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# LONG-TERM STABILITY ESTIMATES AND EXISTENCE OF A GLOBAL ATTRACTOR IN A FINITE ELEMENT APPROXIMATION OF THE NAVIER–STOKES EQUATIONS WITH NUMERICAL SUBGRID SCALE MODELING*

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**Abstract.** Variational multiscale methods lead to stable finite element approximations of the Navier–Stokes equations, dealing with both the indefinite nature of the system (pressure stability) and the velocity stability loss for high Reynolds numbers. These methods enrich the Galerkin formulation with a subgrid component that is modeled. In fact, the effect of the subgrid scale on the captured scales has been proved to dissipate the proper amount of energy needed to approximate the correct energy spectrum. Thus, they also act as effective large-eddy simulation turbulence models and allow one to compute flows without the need to capture all the scales in the system. In this article, we consider a dynamic subgrid model that enforces the subgrid component to be orthogonal to the finite element space in the $L^2$ sense. We analyze the long-term behavior of the algorithm, proving the existence of appropriate absorbing sets and a compact global attractor. The improvements with respect to a finite element Galerkin approximation are the long-term estimates for the subgrid component, which are translated to effective pressure and velocity stability. Thus, the stabilization introduced by the subgrid model into the finite element problem does not deteriorate for infinite time intervals of computation.

**Key words.** Navier–Stokes problem, long-term stability, absorbing set, global attractor, stabilized finite element methods, subgrid scales

**AMS subject classifications.** 65N30, 35Q30

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## 1. Introduction.

The dynamics of Newtonian incompressible flows is governed by the Navier–Stokes equations, a dynamical system that consists in a set of nonlinear partial differential equations with a dissipative structure. For two-dimensional problems, the energy of this system has been proved to be bounded by the data (external forces and initial conditions) for all times. It is also possible to bound the $H^1(\Omega)$-norm of the fluid velocity, which, together with the Rellich–Kondrachov theorem, allows one to prove that any fluid velocity orbit converges to a finite-dimensional set, the so-called global attractor, as the time variable goes to infinity (see [28, 53]). Fractal and Hausdorff dimensions of the global attractor have been estimated using Lyapunov exponents in dimension 2 and 3 [20, 29].

An accurate numerical approximation of the Navier–Stokes equations should mimic their long-term behavior. For direct numerical simulation (DNS), a crude Galerkin approximation using inf-sup stable finite elements admits a numerical global attractor, whose dimension has been estimated in [44]. The convergence of the

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numerical global attractor to that of the Navier–Stokes equations has been analyzed in [35]. Similar results have been proved for finite differences [56].

The finite element approximation of the Navier–Stokes equations for large Reynolds numbers (Re) presents two main difficulties that can make their numerical approximations meaningless: one is the indefinite nature of the system, and the other is the stability loss due to convection dominant regimes. The first problem can be cured by using appropriate velocity-pressure finite element spaces satisfying a discrete version of the Ladyzhenskaya–Babuška–Brezzi condition (see [8]). These finite element pairs are usually called inf-sup stable elements and do not include many spaces that would be interesting for their simplicity and/or efficiency. When using Galerkin approximations and finite elements, the only way to solve the velocity stability loss is to capture all the spatial scales of the flow, i.e., to reduce the computational mesh size up to the Kolmogorov microscale $\lambda_{\text{Kol}}$, below which are the smallest dissipative structures of the flow. This approach, known as direct numerical simulation, requires in dimension 3 $\mathcal{O}(\text{Re}^{2.25})$ mesh nodes. Unsurprisingly, this dimension is also related to the dimension of the continuous global attractor (see [20, 29, 53]). The memory usage grows so fast with respect to Re that DNS computations are unaffordable in most industrial applications, even at moderate Reynolds numbers. Anyway, DNS is a valuable tool in theoretical turbulence research: it allows a deeper understanding of this phenomenon and helps to validate turbulence models. At this point, let us also mention the nonlinear Galerkin method, which consists in a modification of the Galerkin formulation with the aim of better approximating the attractor for long-term analyses (see, e.g., [52, 45, 2, 50]). To approximate the attractor, inertial manifolds (or approximate inertial manifolds) have been developed (see, e.g., [27, 54]). Numerical results supporting this approach for long-term analyses can be found in [22, 40, 41].

Both pressure instability and velocity stability loss for convection dominant regimes can be solved by using finite element stabilization techniques (see, e.g., [9, 38, 13, 15, 19, 3]). In fact, stabilization is essential for the finite element approximation of high Re flows. The common feature of this family of algorithms is to introduce consistent terms to the formulation that would improve the stability properties of the numerical system without spoiling accuracy. Initially, these stabilization techniques were developed without a sound motivation until they were justified by a multiscale decomposition of the continuous solution into resolved (finite element) and unresolved (subgrid) scales. Using this decomposition in the variational form of the problem and modeling the effect of the subscales into the finite element problem, we end up with numerical methods that exhibit enhanced stability properties. We refer to [37, 39] for a detailed exposition of this approach, coined the variational multiscale (VMS) method. Applied to the Navier–Stokes equations, stabilized finite elements lead to stable formulations without the need to represent all the scales of the flow. Thus, coarser meshes can be used, drastically reducing the computational effort of DNS.

VMS subgrid scale models have been motivated by numerical purposes (stability and convergence of the numerical algorithms), but they have also been proved to introduce a numerical dissipation that approximates well the physical dissipation at the unresolved scales [31, 15, 19, 36, 21, 47, 5, 10]. Thus, these methods can be understood as large-eddy simulation turbulence models that properly account for the effect of the smaller universal scales on the large scale motions of the flow that can be captured by the mesh (see, e.g., [49]).

The outline of the article is as follows. In section 2, we state the continuous problem and the basic results that describe its long-term behavior. In section 3 we consider the semidiscrete in space finite element Galerkin approximation and how to stabilize it using our favored VMS subgrid model. For the VMS formulation, we prove existence and uniqueness of solutions. In section 4, we prove the existence of an absorbing set in $\boldsymbol{L}^2(\Omega)$, with particular emphasis on the new bounds due to stabilization. Finally, in section 5, we prove the existence of an absorbing set in $\boldsymbol{H}^1(\Omega)$ and a numerical global attractor in the two-dimensional case. We end up with some conclusions in section 6.

## 2. Problem statement.

**2.1. Notation.** Let $\Omega$ be any open set of $\mathbb{R}^d$, $d = 2$ or 3. As usual, $L^p(\Omega)$ denotes the space of $p$th-power integrable real-valued functions defined on $\Omega$, whereas $L^\infty(\Omega)$ is the space of essentially bounded real-valued functions. This space is a Banach space endowed with the norm $\|v\|_{L^p(\Omega)} = (\int_\Omega |v(\mathbf{x})|^p \,\mathrm{d}\mathbf{x})^{1/p}$ (or $\|v\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{\mathbf{x}\in\Omega} |v(\mathbf{x})|$, respectively). In the particular case $p = 2$, $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = \int_\Omega u(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x}.$$

On the other hand, $L^p_{\mathrm{loc}}(\Omega)$ contains all the real-valued functions defined on $\Omega$ which belong to $L^p(\omega)$ for any compact subset $\omega$ of the open set $\Omega$.

For $m$ a nonnegative integer and $p \geq 1$, we define the classical Sobolev spaces as

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega)\,;\, \partial^k v \in L^p(\Omega)\ \forall\ |k| \leq m\},$$

associated to the norm

$$\|v\|_{W^{m,p}(\Omega)} = \left[\sum_{0\leq|k|\leq m} \|\partial^k v\|^p_{L^p(\Omega)}\right]^{\frac{1}{p}},$$

where $k$ is a multi-index; we will write this norm in compact form as $\|\cdot\|_{m,p}$. In the particular case $p = 2$, $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$, which is a Hilbert space with the obvious inner product and its associated norm $\|\cdot\|_m$. We will use boldface letters for spaces of vector functions.

Let $\mathcal{C}_0^\infty(\Omega)$ be the space of infinitely differentiable functions with compact support in $\Omega$. We denote by $\mathcal{D}(\Omega)$ the topological space of test functions in $\Omega$. Its dual space, the space of distributions, is denoted by $\mathcal{D}'(\Omega)$. The closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$ is defined by $W_0^{m,p}(\Omega)$ (analogously, $H_0^m(\Omega)$ when $p = 2$). The dual space of $W_0^{m,p}(\Omega)$ is identified by $W^{-m,q}(\Omega)$, $q$ being the conjugate index to $p$, i.e., $\frac{1}{q} + \frac{1}{p} = 1$; analogously, we define $H^{-m}$ as the dual space of $H^m(\Omega)_0$. In general, duality pairings will be indicated by the symbol $\langle\cdot,\cdot\rangle$. We will make use of the following space of vector fields:

$$\boldsymbol{H}_0(\mathrm{div}\ 0, \Omega) := \left\{\mathbf{v} \in \boldsymbol{L}^2(\Omega) \text{ such that } \nabla\cdot\mathbf{v} = 0,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\right\},$$

where $\mathbf{n}$ is the outward normal to $\Omega$ on $\partial\Omega$.

Let $-\infty \leq a < b \leq +\infty$, and let $X$ be a Banach space. Then $L^p(a, b; X)$ denotes the space of $X$-valued functions on $(a, b)$ such that $\int_a^b \|f(s)\|^p_X \mathrm{d}s < \infty$ for $1 \leq p < \infty$ or $\operatorname{ess\,sup}_{s\in(a,b)} \|f(s)\|_X < \infty$ for $p = \infty$. $\mathcal{C}([a, b]; X)$ is the space of continuous $X$-valued functions such that $\sup_{t\in[a,b]} \|f(t)\|_X < \infty$. Analogously, $\mathcal{D}'(a, b; X)$ is the space of functions such that their $X$-norms have a distributional sense in $(a, b)$.

Finally, $L^2(\Omega)/\mathbb{R}$ is the quotient space of $L^2(\Omega)$ functions and a constant with the norm $\|p\|_{L^2(\Omega)/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|p+c\| = \|p - \int_\Omega p(\mathbf{x})\mathrm{d}\mathbf{x}\|$.

In what follows, $C$ denotes a positive constant independent of the physical parameters but possibly depending on the size of the domain $\Omega$. When dealing with the finite element problem, $C$ also will be independent of the mesh size $h$. The value of $C$ may be different at different occurrences. We will use the notation $A \gtrsim B$ and $A \lesssim B$ to indicate that $A \geq CB$ and $A \leq CB$, respectively, where $A$ and $B$ are expressions depending on functions that in the discrete case may depend on $h$ as well.

**2.2. The continuous problem.** Let $\Omega$ be a bounded, open set of $\mathbb{R}^d$, $d = 2$ or 3, and let $(0,T)$ be the time interval, with $T \leq \infty$. We denote by $Q = \Omega \times (0,T)$ the cylindrical space-time domain. The flow of a viscous, incompressible, Newtonian fluid is described by the Navier–Stokes equations:

$$\left\{ \begin{aligned} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } Q, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } Q. \end{aligned} \right. \tag{2.1a}$$

The unknowns are the fluid velocity $\mathbf{u}(\mathbf{x},t) : Q \to \mathbb{R}^d$ and the fluid pressure $p(\mathbf{x},t) : Q \to \mathbb{R}$. The physical parameter $\nu > 0$ is the kinematic viscosity, and $\mathbf{f}$ is the external volume force applied to the fluid confined in $\Omega$. These equations are supplemented with the initial condition $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0$ in $\Omega$ and the homogeneous Dirichlet boundary condition $\mathbf{u}(\mathbf{x},t) = \mathbf{0}$ on $\partial\Omega$. We can also state the Navier–Stokes equations in weak or variational form. We seek for $[\mathbf{u}(\mathbf{x},t), p(\mathbf{x},t)] \in L^2(0,T;\boldsymbol{H}_0^1(\Omega)) \times \mathcal{D}'(0,T;L^2(\Omega)/\mathbb{R})$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u},\mathbf{v}) + \langle(\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}\rangle + \nu(\nabla\mathbf{u},\nabla\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) = \langle\mathbf{f},\mathbf{v}\rangle, \tag{2.2a}$$

$$(q,\nabla\cdot\mathbf{u}) = 0 \tag{2.2b}$$

in $\mathcal{D}'(0,T)$, for any $[\mathbf{v},q] \in \boldsymbol{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$, satisfying also the initial condition. The problem is posed with $\mathbf{u}_0 \in \boldsymbol{H}_0(\mathrm{div}\ 0,\Omega)$ and force term $\mathbf{f} \in L^2(0,T,\boldsymbol{H}^{-1}(\Omega))$.

Existence and uniqueness for (2.2) is an open problem in three dimensions. There are some partial results, such as the existence of weak solutions; problem (2.2) has at least one weak solution that satisfies the energy inequality (Leray inequality)

$$\frac{1}{2}\|\mathbf{u}(\mathbf{x},t)\|^2 + \nu\int_0^t \|\nabla\mathbf{u}(\mathbf{x},s)\|^2\mathrm{d}s \lesssim \frac{1}{2}\|\mathbf{u}(\mathbf{x},0)\| + \int_0^t \langle\mathbf{f}(\mathbf{x},s),\mathbf{u}(\mathbf{x},s)\rangle\mathrm{d}s,$$

which implies

$$\|\mathbf{u}(\mathbf{x},t)\|^2 + \nu\int_0^t \|\nabla\mathbf{u}(\mathbf{x},s)\|^2\mathrm{d}s \lesssim \frac{1}{\nu}\int_0^t \|\mathbf{f}\|_{-1}^2\mathrm{d}s + \|\mathbf{u}_h(\mathbf{x},0)\|^2. \tag{2.3}$$

Thus, $\mathbf{u} \in L^2(0,T;\boldsymbol{H}_0^1(\Omega)) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ for all $0 < T < \infty$, under the regularity of the data indicated above.

