Programa de Doctorado "Matemáticas"

PhD Dissertation

ORTHOGONAL MATRIX POLYNOMIALS AND DIFFERENTIAL, DIFFERENCE AND q-DIFFERENCE MATRIX OPERATORS.

Author
Ana Martínez de los Ríos

Supervisor
Prof. Dr. *Antonio Durán Guardeño*

Co-supervisor
Dr. *Renato Álvarez-Nodarse*

May 22, 2014

Contents

Introduction v

1 Preliminaries. 1

1.1 Hypergeometric functions. 1

1.1.1 Hypergeometric functions 1

1.1.2 Basic hypergeometric functions. 3

1.2 Matrix Orthogonal Polynomials. 4

1.3 Matrix Operators. 9

1.3.1 Matrix differential operators 12

1.3.2 Matrix difference operators 16

1.3.3 Matrix q-difference operators 19

2 Discrete orthogonal polynomials 23

2.1 A method to solve the symmetry equations. 25

2.2 Examples. 26

2.2.1 Example with $\mathrm{dgr}(F_1) = 0$ 26

2.2.2 Example with $\mathrm{dgr}(F_1) = 1$ and unbounded support. 28

2.2.3 Example with $\mathrm{dgr}\,(F_1) = 1$ and finite support 30

2.2.4 Example with $\mathrm{dgr}(F_1) = 2$ 31

3 The convex cone. 37

3.1 Two methods. 37

3.2 Examples. 40

3.2.1 Examples constructed by using the first method 40

3.2.2 Examples constructed by using the second method 44

4 Matrix q-polynomials. 51

4.1 A family of q-polynomials. 52

4.2 Matrix q-hypergeometric function. 56

4.3 The 2×2 case. 57

4.3.1 The little q-Jacobi basis. 58

4.3.2 The matrix hypergeometric function. 60

4.3.3 The TTRR and the Rodrigues' formula 60

5 Continuous matrix polynomials 67
5.1 The weight matrix 67
5.2 Two differential operators. 73
5.3 Matrix hypergeometric function. 84
5.4 Three-term recurrence relation 86
5.5 Gegenbauer and Racah polynomials. 88

Introduction

In this thesis we present a series of result that are framed in the theory of Matrix Orthogonal Polynomials, a branch of the very celebrated subject of Orthogonal Polynomials.

The theory of orthogonal polynomials dates back from the end of the 19th century and its birth used to be attributed to the works of T. J. Stieltjes (1856-1894) and P. L. Chebyshev (1821-1894). The first discovered the orthogonality relation of the denominators of certain continuous fractions and to him is also attributed a primitive version of Farvard's theorem. He also introduced what is now known as the moments problem, given a sequence $(\mu_n)_n$ find a measure $d\mu$ such that $\mu_n = \int x^n d\mu(x)$.

Chebyshev, among other results, introduced the Chebyshev polynomials (nowadays known as Chebyshev polynomials of the first kind) T_n in order to find the best polynomial approximation for a continuous function.

However several authors had already introduced particular examples that had came to them via physical or mathematical problems. For example, Adrien Marie Legendre (1752-1833) introduced the Legendre polynomials (later generalized by Jacobi) related with the resolution of planetary motion problems. After him, Charles Hermite (1822-1901) defined the Hermite polynomial for the study of expansion series in $\mathbb{R}$, although these polynomials had already been considered and studied by Laplace in the context of probability theory. In relation with the integral

$$\int_x^\infty \frac{e^{-x}}{x} dx,$$

E. N. Laguerre (1834-1886) introduced the Laguerre polynomials, that were generalized by N. Sonin. In 1859 a work by K. G. J. Jacobi (1804-1851) gave a generalization of the Legendre polynomials, known as Jacobi polynomials.

The first monograph dedicated to the subject of orthogonal polynomials appeared in 1939, Orthogonal polynomials by Gabor Szegö [97].

All these families of orthogonal polynomials happened to have some interesting properties in common, in particular they are all eigenfunctions of a second order differential operator of the form

$$D(\cdot) = \sigma(x)\frac{d^2}{dx^2} + \tau(x)\frac{d}{dx}, \tag{1}$$

where σ is a polynomial of degree at most two and τ is a polynomial of degree exactly one. This property characterizes the families of Jacobi, Hermite and Laguerre polynomials, as it was proven by S. Bochner in 1929, [8]. Some other properties were shown to characterize these families. In 1887 Sonin proved that the only families of orthogonal polynomials whose derivatives are also orthogonal were the Jacobi, Hermite and Laguerre polynomials.

We also want to mention a third characterization proposed by F. Tricomi, that is the fact that these families can be expressed in terms of a Rodrigues formula.

$$p_n(x) = \frac{b_n}{\rho(x)} \frac{d^n}{dx^n} (\sigma^n(x)\rho(x)),$$

where $\rho(x)$ is the weight with respect to which the sequence is orthogonal. A very handy characterizing property of these families of orthogonal polynomials is a distributional one, the weights associated to these families satisfy a Pearson equation

$$(\sigma(x)\rho(x))' = \tau(x)\rho(x).$$

These families are called classical orthogonal polynomials and their associated weights classical weights.

The classical orthogonal polynomials constitute a very important class of families of orthogonal polynomials because of their properties and applications. They can be expressed in terms of hypergeometric functions

$${}_2F_1\left(\begin{matrix} -n, a \\ b \end{matrix} \,\middle|\, x\right) = \sum_{k=0}^{n} \frac{(-n)_k (a)_k}{(b)_k} \frac{x^k}{k!}.$$

In 1949 two authors published what we could see as the seminal works for this thesis. The first one, W. Hahn (1911-1998) extended with *Über Orthogonalpolynome die q-Differenzengleichungen genügen*, [73], the classical orthogonal polynomials by generalizing their properties. For q and w fixed numbers with $q \neq 1$ let L be the linear operator

$$\mathrm{L}\, f(x) = \frac{f(qx+w) - f(x)}{(q-1)x + w}.$$

Hahn proposed the following problem[1]: determine all sequences of orthogonal polynomials $(p_n)_n$ such that

1. $(\mathrm{L}\, p_n(x))_n$ is also a sequence of orthogonal polynomials.

2. For all $n \geq 0$, $p_n(x)$ satisfies an operator equation of the form

$$\tilde{\sigma}(x)\, \mathrm{L}^2\, p_n(x) + \tilde{\tau}(x)\, \mathrm{L}\, p_n(x) = \lambda_n p_n(x) \quad \tilde{\sigma}(x), \tilde{\tau}(x) \in \mathbb{C}[x], \lambda_n \in \mathbb{C}.$$

3. $p_n(x) = \mathrm{L}^n \left(R_0(x) R_1(x) \cdots R_n(x) \rho(x)\right) \rho(x)^{-1}$, where $R_0(x)$ is a polynomial and $R_i(x) = R_{i+1}(qx+w)$ and $\rho(x)$ is the weight with respect to which $(p_n)_n$ is orthogonal.

In this article Hahn proved that these properties determine one and the same class of orthogonal polynomials and these polynomials can be constructed in terms of basic hypergeometric functions.

For the particular case $q = 1$, $w = 1$ we recover what it is known as the classical discrete orthogonal polynomials. The difference equation satisfied by these polynomials,

$$\bar{\sigma}(x)\Delta\nabla p(x) + \bar{\tau}(x)\Delta p(x) = \lambda p(x),$$

[1] Hahn's problem involves two more equivalent properties that we have skipped because we do not consider them in the matrix case.

where $\Delta(p)(x) = p(x+1) - p(x)$, $\nabla(p)(x) = p(x) - p(x-1)$, is a discretization in the lattice $x(s) = s$ of the differential equation satisfied by the classical orthogonal polynomials. This particular problem was considered separately by O. E. Lancaster [83], and P. Lesky [86]. They showed that the only families of classical discrete orthogonal polynomials are the Charlier, Meixner, Krawchuck and Hahn polynomials. These families satisfy a discrete orthogonality property

$$\sum_{k=0}^{\kappa} p_n(x) p_m(x) \rho(x) = h_n \delta_{n,m}, \quad \kappa \in \mathbb{N} \text{ or } \kappa = \infty.$$

As the classical orthogonal polynomials they can be expressed in terms of hypergeometric functions and their orthogonalizing weights satisfy a Pearson equation

$$\Delta\left(\bar{\sigma}(x)\rho(x)\right) = \bar{\tau}(x)\rho(x).$$

By setting $q \in (0,1)$ and $w = 0$ in Hahn's problem we recover one of the first families of q-classical orthogonal polynomials discovered, the big q-Jacobi polynomials. The q-classical[2] orthogonal polynomials are families of orthogonal polynomials that satisfy a second order q-differential equation,

$$\tilde{\sigma}(q^x) D_q D_{q^{-1}} p(q^x) + \tilde{\tau}(q^x) D_q p(q^x) = \lambda p(q^x),$$

where $D_q f(x) = \dfrac{f(qx) - f(x)}{(q-1)x}$ is the q-differential operator.

