{L^2/\mathbb{R}}$, where q_h is that in (2.11). Using (2.14) it only remains to bound the first term above. By taking into account that

$$(u_t,\phi_h) + (\nabla s_h,\nabla\phi_h) + b(u,u,\phi_h) + (\nabla q_h,\phi_h) = (f,\phi_h) \quad \forall\phi_h \in X_{h,r},$$

$$(\dot{v}_h,\phi_h) + (\nabla v_h,\nabla\phi_h) + b(u,u,\phi_h) + (\nabla g_h,\phi_h) = (f,\phi_h) \quad \forall\phi_h \in X_{h,r},$$

and arguing as in the proof of Theorem 4.18, we deduce that

$$\|g_h - q_h\|_{L^2/\mathbb{R}} \le C\|v_h - s_h\|_1 + \|\dot{v}_h - u_t\|_{-1}.$$

To bound the first term we decompose $\|v_h - s_h\|_1 \le \|z_h\|_1 + \|\Pi_h u - u\|_1 + \|s_h - u\|_1$ and then apply Lemma 4.3, (2.7), and (2.13). To bound the second one, by using (2.8) we get

$$\|\dot{v}_h - u_t\|_{-1} \le \|\dot{z}_h\|_{-1} + \|\Pi_h u_t - u_t\|_{-1} \le \|\dot{z}_h\|_{-1} + Ch^{r-1}\|u_t\|_{r-2}.$$

Finally, since $\|\dot{z}_h\|_{-1} \le C\|A_h^{-1/2}\dot{z}_h\|_0$, by using (4.6) we get

$$\|\dot{z}_h\|_{-1} \le C\|z_h\|_1 + \|\theta_h\|_1,$$

and we conclude by applying Lemma 4.3, (2.7), and (2.13). □

Remark 4.3. We want to point out that the error bounds for the Galerkin approximation to the velocity in Corollaries 4.8 and 4.16 are analogous to those obtained in [37, Theorem 3.1]. If we compare the error bound for the pressure in Corollary 4.19 with that of [37, Theorem 3.1], we can observe that the singular behavior of pressure estimate (4.70) as t tends to zero behaves like $t^{-(r-2)/2}$ improving the estimate in [37, Theorem 3.1] that behaves like $t^{-(r-1)/2}$. Recently, in [7] improved convergence of order $O(t^{(-1/4)-\delta})$ for any $\delta > 0$ is shown for the approximation to the velocity using quadratic elements. Error estimates for the pressure are not considered in [7].

5. Analysis of the postprocessed method. We require some error estimates for the time derivative that we now prove.

LEMMA 5.1. *Let (u,p) be the solution of* (1.1)–(1.2). *There exists a positive constant C such that the Hood–Taylor element approximation to u, u_h, satisfies the following bounds:*

$$\|u_t - \dot u_h(t)\|_0 \le \frac{C}{t^{(l+1)/2}} h^{1+l}\,|\log(h)|\,, \quad t\in(0,T], \tag{5.1}$$

$$\|u_t - \dot u_h(t)\|_{-1} \le \frac{C}{t^{(l+1)/2}} h^{2+l}\,|\log(h)|\,, \quad t\in(0,T], \tag{5.2}$$

$$\|A^{-1}\Pi(u_t - \dot u_h(t))\|_0 \le \frac{C}{t^{(l+1)/2}} h^{3+l}\,|\log(h)|\,, \quad t\in(0,T], \tag{5.3}$$

where $l=1$ in the case $r=3$ and $l=2$ in the case $r=4$.