Pressure stability can be obtained from the inf-sup condition

$$\inf_{q\in L^2(\Omega)} \sup_{\mathbf{v}\in\boldsymbol{H}_0^1(\Omega)} \frac{(q,\nabla\cdot\mathbf{v})}{\|q\|\|\mathbf{v}\|_1} \geq \beta > 0, \tag{2.4}$$

which is a consequence of the surjectivity of the divergence operator from $\boldsymbol{H}_0^1(\Omega)$ to $L^2(\Omega)$ (see [42]). Even for the linear transient Stokes problem, in the most general setting in which the problem is well-posed, pressure stability in time is unclear (see [23]). Most of the mathematical analyses of the transient Navier–Stokes equations are obtained using divergence-free velocity spaces that allow one to get rid of the pressure [51, 33, 34]. However, in some engineering applications pressure values are more important than fluid velocities, e.g., in fluid-structure interaction phenomena.

The previous results can be meaningless since the right-hand side of (2.3) can blow up as $t \to \infty$. Thus, new results have been obtained in order to understand the long-term behavior of (2.2). Let us assume that problem (2.1) is well-posed for all $t \geq 0$ and $\mathbf{f}$ is time-independent. We can describe this autonomous infinite-dimensional dynamical system by means of the semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$, i.e., the family of operators

$$\mathcal{S}(t) : \boldsymbol{L}^2(\Omega) \longrightarrow \boldsymbol{L}^2(\Omega), \qquad \mathbf{u}_0(\mathbf{x}) \longrightarrow \mathbf{u}(\mathbf{x}, t), \qquad t \geq 0.$$

The orbit associated to a given initial value is the set $\bigcup_{t\geq 0} \mathcal{S}(t)\mathbf{u}_0$. In dimension 2, it is known that the transient Navier–Stokes equations exhibit an absorbing set $\mathcal{B} \subset \boldsymbol{L}^2(\Omega)$; i.e., for any $\mathbf{u}_0 \in \boldsymbol{L}^2(\Omega)$ there exists a time value $t_*(\mathbf{u}_0)$ such that $\bigcup_{t\geq t_*} \mathcal{S}(t)\mathbf{u}_0 \subset \mathcal{B}$ (see [53]). In fact, it is also possible to prove that there exists an absorbing set in $\boldsymbol{H}^1(\Omega)$. Due to the compactness of the $\boldsymbol{H}^1(\Omega)$ ball in $\boldsymbol{L}^2(\Omega)$, $\mathcal{S}(t)$ turns out to be uniformly compact. In the asymptotic regime $t \to \infty$, it has been proved that all the orbits are attracted by a compact set $\mathcal{A}$ of finite dimension, the global attractor [28, 53].

## 3. Finite element approximation.

**3.1. The Galerkin problem.** From now on, we assume that $\Omega$ is a subset of $\mathbb{R}^d$ ($d = 2$ or 3) having a polygonal or polyhedral Lipschitz-continuous boundary, and $\{\mathcal{T}_h\}_{h>0}$ is a quasi-uniform family of triangulations of $\bar{\Omega}$, that is, $\bar{\Omega} = \cup_{K\in\mathcal{T}_h} K$, with mesh size $h = \max_{K\in\mathcal{T}_h} h_K$, $h_K$ being the diameter of element $K$.

In order to get a conforming finite element approximation of the Navier–Stokes problem, we consider conforming finite element spaces $V_h \subset \boldsymbol{H}_0^1(\Omega)$ and $Q_h \subset L^2(\Omega)/\mathbb{R}$ for velocity and pressure, respectively, with the optimal interpolation properties (see, e.g., [7]). In particular, we assume the following.

Assumption 3.1. *The finite element spaces $V_h$ and $Q_h$ satisfy*

$$\inf_{\mathbf{v}_h\in V_h} \|\mathbf{v} - \mathbf{v}_h\|_s \lesssim h^{j-s}\|\mathbf{v}\|_{H^j(\Omega)} \quad \textit{for} \quad j = 1, 2 \quad \textit{and} \quad s = 0, 1, \tag{3.1a}$$

$$\inf_{q_h\in Q_h} \|q - q_h\| \lesssim h\|q\|_{H^1(\Omega)}. \tag{3.1b}$$

To simplify the exposition, we will consider $Q_h \subset \mathcal{C}^0(\Omega)$. Otherwise, interelement boundary terms involving pressure jumps would be required (see, e.g., [4]).

We will consider the basis $\{\boldsymbol{\phi}_i\}_{i=1,\ldots,\mathrm{n_u}}$ and $\{\pi_i\}_{i=1,\ldots,\mathrm{n}_p}$ for $V_h$ and $Q_h$, respectively. Thus, $\mathrm{n_u}$ and $\mathrm{n}_p$ denote the space dimension for $V_h$ and $Q_h$.

For quasi-uniform partitions, there is a constant $C_{\mathrm{inv}}$, independent of the mesh size $h$ (the maximum of all the element diameters), such that

$$\|\nabla v_h\|_{L^2(K)} \leq C_{\mathrm{inv}} h^{-1}\|v_h\|_{L^2(K)}, \quad \|\Delta v_h\|_{L^2(K)} \leq C_{\mathrm{inv}} h^{-1}\|\nabla v_h\|_{L^2(K)} \tag{3.2}$$

for all finite element functions $v_h$ defined on $K \in \mathcal{T}_h$. These inequalities can be used for scalars, vectors, or tensors.

We use the skew-symmetric form of the convective trilinear form (see [51] and [11] for related numerical aspects), so that instead of the nonlinear term in (2.2a) we use

$$b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}) = \langle (\mathbf{u}_h \cdot \nabla)\mathbf{v}_h, \mathbf{w} \rangle + \frac{1}{2} \langle \nabla \cdot \mathbf{u}_h, \mathbf{v}_h \cdot \mathbf{w} \rangle .$$

Let us denote by $P_h(\cdot)$ and $P_{Q_h}(\cdot)$ the $L^2(\Omega)$-orthogonal projections onto $V_h$ and $Q_h$, respectively, with optimal interpolation properties. We also denote by $P_h^{\perp}(\cdot) := \mathrm{Id}(\cdot) - P_h(\cdot)$ the projection onto $V_h^{\perp}$, the space $L^2(\Omega)$-orthogonal to $V_h$. Then, the semidiscrete problem in space consists in finding $[\mathbf{u}_h, p_h] \in H^1(0,T;V_h) \times L^2(0,T;Q_h)$ such that

$$(\partial_t \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle, \tag{3.3}$$

$$(q_h, \nabla \cdot \mathbf{u}_h) = 0, \tag{3.4}$$

a.e. in time for any $[\mathbf{v}_h, q_h] \in V_h \times Q_h$, also satisfying an initial boundary condition $\mathbf{u}_h(0) = \mathbf{u}_{0h}$. Analogously to the continuous problem, it is easy to prove that the semidiscrete system (3.3) satisfies

$$\|\mathbf{u}_h(\mathbf{x},t)\|^2 + \nu \int_0^t \|\nabla \mathbf{u}_h(\mathbf{x},s)\|^2 \mathrm{d}s \lesssim \frac{1}{\nu} \int_0^t \|\mathbf{f}\|_{-1}^2 \mathrm{d}s + \|\mathbf{u}_h(\mathbf{x},0)\|^2.$$

Pressure stability for the Galerkin approximation of the Navier–Stokes equations cannot be attained from energy bounds. In order to mimic the mathematical structure of the continuous problem, we can build velocity-pressure finite element spaces satisfying a discrete inf-sup condition

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|q_h\| \|\mathbf{v}_h\|_1} \geq \beta^* > 0, \tag{3.5}$$

where $\beta^*$ is uniform with respect to $h$. Obviously, the discrete inf-sup condition is not a direct consequence of (2.4). In fact, some interesting velocity-pressure pairs, such as equal-order velocity-pressure approximations, fail to satisfy this condition, because $\beta^* > 0$ is not uniform with respect to $h$, leading to pressure instabilities.

**3.2. A singularly perturbed problem.** It is well known that the Galerkin finite element approximation of the Navier–Stokes equation (3.3) exhibits numerical instabilities for large Reynolds numbers, where the Reynolds number is defined as

$$\mathrm{Re} = \frac{UL}{\nu},$$

where $U$ and $L$ are characteristic velocity and length scales used in the adimensionalization of the system.[1]

For the continuous problem, the second term on the left-hand side of (2.3) represents the dissipation of kinetic energy. The larger scales of turbulent flows contain most of the kinetic energy of the system, which is transferred to smaller scales via the nonlinear term by an inertial and essentially inviscid mechanism. This process continues creating smaller and smaller scales until forming eddies in which the viscous

[1] This definition is not fully satisfactory in turbulent flows, since there is no suitable characteristic velocity for the definition of the Reynolds number. This definition, as well as other notions about turbulence introduced below, is vaguely used with the aim of giving a physical explanation of the energy pile-up for the Galerkin method (see also [49, 46] for a physical viewpoint of turbulence). We refer to [24, 48] for a precise mathematical description of these concepts.

dissipation of energy finally takes place, i.e., $\nu\|\nabla\mathbf{u}(\mathbf{x},s)\|^2$ becomes dominant. This process is known as the energy cascade (see [26] for a mathematical description of this phenomenon and [49] for a physical one). Even for high Re, the viscous dissipative term of the continuous problem in (2.1) becomes dominant at the smallest scales of the flow; viscous effects extract energy from the system at the smallest scales, "killing" any fluctuation under a certain level, the Kolmogorov microscale $\lambda_{\mathrm{Kol}}$ (see [43, 46]).

From a numerical point of view, $\lambda_{\mathrm{Kol}}$ is obviously related to the number of nodes that are needed in a DNS computational mesh, since all the scales of the flow must be captured in such computations. When the computational mesh is substantially coarser than a DNS mesh, the smallest scales have a size $h \gg \lambda_{\mathrm{Kol}}$; i.e., they belong to the inertial range. On the other hand, following the energy cascade, the energy from larger scales is transferred to the smallest scales. Since eddies in the range $\mathcal{O}(h)$ are much larger than the dissipative eddies that exist at Kolmogorov scales, kinetic energy is essentially not dissipated in this range. The viscous dissipation term $\nu\|\nabla\mathbf{u}_h\|^2$ never becomes important and, as a result, the smallest scales exhibit an energy pile-up (see [32, 10]), leading to space instabilities.

**3.3. Scale splitting and approximation of the subgrid scales.** The formulation we analyze in this work belongs to the framework of VMS methods, the key idea being a decomposition of the unknowns into a resolvable, finite element component and an unresolvable, subgrid scale component. The splitting for the velocity can thus be written as $\mathbf{u} = \mathbf{u}_h + \tilde{\mathbf{u}}$, an approximation being required for the subgrid scale velocity. For the pressure we will assume that its subgrid component is $\tilde{p} = 0$, since the contribution obtained from this component is not essential for the good performance of the algorithm (see, e.g., [14]).

Using VMS stabilized finite element approximations, we get numerical methods with enhanced stability properties for which there is the hope that they can act as turbulence models. Pressure stability does not rely on a discrete inf-sup condition, and fluid velocity bounds remain effective at high Re for mesh sizes $h \gg \lambda_{\mathrm{Kol}}$, placed in the inertial range. Furthermore, the effect of the unresolved scales, i.e., scales in the range $(0,h)$, on the captured scales is properly modeled, in particular, the viscous dissipation that takes place at the smallest unresolved scales $(0,\lambda_{\mathrm{Kol}})$. In fact, it has been proved that the energy spectra of VMS-based algorithms approximate accurately the continuous spectra up to $\mathcal{O}(h)$ scales (see [31, 19, 47, 5]).

We do not include here the motivation of these algorithms, which can be found elsewhere (see [15, 37, 39]). The particular feature of the VMS formulation analyzed herein is the fact that we consider the subgrid velocity to be $L^2$-orthogonal to the finite element velocity and dynamic; by dynamic model we mean that the subgrid time derivatives are properly accounted for. We refer the interested reader to [13, 15] for a discussion about the benefits of using orthogonal subscales and to [3, 17, 19, 47] for some works showing the gain from using dynamic subgrid scales.

Let us describe our finite element approximation. For the sake of conciseness in the following exposition, let us introduce the operator

$$\mathcal{N}(\mathbf{u}_h,\mathbf{v}_h) = (\mathbf{u}_h\cdot\nabla)\mathbf{v}_h + \frac{1}{2}(\nabla\cdot\mathbf{u}_h)\mathbf{v}_h.$$

In order to state the problem, we introduce the subgrid velocity component $\tilde{\mathbf{u}}$, which is modeled as

$$\partial_t\tilde{\mathbf{u}} + \tau^{-1}\tilde{\mathbf{u}} = -P_h^{\perp}(\mathcal{N}(\mathbf{u}_h,\mathbf{u}_h) + \nabla p_h), \tag{3.6}$$

where $\tau^{-1}\tilde{\mathbf{u}}$ is an approximation to $-\nu\Delta\tilde{\mathbf{u}} + \mathbf{u}\cdot\nabla\tilde{\mathbf{u}}$ and the right-hand-side is the projection of the residual of the finite element component (see Remark 3.2). We compute the so-called stabilization parameter $\tau$ as

$$\tau = \left(\frac{C_{\rm s}\nu}{h^2} + \frac{C_{\rm c}\|\mathbf{u}_h\|_{0,\ell}}{h|\Omega|^{\frac{1}{\ell}}}\right)^{-1}. \tag{3.7}$$

$C_{\rm s}$ are $C_{\rm c}$ are algorithmic constants independent of physical and numerical parameters that are usually motivated from the analysis of one-dimensional tests (see, e.g., [14]). In the following, we assume that $2 \le \ell \le \infty$. For practical purposes, a nonconstant $\tau(\mathbf{x})$ is usually implemented, in which the global velocity norm is replaced by its pointwise modulus. The use of a variable stabilization parameter introduces some technical complications in the numerical analysis that have been faced in [16] for the linearized Oseen problem.