More characterizing properties of the q-classical orthogonal polynomials were accomplished at the end of the 20th century and the beginning of the 21st, as well as a complete classification of them (see [3] and [90]). In particular the q-classical polynomials were shown to satisfy a Rodrigues equation and their associated weights a Pearson equation. These orthogonal polynomials can be described in terms of basic hypergeometric series

$${}_r\varphi_s\left[\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, x\right] = \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_k}{(b_1, \cdots, b_s; q)_k} \frac{x^k}{(q;q)_k}.$$

According to the representation of the classical and the classical discrete orthogonal polynomials in terms of hypergeometric functions the Askey-scheme provides a hierarchical classification of them, establishing also some limit relations among the different families of orthogonal polynomials (see for instance [4]).

The families of classical orthogonal polynomials can be obtained as limiting cases of certain q-polynomials written in terms of ${}_4\varphi_3$, the Askey Wilson polynomials, what entailed the appearance of the q-Askey-scheme, where polynomials expressed in terms of basic hypergeometric functions are treated and limit relations among them are shown (see [77]).

A different treatment of the classical discrete and q-classical orthogonal polynomials is due to A. F. Nikiforov and V. B. Uvarov and is based on the analysis of the discretization

[2]The term q-classical makes reference to a bigger class of orthogonal polynomials, those that satisfy an operator equation that results form discretizing the second order differential equation in the q-quadratic lattice. However in the present work we accomplished the term q-classical to this subclass, that used to be referred as Hahn class.

of the second order differential equations satisfied by the classical orthogonal polynomials in the uniform lattice $x(s) = s$, and the non-uniform lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$, [93].

The other author whose work is an starting point for this memory is M. G. Krein (1907-1989). His publications from 1949, *The fundamental propositions of the theory of representations of Hermitian operators with deficiency index* (m, m), [82], and *Infinite J-matrices and a matrix-moment problem*, [81], are considered as the beginning of the theory of matrix orthogonal polynomials.

In this theory one deals with the sesquilinear form defined by a matrix of measures W in the space of matrix polynomials:

$$\langle P, Q \rangle_W = \int P dW Q^*. \tag{2}$$

Under mild assuptions on the matrix of measures W, one can always construct sequences of orthogonal matrix polynomials with respect to W. These matrices of measures will be called weight matrices.

Since its birth and until the last decade of the previous century, the works concerning the theory of matrix orthogonal polynomials were somehow sporadic (see for instance [10], [65]). This situation has radically changed in the last two decades, and since the end of the 20th century more systematic works have appeared, bringing a solid and structured knowledge concerning this theory.

Many basic results from the scalar theory of orthogonal polynomials have been extended to the matrix theory. Among them, there have been discovered recurrence formulas, with extension of Favard's theorem, algebraic aspects, properties of the zeros and Gaussian quadrature formulas (see e.g [22], [25], [26], [27], [47], [54], [58], [96]). Also some density problems and problems regarding matrix moments have been studied (e. g. [48], [49], [50], [53], [84], [85]). The study of asymptotic properties for the matrix orthogonality has also been considered (see e.g. [20], [21], [29], [55], [87], [88]). It is worth mentioning that these properties haven't been obtained by using tools derived from Riemmann-Hilbert, but from quadrature formulas. However taking into account the efficiency of such methods for the scalar case it would be convenient to develop the basis of these tools in the matrix setting that allow its use in the matrix case. The construction and study of examples of matrix orthogonal polynomials satisfying second order differential equations (e.g. [9], [12], [13], [14], [15], [31], [30], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [66], [70], [71], [95], [72], [94]), have shown that these families also posses the wealth of structural properties, as their scalar relatives (e.g. [32], [33], [34], [39], [52]).

The theory of matrix orthogonal polynomials is linked to different areas of the mathematics, pointing out the power of this theory. For instance, in [58] the authors established the link between matrix orthogonal polynomials and scalar polynomials satisfying $2N+1$-recurrence relations. Other works, as e.g. [68], show the connection with probability theory, and random matrices [74]. More applications of the theory of matrix orthogonal polynomials can be found in e.g. [96].

For a more complete view of the matrix orthogonal polynomials theory we reefer the readers to [19], [51] and references therein.

The existence of singular matrices as well as the non-commutativity of the matrix product difficult the treatment of these objects and force us to developed new ideas and techniques for its study.

As we have already mentioned, the classical, classical discrete and q-classical polynomials play an important role in the theory of scalar orthogonal polynomials, and one of the characterizing properties that makes them so interesting is that of being eigenfunctions of certain operators. In the theory of matrix orthogonal polynomials, the search of matrix examples that satisfy an analogue property is a complicate issue. Such a complexity can be seen in the fact that although the theory of matrix orthogonal polynomials was born in the fifties it wasn't until the last decade that the first examples of non-trivial matrix orthogonal polynomials being eigenfunctions of matrix operators (in particular second order differential operators) have appeared.

As a first and primary consequence of the effect that non-commutativity has in the treatment of this theory, we observe that when one wants to study matrix orthogonal polynomials which are eigenfunctions of differential, difference or q-difference equations, we need to take care of the order of multiplication of the coefficients and eigenvalues, as it is explained in [28]. In the case of matrix differential operators, in order to avoid examples which can be reduced to scalar ones, one needs to consider differential operators of the form

$$\sum_{k=0}^{M} \frac{d^k}{dx^k} P_n(x) F_k(x) = \Lambda_n P_n(x), \quad F_k(x) \in \mathbb{C}^{N\times N}[x], \ \Lambda_n \in \mathbb{C}^{N\times N},$$

that is, the matrix coefficients are multiplying on the right and the eigenvalues on the left.

In the study of matrix operators with sequences of matrix orthogonal polynomials as eigenfunctions, the concept of symmetry is a very important one. A matrix operator D is said to be symmetric with respect to a weight matrix W if for all matrix polynomials P and Q, $\langle D(P), Q\rangle_W = \langle P, D(Q)\rangle_W$, where $\langle \cdot, \cdot \rangle_W$ is defined by (2).

It happens that symmetric operators taking matrix polynomials into matrix polynomials and such that they do not raise the degree of polynomials always have as eigenfunctions a sequence of orthonormal polynomials (see Lemma 1.3.3). This apply to any kind of matrix operators, whatever it is the nature of them.

Starting with the works of Durán, Grünbaum, Tirao and Pacharoni, [38], [66], [70], [71], in the last decade numerous examples of matrix orthogonal polynomials satisfying second order differential equations have been constructed. One of the starting points for the discovery of those examples is the symmetry of a differential operator with respect to a weight matrix. This symmetry can be reached through a set of commuting and differential equations, under certain boundary conditions, for the coefficients of the matrix differential operator and the weight matrix (see [28], [30] and Section 1.3 of the present thesis).

The solution of these set of matrix differential equations is far away from being direct or simple, however different methods have been accomplished in the last years and have led to different shorts of examples (see e.g. [38], [70]).

The link between representation theory and matrix-valued spherical functions, see [60], gives another way to construct matrix-valued orthogonal polynomials that satisfy second order differential equation. This approach was initiated by Grünbaum, Pacharoni and Tirao [71] using invariant differential operators on SU(3)/U(2). Since then several papers have appeared following this approach (see [67], [78], [94], [95], [101] etc.), showing the powerful connection between matrix orthogonal polynomials and representation theory, as happened in the scalar case (see e.g. [100]).

The families of orthogonal matrix polynomials mentioned in the last paragraphs have allowed to discover many new and interesting phenomena which are absent in the well known scalar theory.

One of the phenomena is the fact that the elements of a family of orthogonal matrix polynomials can be eigenfunctions of several linearly independent second order differential or difference operators (while in the scalar case, the symmetric second order differential operator is unique up to multiplicative and additive constants, see e.g. [14], [30], [35], [72], [94]).

As a consequence of this phenomenon, the algebra of differential and difference operators are receiving some attention ([35], [72]). For the classical and the classical discrete families this algebra is isomorphic to $\mathbb{C}[x]$. In the matrix case the structure of these algebras in much more complicated, and mostly we only have conjectures on them (see [14], [35], [37], [99]).

As the dual situation to that mentioned in the previous paragraph we find that a differential operator D can have different families of orthogonal polynomials as eigenfunction, and can also have different weights with respect to which it is symmetric, bringing into the picture the two sets of weight matrices:

$$\mathfrak{X}(D) = \{W | DP_n^W = \Lambda_n P_n^W\},$$
$$\Upsilon(D) = \{W | D \text{ is symmetric with respect to } W\}.$$

It happens that $\mathfrak{X}(D)$ is a cone and $\Upsilon(D)$ a convex cone. In [36] the authors provided a method to construct new weights in the convex cone of a differential operator.