Proof. By writing

$$u_t - \dot u_h = (u_t - \Pi_h u_t) + (\Pi_h u_t - \dot v_h) + (\dot v_h - \dot u_h) = (u_t - \Pi_h u_t) + \dot z_h + \dot e_h$$

and applying (2.7) and (2.8) to bound the first term and Lemma 4.17 for the third one, we only need to bound the different norms of $\dot z_h$. Since the norms $\|A_h^{-1/2}\dot z_h\|_0$ and $\|\dot z_h\|_{-1}$ are equivalent and since in view of (2.20) $\|A^{-1}\Pi\dot z_h\|_0$ can be bounded in terms of $\|\dot z_h\|_0$ and $\|A_h^{-1}\dot z_h\|_0$, we only have to bound $\|A_h^{-k/2}\dot z_h\|_0$ for $k=0,1,2$. For this purpose, we will argue as in the proof of (4.10). We notice that $\ddot z_h + A_h\dot z_h = A_h\dot\theta_h$, and if we let j be $j=(1+l)/2$ for $y_h = t^j A_h^{-1}\dot z_h$, we have $\dot y_h + A_h y_h = t^j\dot\theta_h + jt^{j-1}A_h^{-1}\dot z_h$, so that

$$A_h^{-1} t^j \dot z_h(t) = \int_0^t e^{-A_h(t-s)} s^j\dot\theta_h(s)\,ds + j\int_0^t e^{-A_h(t-s)} A_h^{-1} s^{j-1}\dot z_h(s)\,ds.$$

By applying (2.10) to both integrals we obtain

$$\left\|A_h^{-1}t^j\dot z_h(t)\right\|_0 \le CB\left(\frac12,\frac12\right) \left(\max_{0\le s\le t} s^{j+1/2}\left\|A_h^{-1/2}\dot\theta_h(s)\right\|_0 + j\max_{0\le s\le t} s^{j-1/2}\left\|A_h^{-3/2}\dot z_h(s)\right\|_0\right).$$

Now, on the one hand, we have that (2.16) and (2.8) are valid if u, s_h, and p are replaced by their time derivatives. On the other hand, we have that $A_h^{-3/2}\dot z_h = A_h^{-1/2}\theta_h - A_h^{-1/2}z_h$, so that in view of (2.16), (2.8), (4.10), and (4.11) the proof of (5.3) is easily finished. Taking into account that $\left\|A_h^{k/2}\right\|_0 \le Ch^{-k}$ for $k=1,2$, then (5.1) and (5.2) follow. ∎

THEOREM 5.2. *Let (u,p) be the solution of* (1.1)–(1.2). *There exists a positive constant C such that the postprocessed MFE approximation to u, $\tilde u_h$, satisfies the following bounds for $j=0,1$:*

(i) *If the postprocessing element is $(\widetilde X,\widetilde Q) = (X_{h',r}, Q_{h',r-1})$, then*

$$\|u(t) - \tilde u_h(t)\|_j \le \frac{C}{t^{1/2}}(h')^{3-j} + \frac{C}{t}h^{4-j}|\log(h)|, \quad t\in(0,T],\ r\ge 3, \tag{5.4}$$

$$\|u(t) - \tilde u_h(t)\|_j \le \frac{C}{t}(h')^{4-j} + \frac{C}{t^{3/2}}h^{5-j}|\log(h)|, \quad t\in(0,T],\ r\ge 4; \tag{5.5}$$

(ii) *if the postprocessing element is* $(\widetilde X, \widetilde Q) = (X_{h,r+1}, Q_{h,r})$, *then*

$$\|u(t) - \tilde u_h(t)\|_j \le \frac{C}{t} h^{4-j} |\log(h)|, \quad t \in (0,T],\ r \ge 3, \tag{5.6}$$

$$\|u(t) - \tilde u_h(t)\|_j \le \frac{C}{t^{3/2}} h^{5-j} |\log(h)|, \quad t \in (0,T],\ r \ge 4. \tag{5.7}$$

Proof. The proof follows the same steps as [6, Theorem 3.14]. Let $\widetilde S_h(u(t)) \in \widetilde V$ be the Stokes projection of the solution of (1.1)–(1.2) at time t. We decompose $\|u(t) - \tilde u_h(t)\|_l \le \|u(t) - \widetilde S_h(u(t))\|_l + \|\widetilde S_h(u(t)) - \tilde u_h(t)\|_l$, $l = 0, 1$. We apply (2.13) to bound the first term, so that we will concentrate now on the second. It is easy to obtain

$$\|\tilde u_h(t) - \widetilde S_h(u(t))\|_1 \le C\|F(u(t), u(t)) - F(u_h(t), u_h(t))\|_{-1} + C\|u_t(t) - \dot u_h(t)\|_{-1}.$$