In (3.6) we can identify the two key features of our formulation: the $L^2$ orthogonality enforced by the projection in the right-hand side and the dynamic model due to the fact that it is an ordinary differential equation. The subscale model is very cheap, since it is a local problem at every finite element of the triangulation. In its numerical implementation, the subgrid component will be simply evaluated by using (3.6) at every integration point of every finite element. See [18] for different aspects related to the implementation of the orthogonal projection.

Let us consider the subgrid space

$$\tilde{V} = \mathrm{span}\{P_h^{\perp}(\mathcal{N}(\boldsymbol{\phi}_i, \boldsymbol{\phi}_j)), P_h^{\perp}(\nabla\pi_k)\}$$

for $i, j = 1, \dots, \mathrm{n_u}$ and $k = 1, \dots, \mathrm{n}_p$. We can easily see that the dimension $\tilde{\mathrm{n}}_{\mathrm{u}}$ of $\tilde{V}$ is less than or equal to $\mathrm{n_u^2} + \mathrm{n}_p < \infty$. So, $\tilde{V} \subset \boldsymbol{L}^2(\Omega)$ is a finite-dimensional space, and we could explicitly construct a basis $\{\tilde{\boldsymbol{\phi}}_i\}_{i=1,\dots,\tilde{\mathrm{n}}_{\mathrm{u}}}$ using, e.g., a Karhunen–Loève decomposition. We denote the $L^2(\Omega)$-orthogonal projection onto $\tilde{V}$ with $\tilde{P}(\cdot)$.

Thus, let us consider the following finite element approximation of the Navier–Stokes equations using a VMS dynamic orthogonal subgrid model: find $\mathbf{u}_h \in H^1(0,T;V_h)$, $p_h \in L^2(0,T;Q_h)$, and $\tilde{\mathbf{u}} \in H^1(0,T;\tilde{V})$ such that

$$(\partial_t\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla\mathbf{u}_h, \nabla\mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) - b(\mathbf{u}_h, \mathbf{v}_h, \tilde{\mathbf{u}}) = \langle \mathbf{f}, \mathbf{v}_h\rangle, \tag{3.8a}$$

$$(\partial_t\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) + b(\mathbf{u}_h, \mathbf{u}_h, \tilde{\mathbf{v}}) + \tau^{-1}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) + (\nabla p_h, \tilde{\mathbf{v}}) = \mathbf{0}, \tag{3.8b}$$

$$(q_h, \nabla\cdot\mathbf{u}_h) - (\tilde{\mathbf{u}}, \nabla q_h) = 0, \tag{3.8c}$$

which hold a.e. in $(0,T)$ and for any $\mathbf{v}_h \in V_h$, $\tilde{\mathbf{v}} \in \tilde{V}$, and $q_h \in Q_h$. These equations are supplemented with the initial condition

$$\mathbf{u}_h(0) = \mathbf{u}_{0h}, \quad \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0. \tag{3.9}$$

It will be shown later that this problem admits a unique solution in the spaces chosen. It has to be noted that the spatial boundedness is not necessarily uniform in $h$.

The initialization of the discrete problem can be obtained by the following projection problem: find $\mathbf{u}_{0h} \in V_h$, $\tilde{\mathbf{u}}_0 \in \tilde{V}$, and $\xi_h \in Q_h$ such that

$$\begin{aligned}
(\mathbf{u}_{0h}, \mathbf{v}_h) - (\xi_h, \nabla\cdot\mathbf{v}_h) &= (\mathbf{u}_0, \mathbf{v}_h) && \forall\ \mathbf{v}_h \in V_h,\\
(\tilde{\mathbf{u}}_0, \tilde{\mathbf{v}}) + (\nabla\xi_h, \tilde{\mathbf{v}}) &= (\mathbf{u}_0, \tilde{\mathbf{v}}) && \forall\ \tilde{\mathbf{v}} \in \tilde{V},\\
(\nabla\cdot\mathbf{u}_{0h}, q_h) - (\nabla q_h, \tilde{\mathbf{u}}_0) &= 0 && \forall\ q_h \in Q_h.
\end{aligned}$$

The nice feature of this choice is the fact that the initial velocity components satisfy (3.8c), which can have important effects on the stability of the fully discrete problem (see [12]).

We can easily see that the pointwise and weak subgrid equations, (3.6) and (3.8b), respectively, are equivalent. Equation (3.8b) can be written as $\tilde{P}(\partial_t \tilde{\mathbf{u}} + \tau^{-1}\tilde{\mathbf{u}}) = -\tilde{P}(\mathcal{N}(\mathbf{u}_h, \mathbf{u}_h) + \nabla p_h)$. So, using the fact that $\tau^{-1}$ is space-independent, we easily recover (3.6). We will use the weak formulation for the subsequent analysis. As far as we know, a weak formulation of the subgrid model is new.

REMARK 3.1. *The $\tilde{\mathbf{u}}$-dependent term on the left-hand side of* (3.8a) *and* (3.8c) *stands for the effect of the subgrid scales on the finite element component. The first one gives enhanced velocity stability, whereas the second provides pressure stability, as we shall see.*

REMARK 3.2. *In general, for a residual-based stabilized method, the right-hand side of* (3.8b) *includes the force and the viscous terms in order to have a consistent method; i.e., it must be*

$$\langle \mathbf{f}, \tilde{\mathbf{v}} \rangle - b\left(\mathbf{u}_h, \mathbf{u}_h, \tilde{\mathbf{v}}\right) - \left(\nabla p_h, \tilde{\mathbf{v}}\right) + \sum_{K \in \mathcal{T}_h} \left(\nu \Delta \mathbf{u}_h, \tilde{\mathbf{v}}\right)_K .$$

*The subscript $K$ in the last term indicates that the Laplacian is considered inside every finite element separately. It is obvious that the viscous term vanishes for piecewise linear approximations. However, for higher order polynomial approximations, this term and the force term do not vanish. In the following, we perform the analysis omitting these two terms. The resulting method has been proved to be optimally convergent, using the fact that $\tilde{V}$ is orthogonal to $V_h$. We refer to* [3, 16] *for more details.*

Let us denote by $V_*$ the space $V_h \oplus \tilde{V}$ and by $J_*$ the finite-dimensional space of functions $\mathbf{v}$ that are the sum of a finite element function $\mathbf{v}_h \in V_h$ and a subgrid scale function $\tilde{\mathbf{v}} \in \tilde{V}$ satisfying the constraint $(q_h, \nabla \cdot \mathbf{v}_h) - (\tilde{\mathbf{v}}, \nabla q_h) = 0$ for any $q_h \in Q_h$. In particular, let us define $\mathbf{u}_* := \mathbf{u}_h + \tilde{\mathbf{u}}$. We make the following assumption.

ASSUMPTION 3.2. *The finite-dimensional space $J^*$ is not reduced to the null element.*

This assumption is very mild, and is satisfied for equal-order finite element spaces for velocity and pressure (see [3]). However, it prevents the pressure space from being arbitrarily large.

**3.4. Preliminary results.** In the next lemma, we prove existence and uniqueness for system (3.8), inspired by the ideas in [33] for the Galerkin approximation (3.3) under the compatibility condition (3.5).

LEMMA 3.1. *The semidiscrete problem* (3.8) *has a unique solution that satisfies $\mathbf{u}_* \in \mathcal{C}([0,T]; \boldsymbol{L}^2(\Omega))$ and its time derivative $\partial_t \mathbf{u}_* \in L^2(0,T;\boldsymbol{L}^2(\Omega))$ for any $T > 0$.*

*Proof.* We can eliminate the pressure in the stabilized semidiscrete problem (3.8) in a similar fashion as for the Galerkin approximation (see [33]). Since $V_h \cap \tilde{V} \equiv \{\mathbf{0}\}$, the decomposition of any function $\mathbf{v}_* \in J_*$ into its finite element and subgrid components, $P_h(\mathbf{v}_*)$ and $\tilde{P}(\mathbf{v}_*)$, respectively, is unique. System (3.8) can now be stated as follows: find $\mathbf{u}_*(t) \in H^1(0,T;J_*)$ such that

$$\begin{aligned} (3.10) \qquad & (\partial_t \mathbf{u}_*, \mathbf{v}_*) + b\left(P_h(\mathbf{u}_*), P_h(\mathbf{u}_*), \mathbf{v}_*\right) + \nu\left(\nabla P_h(\mathbf{u}_*), \nabla P_h(\mathbf{v}_*)\right) \\ & + \tau^{-1}(\tilde{P}(\mathbf{u}_*), \tilde{P}(\mathbf{v}_*)) - b(P_h(\mathbf{u}_*), P_h(\mathbf{v}_*), \tilde{P}(\mathbf{u}_*)) = \langle \mathbf{f}, P_h(\mathbf{v}_*) \rangle \end{aligned}$$

for any $\mathbf{v}_* \in J_*$, a.e. in $(0,T)$, with the initial condition $\mathbf{u}_*(0) = \mathbf{u}_{0h} + \tilde{\mathbf{u}}_0$. Using the theory of ordinary differential equations, we can prove that in fact $\mathbf{u}_*(t)$ and

$\partial_t \mathbf{u}_*(t)$ are square integrable functions in some time interval $[0, t_h)$ for $t_h > 0$ small enough, since $\langle \mathbf{f}, P_h(\mathbf{v}_*)\rangle$ belongs to $L^2(0, t_h)$ for $t_h < \infty$ (see, e.g., [51, Theorem 3.1, Chapter 3]). On the other hand, $\mathbf{u}_* \in \mathcal{C}([0, t_h); \boldsymbol{L}^2(\Omega))$, $\partial_t \mathbf{u}_* \in L^2(0, t_h; \boldsymbol{L}^2(\Omega))$ (see [51, Lemma 1.1, Chapter 3]). Thus, the initial condition is meaningful. Now, we are in a position to test (3.10) against $\mathbf{u}_*$. By using the fact that

$$b(P_h(\mathbf{u}_*), P_h(\mathbf{u}_*), \mathbf{u}_*) - b(P_h(\mathbf{u}_*), P_h(\mathbf{u}_*), \tilde{P}(\mathbf{u}_*)) = 0,$$

which comes from the skew-symmetry of $b$, and treating the time derivative term using [51, Lemma 1.2, Chapter 3], we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_*\|^2 + \nu\|\nabla P_h(\mathbf{u}_*)\|^2 + \tau^{-1}\|\tilde{P}(\mathbf{u}_*)\|^2 \leq \langle \mathbf{f}, P_h(\mathbf{u}_*)\rangle, \tag{3.11}$$

and so $\|\mathbf{u}_*(t)\| \lesssim \int_0^t \|\mathbf{f}(s)\|_{-1}\mathrm{d}s + \|\mathbf{u}_{*0}\|$ (see also [33]). This energy bound allows us to extend the regularity results for $\mathbf{u}_*$ over the whole time interval $[0, T]$ (see [51, Theorem 3.1, Chapter 3] for more details).

Given $\mathbf{u}_*(t)$, the problem for the pressure now reads as follows: find $p_h(t) \in Q_h$ such that

$$\begin{aligned}(3.12)\qquad & (p_h, \nabla\cdot P_h(\mathbf{v}_*)) - (\tilde{P}(\mathbf{v}_*), \nabla p_h)\\ &= (\partial_t \mathbf{u}_*, \mathbf{v}_*) + \nu\,(\nabla P_h(\mathbf{u}_*), \nabla P_h(\mathbf{v})) + \tau^{-1}(\tilde{P}(\mathbf{u}_*), \tilde{P}(\mathbf{v}_*))\\ &\quad + b\,(P_h(\mathbf{u}_*), P_h(\mathbf{u}_*), \mathbf{v}_*) - b(P_h(\mathbf{u}_*), P_h(\mathbf{v}_*), \tilde{P}(\mathbf{u}_*)) - \langle \mathbf{f}, P_h(\mathbf{v}_*)\rangle\end{aligned}$$

a.e. in $(0, T)$ for any $\mathbf{v}_* \in V_*$. Since the right-hand side is square integrable in $(0, T)$, so is $p_h$. We have that $\nabla Q_h \subset V_*$ and $J_* \equiv (\nabla Q_h)^{\perp} \cap V_*$ (where the orthogonality is understood in the $L^2$ sense) by construction of these finite-dimensional spaces. On the one hand, problem (3.12) for any $\mathbf{v}_* \in \nabla Q_h$ is equivalent to that of finding a $p_h \in Q_h$ solution of the finite element approximation of a Laplacian problem with Neumann boundary conditions, whose existence and uniqueness is easily obtained (see, e.g., [30]). On the other hand, for $\mathbf{v}_* \in J_*$ both the left-hand side and the right-hand side of (3.12) vanish, yielding the identity $0 = 0$. Thus problem (3.12) has a unique solution $(\mathbf{u}_*, p_h)$. This proves the lemma. ∎

Let us prove some preliminary results that will be needed in the following sections. First, we analyze the approximation properties of the VMS stabilized finite element approximation of the steady Stokes problem using orthogonal subscales. The Stokes problem reads as follows: find $\mathbf{a} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega)$ and $\varphi \in L_0^2(\Omega)/\mathbb{R} \cap H^1(\Omega)$ such that

$$-\nu\Delta\mathbf{a} + \nabla\varphi = \mathbf{g} \quad \text{in } \Omega, \tag{3.13a}$$

$$\nabla\cdot\mathbf{a} = 0 \quad \text{in } \Omega \tag{3.13b}$$

for any $\mathbf{g} \in \boldsymbol{L}^2(\Omega)$. Let us make the following assumption, which is known to be true when $\Omega$ satisfies some regularity properties (see, e.g., [30]).