In the matrix case one misses some of the equivalences among the characterizing properties for the classical orthogonal polynomials. For instance, for a sequence of matrix orthogonal polynomials which are eigenfunctions of a second order differential or difference operator, it is not longer true that the sequence formed by its derivatives is again a sequence of matrix orthogonal polynomials (for more details see [12], [39] and [57], [42], [46]).

Finding structural formulas for the matrix orthogonal polynomials is in general much more complicated in the matrix case that in the scalar one, for obvious reasons. Most of the formulas that one can find in the literature are usually valid for lower dimension cases (e.g. [39], [52]), although there are some examples that have been fully studied for arbitrary size, [34].

As it has already been pointed out, the search for examples of matrix polynomials which are also eigenfunctions of certain matrix operators becomes a rather difficult issue. This difficulty is due to several reasons. The most important of these reasons is the increase of the difficulties in the computations related with the non-commutativity of the matrix product, and the existence of singular matrices. However, having a wide set of examples of matrix orthogonal polynomials has shown to be decisive in the study and discovery of new phenomena happening in the matrix orthogonality, as those that we have mentioned above. This has been the case with the examples of matrix orthogonal polynomials satisfying differential equations. An example of the increasing knowledge about these families of orthogonal polynomials is shown in the last chapter of this memory. There lot of tools developed in the last decades are used to build and study in deep an interesting family of matrix orthogonal polynomials satisfying second order differential equations. Moreover these families are shown to satisfy first order differential equations as well.

In the case of matrix orthogonal polynomials satisfying difference equations, very little was known. Apart from the examples in size 2×2 in [35] (and some others reducible to the scalar case) there were no examples of such matrix orthogonal polynomials. With this thesis this lack of examples starts to be solved. Moreover, we introduce a method to construct examples of matrix orthogonal polynomials satisfying second order difference equations, and by making use of it we give a variety of examples. Having such a method is of the main importance, because we skip the complexity in the computations that made the search of examples so difficult, and now dealing with matrix orthogonal polynomials and matrix difference equations becomes much easier to handle. The method profits of the factorization of a weight matrix and the symmetry equations obtained in [35] for a discrete weight matrix and a difference operator. These symmetry equations are the starting point to develop the method. Also, as in the continuous case, the availability of examples has already been used to explore new features and properties satisfied by this objects (see Section 3, [57], [56]).

For the case of matrix orthogonal polynomials satisfying q-difference equations, even less was known. In this thesis we establish the symmetry equations for the q-difference case, and we adapt the method developed for the difference case to obtain the first non-trivial examples of matrix orthogonal polynomials satisfying second order q-difference operator. That emphasizes the power of the method to build examples for the difference case. With our method we construct an example of matrix orthogonal polynomials satisfying q-difference equations, but this method can easily be used to get a wider class of examples and to explore their properties.

With the content of this thesis we get a more complete view of the theory of matrix orthogonal polynomials, and many questions can now be tackled, such as those concerning limiting relations among matrix orthogonal polynomials satisfying second order difference equations (or q-difference) and matrix orthogonal polynomials satisfying second order differential equations.

The contents of this memory is as follows.

In **Chapter 1** we show some background on the theory of matrix orthogonal polynomials and matrix operators that are needed for the understanding of the foregoing work. We also review some basic concepts and display some known results concerning hypergeometric functions that will be used later.

Chapters 2, 3, 4 and 5 are formed by the original contents of this thesis.

Chapter 2 is devoted to the study of second order difference operators of the form

$$D(P)(x) = P(x-1)F_{-1}(x) + P(x)F_0(x) + P(x+1)F_1(x)$$

with a sequence of matrix orthogonal polynomials $(P_n)_n$ as eigenfunctions, $D(P_n)(x) = \Lambda_n P_n(x)$. As for differential operators, in the discrete case similar considerations about the order of multiplication of the coefficients and eigenvalues also applies, so we will just consider difference operators with coefficients multiplying on the right and eigenvalues multiplying on the left.

The symmetry of a generic difference operator D with respect to a discrete weight matrix W can be achieved through a set of commuting and difference equations (the symmetry equations) and boundary conditions regarding the coefficients of D and W, (see Theorem 1.3.19 or [35]). For a second order difference operator D and a discrete weight matrix W supported on a subset of the natural numbers $\mathfrak{S} \subseteq N$ the symmetry equations are (see Theorem 1.3.7)

$$F_1(x)W(x) = W(x+1)F_{-1}(x+1)^*, \quad x \in \mathfrak{S} \cap (-1+\mathfrak{S}), \tag{3}$$

$$F_0(x)W(x) = W(x)F_0(x)^*, \quad x \in \mathfrak{S}. \tag{4}$$

In Chapter 2 we provide a method to solve these equations. For the benefit of the reader we exhibit here Theorem 2.1.1, that shows the method.

Theorem 0.0.1. *Let κ be either a positive integer or infinite, and F_1 and F_{-1} matrix polynomials. Assume that there exists a scalar function $s(x)$ such that for $x = 1, \cdots, \kappa$, $s(x) \neq 0$ and*

$$F_1(x-1)F_{-1}(x) = |s(x)|^2 I, \quad x \in \{1, \cdots, \kappa\}.$$

Write T for the solution of the first order difference equation

$$T(x-1) = \frac{F_{-1}(x)}{s(x)} T(x), \quad \text{for } x \in \{1, \cdots, \kappa\}, \quad T(0) = I.$$

Then, the weight matrix

$$W = \sum_{x=0}^{\kappa} T(x)T^*(x)\delta_x,$$

satisfies the difference equation (3). Moreover if the matrix $T(x)^{-1}F_0(x)T(x)$ is Hermitian then, the difference equation (4) also holds.

With this method we construct four illustrative examples of matrix difference operators that are symmetric with respect to discrete weight matrices. We display here an example of size $N \times N$ that will be treated with more detail in Chapter 2.

Let a be a positive real number, and W the discrete weight matrix given by

$$W = \sum_{x \in \mathbb{N}} \frac{a^x}{x!} (I+A)^x (I+A^*)^x \delta_x, \tag{5}$$

where A is the $N \times N$ nilpotent matrix given by

$$A = \begin{pmatrix} 0 & v_1 & 0 & \cdots & 0 \\ 0 & 0 & v_2 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & \cdots & 0 & v_{N-1} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

and I the identity matrix. Then, the sequence of monic orthogonal polynomials $(P_n)_n$, with respect to W, satisfies the difference equation

$$aP_n(x+1)(I+A) + P_n(x)\left(-J - (I+A)^{-1}x\right) + P_n(x-1)(I+A)^{-1}x = \Lambda_n P_n(x),$$

where

$$\Lambda_n = \left(a(I+A) - J - n(I+A)^{-1}\right),$$

and J is the $N \times N$ diagonal matrix given by

$$J = \begin{pmatrix} N-1 & 0 & 0 & \cdots & 0 \\ 0 & N-2 & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The content of this chapter has been published in [2].

Chapter 3 is devoted to the study of the convex cone associated to a difference operator D. For a fixed difference operator D of the form (1.33) we define the set of weight matrices

$$\Upsilon(D) = \{W : D \text{ is symmetric with respect to } W\}. \tag{6}$$

One straightforwardly has that if $\Upsilon(D) \neq \emptyset$ then, it is a convex cone: if $W_1, W_2 \in \Upsilon(D)$ and $\gamma, \zeta \geq 0$ (one of them non null), then $\gamma W_1 + \zeta W_2 \in \Upsilon(D)$. The convex cone associated to a second order differential operator was studied in [36].

When $\Upsilon(D) \neq \emptyset$, it contains, at least, a half line: γW, $\gamma > 0$. In the scalar case, the convex cone of positive measures associated to a second order difference operator always reduces to the empty set except for those operators associated to the classical discrete measures in which case the convex cone is the half line defined by the classical measure itself. The situation is again rather different in the matrix orthogonality: in chapter 3 we show examples of second order difference operators D for which $\Upsilon(D)$ is a higher dimensional convex cone.

We provide two methods to find such examples and show a collection of instructive examples. We remark that the convex cones generated by both methods have a completely different structure.

The first method consists in the following. Consider a weight matrix W factorized in the form $W(x) = T(x)T^*(x)$, $x \in \operatorname{Supp}(W)$. We then form a new weight matrix W_S by inserting between the factor T and its adjoint T^* a diagonal matrix of numbers S with positive entries: $W_S(x) = T(x)ST^*(x)$. Although the diagonal matrix S does not depend on x, its effect on the orthogonal polynomials with respect to W_S can be highly non lineal (as we show with an example); of course, this is due to the non commutativity of the matrix product. We then find suitable conditions on a second order difference operator D and on the weight matrix factor T which guarantee the symmetry of D with respect to all the weight matrices W_S.