For the first term above we apply (4.20) and Corollaries 4.8 and 4.16 to obtain

$$\|F(u(t), u(t)) - F(u_h(t), u_h(t))\|_{-1} \le C\|u(t) - u_h(t)\|_0 \le \frac{C}{t^{(r-1)/2}} h^r.$$

The second term is already bounded in (5.2). Then the proof for the H^1 norm is complete. We next deal with the estimate in the L^2 norm. We first observe that

$$\widetilde A_h(\tilde u_h(t) - \widetilde S_h(u(t))) = \widetilde\Pi_h(F(u(t), u(t)) - F(u_h(t), u_h(t))) + \widetilde\Pi_h(u_t(t) - \dot u_h(t)).$$

Then, by applying $\widetilde A_h^{-1}$ to both sides of the above equation, we obtain

$$\|\tilde u_h - \widetilde S_h(u)\|_0 \le \|\widetilde A_h^{-1}\widetilde\Pi_h(F(u(t), u(t)) - F(u_h(t), u_h(t)))\|_0 + \|\widetilde A_h^{-1}\widetilde\Pi_h(u_t(t) - \dot u_h(t))\|_0.$$

As regards the nonlinear term, by taking into account (2.19) and applying Lemma 4.1 and (4.19) we get

$$\begin{aligned}&\|\widetilde A_h^{-1}\widetilde\Pi_h(F(u, u) - F(u_h, u_h))\|_0\\ &\qquad \le C\tilde h^2\|u - u_h\|_1 + C(\|u - u_h\|_{-1} + \|u - u_h\|_0\|u - u_h\|_1).\end{aligned}$$

The estimates for the L^2 and H^1 norms of the errors of the Galerkin approximation to the velocity are granted by Corollaries 4.8 and 4.16. As regards the estimate in the H^{-1} norm, we use the decomposition $\|u - u_h\|_{-1} \le \|u - \Pi_h u\|_{-1} + \|z_h\|_{-1} + \|v_h - u_h\|_0$ and apply (2.8) together with Lemma 4.3 and Theorems 4.7 and 4.15. Finally, to bound $\|\widetilde A_h^{-1}\widetilde\Pi_h(u_t(t) - \dot u_h(t))\|_0$ we apply (2.19) again and then use estimates (5.1) and (5.3) from Lemma 5.1. ∎

THEOREM 5.3. *Let* (u, p) *be the solution of* (1.1)–(1.2). *There exists a positive constant* C *such that the postprocessed MFE approximation to* p, $\tilde p_h$, *satisfies the following bounds:*

(i) *If the postprocessing element is* $(\widetilde X, \widetilde Q) = (X_{h',r}, Q_{h',r-1})$, *then*

$$\|p(t) - \tilde p_h(t)\|_{L^2/\mathbb R} \le \frac{C}{t^{(r-2)/2}} (h')^{r-1} + \frac{C}{t^{(r-1)/2}} h^r |\log(h)|, \quad t \in (0,T];$$

(ii) *if the postprocessing element is* $(\widetilde X, \widetilde Q) = (X_{h,r+1}, Q_{h,r})$, *then*

$$\|p(t) - \tilde p_h(t)\|_{L^2/\mathbb R} \le \frac{C}{t^{(r-1)/2}} h^r |\log(h)|, \quad t \in (0,T]. \tag{5.8}$$

Proof. The proof is reached by reasoning as in [6, Theorem 3.15]. Let us denote by $\tilde{q}_h(t)$ the approximation to the pressure $p(t)$ such that $(\widetilde{S}_h(u(t)), \tilde{q}_h(t)) \in (\widetilde{X}, \widetilde{Q})$ solves (2.11)–(2.12) with $g = f - u_t - u \cdot \nabla u$. Then we use the decomposition

$$\|p(t) - \tilde{p}_h(t)\|_{L^2/\mathbb{R}} \le \|p(t) - \tilde{q}_h(t)\|_{L^2/\mathbb{R}} + \|\tilde{q}_h(t) - \tilde{p}_h(t)\|_{L^2/\mathbb{R}}.$$