ASSUMPTION 3.3. *The solution of system* (3.13) *satisfies the elliptic regularity assumption*

$$\nu\|\mathbf{a}\|_2 + \|\varphi\|_1 \leq \|\mathbf{g}\|. \tag{3.14}$$

The stabilized finite element approximation of the Stokes problem, using orthogonal subscales, reads as (see [3]) follows: find $\mathbf{a}_h \in V_h$, $\varphi_h \in Q_h$, and $\tilde{\mathbf{a}} \in \tilde{V}$ such that

$$\nu\left(\nabla \mathbf{a}_h, \nabla \mathbf{v}_h\right) - \left(\varphi_h, \nabla \cdot \mathbf{v}_h\right) = \left\langle \mathbf{g}, \mathbf{v}_h \right\rangle, \tag{3.15a}$$

$$\left(q_h, \nabla \cdot \mathbf{a}_h\right) - \left(\tilde{\mathbf{a}}, \nabla q_h\right) = 0, \tag{3.15b}$$

$$\tau_\nu^{-1}\left(\tilde{\mathbf{a}}, \tilde{\mathbf{v}}\right) + \left(\nabla \varphi_h, \tilde{\mathbf{v}}\right) = \left\langle \mathbf{g}, \tilde{\mathbf{v}} \right\rangle, \tag{3.15c}$$

where $\tau_\nu^{-1} := \frac{C_{\mathrm{s}}\nu}{h^2}$ and $\tilde{V}$ is designed as before, but without the contribution from the convective term in the Navier–Stokes equations, i.e., $\tilde{V} := \operatorname{span}\{P_h^{\perp}(\nabla \pi_k)\}$, for $k = 1, \ldots, \mathrm{n}_p$.

Lemma 3.2 (error estimates for (3.15)). *Let us assume that Assumption* 3.3 *holds. Then, the solution* $(\mathbf{a}_h, \varphi_h, \tilde{\mathbf{a}})$ *of problem* (3.15) *and the continuous solution* $(\mathbf{a}, \varphi)$ *of problem* (3.13) *satisfy the error estimates*

$$\nu^{\frac{1}{2}}\|\nabla(\mathbf{a} - \mathbf{a}_h)\| + \frac{\nu^{\frac{1}{2}}}{h}\|\tilde{\mathbf{a}}\| + \frac{1}{\nu^{\frac{1}{2}}}\|\varphi - \varphi_h\| \lesssim \frac{h}{\nu^{\frac{1}{2}}}\|\mathbf{g}\|. \tag{3.16}$$

*Proof.* We indicate the finite element component of the error functions with

$$\mathbf{e}_h = P_h(\mathbf{a}) - \mathbf{a}_h, \qquad \psi_h = P_{Q_h}(\varphi) - \varphi_h.$$

Subtracting the weak form of system (3.13) and (3.15), we obtain the error system

$$\nu\left(\nabla \mathbf{e}_h, \nabla \mathbf{v}_h\right) - \left(\psi_h, \nabla \cdot \mathbf{v}_h\right) = \langle \mathcal{E}^1, \mathbf{v}_h \rangle, \tag{3.17a}$$

$$\left(\nabla \cdot \mathbf{e}_h, q_h\right) + \left(\tilde{\mathbf{a}}, \nabla q_h\right) = \langle \mathcal{E}^2, q_h \rangle, \tag{3.17b}$$

with

$$\langle \mathcal{E}^1, \mathbf{v}_h \rangle := -\nu\left(\nabla(\mathbf{a} - P_h(\mathbf{a})), \nabla \mathbf{v}_h\right) + \left(\varphi - P_{Q_h}(\varphi), \nabla \cdot \mathbf{v}_h\right),$$
$$\langle \mathcal{E}^2, q_h \rangle := -\left(\nabla \cdot (\mathbf{a} - P_h(\mathbf{a})), q_h\right).$$

Let us rewrite the subscale equation as follows:

$$\frac{C_{\mathrm{s}}\nu}{h^2}\left(\tilde{\mathbf{a}}, \tilde{\mathbf{v}}\right) - \left(\nabla(P_{Q_h}(\varphi) - \varphi_h), \tilde{\mathbf{v}}\right) = \left(\mathbf{g} - \nabla P_{Q_h}(\varphi), \tilde{\mathbf{v}}\right) =: \langle \mathcal{E}^3, \tilde{\mathbf{v}} \rangle. \tag{3.18}$$

We denote by $\varepsilon_i(v) := |v - P_h(v)|_i$, where $|\cdot|_i$ denotes the seminorm in $H^i(\Omega)$. We can easily bound the right-hand side of system (3.17)–(3.18) using integration by parts and invoking the momentum equation in (3.13a) as follows:

$$\langle \mathcal{E}^1, \mathbf{v}_h \rangle \lesssim \left(\nu^{\frac{1}{2}}\varepsilon_1(\mathbf{a}) + \nu^{-\frac{1}{2}}\varepsilon_0(\varphi)\right)\nu^{\frac{1}{2}}\|\nabla \mathbf{v}_h\|,$$
$$\langle \mathcal{E}^2, q_h \rangle \lesssim \nu^{\frac{1}{2}}h^{-1}\varepsilon_0(\mathbf{a})\nu^{-\frac{1}{2}}h\|P_h^{\perp}(\nabla q_h)\|,$$
$$\langle \mathcal{E}^3, \tilde{\mathbf{v}} \rangle \lesssim h\nu^{-\frac{1}{2}}\left(\nu\|\Delta \mathbf{a}\| + \|\nabla(\varphi - P_{Q_h}(\varphi))\|\right)\nu^{\frac{1}{2}}h^{-1}\|\tilde{\mathbf{v}}\|.$$

Let us define the interpolation and consistency error function

$$E(h) := \nu^{\frac{1}{2}}h^{-1}\varepsilon_0(\mathbf{a}) + \nu^{\frac{1}{2}}\varepsilon_1(\mathbf{a}) + h\nu^{\frac{1}{2}}\|\Delta \mathbf{a}\| + \nu^{-\frac{1}{2}}\varepsilon_0(\varphi) + h\nu^{-\frac{1}{2}}\varepsilon_1(\varphi).$$

Now, we take $\mathbf{v}_h = \mathbf{e}_h$, $q_h = \psi_h$, and $\tilde{\mathbf{v}} = \tilde{\mathbf{a}}$ in (3.17)–(3.18), respectively. We obtain

$$\nu\|\nabla \mathbf{e}_h\|^2 + \frac{\nu}{h^2}\|\tilde{\mathbf{a}}\|^2 \lesssim E(h)\left(\nu^{\frac{1}{2}}\|\nabla \mathbf{e}_h\| + \frac{h}{\nu^{\frac{1}{2}}}\|P_h^{\perp}\nabla \psi_h\| + \frac{\nu^{\frac{1}{2}}}{h}\|\tilde{\mathbf{a}}\|\right). \tag{3.19}$$

We can find a bound for $h\nu^{-\frac{1}{2}}\|P_h^{\perp}\nabla\psi_h\|$ using the subscale equation (3.15c) in its pointwise sense and (3.13a), getting

(3.20)
$$\|P_h^{\perp}(\nabla\psi_h)\| \lesssim \frac{\nu}{h^2}\|\tilde{\mathbf{a}}\| + \|\mathbf{g} - \nabla\varphi\| + \|\nabla(\varphi - P_{Q_h}(\varphi))\| \lesssim \frac{\nu}{h^2}\|\tilde{\mathbf{a}}\| + \nu\|\Delta\mathbf{a}\| + \|\nabla\varphi\|,$$

where we have used the $H^1(\Omega)$-stability of $P_h(\cdot)$ for quasi-uniform meshes (see [7]). This expression is now incorporated into (3.19) to get

$$\nu\|\nabla\mathbf{e}_h\|^2 + \frac{\nu}{h^2}\|\tilde{\mathbf{a}}\|^2 \lesssim E(h)\left(\nu^{\frac{1}{2}}\|\nabla\mathbf{e}_h\| + h\nu^{\frac{1}{2}}\|\Delta\mathbf{a}\| + \frac{h}{\nu^{\frac{1}{2}}}\|\nabla\varphi\| + \frac{\nu^{\frac{1}{2}}}{h}\|\tilde{\mathbf{a}}\|\right).$$

The regularity assumptions in the statement of Lemma 3.2 allow us to obtain $\nu\|\mathbf{a}\|_2 + \|\varphi\|_1 \leq \|\mathbf{g}\|$. This leads to

$$\nu^{\frac{1}{2}}\|\nabla\mathbf{e}_h\| + h^{-1}\nu^{\frac{1}{2}}\|\tilde{\mathbf{a}}\| \lesssim h\nu^{-\frac{1}{2}}\|\mathbf{g}\|,$$

where we have used the fact that $E(h) \lesssim h\nu^{-\frac{1}{2}}\|\mathbf{g}\|$, a direct consequence of the interpolation results in (3.1) and the stability of $P_h(\cdot)$ in $H^1(\Omega)$. Global errors (3.16) are obtained using the standard interpolation results (3.1) and the triangle inequality.

In order to get stability bounds over the pressure, we test (3.17a) with $\mathbf{v}_h = P_h(\nabla\varphi - \nabla\varphi_h)$. After rearranging the resulting equality and invoking an inverse inequality, we get

$$h\nu^{-\frac{1}{2}}\|P_h(\nabla\varphi - \nabla\varphi_h)\| \lesssim \nu^{\frac{1}{2}}\|\nabla(\mathbf{a} - \mathbf{a}_h)\| \lesssim E(h).$$

So, using (3.20), the definition of $E(h)$, and the stability of $P_{Q_h}(\cdot)$, we get

$$\begin{aligned} h\nu^{-\frac{1}{2}}\|\nabla\varphi - \nabla\varphi_h\| &\leq h\nu^{-\frac{1}{2}}\|P_h(\nabla\varphi - \nabla\varphi_h)\| + h\nu^{-\frac{1}{2}}\|P_h^{\perp}(\nabla\varphi - \nabla\varphi_h)\| \\ &\lesssim E(h) + h\nu^{-\frac{1}{2}}\|P_h^{\perp}(\nabla\psi_h)\| + h\nu^{-\frac{1}{2}}\|P_h^{\perp}(\nabla\varphi - \nabla P_{Q_h}(\varphi))\| \\ &\lesssim h\nu^{-\frac{1}{2}}\|\mathbf{g}\|. \end{aligned} \tag{3.21}$$

On the other hand, for all $q \in L^2(\Omega)$, there exists $\mathbf{v}_q \in \boldsymbol{H}_0^1(\Omega)$ such that

$$(q, \nabla\cdot\mathbf{v}_q) \gtrsim \|q\|\|\mathbf{v}_q\|_1,$$

due to (2.4). Therefore, for $Q_h \subset \mathcal{C}^0(\Omega)$ we can find $\mathbf{v}_e$ such that

$$\begin{aligned} \|\varphi - \varphi_h\|\|\mathbf{v}_e\|_1 &\lesssim (\nabla(\varphi - \varphi_h), \mathbf{v}_e) \\ &\leq (\nabla(\varphi - \varphi_h), \mathbf{v}_e - P_h(\mathbf{v}_e)) + (\nabla(\varphi - \varphi_h), P_h(\mathbf{v}_e)) \\ &\lesssim (\nabla(\varphi - \varphi_h), \mathbf{v}_e - P_h(\mathbf{v}_e)) - \nu(\nabla(\mathbf{a} - \mathbf{a}_h), \nabla P_h(\mathbf{v}_e)) \\ &\lesssim \|\nabla(\varphi - \varphi_h)\|h\|\mathbf{v}_e\|_1 + \nu\|\nabla(\mathbf{a} - \mathbf{a}_h)\|\|\mathbf{v}_e\|_1. \end{aligned}$$

We easily get $\nu^{-\frac{1}{2}}\|\varphi - \varphi_h\| \lesssim h\nu^{-\frac{1}{2}}\|\mathbf{g}\|$. ∎

Finally, let us prove a discrete version of a well-known interpolation inequality (see [1]) that will be required for the treatment of the nonlinear terms. In order to prove the following lemma, we assume that the regularity of the Poisson–Dirichlet problem holds.

ASSUMPTION 3.4. *Let $\Omega$ be such that $\|\mathbf{u}\|_2 \lesssim \|\Delta\mathbf{u}\|$ for any $\mathbf{u} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega)$.*

Let us also introduce the discrete Laplacian $\Delta_h \mathbf{u}_h \in V_h$, solution of

$$(\Delta_h \mathbf{u}_h, \mathbf{v}_h) = (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) \qquad \forall\ \mathbf{v}_h \in V_h.$$

LEMMA 3.3. *Let $\Omega \subset \mathbb{R}^2$ satisfy Assumption* 3.4*, and consider a quasi-uniform family of finite element meshes. For any $\mathbf{u}_h \in V_h$, the following inequality holds:*

$$\|\nabla \mathbf{u}_h\|_{0,4} \lesssim \|\nabla \mathbf{u}_h\|^{\frac{1}{2}} \|\Delta_h \mathbf{u}_h\|^{\frac{1}{2}}.$$

*Proof.* Let us consider $\mathbf{u} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega)$ such that $\Delta \mathbf{u} = \Delta_h \mathbf{u}_h$. Assuming regularity of the domain, e.g., a convex domain $\Omega$, we get the classical error estimates

$$\|\mathbf{u} - \mathbf{u}_h\| + h\|\nabla(\mathbf{u} - \mathbf{u}_h)\| \lesssim h^2 \|\Delta \mathbf{u}\|, \tag{3.22}$$

where the error estimate in the $L^2(\Omega)$ norm is proved using Aubin–Nitsche duality arguments (see, e.g., [23]). In particular, we get the error estimate

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,4} \lesssim h^{\frac{3}{2}} \|\Delta \mathbf{u}\|,$$

due to the Gagliardo–Nirenberg inequality (see [1]).