The second method shows once again the differences with what happens in the scalar orthogonality. We take a weight matrix W having several linearly independent symmetric second order difference operators. We then add to W a Dirac distribution $M(x_0)\delta_{x_0}$, where the real number x_0 and the mass $M(x_0)$ (a Hermitian positive semidefinite matrix) are carefully chosen. We show in Section 3.1 that, for certain numbers x_0, we can produce, under certain mild hypothesis, a positive semidefinite matrix $M(x_0)$ and a second order difference operator D symmetric with respect to W and to any weight matrix of the form $\gamma W + \zeta M(x_0)\delta_{x_0}$, $\gamma > 0, \zeta \geq 0$. We illustrate this method with two examples, one of them in size 2×2 and the other in arbitrary size. The latter can be considered as a matrix relative of the Charlier scalar family. In this case, we add a Dirac delta at $x_0 = 0$. The situation is different to that of the scalar case. Indeed, when we add a mass point at 0 to the classical weight of Charlier the existence of a symmetric finite order difference operator automatically disappears (see [7]).

For the benefit of the reader, we exhibit now a couple of examples. For $a > 0$ and $b \in \mathbb{R}\backslash\{0\}$, we consider the second order difference operator

$$D_1(\cdot) = \mathfrak{s}_1(\cdot)\begin{pmatrix} -a & -ab \\ 0 & -a \end{pmatrix} + \mathfrak{s}_0(\cdot)\begin{pmatrix} x+1 & -bx \\ 0 & x \end{pmatrix} + \mathfrak{s}_{-1}\begin{pmatrix} -x & xb \\ 0 & -x \end{pmatrix}.$$

Using the first method mentioned above, we will show that its convex cone is formed by the weight matrices γW_ξ, $\gamma, \xi > 0$, where

$$W_\xi = \sum_{x \in \mathbb{N}} \frac{a^x}{x!}\begin{pmatrix} \xi + b^2x^2 & bx \\ bx & 1 \end{pmatrix}\delta_x$$

(notice that for $\xi = 1$ we recover the example exhibited above (5) for the case $N = 2$). On the other hand, the convex cone for the second order difference operator D_2 defined by

$$\begin{aligned} D_2(\cdot) =& \mathfrak{s}_{-1}(\cdot)\begin{pmatrix} 0 & \frac{x}{a} - b^2x(x-1) \\ 0 & -bx \end{pmatrix} + \mathfrak{s}_1(\cdot)\begin{pmatrix} b(x+1) & -b^2(x+1)(x+a) \\ 1 & -b(x+a) \end{pmatrix} \\ &+ \mathfrak{s}_0(\cdot)\begin{pmatrix} \frac{1}{ab} - b(x+a) & -(\frac{x}{a}+1) + b^2x(2x+a) \\ -1 & 2bx \end{pmatrix} \end{aligned}$$

will be constructed using the second method. We will show that

$$\Upsilon(D_2) = \{\gamma W + \xi\begin{pmatrix} ab & 1 \\ 1 & 1/(ab) \end{pmatrix}\delta_0, \quad \gamma > 0, \xi \geq 0\},$$

where

$$W = \sum_{k=0}^{\infty} \frac{a^x}{x!}\begin{pmatrix} 1 + b^2x^2 & bx \\ bx & 1 \end{pmatrix}.$$

Notice that the weight matrix W belongs to both convex cone (that is, both operators D_1 and D_2 are symmetric with respect to W).

The content of this chapter has been published in [56].

In **Chapter 4** we study matrix orthogonal polynomials that are eigenfunctions of a matrix q-difference operator of the form

$$D(P)(x) = \sum_{k=r}^{-r} P(q^k x)F_k(x), \quad F_k(x) \in \mathbb{C}^{N \times N}[x^{-1}].$$

We also remark here that the considerations on the order of multiplication of the coefficients and eigenvalues also applies in the q-matrix case.

To find examples of matrix orthogonal polynomials satisfying q-difference equations we exploit Lemma (1.3.3), so in fact we find examples of q-difference operators D that are symmetric with respect to a weight matrix W.

We establish a set of commuting and q-difference equations that, together with certain boundary conditions, assure the symmetry of a q-difference operator D of the form (1.34) with respect to a weight matrix W of the form (1.18) (see Theorem 1.3.21). For the particular case of a second order q-difference operator

$$D(P)(q^x) = P(q^{x+1})F_1(q^x) + P(q^x)F_0(q^x) + P(q^{x-1})F_{-1}(q^x), \quad F_k(x) \in \mathbb{C}^{N\times N}[x^{-1}],$$

the symmetry equations are given by

$$\begin{aligned} F_0(q^x)W(q^x) &= W(q^x)F_0(q^x)^*, \quad x \in \mathbb{N}, \\ F_1(q^{x-1})W(q^{x-1}) &= qW(q^x)F_{-1}(q^x)^*, \quad x \in \mathbb{N}\backslash\{0\}, \end{aligned}$$

and the boundary condition

$$\begin{aligned} W(1)F_{-1}(1)^* &= 0. \\ q^{2x}F_1(q^x)W(q^x) &\to 0, \qquad \text{as } x \to \infty, \\ q^x\left(F_1(q^x)W(q^x) - W(q^x)F_1(q^x)^*\right) &\to 0, \qquad \text{as } x \to \infty. \end{aligned}$$

We provide a method to construct second order q-difference operators and q-weights that satisfy the symmetry conditions. This method is based on that developed for the discrete case and profits of the decomposition of the q-weight matrix $W(q^x) = T(q^x)T(q^x)^*$.

By making use of this method we construct an example of arbitrary size that constitute, as long as we know, the first non trivial example of matrix q-polynomials satisfying q-difference equations.

As an extension of the matrix hypergeometric functions introduced by Tirao [98] we define a matrix basic hypergeometric function. Such a matrix function can be used to describe the rows of the matrix orthogonal polynomials which are eigenfunctions of a particular second order q-difference operator, whose coefficients are related with the parameters of the matrix basic hypergeometric function.

In Chapter 4 we also carry an extensive study of the particular case of size $N = 2$, exhibited below.

Let a and b be two real numbers such that $0 < a < q^{-1}$ and $b < q^{-1}$. Consider the matrices M, L and U given by

$$U = \begin{pmatrix} 0 & v(1-q)(q^{-1}-a) \\ 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & v \\ 0 & 0 \end{pmatrix}, \qquad L = e^{\log(q)M} = \begin{pmatrix} q & -v(1-q) \\ 0 & 1 \end{pmatrix}.$$

The monic orthogonal polynomials $(P_n)_n$ with respect to

$$W(q^x) = a^x \frac{(bq;q)_x}{(q;q)_x} L^x (L^*)^x,$$

satisfy then the matrix q-difference equation

$$P_n(q^{x-1})(q^{-x}-1)L^{-1} + P_n(q^x)\left(U - q^{-x}(L^{-1}+aL)\right) + P_n(q^{x+1})(aq^{-x}-abq)L = \Lambda_n P_n(q^x),$$

where

$$\Lambda_n = \begin{pmatrix} -q^{-n-1} - abq^{n+2} & v(1-q)(abq^{n+1} - q^{1-n} + q^{-1} - a) \\ 0 & -q^{-n} - abq^{n+1} \end{pmatrix}.$$

By profiting the factorization of the weight matrix, we provide formulas for the matrix polynomials P_n in terms of little q-Jacobi polynomials. We also make use of the matrix basic hypergeometric function introduced in Section 4.2 to give an alternative expression of the polynomials, and by making use of this expression we find a three term recurrence relation for the 2×2 polynomials. Also a Rodrigues formula is found.

The content of this chapter can be found in [1].

Chapter 5 is related to the already mentioned connection between representation theory and matrix orthogonal polynomials is showing to be a very fruitful one. In [78], and [79] the authors studied the full spherical functions associated to the particular case $(SU(2) \times SU(2), SU(2))$ and related them with a family matrix orthogonal polynomials. By profiting of the machinery of the representation theory the authors of [78] and [79] showed that these matrix orthogonal polynomials satisfy a second order differential equation, and also a first order differential equation.

In Chapter 5 we introduce a family of weight matrices labeled by a positive number $\left(W^{(\nu)}\right)_{\nu>0}$, and their sequences of orthogonal polynomials $(P_n^{(\nu)})_n$. For the particular case of $\nu = 1$ we recover the example in [78].

We define the weight matrices via its LDU-decomposition and we prove another equivalent and useful expression for the weights in terms of Gegenbauer polynomials.