The first term is estimated by applying (2.14). Let us now bound the second term. Using the inf-sup condition (2.6) it is easy to deduce that

$$\beta\|\tilde{p}_h - \tilde{q}_h\|_{L^2/\mathbb{R}} \le \|\tilde{u}_h - \widetilde{S}_h(u)\|_1 + \|F(u_h, u_h) - F(u, u)\|_{-1} + \|u_h - u_t\|_{-1}.$$

By applying Theorem 5.2 to bound the first term, (4.20) together with Corollaries 4.8 and 4.16 for the second, and (5.2) for the third, the conclusion is reached. ∎

6. Remarks on fully discrete postprocessing. In practice, the finite-element approximations u_h and p_h, being solutions of system (3.1)–(3.2), can rarely be computed analytically, and one has to compute approximations $u_h^{(n)} \approx u_h(t_n)$ and $p_h^{(n)} \approx p_h(t_n)$ on some time levels $0 = t_0 < t_1 \ldots < t_N = T$ by means of a time integrator. Consequently, instead of the postprocessed approximations $\tilde{u}_h(t_n)$ and $\tilde{p}_h(t_n)$, one obtains $\tilde{u}_h^{(n)}$ and $\tilde{p}_h^{(n)}$ as solutions of

$$\big(\nabla\tilde{u}_h^{(n)}, \nabla\tilde{\phi}\big) + \big(\nabla\tilde{p}_h^{(n)}, \tilde{\phi}\big) = (f, \tilde{\phi}) - b(u_h^{(n)}, u_h^{(n)}, \tilde{\phi}) - (d_t u_h^{(n)}, \tilde{\phi}) \quad \forall\, \tilde{\phi} \in \widetilde{X}, \tag{6.1}$$

$$\big(\nabla \cdot \tilde{u}_h^{(n)}, \tilde{\psi}\big) = 0 \quad \forall\, \tilde{\psi} \in \widetilde{Q}, \tag{6.2}$$

where $(\widetilde{X}, \widetilde{Q})$ is as in (3.3)–(3.4), and $d_t u_h^{(n)}$ is an approximation to $\dot{u}_h(t_n)$ computed in terms of as many values $u_h^{(n-j)}$, $j = 0, 1, \ldots, N - n$, as needed. Recent research in [22] suggests

$$d_t u^{(n)} = \Pi_h f - A_h u_h^{(n)} - B_h(u_h^{(n)}, u_h^{(n)}) \tag{6.3}$$

as a convenient and adequate approximation.

Notice then that the error $u(t_n) - \tilde{u}_h^{(n)}$ can be expressed as $u(t_n) - \tilde{u}_h^{(n)} = (u(t_n) - \tilde{u}_h(t_n)) + (\tilde{u}_h(t_n) - \tilde{u}_n^{(n)})$. The first term on the right-hand side is the spatial discretization error which we have analyzed in the previous sections. The second one is the error arising from temporal discretization that will be analyzed now.

Let us denote by $e_h^{(n)} = u_h(t_n) - u_h^{(n)}$ and $\tilde{e}_h^{(n)} = \tilde{u}_h(t_n) - \tilde{u}_h^{(n)}$ the errors induced by the temporal discretization in the Galerkin and postprocessed approximations, respectively. We estimate here $\tilde{e}_h^{(n)}$ in terms of $e_h^{(n)}$, which depends on the particular time integrator used to approximate the solution of (3.1)–(3.2). Estimates of $e_h^{(n)}$ (when $t_n = nk$, $n = 0, 1, \ldots, N = T/k$) for the Crank–Nicolson method can be found in [38], with $O(t_n^{-1}k^2)$ and $O(t_n^{-1}k)$ estimates in the L^2 and H^1 norms, respectively, (see also [31]). For the second-order backward differentiation formula (BDF), similar estimates are obtained in [13] (for the limit case $h \to 0$), with a $k^{1/2}$ improvement in the H^1 estimate with respect to that in [38]. Also, from results in [41], it is straightforward to obtain $O(k)$ estimates in both the L^2 and the H^1 norms for the implicit Euler method. Notice that all of these estimates together with the boundedness of $\|u_h(t)\|_1$, $0 \le t \le T$, and h sufficiently small [37] imply that $\|u_h^{(n)}\|_1$ is bounded for $0 \le n \le N$ and h sufficiently small. Further estimates for other integrators are outside the scope of the present paper.