Using the inverse inequality $\|\nabla \mathbf{v}_h\|_{0,p} \lesssim h^{-1} \|\mathbf{v}_h\|_{0,p}$ (for $1 \le p \le \infty$) and the definition of $\Delta_h$, we easily get

$$\|\Delta_h \mathbf{u}_h\| \lesssim h^{-1} \|\nabla \mathbf{u}_h\| \lesssim h^{-2} \|\mathbf{u}_h\|.$$

These inverse estimates, together with the error estimates (3.22) and the definition of $\mathbf{u}$, lead to

$$\|\mathbf{u}\| \lesssim \|\mathbf{u}_h\|, \qquad \|\nabla \mathbf{u}\| \lesssim \|\nabla \mathbf{u}_h\|.$$

Let us introduce the Scott–Zang interpolation operator $\mathcal{SZ}_{V_h}(\cdot)$ with regard to $V_h$ (see [23, 7]). Using Assumption 3.4 and the previous inequalities, we obtain

$$\begin{aligned}
\|\nabla \mathbf{u}_h\|_{0,4} &\le \|\nabla(\mathcal{SZ}_{V_h}(\mathbf{u}) - \mathbf{u}_h)\|_{0,4} + \|\nabla \mathcal{SZ}_{V_h}(\mathbf{u})\|_{0,4} \\
&\lesssim h^{-1} \|\mathcal{SZ}_{V_h}(\mathbf{u}) - \mathbf{u}_h\|_{0,4} + \|\nabla \mathbf{u}\|_{0,4} \\
&\lesssim h^{-1} \left( \|\mathcal{SZ}_{V_h}(\mathbf{u}) - \mathbf{u}\|_{0,4} + \|\mathbf{u} - \mathbf{u}_h\|_{0,4} \right) + \|\nabla \mathbf{u}\|_1^{\frac{1}{2}} \|\nabla \mathbf{u}\|^{\frac{1}{2}} \\
&\lesssim h^{\frac{1}{2}} \|\Delta_h \mathbf{u}_h\|^{\frac{1}{2}} \|\Delta_h \mathbf{u}_h\|^{\frac{1}{2}} + \|\Delta_h \mathbf{u}_h\|^{\frac{1}{2}} \|\nabla \mathbf{u}_h\|^{\frac{1}{2}} \\
&\lesssim \|\nabla \mathbf{u}_h\|^{\frac{1}{2}} \|\Delta_h \mathbf{u}_h\|^{\frac{1}{2}}.
\end{aligned}$$

For the bound in the second line we have used the $W^{1,p}(\Omega)$-stability of the Scott–Zang interpolation (see [7, Theorem 9.8.15]). Then, in order to obtain the bounds in the third and fourth lines, we have invoked a Gagliardo–Niremberg inequality, the bound (3.22), and the interpolation properties of the projector (see [7, Theorem 9.8.12]). In particular, we have used the bound

$$\|\mathcal{SZ}_{V_h}(\mathbf{u}) - \mathbf{u}\|_{0,4} \lesssim \|\mathcal{SZ}_{V_h}(\mathbf{u}) - \mathbf{u}\|_1^{\frac{1}{2}} \|\mathcal{SZ}_{V_h}(\mathbf{u}) - \mathbf{u}\|^{\frac{1}{2}} \lesssim h^{\frac{3}{2}} \|\Delta \mathbf{u}\|.$$

This proves the lemma. ∎

**4. Long-term stability in $\boldsymbol{L^\infty(0,\infty;L^2(\Omega))}$.** Our first result proves that the VMS finite element approximation of the Navier–Stokes equations (3.8) exhibits an absorbing set in $\boldsymbol{L}^2(\Omega)$. A key difference with respect to previous analysis is the proof of an $\boldsymbol{L}^2(\Omega)$ absorbing set for the subgrid component too. We prove the existence of the $\boldsymbol{L}^2(\Omega)$ absorbing set and some long-term stability bounds in the next theorem that holds in two and three dimensions. When there is no confusion, we will omit the time label for the unknowns.

Let us start this section with short-term stability bounds that are straightforward from (3.11) for $T < \infty$.

THEOREM 4.1 (short-term stability). *Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3. When the time domain is bounded, i.e., $T < \infty$, system (3.8) with $\mathbf{u}_0 \in \boldsymbol{H}_0(\text{div } 0, \Omega)$, $\mathbf{f} \in L^2(0,T;\boldsymbol{L}^2(\Omega))$ satisfies the energy-type inequality*

$$\left(\|\mathbf{u}_h(t)\|^2 + \|\tilde{\mathbf{u}}(t)\|^2\right) + \int_0^T \left(\nu\|\nabla\mathbf{u}_h\|^2 + \tau^{-1}\|\tilde{\mathbf{u}}\|^2\right) \mathrm{d}s \lesssim \int_0^T \left(\frac{1}{2\nu}\|\mathbf{f}\|_{-1}^2\right) \mathrm{d}s + \|\mathbf{u}_h(0)\|^2 \qquad \text{a.e. for } t \in [0,T],$$

*which implies that*

$$\mathbf{u}_h \text{ is bounded in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)), \qquad \tilde{\mathbf{u}} \text{ is bounded in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)),$$
$$\nabla\mathbf{u}_h \text{ is bounded in } L^2(0,T;\boldsymbol{L}^2(\Omega)), \qquad \tau^{-\frac{1}{2}}\tilde{\mathbf{u}} \text{ is bounded in } L^2(0,T;\boldsymbol{L}^2(\Omega)).$$

The previous stability results are obtained with the minimum requirement that the body force $\mathbf{f} \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$. However, those stability results are not uniform with respect to $T$, since $\int_0^\infty \|\mathbf{f}\|_{-1}^2 \mathrm{d}s = \infty$ for a constant body force, e.g., the gravity force. In the next theorem, we will obtain long-term stability estimates that remain effective when $T \to \infty$. In order to obtain these results, a slightly more regular body force is needed, i.e., $\mathbf{f} \in L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$.

Let us introduce the dimensionless number

$$\mathrm{G} := \frac{|\Omega|^{\frac{2}{d}}}{\nu^2}\|\mathbf{f}\|_{L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))},$$

defined in [25] as the dimensionless Grashof number; G can also be interpreted as $\mathrm{Re}^2$. In the next theorems, we make use of $\rho := \nu\mathrm{G}$. Since the forcing term is understood in a Lebesgue sense, we understand the pointwise (in)equalities a.e. in time. For the sake of conciseness, we will usually omit this indication and implicitly assume distributional sense from here onwards.

THEOREM 4.2 (long-term stability in $L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$). *Let us assume that Assumption 3.3 holds. Then, the solution of problem (3.8) for $d = 2,\ 3$ satisfies*

$$\mathbf{u}_h \text{ is bounded in } L^\infty(0,\infty;\boldsymbol{L}^2(\Omega)), \qquad \tilde{\mathbf{u}} \text{ is bounded in } L^\infty(0,\infty;\boldsymbol{L}^2(\Omega)),$$
$$\nabla\mathbf{u}_h \text{ is bounded in } L^2_{\mathrm{loc}}(0,\infty;\boldsymbol{L}^2(\Omega)), \qquad \tau^{-\frac{1}{2}}\tilde{\mathbf{u}} \text{ is bounded in } L^2_{\mathrm{loc}}(0,\infty;\boldsymbol{L}^2(\Omega))$$

*for $\mathbf{u}_0 \in \boldsymbol{H}_0(\text{div } 0,\Omega)$ and $\mathbf{f} \in L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$. On the other hand, the following inequality holds:*

$$\limsup_{t\to\infty} \left(\|\mathbf{u}_h(t)\|^2 + \|\tilde{\mathbf{u}}(t)\|^2\right) \lesssim \frac{|\Omega|^{\frac{4}{d}}}{\nu^2}\|\mathbf{f}\|^2_{L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))}, \tag{4.1}$$

*which implies the existence of an absorbing set in $\boldsymbol{L}^2(\Omega)$.*

*Proof.* Let us start from the equality (3.11), rewritten as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{u}_h\|^2+\|\tilde{\mathbf{u}}\|^2\right)+\nu\|\nabla\mathbf{u}_h\|^2+\tau^{-1}\|\tilde{\mathbf{u}}\|^2=(\mathbf{f},\mathbf{u}_h)\,. \tag{4.2}$$

In order to bound the right-hand side, we use Hölder and Poincaré inequalities, the latter in the form $\|v\|^2 \le C_{\mathrm{P}}|\Omega|^{\frac{2}{d}}\|\nabla v\|^2$ for any $v \in H_0^1(\Omega)$. We obtain

$$(\mathbf{f},\mathbf{u}_h)\lesssim C_{\mathrm{P}}\frac{|\Omega|^{\frac{2}{d}}}{2\nu}\|\mathbf{f}\|^2+\frac{\nu}{2}\|\nabla\mathbf{u}_h\|^2. \tag{4.3}$$

Combining (4.2) and (4.3), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{u}_h\|^2+\|\tilde{\mathbf{u}}\|^2\right)+\nu\|\nabla\mathbf{u}_h\|^2+\tau^{-1}\|\tilde{\mathbf{u}}\|^2\lesssim\frac{|\Omega|^{\frac{2}{d}}}{\nu}\|\mathbf{f}\|^2, \tag{4.4}$$

which, integrated over $[t_0, t]$, leads to

$$\begin{aligned}&\|\mathbf{u}_h(t)\|^2+\|\tilde{\mathbf{u}}(t)\|^2+\int_{t_0}^{t}\left(\nu\|\nabla\mathbf{u}_h\|^2+\tau^{-1}\|\tilde{\mathbf{u}}\|^2\right)\mathrm{d}s\\ &\quad\lesssim\int_{t_0}^{t}\frac{|\Omega|^{\frac{2}{d}}}{\nu}\|\mathbf{f}\|^2\mathrm{d}s+\left(\|\mathbf{u}_h(t_0)\|^2+\|\tilde{\mathbf{u}}(t_0)\|^2\right).\end{aligned} \tag{4.5}$$

On the other hand, using the Poincaré inequality in (4.2), the inequality (4.3), the fact that $h < |\Omega|^{\frac{1}{d}}$, and the expression for $\tau \lesssim h^2\nu^{-1}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{u}_h\|^2+\|\tilde{\mathbf{u}}\|^2\right)+\nu|\Omega|^{-\frac{2}{d}}\left(\|\mathbf{u}_h\|^2+\|\tilde{\mathbf{u}}\|^2\right)\lesssim\frac{|\Omega|^{\frac{d}{2}}}{2\nu}\|\mathbf{f}\|^2.$$

Now, we can use the classical Gronwall lemma (see [51]), obtaining

$$\begin{aligned}\left(\|\mathbf{u}_h(t)\|^2+\|\tilde{\mathbf{u}}(t)\|^2\right)&\lesssim\left(1-\exp\left(-\nu|\Omega|^{-\frac{2}{d}}t\right)\right)\frac{|\Omega|^{\frac{4}{d}}}{\nu^2}\|\mathbf{f}\|^2_{L^\infty(0,t;\boldsymbol{L}^2(\Omega))}\\ &\quad+\exp\left(-\nu|\Omega|^{-\frac{2}{d}}t\right)\left(\|\mathbf{u}_h(0)\|^2+\|\tilde{\mathbf{u}}(0)\|^2\right).\end{aligned}$$

The previous inequality proves the $L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$-stability results and the existence of the $\boldsymbol{L}^2(\Omega)$ absorbing set, such that the orbit associated to any $\mathbf{u}_0 \in \boldsymbol{H}_0(\mathrm{div}\ 0,\Omega)$ enters this subset at some time $t^*(\rho,\mathbf{u}_0)$. Now, taking the limit superior for $t \to \infty$, we get

$$\limsup_{t\to\infty}\left(\|\mathbf{u}_h(t)\|^2+\|\tilde{\mathbf{u}}(t)\|^2\right)\lesssim\frac{|\Omega|^{\frac{4}{d}}}{\nu^2}\|\mathbf{f}\|^2_{L^\infty(0,t;\boldsymbol{L}^2(\Omega))}.$$

This proves the second part of the theorem. On the other hand, we get from (4.5) that

$$\int_{t_0}^{t}\left(\nu\|\nabla\mathbf{u}_h\|^2+\tau^{-1}\|\tilde{\mathbf{u}}\|^2\right)\mathrm{d}s\lesssim\left(\frac{|\Omega|^{\frac{2}{d}}}{\nu}+(t-t_0)\right)\frac{|\Omega|^{\frac{2}{d}}}{\nu}\|\mathbf{f}\|^2_{L^\infty(0,t;\boldsymbol{L}^2(\Omega))},$$

which proves the $L^2_{\mathrm{loc}}(0,\infty;\boldsymbol{L}^2(\Omega))$-stability results. ∎

REMARK 4.1. *The previous theorem proves the existence of an absorbing set for* $[\mathbf{u}_h, \tilde{\mathbf{u}}]$ *in* $\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)$ *of radius of the order of* Re. *Let us stress the fact that any* *stabilized finite element formulation without a dynamic subgrid model does not exhibit the subgrid absorbing set and the $L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$ subgrid stability bounds.*

The subgrid component is related to the part of the pressure gradient and convective term $L^2$-orthogonal to the finite element space. The goal of the VMS approach is to provide pressure stability without the need of an inf-sup condition and a numerical dissipation that will prevent energy pile-up at the smallest scales, effective as $\nu \to 0$. In the next theorem, we give a precise mathematical description of this fact; the idea is to translate the subgrid stability estimates in terms of the finite element components, as is usual for stabilized methods. The extra estimates for scheme (3.8) in the next theorem, which the Galerkin finite element method (FEM) does not provide, are weighted with a time-independent parameter $\tau_0 = \inf_{t\in(0,\infty)} \tau(t)$; i.e.,

$$\tau_0^{-1} = \frac{C_{\rm s}\nu}{h^2} + \frac{C_{\rm c}\sup_{t\in(0,\infty)}\|\mathbf{u}_h(t)\|_{0,\ell}}{h|\Omega|^{\frac{1}{\ell}}}.$$

Observe that the parameter $\tau_0^{-1}$ is well defined for a fixed $h > 0$ by using an inverse inequality $\|\mathbf{v}_h\|_{0,\ell} \lesssim h^{-(\frac{1}{2}-\frac{1}{\ell})}\|\mathbf{v}_h\|$ (for $2 \le \ell \le \infty$) and estimate (4.1). Thus, $\tau_0$ does not degenerate to 0. Let us stress the fact that the introduction of the weighting parameter $\tau_0$ comes from technical aspects in the subsequent analysis, but the results apply to system (3.8) with the time-dependent expression of $\tau$ in (3.7).