These weight matrices can be reduced to lower size, explicitly

$$W^{(\nu)}(x) = Y_{\frac{N}{2}}^t \begin{pmatrix} W_+^{(\nu)}(x) & 0 \\ 0 & W_-^{(\nu)}(x) \end{pmatrix} Y_{\frac{N}{2}},$$

where $Y_{\frac{N}{2}}$ are the matrices given by

$$Y_{p+\frac{1}{2}} = \frac{1}{2}\sqrt{2}\begin{pmatrix} I_{p+1} & \mathfrak{J}_{p+1} \\ -\mathfrak{J}_{p+1} & I_{p+1} \end{pmatrix}, \quad Y_p = \frac{1}{2}\sqrt{2}\begin{pmatrix} I_p & 0 & \mathfrak{J}_p \\ 0 & \sqrt{2} & 0 \\ -\mathfrak{J}_p & 0 & I_p \end{pmatrix}, \quad p \in \mathbb{N},$$

where I_p denotes the identity matrix of size p and

$$\mathfrak{J}_p = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ & & \iddots & & \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0. \end{pmatrix}$$

It is also worth to notice that $W_+^{(\nu)}(x) \in \mathbb{C}^{r_1 \times r_1}$ and $W_-^{(\nu)}(x) \in \mathbb{C}^{r_2 \times r_2}$, where $r_1 = \lceil \frac{N}{2} \rceil$, $r_2 = \lfloor \frac{N}{2} \rfloor$, and $\lceil\ \rceil$ and $\lfloor\ \rfloor$ are the ceiling and floor functions,

$$\lceil x \rceil = \min\{m \in \mathbb{Z} \,|\, m \geq x\}, \quad \lfloor x \rfloor \max\{m \in \mathbb{Z} \,|\, m \leq x\}.$$

Moreover they admit no more reduction.

For each $\nu > 0$ we construct a pair of differential operators, one of order two $D^{(\nu)}$ and the other of order one $E^{(\nu)}$, that satisfy the symmetry equations and boundary conditions for $W^{(\nu)}$. The operator $D^{(\nu)}$ can also be reduced to lower size via the composition with the matrices Y_ℓ, but the operator $E^{(\nu)}$ does not reduce to lower size.

These operators can be combined to obtain a third differential operator,

$$D_{\Phi,\Psi}^{(\nu)} = \frac{d^2}{dx^2}\Phi^{(\nu)} + \frac{d}{dx}\Psi^{(\nu)}.$$

The existence of such operator implies the Hermiticity of $\Phi(x)^{(\nu)}W^{(\nu)}(x)$ and that the weight matrices $W^{(\nu)}$ satisfy a Pearson-type equation as in [12],

$$\left(\Phi(x)^{(\nu)}W^{(\nu)}(x)\right)' = \Psi^{(\nu)}(x)W^{(\nu)}(x).$$

By exploiting the expression of the weights in terms of scalar Gegenbauer polynomials, we can prove that in fact $W^{(\nu+1)}(x) = \Phi(x)^{\nu}W^{\nu}(x)$, connecting the orthogonal polynomials $(P_n^{(\nu)})_n$ with $(P_n^{(\nu+1)})_n$. Moreover we can prove that the basic differential operator $\frac{d}{dx}$ is a forward shift operator, i.e. if $(P_n^{(\nu)})_n$ is the sequence of monic orthogonal polynomials with respect to $W^{(\nu)}$,

$$\frac{d}{dx}\left(P_n^{(\nu)}\right) = nP_{n-1}^{(\nu+1)}.$$

And by carrying an integration by parts on $\langle \frac{d}{dx}P, Q\rangle^{(\nu)}$ we can also give an operator,

$$\left(T^{(\nu)}Q\right)(x) = \frac{dQ}{dx}(x)\left(\Phi^{(\nu)}(x)\right)^* + Q(x)\left(\Psi^{(\nu)}(x)\right)^*,$$

that satisfies $T^{(\nu)}P_{n-1}^{(\nu+1)} = K_n^{(\nu)}P_n^{(\nu)}$, for certain matrices $K_n^{(\nu)}$ that can be determined from $\Phi^{(\nu)}(x)$ and $\Psi^{(\nu)}(x)$, that is $T^{(\nu)}$ is a backward shift operator.

By combining the forward shift and backward shift operators we can get a compact Rodrigues formula for the sequences of orthogonal polynomials $(P_n^{(\nu)})_n$,

$$P_n^{(\nu)}(x) = G_n^{(\nu)}\frac{d^n}{dx^n}\left(W^{(\nu+n)}(x)\right)W^{(\nu)}(x)^{-1},$$

where $G_n^{(\nu)}$ are diagonal matrices.

We also provide expressions of the matrix orthogonal polynomials in terms of matrix-valued hypergeometric functions as in [98]. From this expression we can calculate the three term recurrence relation satisfied by the polynomials. From the LDU decomposition of the weight matrix (5.5), and by performing an appropriate change of variables, we find the expression of the matrix orthogonal polynomials $(P_n^{(\nu)})_n$ in terms of Gegnbauer and Racah polynomials.

We end this introduction by displaying an explicit example of size $N = 3$ of the weight matrices $W^{(\nu)}$ and the differential operators $D^{(\nu)}$ and $E^{(\nu)}$. For $\nu > 0$ we define the weight matrix $W^{(\nu)}$ in $(-1, 1)$ by

$$W^{(\nu)}(x) = (1-x^2)^{(\nu-1/2)}(2+\nu)\begin{pmatrix} 1 & x & \frac{2x^2(\nu+1)-1}{2\nu+1} \\ x & \frac{x^2\nu+\nu+1}{2\nu+1} & x \\ \frac{2x^2(\nu+1)-1}{2\nu+1} & x & 1 \end{pmatrix}.$$

The weight matrix $W^{(\nu)}$ can be decomposed in a two block matrix:

$$W^{(\nu)}(x) = (1-x^2)^{(\nu-1/2)}(2+\nu)Y_{\frac{3}{2}}\begin{pmatrix} \frac{x^2(\nu+1)+\nu}{2\nu+1} & \sqrt{2}x & 0 \\ \sqrt{2}x & \frac{\nu x^2+\nu+1}{2\nu+1} & 0 \\ 0 & 0 & -\frac{2(x-1)(x+1)(1+\nu)}{2\nu+1} \end{pmatrix}Y_{\frac{3}{2}}^{+},$$

where

$$Y_{\frac{3}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

The sequence of monic orthogonal polynomials $(P_n^{(\nu)})_n$ with respect to $W^{(\nu)}$ satisfy the following differential equations:

(7)

$$(1-x^2)\left(P_n^{(\nu)}(x)\right)'' + (2\nu+3)\left(P_n^{(\nu)}(x)\right)'\begin{pmatrix} -x(2\nu+3) & 2 & 0 \\ 1 & -x(2\nu+3) & 0 \\ 0 & 2 & -x(2\nu+3) \end{pmatrix}$$

(8)

$$+ P_n^{(\nu)}(x)\begin{pmatrix} -(v-1)(3+\nu) & 0 & 0 \\ 0 & 1-(\nu-1)(3+\nu) & 0 \\ 0 & 0 & -(\nu-1)(3+\nu) \end{pmatrix} = \Lambda_n(D^{(\nu)})P_n^{(\nu)}(x),$$

(9)

$$\left(P_n^{(\nu)}(x)\right)'\begin{pmatrix} -x & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 0 & -1 & x \end{pmatrix} + P_n^{(\nu)}(x)\begin{pmatrix} -3-\nu & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \nu-1 \end{pmatrix} = \Lambda_n(E^{(\nu)})P_n^{(\nu)}(x)$$

where

$$\Lambda_n(D^{(\nu)}) = \begin{pmatrix} (2\nu+3)-1-(\nu-1)(3+\nu) & 0 & 0 \\ 0 & (2\nu+3)-(\nu-1)(3+\nu) & 0 \\ 0 & 0 & (2\nu+3)-1-(\nu-1)(3+\nu) \end{pmatrix},$$

$$\Lambda_n(E^{(\nu)}) = \begin{pmatrix} -4-\nu & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \nu \end{pmatrix}.$$

Moreover the sequences of monic orthogonal polynomials $(P_n^{(\nu)})$ satisfy

$$\frac{d^k}{dx^k}P_n^{(\nu)}(x) = (n-k+1)_k P_{n-k}^{(\nu+k)}(x), \quad \text{for all } k \le n.$$

The content of this chapter can be found in [80].

Chapter 1

Preliminaries

In this chapter we introduce some results concerning the theory of matrix orthogonal polynomials that will be used in the forward chapters. We also provide some useful formulas and establish some notation that will be used in the rest of the thesis.