From (6.3) it follows that

$$\dot{u}_h(t_n) - d_t u_h^{(n)} = -A_h e_h^{(n)} + \Pi_h\big(F(u_h^{(n)}, u_h^{(n)}) - F(u_h(t_n), u_h(t_n))\big), \tag{6.4}$$

so that by subtracting (6.1) from (3.3) we get

$$\widetilde{A}_h \tilde{e}_h^{(n)} = (\widetilde{\Pi}_h - \Pi_h)\big(F(u_h^{(n)}, u_h^{(n)}) - F(u_h(t_n), u_h(t_n))\big) - \widetilde{\Pi}_h A_h e_h^{(n)}.$$

On the one hand, we may write $\widetilde{A}_h^{-1}\widetilde{\Pi}_h A_h e_h^{(n)} = e_h^{(n)} + \big(\widetilde{A}_h^{-1}\widetilde{\Pi}_h - A_h^{-1}\big)A_h e_h^{(n)}$, so that by using (2.17) we get

$$\|\tilde{A}_h^{-1}\tilde{\Pi}_h A_h e_h^{(n)}\|_j \le \|e_h^{(n)}\|_j + Ch^{2-j}\|A_h e_h^{(n)}\|_0, \quad j = 0, 1.$$

On the other hand, a simple duality argument and (2.17) show that $\|\widetilde{A}_h^{-1}(\widetilde{\Pi}_h - \Pi_h)f\|_j \le Ch^{2-j}\,\|f\|_0$ for $f \in L^2(\Omega)^d$, $j = 0, 1$, so that first applying this inequality to $f = F(u_h^{(n)}, u_h^{(n)}) - F(u_h(t_n), u_h(t_n))$ and then well-known inequalities (e.g., equation [38, (3.7)]), we conclude that

$$\big\|\tilde{e}_h^{(n)}\big\|_j \le \big\|e_h^{(n)}\big\|_j + Ch^{2-j}\Big(\big\|e_h^{(n)}\big\|_1 + \big\|A_h e_h^{(n)}\big\|_0\Big), \tag{6.5}$$

where the constant C depends on $\|u_n^{(n)}\|_1$ (which, as mentioned before, is bounded for h sufficiently small) and on $\|A_h u_h(t_n)\|_0$, which can be easily seen to be bounded for $0 \le t \le T$ and h sufficiently small. Taking into account that $h^{2-j}\big\|A_h e_h^{(n)}\big\|_0 \le C\|e_h^{(n)}\|_j$, we conclude that the temporal error $\tilde{e}_h^{(n)}$ of the fully discrete postprocessed method is proportional to temporal error $e_h^{(n)}$ of the mixed finite-element approximation. Furthermore, for the second-order BDF formula, besides the estimates in [13], in [23] we obtain $O(t_n^{-3/2}k^2)$ and $O(t_n^{-2}k^2)$ estimates for $\|e_h^{(n)}\|_1$ and $\|A_h e_h^{(n)}\|_0$, respectively, so that, in view of (6.5), the two errors $\tilde{e}_h^{(n)}$ and $e_h^{(n)}$ are asymptotically equivalent as $h \to 0$.

Finally, for the pressure, arguing as in the proof of Theorem 5.3 and using the same arguments we have used for $\tilde{e}_h^{(n)}$, one can easily show that also

$$\big\|\tilde{p}_h(t_n) - \tilde{p}_h^{(n)}\big\|_{L^2/\mathbb{R}} \le C\big\|e_h^{(n)}\big\|_1.$$

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