THEOREM 4.3. *Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3. Algorithm* (3.8) *with $2 < \ell \le \infty$ in* (3.7) *satisfies, for any $\varpi \ge 0$ and $t_0 \ge 0$,*

$$\tau_0^{\frac{1}{2}}\|\nabla p_h + \mathcal{N}(\mathbf{u}_h,\mathbf{u}_h)\|_{H_0^{-1}(t_0,t_0+\varpi;\boldsymbol{L}^{q'}(\Omega))} \le C$$

*for $q' = \frac{2\ell}{\ell-2}$. In the case $\ell = 2$, there holds*

$$\tau_0^{\frac{1}{2}}\|\nabla p_h + \mathcal{N}(\mathbf{u}_h,\mathbf{u}_h)\|_{H_0^{-1}(t_0,t_0+\varpi;\boldsymbol{W}_0^{-1,(d+\varepsilon)'})} \le C$$

*for a fixed $\varepsilon > 0$, where $(d+\varepsilon)'$ denotes the conjugate exponent of $d+\varepsilon$, and $C$ is a constant that depends on $(\mathbf{u}_0,\rho,\Omega,\varpi)$. In particular, for $t_0 \to \infty$, $C$ depends only on $(\rho,\Omega,\varpi)$.*

*Proof.* Recall that $P_h(\cdot)$ is the orthogonal projection operator with respect to the $L^2$ inner product. Let us set $\bar{t} := t_0 + \varpi$. We take $\mathbf{v}_h = P_h(\mathbf{v})$ in the finite element equation (3.8a), where the regularity of $\mathbf{v}$ will be defined later on, and integrate it over a finite interval $[t_0,\bar{t}]$ and multiply the resulting equation by the scalar value $\tau_0^{\frac{1}{2}}$. For simplicity, let us also consider that $\mathbf{v}(\bar{t}) = \mathbf{v}(t_0) = 0$. We get

$$\begin{aligned}&\int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}}\left(P_h(\nabla p_h + \mathcal{N}(\mathbf{u}_h,\mathbf{u}_h)),\mathbf{v}\right){\rm d}s\\ &= -\int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}}\left\{(\partial_t\mathbf{u}_h,\mathbf{v}_h) + \nu\,(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h) - b\,(\mathbf{u}_h,\mathbf{v}_h,\tilde{\mathbf{u}})\right\}{\rm d}s.\end{aligned} \tag{4.6}$$

In the following, we bound the right-hand side terms in the finite element equation (4.6). The first term can be bounded using integration-by-parts in time and the definition of $\mathbf{v}$ in order to obtain

$$\begin{aligned}-\int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}}\,(\partial_t\mathbf{u}_h,\mathbf{v}_h)\,{\rm d}s &= \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}}\,(\mathbf{u}_h,\partial_t\mathbf{v}_h)\,{\rm d}s \lesssim \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}}\|\mathbf{u}_h\|\|\partial_t\mathbf{v}_h\|{\rm d}s\\ &\lesssim \tau_0^{\frac{1}{2}}\left(\int_{t_0}^{\bar{t}}\|\mathbf{u}_h\|^2{\rm d}s\right)^{\frac{1}{2}}\left(\int_{t_0}^{\bar{t}}\|\partial_t\mathbf{v}_h\|^2{\rm d}s\right)^{\frac{1}{2}}.\end{aligned} \tag{4.7}$$

The convective term is bounded using Hölder's inequality for mixed norms (see [1]) as follows:

$$
\begin{aligned}
-\int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} b\left(\mathbf{u}_h, \mathbf{v}_h, \tilde{\mathbf{u}}\right) \mathrm{d}s &\lesssim \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} \|\tilde{\mathbf{u}}\| h^{-1} \|\mathbf{u}_h\|_\ell \|\mathbf{v}_h\|_q \mathrm{d}s \\
&\lesssim \int_{t_0}^{\bar{t}} \tau^{-\frac{1}{2}} \|\tilde{\mathbf{u}}\| |\Omega|^{\frac{1}{\ell}} \|\mathbf{v}_h\|_q \mathrm{d}s \\
&\lesssim |\Omega|^{\frac{1}{\ell}} \left( \int_{t_0}^{\bar{t}} \tau^{-1} \|\tilde{\mathbf{u}}\|^2 \mathrm{d}s \right)^{\frac{1}{2}} \left( \int_{t_0}^{\bar{t}} \|\mathbf{v}_h\|_q^2 \mathrm{d}s \right)^{\frac{1}{2}}, \qquad (4.8)
\end{aligned}
$$

where we recall that $2 \leq \ell \leq \infty$ in the definition (3.7) of $\tau$, whereas $q = \frac{2\ell}{\ell - 2}$. Let us observe that $q \geq 2$. Finally, using an inverse inequality, we obtain

$$
\begin{aligned}
-\int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} \nu \left(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h\right) \mathrm{d}s &\lesssim \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} h^{-1} \nu \|\nabla \mathbf{u}_h\| \|\mathbf{v}_h\| \mathrm{d}s \\
&\lesssim \left( \int_{t_0}^{\bar{t}} \nu \|\nabla \mathbf{u}_h\|^2 \mathrm{d}s \right)^{\frac{1}{2}} \left( \int_{t_0}^{\bar{t}} \|\mathbf{v}_h\|^2 \mathrm{d}s \right)^{\frac{1}{2}}. \qquad (4.9)
\end{aligned}
$$

Combining (4.7)–(4.9), we get

$$
\begin{aligned}
\int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} \left( P_h(\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)), \mathbf{v} \right) \mathrm{d}s \leq \tau_0^{\frac{1}{2}} &\left( \int_{t_0}^{\bar{t}} \|\mathbf{u}_h\|^2 \mathrm{d}s \right)^{\frac{1}{2}} \left( \int_{t_0}^{\bar{t}} \|\partial_t \mathbf{v}_h\|^2 \mathrm{d}s \right)^{\frac{1}{2}} \\
&+ |\Omega|^{\frac{1}{\ell}} \left( \int_{t_0}^{\bar{t}} \tau^{-1} \|\tilde{\mathbf{u}}\|^2 \mathrm{d}s \right)^{\frac{1}{2}} \left( \int_{t_0}^{\bar{t}} \|\mathbf{v}_h\|_q^2 \mathrm{d}s \right)^{\frac{1}{2}} \\
&+ \left( \int_{t_0}^{\bar{t}} \nu \|\nabla \mathbf{u}_h\|^2 \mathrm{d}s \right)^{\frac{1}{2}} \left( \int_{t_0}^{\bar{t}} \|\mathbf{v}_h\|^2 \mathrm{d}s \right)^{\frac{1}{2}}.
\end{aligned}
$$

In view of the above discussion, we consider $\mathbf{v} \in H_0^1(t_0, \bar{t}; \boldsymbol{L}^q(\Omega))$ to conclude that

$$
\tau_0^{\frac{1}{2}} \|P_h(\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h))\|_{H_0^{-1}(t_0, \bar{t}; \boldsymbol{L}^{q'}(\Omega))} \leq C, \qquad (4.10)
$$

with $q'$ being the conjugate of $q$ and $C$ involving the problem data $(\mathbf{u}_0, \rho, \Omega, \varpi)$, by using the fact that $P_h(\cdot)$ is a stable operator in $L^s$, with $1 \leq s \leq \infty$. In particular, for $t_0 \to \infty$, $C$ depends only on $(\rho, \Omega, \varpi)$ . Note that when $\ell = 2$, we have $q = \infty$, whose dual space is not identified with $L^1(\Omega)$. To bypass this problem, we use the Sobolev embedding $W_0^{1,d+\varepsilon}(\Omega) \hookrightarrow L^\infty(\Omega)$, where $d$ is the space dimension and $\varepsilon > 0$ is a fixed number. Therefore, we have that

$$
\tau_0^{\frac{1}{2}} \|P_h(\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)\|_{H_0^{-1}(t_0, \bar{t}; \boldsymbol{W}_0^{-1,(d+\varepsilon)'}(\Omega))} \leq C
$$

when $\ell = 2$ and $(d + \varepsilon)'$ is the conjugate of $(d + \varepsilon)$.

Our next step is to find a bound for the subscale part of $\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)$. For this we multiply the subscale equation by $\tau_0^{\frac{1}{2}}$ and integrate it over a finite interval $[t_0, \bar{t}]$. We get

$$
\int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} \left( P_h^{\perp}(\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)), \mathbf{v} \right) \mathrm{d}s = -\int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} \left( \partial_t \tilde{\mathbf{u}} + \tau^{-1} \tilde{\mathbf{u}}, \mathbf{v} \right) \mathrm{d}s. \qquad (4.11)
$$

For the right-hand side terms in the subgrid equation (4.11), we proceed as follows:

$$-\int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\partial_t \tilde{\mathbf{u}}, \mathbf{v})\, \mathrm{d}s = \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\tilde{\mathbf{u}}, \partial_t \mathbf{v})\, \mathrm{d}s \lesssim \int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} \|\tilde{\mathbf{u}}\| \|\partial_t \mathbf{v}\| \mathrm{d}s$$
$$\lesssim \tau_0^{\frac{1}{2}} \left( \int_{t_0}^{\bar{t}} \|\tilde{\mathbf{u}}\|^2 \mathrm{d}s \right)^{\frac{1}{2}} \left( \int_{t_0}^{\bar{t}} \|\partial_t \mathbf{v}\|^2 \mathrm{d}s \right)^{\frac{1}{2}},$$
$$-\int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} (\tau^{-1}\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \mathrm{d}s \lesssim \left( \int_{t_0}^{\bar{t}} \tau^{-1} \|\tilde{\mathbf{u}}\|^2 \mathrm{d}s \right)^{\frac{1}{2}} \left( \int_{t_0}^{\bar{t}} \|\tilde{\mathbf{v}}\|^2 \mathrm{d}s \right)^{\frac{1}{2}}.$$

Therefore,

$$\int_{t_0}^{\bar{t}} \tau_0^{\frac{1}{2}} \left( P_h^{\perp}(\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)), \mathbf{v} \right) \mathrm{d}s \lesssim \tau_0^{\frac{1}{2}} \left( \int_{t_0}^{\bar{t}} \|\tilde{\mathbf{u}}\|^2 \mathrm{d}s \right)^{\frac{1}{2}} \left( \int_{t_0}^{\bar{t}} \|\partial_t \tilde{\mathbf{v}}\|^2 \mathrm{d}s \right)^{\frac{1}{2}}$$
$$+ \left( \int_{t_0}^{\bar{t}} \tau^{-1} \|\tilde{\mathbf{u}}\|^2 \mathrm{d}s \right)^{\frac{1}{2}} \left( \int_{t_0}^{\bar{t}} \|\tilde{\mathbf{v}}\|^2 \mathrm{d}s \right)^{\frac{1}{2}}.$$

Thus, we have proved by selecting $\mathbf{v} \in H_0^1(t_0, \bar{t}; \boldsymbol{L}^2(\Omega))$ that

$$\tau_0^{\frac{1}{2}} \|P_h^{\perp}(\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h))\|_{H_0^{-1}(t_0, \bar{t}; \boldsymbol{L}^2(\Omega))} \leq C. \tag{4.12}$$

Then, it is clear that from (4.10) and (4.12) that we have

$$\tau_0^{\frac{1}{2}} \|\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)\|_{H_0^{-1}(t_0, \bar{t}; \boldsymbol{L}^{q'}(\Omega))} \leq C,$$

where $C$ depends only on $(\rho, \Omega, \varpi)$. Analogously, for $\ell = 2$, we arrive at

$$\|\nabla p_h + \mathcal{N}(\mathbf{u}_h, \mathbf{u}_h)\|_{H_0^{-1}(t_0, \bar{t}; \boldsymbol{W}_0^{-1,(d+\varepsilon)'})} \leq C$$

for a fixed $\varepsilon > 0$. ∎

REMARK 4.2. *The previous result proves the effectiveness of algorithm* (3.8) *as a stabilization technique. Both pressure stability and velocity stability that do not vanish with $\nu \to 0$ have been proved at all times.*

REMARK 4.3. *The previous results bound a sum of pressure and convection terms, whereas it would be desirable to have separate control of these two terms. This kind of result is not specific to our formulation, being a common feature of residual-based stabilization techniques. In fact, this is the case even for the steady Navier–Stokes equations (see, e.g.,* [14]*). Numerical evidence shows the effectiveness of residual-based stabilization techniques, even though separate bounds that would be effective for large* Re *have not been proved so far. A partial remedy could be the split version of the stabilization terms proposed in* [16].