1.1 Hypergeometric and basic hypergeometric function.

Far away from being a deep treatment in hypergeometric and basic hypergeometric functions, we establish here some notation and define some important functions that will be used in this thesis. We also list a few interesting and useful summation formulas that will be needed in the forthcoming chapters. For a deeper understanding and study we refer the reader to [6], [63], [76], [77], among others.

1.1.1 Hypergeometric functions

For $a \in \mathbb{C}$, $k \in \mathbb{N}$ we set $(a)_k$ for the ascending Pochhammer symbol, that is

$$(a)_k = a(a+1)(a+2),\cdots,(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

We can extend this function for negative integers by the convention

$$(a)_{-k} = \begin{cases} 0 & \text{if } k \geq a+1 \\ \dfrac{\Gamma(a+1)}{\Gamma(a-k+1)} & \text{if } k < a+1 \end{cases}$$

For $a, b, c \in \mathbb{C}$ the Gauss hypergeometric function is defined by

$$1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!},$$

whenever it makes sense. Notice that if $c \in \mathbb{Z}$ with $c < 0$ then, this function is not defined unless $a \in \mathbb{Z}$ with $a < 0$ and $c \leq a$ (or the same but with b). The Gauss hypergeometric function converges for $|x| < 1$ and for $x = 1$ when $\Re(c-a-b+1) > 0$. It satisfies the second order differential equation

$$x(1-x)\frac{d^2}{dx^2}(f(x)) + (c-(a+b+1)x)\frac{d}{dx}(f(x)) - abf(x) = 0. \tag{1.1}$$

We will denote the Gauss hypergeometric function by

$$ {}_2F_1(a,b;c|x) = {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix} \Big| x\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}. $$

If $n \in \mathbb{N}$ then, ${}_2F_1(\begin{smallmatrix} -n,b \\ c \end{smallmatrix}|x)$ is a polynomials in x of degree n that satisfies the second order differential equation (1.1).

For $a_1, a_2, \cdots, a_r \in \mathbb{C}$ and $b_1, b_2, \cdots, b_p \in \mathbb{C}$ the generalized hypergeometric function is given by

$$ {}_rF_p\left(\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_p \end{matrix} \Big| x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_r)_k}{(b_1)_k(b_2)_k\cdots(b_p)_k}\frac{x^k}{k!}. \tag{1.2} $$

We now define *balanced*, *well-poissed* and *very-well-poised* series.

Definition 1.1.1. *A hypergeometric series* ${}_{p+1}F_p(\begin{smallmatrix} a_1,a_2,\cdots,a_{p+1} \\ b_1,b_2,\cdots,b_p \end{smallmatrix}|x)$ *is called* balanced *(or* Saalschützian*) if it satisfies*

$$ 1 + \sum_{k=0}^{p+1} a_k = \sum_{k=0}^{p} b_k. $$

We say that ${}_{p+1}F_p$ *is* well-poised *if it satisfies*

$$ 1 + a_1 = b_1 + a_2 = b_2 + a_3 = \cdots = b_p + a_{p+1}. $$

And we say that ${}_{p+1}F_p$ *is a very-well-poised series if it is well-poised and* $a_2 = \frac{a_1}{2} + 1$.

Summation formulas

We list here some summation formulas for hypergeometric series. They can be found in e.g. [5], [6].

- *Gauss summation formula.* For $\Re(c-a-b) > 0$

$$ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix} \Big| x\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.3} $$

- *Kummer's summation formula*

$$ {}_2F_1\left(\begin{matrix} a,b \\ 1+a-b \end{matrix} \Big| -1\right) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}. \tag{1.4} $$

- *Pfaff-Saalschütz formula.* For a ${}_3F_2$ balanced series we have

$$ {}_3F_2\left(\begin{matrix} -n,a,b \\ c,a+b-n-c+1 \end{matrix} \Big| 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}. \tag{1.5} $$

- *Whipple's transformations.* For a ${}_4F_3$ balanced series we have

$$ {}_4F_3\left(\begin{matrix} -n,a,b,c \\ d,e,f \end{matrix} \Big| 1\right) = \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n}{}_4F_3\left(\begin{matrix} -n,a,d-b,d-c \\ d,a+1-n-e,a+1-n-f \end{matrix} \Big| 1\right). \tag{1.6} $$

- *Dougall summation formula.*

$$ {}_5F_4\left(\begin{matrix} a,a/2+1,c,d,-m \\ a/2,a-c+1,a-d+1,a+m+1 \end{matrix} \Big| 1\right) = \frac{(a+1)_m(a-c-d+1)_m}{(a-c+1)_m(a-d+1)_m}. \tag{1.7} $$

1.1.2 Basic hypergeometric functions.

In forthcoming chapters and sections we will occasionally work with q-orthogonal polynomials, for what we need to introduce some notation and definitions.

For the rest of the thesis q stands for a real number with $0 < q < 1$.

For $m \in \mathbb{N}$ the q-shifted factorial, (or simply q-factorial or q-Pochhammer symbol), is defined by

$$(\alpha;q)_m = (1-\alpha)(1-q\alpha)(1-q^2\alpha)\cdots(1-q^{m-1}\alpha), \quad (\alpha;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \tag{1.8}$$

and the multiple q-shifted factorials

$$(\alpha_1,\alpha_2,\cdots,\alpha_p;q)_m = \prod_{k=1}^{p}(\alpha_k;q)_m.$$

We can extend the definition of q-shifted factorial for $m \in \mathbb{Z}$ by using the formula

$$(\alpha q;q)_m = \frac{(\alpha;q)_\infty}{(\alpha q^m;q)_\infty}, \tag{1.9}$$

valid for a negative integer m provided $\alpha q^m \neq q^{-n}$ for any $n \in \mathbb{N}$.

The q-binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

We define the *basic hypergeometric serie* of parameters $\alpha_1,\alpha_2,\cdots,\alpha_p \in \mathbb{C}$ and $\beta_1,\beta_2,\cdots,\beta_r \in \mathbb{C}$ by

$${}_p\phi_r\left[\begin{matrix}\alpha_1,\alpha_2,\cdots,\alpha_p \\ \beta_1,\beta_2,\cdots,\beta_q\end{matrix};q,x\right] = \sum_{k=0}^{\infty}\frac{(\alpha_1,\alpha_2,\cdots,\alpha_p;q)_k}{(\beta_1,\beta_2,\cdots,\beta_r;q)_k}\frac{x^k}{(q;q)_k},$$

see [63].

If for some $k = 1,\cdots,r$, $\beta_k = q^{-n}$ for some $n \in \mathbb{N}$, then the function ${}_p\varphi_r$ is not defined unless $\alpha_j = q^{-m}$ with $m < n$ for some $j = 1,\cdots,p$. Notice that if $\alpha_j = q^{-m}$ for some $j = 1,\cdots,r$ then,

$${}_p\phi_r\left[\begin{matrix}\alpha_1,\alpha_2,\cdots,\alpha_p \\ \beta_1,\beta_2,\cdots,\beta_q\end{matrix};q,x\right]$$

is a polynomial in x of degree at most m.

There is a q-analogue of the gamma function. *The q-gamma function* is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(1-q)^{x-1}(q^x;q)_\infty}.$$

It satisfies the functional equation

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x).$$

There is also a notion of q-derivation and q-integration. We define the q-derivative of a function with derivative at $x = 0$ by

$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x} & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0. \end{cases} \tag{1.10}$$

The q-integral or Jackson integral is the inverse of the q-derivative,

$$\int_0^a f(x) d_q x = \sum_{k=0}^{\infty} (aq^k - aq^{k+1}) f(aq^k), \quad \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{1.11}$$

1.2 Matrix orthogonal polynomials

Let us denote by $\mathbb{C}^{N_1 \times N_2}$ the vector space of matrices of size $N_1 \times N_2$ whose entries are complex numbers, and write $\mathbb{C}^N$ for the vector space formed by the complex vectors of size N, and we will specify whenever it is needed whether we are working with row or column vectors.

Definition 1.2.1. *A matrix polynomial $P(x)$ is a polynomial whose coefficients are matrices, all of them of the same size. We denote by $\mathbb{C}^{N_1 \times N_2}[x]$ the vector space of all matrix polynomials of size $N_1 \times N_2$,*

$$\mathbb{C}^{N_1 \times N_2}[x] = \left\{ \sum_{k=0}^{m} M_k x^k \,\middle|\, M_k \in \mathbb{C}^{N_1 \times N_2},\, m \in \mathbb{N} \right\}.$$

For $k \in \mathbb{N}$ we denote by $\mathbb{C}_k^{N_1 \times N_2}[x] \subset \mathbb{C}^{N_1 \times N_2}[x]$ the set of all matrix polynomials of size $N_1 \times N_2$ with degree at most k.

In this work we focus on the particular case of square matrix polynomials, $\mathbb{C}^{N \times N}[x]$ and we will restrict the variable x to the real line. There are also very interesting works on the theory of matrix orthogonal polynomials on the complex plane, in particular on the unit circle (see e.g. [23], [64], [89]), but it is out of the scope of this thesis.

The identity matrix in $\mathbb{C}^{N \times N}$ is represented by I. The symbol $\mathcal{E}_{i,j}$ stands for the $N \times N$ matrix with entry (i,j) equal to 1 and 0 otherwise. It is not difficult to see that these matrices satisfy the following property

$$\mathcal{E}_{i,j} \mathcal{E}_{k,l} = \mathcal{E}_{i,l} \delta_{j,k}. \tag{1.12}$$

With this notation a matrix $M \in \mathbb{C}^{N \times N}$ can be written as $M = \sum_{i,j=1}^{N} m_{ij} \mathcal{E}_{ij}$, where $m_{ij} \in \mathbb{C}$.

For a matrix $M \in \mathbb{C}^{N \times N}$, we denote by M^* its Hermitian conjugate, that is if $M = (m_{i,j})_{i,j=1}^N$ then, $M^* = (\overline{m}_{j,i})_{i,j=1}^N$. The spectrum of a matrix M is denoted by $\sigma(M)$,

$$\sigma(M) = \{\lambda_i \in \mathbb{C} | \det(M - \lambda_i I) = 0\},$$

and we will denote by $\rho(M)$ its spectral radius, that is

$$\rho(M) = \max_{\lambda \in \sigma}(\lambda). \tag{1.13}$$

The orthogonality will be defined with respect to a *weight matrix*.

Definition 1.2.2. *An Hermitian square matrix of Borel measures W, all of them supported in the Borel set $\mathfrak{S} \subseteq \mathbb{R}$, is said to be a weight matrix if it satisfies*

1. *For any Borel set $B \subseteq \mathfrak{S}$, $W(B)$ is a positive semi-definite matrix.*
2. *W has finite moments of every order, i.e., for all $n \in \mathbb{N}$, $\mu_n = \int_{\mathfrak{S}} x^n dW(x) \in \mathbb{C}^{N\times N}$, where the integral is taken entrywise.*
3. *If $P(x)$ is a matrix polynomial whose leading coefficient is non singular, $\int_{\mathfrak{S}} P(x)dW(x)P(x)^*$ is also non singular*

We associate to a weight matrix W the Hermitian sesquilinear form

$$\langle\cdot,\cdot\rangle_W : \mathbb{C}^{N\times N}[x] \to \mathbb{C}^{N\times N}$$

defined by

$$\langle P, Q\rangle_W = \int_{\mathfrak{S}} P(x)dW(x)Q(x)^*, \quad P, Q \in \mathbb{C}^{N\times N}[x]. \tag{1.14}$$

This form satisfies the following properties, for all $A_1, A_2 \in \mathbb{C}^{N\times N}$ and $P, Q, R \in \mathbb{C}^{N\times N}[x]$

- $\langle A_1 P + A_2 R, Q\rangle_W = A_1\langle P, Q\rangle_W + A_2\langle R, Q\rangle_W$
- $\langle P, Q\rangle_W = (\langle Q, P\rangle_W)^*$.
- $\langle P, P\rangle_W \geq 0$, and $\langle P, P\rangle_W = 0$ if and only if $P = 0$,

where $A \geq 0$ for a matrix $A \in \mathbb{C}^{N\times N}$ means that it is semi-definite positive. Abusing of terminology we will say that the Hermitian sesquilinear form $\langle\cdot,\cdot\rangle_W$ is the inner product associated to W.

Notice that, since the matrix product is non commutative, there is an implicit choice in the inner product we have just presented. One could also consider the following inner product

$$\langle P, Q\rangle_W^R = \int_{\mathfrak{S}} Q(x)^* dW(x)P(x). \tag{1.15}$$

Both inner products are related by the formula

$$\langle P, Q\rangle_W^R = (\langle P^*, Q^*\rangle_W)^*,$$

so the theories that can be developed by choosing one or another inner product are completely analogous.

In this thesis we deal with three particular types of weights: weights with a differentiable density with respect to the Lebesgue measure,

$$dW(x) = W(x)dx, \quad \langle P, Q\rangle_W = \int_{\mathfrak{S}} P(x)W(x)Q(x)^* dx, \quad W \in \mathcal{C}^\infty(\mathfrak{S}), \tag{1.16}$$

discrete weight matrices

$$W = \sum_{x=0}^{\kappa} W(x)\delta_x, \quad \langle P, Q\rangle_W = \sum_{x=0}^{\kappa} P(x)W(x)Q(x)^*, \tag{1.17}$$

where $\kappa \in \mathbb{N}$, or, $\kappa = \infty$, and q-weight matrices

$$W = \sum_{x=0}^{\infty} q^x W(q^x)\delta_{q^x}, \quad \langle P, Q\rangle_W = \sum_{x=0}^{\infty} q^x P(q^x)W(q^x)Q(q^x)^*, \quad 0 < q < 1. \tag{1.18}$$

Notice that in the previous expressions we have used W to refer both the weight matrix and the function that defines it. However the context will make clear which one we are dealing with.

From now on whenever there is no place for confusion we leave out the W in the inner product and we will simply write $\langle P, Q\rangle$ to denote the inner product of P and Q (1.14).

We can now introduce the concept of *matrix orthogonal polynomials.*

Definition 1.2.3. *A sequence of matrix polynomials* $(P_n)_n \subset \mathbb{C}^{N\times N}[x]$ *is said to be a sequence of matrix orthogonal polynomials with respect to* W *if it satisfies*

1. *For all* n, $P_n(x)$ *is of degree* n *with non-singular leading coefficient,* $\det(P_n^n) \neq 0$.
2. *For all* $n, m \in \mathbb{N}$, $\langle P_n, P_m\rangle = \Gamma_n \delta_{n,m}$, *where* $\delta_{n,m}$ *is the Kronecker delta and* $\Gamma_n \in \mathbb{C}^{N\times N}$ *is an Hermitian matrix.*

For any weight matrix W one can always construct a family of orthogonal polynomials with respect to it (see e.g. [19], [47] [92]). Such a sequence is unique up to multiplication on the left by a sequence of non singular matrices. To see different computational methods to construct such a sequence we refer the reader to [92].

If $\langle P_n, P_n\rangle = I$ for all n we say that $(P_n)_n$ is a sequence of *orthonormal* polynomials with respect to W, and we say that it is a sequence of *monic* orthogonal polynomials if $P_n^n = I$, for all $n \geq 0$, where P_n^n is the leading coefficient of P_n.

Given a sequence of polynomials $(P_n)_n$ orthogonal with respect to a weight matrix W, we can construct a sequence of orthonormal polynomials with respect to W. If $\Gamma_n = \langle P_n, P_n\rangle$ then, because of the properties of a weight matrix, Γ_n is a non-singular Hermitian matrix and it can be factorized as $\Gamma_n = G_n G_n^*$, where G_n is a non-singular matrix. The sequence $(G_n^{-1}P_n)_n$ is a sequence of orthonormal polynomials:

$$\langle G_n^{-1}P_n, G_m^{-1}P_m\rangle = G_n^{-1}\Gamma_n (G_n^{-1})^*\delta_{mn} = G_n^{-1}G_nG_n^*(G_n^{-1})^*\delta_{mn} = I\delta_{mn}.$$

Notice that if (P_n) is a sequence of orthonormal polynomials then, for any sequence of unitary matrices $(U_n)_n$ the sequence $(U_nP_n)_n$ is also a sequence of orthonormal polynomials with respect to W.

As in the scalar case the orthonormality of a sequence of polynomials is equivalent to a *three term recurrence relation.*

Proposition 1.2.4. *[26, 47] Let* $(P_n)_n$ *be a sequence of matrix polynomials with initial conditions* $P_{-1} = 0$ *and* P_0 *a non singular matrix, such that it satisfies the recurrence relation*

$$xP_n(x) = A_{n+1}P_{n+1}(x) + B_nP_n(x) + A_n^*P_{n-1}(x), \quad n = 0, 1, 2, \cdots \tag{1.19}$$

where $A_n \in \mathbb{C}^{N\times N}$, $n \in \mathbb{N}$ *are non singular matrices and* $B_n \in \mathbb{C}^{N\times N}$, $n \in \mathbb{N}$ *are Hermitian matrices. Then, there exists a weight matrix* W *such that* $(P_n)_n$ *is the sequence of orthonormal polynomials with respect to it.*

Conversely, for any sequence of orthonormal polynomials with respect to a weight matrix, $(P_n)_n$, there exist a sequence of Hermitian matrices, $B_n \subset \mathbb{C}^{N\times N}$, $n \in \mathbb{N}$, and a sequence of non singular matrices, $A_n \in \mathbb{C}^{N\times N}$, $n \in \mathbb{N}$, such that (1.19) holds.