Connected to Remark 4.3, the next result provides an estimate for the convective term independent of the pressure term when $2 < \ell \leq \infty$. For the linearized problem and with divergence-free advection velocities, this would allow us to obtain as well an estimate for the pressure gradient alone using Theorem 4.3. For the problem we consider, this could also be achieved by introducing the pressure subgrid scale and the additional control on the velocity divergence it provides, although we will not exploit this here (see [15, 16]). As far as we know, even though it is rather weak, this is the first time that a result of this kind has been established. Similar weak norms have been investigated in the framework of a posteriori error estimation for convection-diffusion equations in [55].

COROLLARY 4.4. *Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3. There holds*

$$\tau^{\frac{1}{2}} \|\nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h)\|_{L^\infty(0,T;\boldsymbol{W}_0^{-1,s'}(\Omega))} \leq C \|\mathbf{u}_h\|_{0,2}^2,$$

*where $s'$ is the conjugate of $s$ such that $s = \frac{2\ell}{\ell-1}$ in two dimensions and $s = \frac{12\ell}{5\ell-6}$ in three dimensions, when $2 < \ell \leq \infty$.*

*Proof.* Let us give the proof for the two-dimensional case only. For each $2 < \ell \leq \infty$, we can find $2 < r < \ell$ such that the interpolation inequality

$$\|\mathbf{u}_h\|_{0,r} \leq \|\mathbf{u}_h\|^{\frac{1}{2}} \|\mathbf{u}_h\|_{0,\ell}^{\frac{1}{2}}$$

holds, with $\frac{1}{2} + \frac{1}{\ell} = \frac{2}{r}$. Therefore, thanks to $\frac{1}{4} + \frac{1}{r} + \frac{1}{s} = 1$, we write, for all $\boldsymbol{\phi} \in \boldsymbol{W}_0^{1,s}(\Omega)$,

$$\begin{aligned} \tau^{\frac{1}{2}} (\nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h), \boldsymbol{\phi}) &\leq \tau^{\frac{1}{2}} \|\mathbf{u}_h\|_{0,r} \|\mathbf{u}_h\|_{0,4} \|\nabla \boldsymbol{\phi}\|_{0,s} \\ &\leq \tau^{\frac{1}{2}} \|\mathbf{u}_h\|^{1/2} \|\mathbf{u}_h\|_{0,\ell}^{1/2} \|\mathbf{u}_h\|_{0,4} \|\nabla \boldsymbol{\phi}\|_{0,s} \\ &\leq \left( \frac{h \|\mathbf{u}_h\|_{0,\ell}}{C_{\mathrm{s}}\nu + C_{\mathrm{c}} h \|\mathbf{u}_h\|_{0,\ell} |\Omega|^{\frac{1}{\ell}}} \right)^{1/2} h^{1/2} \|\mathbf{u}_h\|_{0,4} \|\boldsymbol{\phi}\|_{1,s} \\ &\lesssim \|\boldsymbol{\phi}\|_{1,s} \|\mathbf{u}_h\|_{0,2}^2, \end{aligned}$$

where in the last line we have used the inverse inequality $\|\mathbf{v}_h\|_{0,4} \lesssim h^{-\frac{1}{2}} \|\mathbf{v}_h\|_{0,2}$. ∎

**5. Absorbing set in $\boldsymbol{H}^1(\Omega)$ and the global attractor for $d = 2$.** In this section, we prove the existence of an absorbing set in $\boldsymbol{H}^1(\Omega)$, which is the key result for the existence of a global attractor for algorithm (3.8). Let us introduce first the uniform Gronwall lemma (see, e.g., [53]).

LEMMA 5.1 (uniform Gronwall lemma). *Let $x$, $\mu$, $f$ be three positive locally integrable functions on $(t_0, \infty)$, such that $\partial_t x$ is locally integrable on $(t_0, \infty)$, which satisfy*

$$\partial_t x \leq \mu x + f \qquad \text{a.e. for } t \geq t_0,$$

$$\int_t^{t+\varpi} \mu(s) \mathrm{d}s \leq a_1, \quad \int_t^{t+\varpi} f(s) \mathrm{d}s \leq a_2, \quad \int_t^{t+\varpi} x(s) \mathrm{d}s \leq a_3 \qquad \text{for } t \geq t_0,$$

*where $\varpi$, $a_1$, $a_2$, $a_3$ are positive constant values. Then,*

$$x(t+\varpi) \leq \left( \frac{a_3}{\varpi} + a_2 \right) \exp(a_1) \qquad \text{a.e. for } t \geq t_0.$$

In order to get the bounds that lead to the existence of the $\boldsymbol{H}^1(\Omega)$ absorbing set, let us introduce the scalar value

$$\tau_U^{-1} = \frac{C_{\mathrm{s}}\nu}{h^2} + \frac{C_{\mathrm{c}} U}{h},$$

where $U > 0$ is a bounded characteristic velocity of the problem. In particular, $U = \sup_{t\in(t_0,\infty)} |\Omega|^{-\frac{1}{\ell}}\|\mathbf{u}_h\|$ is a possible choice, since $\ell \geq 2$ and $\sup_{t\in(t_0,\infty)}\|\mathbf{u}_h\|$ has been bounded in Theorem 4.2. The long-term stability of the subgrid velocity in the next theorem is weighted by $\tau_U^{-1}$. Again, the introduction of the weighting parameter $\tau_U$ is purely technical, and the following results apply to system (3.8) with the time-dependent expression of $\tau$ in (3.7). Let us recall that it is only the definition of $\tau$ that depends on $\ell$.

THEOREM 5.2 ($\boldsymbol{H}^1(\Omega)$ absorbing set). *Let $\Omega \subset \mathbb{R}^2$ be such that Assumptions 3.3 and 3.4 hold. Then, the solution $(\mathbf{u}_h, p_h, \tilde{\mathbf{u}})$ of problem (3.8), for $2 \leq \ell < \infty$, satisfies the long-term stability bound*

$$\limsup_{t\to\infty}\left(\nu\|\nabla\mathbf{u}_h\|^2 + \tau_U^{-1}\|\tilde{\mathbf{u}}\|^2\right) \lesssim \left(a_3 + \frac{a_2}{\varpi}\right)\exp\left(a_1\right),$$

*with*

$$\int_t^{t+\varpi}\nu^{-2}\left(\nu^{-2}\|\mathbf{u}_h\|^2+1\right)\left(\nu\|\nabla\mathbf{u}_h\|^2+\tau_\nu^{-1}\|\tilde{\mathbf{u}}\|^2\right)\mathrm{d}s \lesssim \left(\nu^{-4}\rho^2+\nu^{-2}\right)a_2 =: a_1,$$

$$\int_t^{t+\varpi}\left(\|\mathbf{f}\|^2+U^4\right)\mathrm{d}s \leq \varpi\left(\|\mathbf{f}\|^2_{L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))}+U^4\right) =: a_2,$$

$$\int_t^{t+\varpi}\left(\nu\|\nabla\mathbf{u}_h\|^2+\tau_U^{-1}\|\tilde{\mathbf{u}}\|^2\right)\mathrm{d}s \lesssim \rho^2\left(1+\frac{\varpi\nu}{|\Omega|}\right) =: a_3$$

*for any fixed $\varpi > 0$. This bound proves the existence of an absorbing set in $\boldsymbol{H}^1(\Omega)$ for the finite element fluid velocity and an absorbing set in $\boldsymbol{L}^2(\Omega)$ for $\tau_U^{-\frac{1}{2}}\tilde{\mathbf{u}}$.*

*Proof.* Let us reformulate system (3.8a), (3.8c), and (3.8b) in an appropriate way for the subsequent analysis, introducing the new variables $\mathbf{z}_h$ and $\tilde{\mathbf{z}}$:

$$\left(\partial_t\mathbf{u}_h,\mathbf{v}_h\right)+b\left(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h\right)+\left(\mathbf{z}_h,\mathbf{v}_h\right)-b\left(\mathbf{u}_h,\mathbf{v}_h,\tilde{\mathbf{u}}\right)=\left(\mathbf{f},\mathbf{v}_h\right), \tag{5.1a}$$

$$\nu\left(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h\right)-\left(p_h,\nabla\cdot\mathbf{v}_h\right)=\left(\mathbf{z}_h,\mathbf{v}_h\right), \tag{5.1b}$$

$$\left(\partial_t\tilde{\mathbf{u}},\tilde{\mathbf{v}}\right)+\left(\tilde{\mathbf{z}},\tilde{\mathbf{v}}\right)+C_{\mathrm{c}}\left(\frac{\|\mathbf{u}_h\|_\ell}{|\Omega|^{\frac{1}{\ell}}h}-\frac{U}{h}\right)\left(\tilde{\mathbf{u}},\tilde{\mathbf{v}}\right)-b\left(\mathbf{u}_h,\mathbf{u}_h,\tilde{\mathbf{v}}\right)=\mathbf{0}, \tag{5.1c}$$

$$\tau_U^{-1}\left(\tilde{\mathbf{u}},\tilde{\mathbf{v}}\right)+\left(\nabla p_h,\tilde{\mathbf{v}}\right)=\left(\tilde{\mathbf{z}},\tilde{\mathbf{v}}\right), \tag{5.1d}$$

$$\left(q_h,\nabla\cdot\mathbf{u}_h\right)-\left(\tilde{\mathbf{u}},\nabla q_h\right)=0. \tag{5.1e}$$

First, we take $\mathbf{v}_h = \mathbf{z}_h$ in (5.1a) and $\tilde{\mathbf{v}} = \tilde{\mathbf{z}}$ in (5.1c) in order to get

$$\left(\partial_t\mathbf{u}_h,\mathbf{z}_h\right)+b\left(\mathbf{u}_h,\mathbf{u}_h,\mathbf{z}_h\right)+\|\mathbf{z}_h\|^2-b\left(\mathbf{u}_h,\mathbf{z}_h,\tilde{\mathbf{u}}\right)=\left(\mathbf{f},\mathbf{z}_h\right), \tag{5.2a}$$

$$\left(\partial_t\tilde{\mathbf{u}},\tilde{\mathbf{z}}\right)+\|\tilde{\mathbf{z}}\|^2=-b\left(\mathbf{u}_h,\mathbf{u}_h,\tilde{\mathbf{z}}\right)-C_{\mathrm{c}}\left(\frac{\|\mathbf{u}_h\|_p}{|\Omega|^{\frac{1}{p}}h}-\frac{U}{h}\right)\left(\tilde{\mathbf{u}},\tilde{\mathbf{z}}\right). \tag{5.2b}$$

From the regularity of the solution in Lemma 3.1, we are allowed to integrate (5.1b) and (5.1b) in the time domain for $\mathbf{v}_h = \partial_t\mathbf{u}_h$ and $\tilde{\mathbf{v}} = \partial_t\tilde{\mathbf{u}}$. Doing that, we finally get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\nu\|\nabla\mathbf{u}_h\|^2-\left(p_h,\nabla\cdot\partial_t\mathbf{u}_h\right)=\left(\mathbf{z}_h,\partial_t\mathbf{u}_h\right), \tag{5.3a}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\tau_U^{-1}\|\tilde{\mathbf{u}}\|^2+\left(\nabla p_h,\partial_t\tilde{\mathbf{u}}\right)=\left(\tilde{\mathbf{z}},\partial_t\tilde{\mathbf{u}}\right). \tag{5.3b}$$

Note that we have used the fact that $\tau_U$ is constant in time; this technical reason prevents us from getting long-term subscale estimates multiplied by the time-dependent stabilization parameter $\tau$ that is used in the algorithm. Now, using [51, Lemma 1.1, Chapter 3] and the regularity of the solution in Lemma 3.1, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}_h + \tilde{\mathbf{u}}, \nabla q_h) = (\partial_t(\mathbf{u}_h + \tilde{\mathbf{u}}), \nabla q_h) = 0 \qquad \forall q_h \in Q_h.$$

We invoke this result in (5.2) and (5.3), obtaining

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\nu}{2}\|\nabla \mathbf{u}_h\|^2 + \frac{\tau_U^{-1}}{2}\|\tilde{\mathbf{u}}\|^2\right) + \|\mathbf{z}_h\|^2 + \|\tilde{\mathbf{z}}\|^2 = (\mathbf{f}, \mathbf{z}_h) \\ &\quad - b\left(\mathbf{u}_h, \mathbf{u}_h, \mathbf{z}_h\right) - b\left(\mathbf{u}_h, \mathbf{u}_h, \tilde{\mathbf{z}}\right) + b\left(\mathbf{u}_h, \mathbf{z}_h, \tilde{\mathbf{u}}\right) - C\left(\frac{\|\mathbf{u}_h\|_p}{|\Omega|^{\frac{1}{p}} h} - \frac{U}{h}\right)(\tilde{\mathbf{u}}, \tilde{\mathbf{z}}). \end{aligned} \tag{5.4}$$

Before controlling the right-hand side of (5.4) we introduce some technical tools. Let us define $\hat{\mathbf{u}} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega)$ as the solution of the following Stokes problem:

$$\begin{cases} -\nu\Delta\hat{\mathbf{u}} + \nabla\hat{p} = \mathbf{g} := \mathbf{z}_h + \tilde{\mathbf{z}} + (\tau_\nu^{-1} - \tau_U^{-1})\tilde{\mathbf{u}} & \text{in } \Omega, \\ \nabla\cdot\hat{\mathbf{u}} = 0 & \text{in } \Omega, \\ \hat{\mathbf{u}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\tau_\nu^{-1} := C_{\mathrm{s}} h^{-2}\nu$ (see (3.15)). From (5.1b), (5.1e), (5.1d), one can write

$$\begin{cases} \nu\left(\nabla\mathbf{u}_h, \nabla\mathbf{v}_h\right) - (p_h, \nabla\cdot\mathbf{v}_h) = (\mathbf{z}_h, \mathbf{v}_h), \\ (q_h, \nabla\cdot\mathbf{u}_h) - (\tilde{\mathbf{u}}, \nabla q_h) = 0, \\ \tau_\nu^{-1}\left(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}\right) + (\nabla p_h, \tilde{\mathbf{v}}) = \left((\tau_\nu^{-1} - \tau_U^{-1})\tilde{\mathbf{u}} + \tilde{\mathbf{z}}, \tilde{\mathbf{v}}\right). \end{cases}$$

From Lemma 3.2, we know that $\nu^{\frac{1}{2}}\|\nabla(\hat{\mathbf{u}} - \mathbf{u}_h)\| + h^{-1}\nu^{\frac{1}{2}}\|\tilde{\mathbf{u}}\| \lesssim h\nu^{-\frac{1}{2}}\|\mathbf{g}\|$. Next, we want to bound $\|\Delta_h\mathbf{u}_h\|$ in terms of $\nu^{-1}\|\mathbf{g}\|$. Indeed, taking $\mathbf{u} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega)$ as the solution of $\Delta\mathbf{u} = \Delta_h\mathbf{u}_h$, we obtain (see [34])

$$\begin{aligned} \|\Delta_h\mathbf{u}_h\|^2 &\leq -\left(\nabla\mathbf{u}_h, \nabla\Delta_h\mathbf{u}_h\right) \\ &\leq -\left(\nabla(\hat{\mathbf{u}} - \mathbf{u}_h), \nabla\Delta_h\mathbf{u}_h\right) + (\Delta\hat{\mathbf{u}}, \Delta_h\mathbf{u}_h) \\ &\leq \|\Delta_h\mathbf{u}_h\|\left(h^{-1}\|\nabla(\hat{\mathbf{u}} - \mathbf{u}_h)\| + \|\Delta\hat{\mathbf{u}}\|\right) \\ &\lesssim \|\Delta_h\mathbf{u}_h\|\nu^{-1}\|\mathbf{g}\|. \end{aligned}$$

This result allows us to say that $\nu\|\Delta_h\mathbf{u}_h\| + h^{-2}\nu\|\tilde{\mathbf{u}}\| \lesssim \|\mathbf{g}\|$. On the other hand, using the expression of $\tau_U$, we find

$$\|\mathbf{g}\| \lesssim \|\mathbf{z}_h\| + \|\tilde{\mathbf{z}}\| + h^{-1}U\|\tilde{\mathbf{u}}\| \lesssim \|\mathbf{z}_h\| + \|\tilde{\mathbf{z}}\| + U^2 + \nu^{-1}\tau_\nu^{-1}\|\tilde{\mathbf{u}}\|^2.$$

Our goal now is to bound the right-hand side of (5.4). For every nonlinear term, we will repeatedly apply the results of Lemma 3.3 and Young's inequality. For the first nonlinear term on the right-hand side of (5.4), we use these ingredients and the fact that $\nu\|\Delta_h\mathbf{u}_h\| \lesssim \|\mathbf{g}\|$:

$$\begin{aligned} b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{z}_h) &\leq \|\mathbf{u}_h\|_{0,4}\|\nabla\mathbf{u}_h\|_{0,4}\|\mathbf{z}_h\| \lesssim \|\mathbf{u}_h\|^{\frac{1}{2}}\|\nabla\mathbf{u}_h\|\|\Delta_h\mathbf{u}_h\|^{\frac{1}{2}}\|\mathbf{z}_h\| \\ &\lesssim \frac{\nu^{-1}}{\delta}\|\mathbf{u}_h\|\|\nabla\mathbf{u}_h\|^2\|\mathbf{g}\| + \delta\|\mathbf{z}_h\|^2 \\ &\lesssim \frac{\nu^{-2}}{\delta^3}\|\mathbf{u}_h\|^2\|\nabla\mathbf{u}_h\|^4 + \delta\|\mathbf{g}\|^2 + \delta\|\mathbf{z}_h\|^2, \end{aligned}$$

and, analogously,

$$b(\mathbf{u}_h, \mathbf{u}_h, \tilde{\mathbf{z}}) \lesssim \frac{\nu^{-2}}{\delta^3}\|\mathbf{u}_h\|^2\|\nabla \mathbf{u}_h\|^4 + \delta\|\mathbf{g}\|^2 + \delta\|\tilde{\mathbf{z}}\|^2.$$

For the third nonlinear term, we use the inverse inequalities in (3.2), the expression for $\tau_\nu$, and the result $\tau_\nu^{-1}\|\tilde{\mathbf{u}}\| \lesssim \|\mathbf{g}\|$, obtaining the following bound:

$$\begin{aligned} b\left(\mathbf{u}_h, \mathbf{z}_h, \tilde{\mathbf{u}}\right) &\leq \|\tilde{\mathbf{u}}\|\|\mathbf{u}_h\|_{0,4}\|\nabla \mathbf{z}_h\|_{0,4} + \|\tilde{\mathbf{u}}\|\|\nabla\cdot\mathbf{u}_h\|_{0,4}\|\mathbf{z}_h\|_{0,4} \\ &\lesssim h^{-1}\|\tilde{\mathbf{u}}\|\|\mathbf{u}_h\|_{0,4}\|\mathbf{z}_h\|_{0,4} \\ &\lesssim h^{-1}\|\tilde{\mathbf{u}}\|\|\mathbf{u}_h\|^{\frac{1}{2}}\|\nabla\mathbf{u}_h\|^{\frac{1}{2}}\|\mathbf{z}_h\|^{\frac{1}{2}}\|\nabla\mathbf{z}_h\|^{\frac{1}{2}} \\ &\lesssim \|\mathbf{u}_h\|^{\frac{1}{2}}h^{-\frac{1}{2}}\|\tilde{\mathbf{u}}\|^{\frac{1}{2}}\|\nabla\mathbf{u}_h\|^{\frac{1}{2}}h^{-1}\|\tilde{\mathbf{u}}\|^{\frac{1}{2}}\|\mathbf{z}_h\| \\ &\lesssim \frac{\nu^{-2}}{\delta^3}\|\mathbf{u}_h\|^2\nu^{-1}\tau_\nu^{-1}\|\tilde{\mathbf{u}}\|^2\|\nabla\mathbf{u}_h\|^2 + \delta\|\mathbf{g}\|^2 + \delta\|\mathbf{z}_h\|^2 \\ &\lesssim \frac{\nu^{-2}}{\delta^3}\|\mathbf{u}_h\|^2\left(\nu^{-2}\tau_\nu^{-2}\|\tilde{\mathbf{u}}\|^4 + \|\nabla\mathbf{u}_h\|^4\right) + \delta\|\mathbf{g}\|^2 + \delta\|\mathbf{z}_h\|^2. \end{aligned}$$

Finally, by using the Sobolev embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$, i.e., $\|\mathbf{u}\|_p \leq C_{\mathrm{P}}|\Omega|^{\frac{1}{p}}\|\nabla\mathbf{u}\|$ for $d = 2$, and the expression of $\tau_\nu$, we bound the last term on the right-hand side of (5.4) as follows:

$$\left(\frac{\|\mathbf{u}_h\|_p}{|\Omega|^{\frac{1}{p}}h} - \frac{U}{h}\right)(\tilde{\mathbf{u}}, \tilde{\mathbf{z}}) \leq \frac{1}{\delta}\|\nabla\mathbf{u}_h\|^4 + \frac{1}{\delta}U^4 + \frac{1}{\delta}\nu^{-2}\tau_\nu^{-2}\|\tilde{\mathbf{u}}\|^4 + \delta\|\tilde{\mathbf{z}}\|^2.$$

For the force term, we simply have

$$(\mathbf{f}, \mathbf{z}_h) \lesssim \frac{1}{\delta}\|\mathbf{f}\|^2 + \delta\|\mathbf{z}_h\|^2.$$

The above bounds applied to (5.4), picking $\delta$ small enough, yield

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\nu\|\nabla\mathbf{u}_h\|^2 + \tau_U^{-1}\|\tilde{\mathbf{u}}\|^2\right) + \|\mathbf{z}_h\|^2 + \|\tilde{\mathbf{z}}\|^2 \\ &\quad\lesssim \nu^{-2}\left(\nu^{-2}\|\mathbf{u}_h\|^2 + 1\right)\left(\nu^2\|\nabla\mathbf{u}_h\|^4 + \tau_\nu^{-2}\|\tilde{\mathbf{u}}\|^4\right) + U^4 + \|\mathbf{f}\|^2 \\ &\quad\lesssim \nu^{-2}\left(\nu^{-2}\|\mathbf{u}_h\|^2 + 1\right)\left(\nu\|\nabla\mathbf{u}_h\|^2 + \tau_\nu^{-1}\|\tilde{\mathbf{u}}\|^2\right)\left(\nu\|\nabla\mathbf{u}_h\|^2 + \tau_U^{-1}\|\tilde{\mathbf{u}}\|^2\right) + U^4 + \|\mathbf{f}\|^2. \end{aligned}$$

We finish the proof using the uniform Gronwall lemma over the previous inequality, with the constants in the statement of Theorem 5.2. ∎

The previous stability bounds lead to an absorbing set in $\boldsymbol{H}^1(\Omega)$ for the finite element component of the velocity. With regard to the subgrid scale, Theorem 5.2 proves that $\tau_U^{-\frac{1}{2}}\tilde{\mathbf{u}}$ also exhibits an absorbing set in $\boldsymbol{L}^2(\Omega)$, which can only be obtained for dynamic subgrid models. With regard to the norms involved, the previous results are stronger than those in Theorem 4.2. However, the radius of the absorbing set in Theorem 4.2 is much smaller than the one for Theorem 5.2 for large Re. Thus, from a numerical point of view, in which constants do matter, the $\boldsymbol{L}^2(\Omega)$ results are stronger.

We conclude the section with the proof of the existence of a global attractor.

COROLLARY 5.3. *Let* $\Omega \subset \mathbb{R}^2$ *satisfy Assumption* 3.3. *For autonomous systems, i.e., if* $\mathbf{f}$ *is time-independent, the* $\boldsymbol{L}^2(\Omega)$ *absorbing set for* $\mathbf{u}_h$ *is a global attractor.*

*Proof.* The existence of an absorbing set in $\boldsymbol{H}^1(\Omega)$ obtained in the previous theorem allows us to say that there is always a ball in $\boldsymbol{H}^1(\Omega)$ that absorbs all orbits for large enough time values. Due to the Rellich–Kondrachov embedding theorem, i.e., $H^1(\Omega) \hookrightarrow L^2(\Omega)$, the $\boldsymbol{L}^2(\Omega)$ absorbing set in Theorem 4.2 is in fact compact in dimension 2. Thus, the operators $\mathcal{S}(t)$ are uniformly compact for $t$ large enough, and the existence of a compact global attractor can be proved (see [53, Theorem 1.1] for details). ∎

**6. Conclusions.** We have presented a finite element approximation of the Navier–Stokes equations with numerical subgrid scale modeling for which the results obtained here are easily summarized: we have been able to prove that the long-term behavior is similar to what is found for the pure Galerkin method, plus additional control on the velocity subgrid scales. In particular, we have shown that $\mathbf{u}_h$ is bounded in $\boldsymbol{L}^2(\Omega)$ for all time *and so is* the velocity subgrid scale $\tilde{\mathbf{u}}$, that in two dimensions the spatial dissipation associated to $\mathbf{u}_h$ is bounded in $L^2(0,\infty)$ *and so is* the dissipation associated to $\tilde{\mathbf{u}}$, and that $\mathbf{u}_h$ has an absorbing set in $\boldsymbol{L}^2(\Omega)$ *and so does* $\tilde{\mathbf{u}}$. In the two-dimensional case, for $\mathbf{u}_h$ the absorbing set can be shown to be a global attractor using classical arguments.

The *benefit* of our approach is that additional control on the pressure and the convective terms can be recovered from the stability obtained for the velocity subgrid scales. The key point, and in some sense the essence of stabilized FEMs for convection dominant flows, is that this control remains meaningful for $\nu \to 0$.

This last issue brings us to discuss the *limitations* of our analysis. As for all stabilized formulations we are aware of, *full* control on the pressure is not obtained (not even for the stationary Oseen problem), but only the sum of the pressure gradient and the convective term can be shown to be stable. In practice, however, this seems to be enough, although, as far as we know, no theoretical explanation has been provided. We have, however, provided a weak estimate in this direction, showing that some control can be proved for the convective term and the pressure gradient alone. Another limitation of our analysis is that we have needed to assume that the advection velocity is $\mathbf{u}_h$, and not $\mathbf{u}_h+\tilde{\mathbf{u}}$, and that we have had to take a constant stabilization parameter, whereas in practice it is computed from local values (at least at the element level).

Let us stress also that *the key* for being able to prove our stability estimates is twofold: the velocity subgrid scale $\tilde{\mathbf{u}}$ *needs to be time-dependent and orthogonal to the finite element space*. These ideas were introduced in [15], and we have used them in an essential way in the analysis presented here, starting with the existence and uniqueness proof. For these reasons, it is not clear whether or not the results proved in this work will be shared by nondynamical stabilized formulations and/or techniques that do not enforce $L^2$-orthogonality (see [17, 10, 12, 6, 36, 39] for examples).

The next issue we wish to consider is the design of time integration schemes that preserve the stability results proved here for the time-continuous case, particularly considering that the time integration of $\mathbf{u}_h$ and of $\tilde{\mathbf{u}}$ will probably have different requirements. This is, however, the subject of future research.

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