It is clear that if $(P_n)_n$ is a sequence of orthonormal polynomials with respect to W then, any other sequence of orthogonal polynomials with respect to W, $(\tilde{P}_n)_n = (G_n P_n)_n$ satisfies a three term recurrence relation of the form

$$x\tilde{P}_n = \tilde{A}_n \tilde{P}_{n+1} + \tilde{B}_n \tilde{P}_n + \tilde{C}_n \tilde{P}_{n-1}, \tag{1.20}$$

where $\tilde{A}_n = G_n A_{n+1}(G_{n+1})^{-1}$, $\tilde{B}_n = G_n B_n (G_n)^{-1}$ and $\tilde{C}_n = G_n A_n^*(G_{n-1})^{-1}$.

A recurrence relation of the form (1.20) can be seen as a matrix second order difference operators acting on matrix functions of two variables. That is, if we set $P(n,x) = \tilde{P}_n(x)$, $A(n) = \tilde{A}_n$, $B(n) = \tilde{B}_n$ and $C(n) = \tilde{C}_n$ then, (1.20) can be written as $\Theta(P(n,x)) = xP(n,x)$ where

$$\Theta(P(n,x)) = A(n)P(n+1,x) + B(n)P(n,x) + C(n)P(n-1,x). \tag{1.21}$$

By writing

$$\mathcal{P}(x) = \begin{pmatrix} P(0,x) \\ P(1,x) \\ P(2,x) \\ \vdots \end{pmatrix}$$

the operator Θ can be seen as a semi-infinite block Jacobi matrix,

$$\Theta = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

satisfying

$$\Theta\mathcal{P}(x) = x\mathcal{P}(x).$$

When working with matrix polynomials a very important concept is that of *similar weights.*

Definition 1.2.5. *Two weight matrices, W_1 and W_2 are* similar *if there exists a non-singular matrix $T \in \mathbb{C}^{N\times N}$ independent of x such that $W_1(x) = TW_2(x)T^*$.*

If $(P_n^1)_n$ is a sequence of matrix orthogonal polynomials with respect to W_1 then, $(P_n^2)_n = (P_n^1 T)_n$ is a sequence of matrix orthogonal polynomials with respect to W_2.

We say that a weight matrix W reduces to lower sizes *if it is similar to a block weight matrix*

$$W(x) = T\begin{pmatrix} Z_1(x) & 0 \\ 0 & Z_2(x) \end{pmatrix} T^*.$$

As a particular case we say that a weight matrix W *reduces to scalar weights* if it is similar to a diagonal weight matrix

$$W(x) = T\begin{pmatrix} w_1(x) & 0 & \cdots & 0 \\ 0 & w_2(x) & \cdots & \\ & & \ddots & 0 \\ 0 & 0 & \cdots & w_N(x) \end{pmatrix} T^*.$$

Notice that if a weight matrix W reduces to scalar weights, and $(P_n)_n$ is a sequence of orthogonal polynomials with respect to W, then the following formula holds, up to multiplication by a sequence of non-singular matrices,

$$P_n(x)T = \begin{pmatrix} p_n^1(x) & & & \\ & p_n^2(x) & & \\ & & \ddots & \\ & & & p_n^N(x) \end{pmatrix}, \quad n \geq 0,$$

where $(p_n^i)_n$ is a sequence of orthogonal polynomials with respect to the measure w_i. Therefore the study of weight matrices that reduce to scalar weights belong to the study of scalar orthogonality more that to the matrix one.

In general it is not an easy task to see whether a weight matrix reduces to scalar weights or not, however in the case when a weight matrix W is of the form (1.16), (1.17) or (1.18), as a consequence of [75, Theorem 4.1.6] (see also [38]) the following proposition gives a very useful criterion to check if we are dealing with weight matrices that reduce to scalar weights or not.

Proposition 1.2.6. *Assume that there exists $a \in \mathfrak{S}$ such that $W(a) = I$, then W reduces to scalar weights if and only if $W(s)W(t) = W(t)W(s)$ for all $s, t \in \mathfrak{S}$.*

We will use that an analytic function f at x_0, with convergent power series given by

$$f(x) = \sum_{i=0}^{\infty} a_i (x - x_0)^i, \quad |x - x_0| < \epsilon,$$

defines the following function over the matrices M with $\rho(M - x_0 I) < \epsilon$ ($\rho(M)$ stands for the spectral radius of a matrix M, 1.13):

$$f(M) = \sum_{i=0}^{\infty} a_i (M - x_0 I)^i.$$

For any two matrices X and Y, we use the standard notation

$$\operatorname{ad}_X^0 Y = Y, \quad \operatorname{ad}_X^1 Y = [X, Y] = XY - YX, \quad \operatorname{ad}_X^{n+1} Y = [X, \operatorname{ad}_X^n Y]. \tag{1.22}$$

For any matrix $G \in \mathbb{C}^{N\times N}$, we denote by G_R and G_L the operators right and left multiplication, i.e. for all $F \in \mathbb{C}^{N\times N}$

$$G_r(F) = FG, \quad G_l(F) = GF. \tag{1.23}$$

We will also use the following formula [91, Lemma 5.3, page 160]: if X and Y are $N \times N$ matrices we have

$$e^{-X} Y e^X = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \operatorname{ad}_X^k Y. \tag{1.24}$$

We denote by A and J the nilpotent and diagonal matrices, respectively, defined by

$$A = \sum_{i=1}^{N-1} v_i \mathcal{E}_{i,i+1}, \quad J = \sum_{i=1}^{N} N - i \mathcal{E}_{i,i}, \tag{1.25}$$

where $v_1, v_2, \cdots, v_{N-1} \in \mathbb{C}$.

1.3 Matrix Operators

The classical orthogonal polynomials are families of orthogonal polynomials characterized for being eigenfunctions of a second order differential operator of the form

$$D(\cdot) = \sigma(x)\frac{d^2}{dx^2}(\cdot) + \tau(x)\frac{d}{dx}(\cdot), \quad \sigma(x), \tau(x) \in \mathbb{C}[x], \, \mathrm{dgr}(\sigma) \leq 2, \, \mathrm{dgr}(\tau) = 1. \tag{1.26}$$

This property is equivalent to a bunch of interesting characterizing properties (see, for instance [16]). For instance a sequence of classical orthogonal polynomials $(p_n)_n$ with respect to a weight $\rho(x)$ can be expressed in terms of a Rodrigues formula

$$p_n = B_n \frac{d^n}{dx^n} \left(\sigma(x)^n \rho(x)\right) \rho(x)^{-1}.$$

Also the weight satisfies a Person equation

$$\frac{d}{dx}(\sigma(x)\rho(x)) = \tau(x)\rho(x),$$

and the sequence of their derivatives $(p_n')_n$ is again a sequence of classical orthogonal polynomials with respect to $\sigma(x)\rho(x)$.

One can consider different discretizations of the operator (1.26) by replacing the derivative by certain approximations on a *lattice*.

Definition 1.3.1. *[93] A lattice is a complex function $x \in \mathcal{C}^2(\Omega)$ where Ω is a complex domain, $\mathbb{N}_0 \subseteq \Omega$, and $x(s)$, $s = 0, 1, \cdots$ are the points where we discretize (1.26).*

By considering the uniform lattice $x(s) = s$ the operator (1.26) becomes

$$D(\cdot) = \bar{\sigma}(x)\Delta\nabla(\cdot) + \bar{\tau}(x)\Delta(\cdot), \quad \bar{\sigma}, \bar{\tau} \in \mathbb{C}[x], \, \mathrm{dgr}(\bar{\sigma}) \leq 2, \, \mathrm{dgr}(\bar{\tau}) = 1, \tag{1.27}$$

where $\Delta(p(x)) = p(x+1) - p(x)$ and $\nabla(p(x)) = p(x) - p(x-1)$ are the forward and backward difference operators respectively. The families of orthogonal polynomials being eigenfunctions of such a second order difference operator are called classical discrete orthogonal polynomials. Like it happened in the continuous case, there are several properties that characterize them, such as a Rodrigues formula

$$p_n = B_n \nabla^n \left(\left[\prod_{m=1}^{n} \sigma(x+m) \right] \rho(x+n) \right), \tag{1.28}$$

a Pearson equation,

$$\Delta(\bar{\sigma}(x)\rho(x)) = \bar{\tau}(x)\rho(x),$$

and that the sequence of their difference polynomials $(\nabla p_n)_n$ is a sequence of classical discrete orthogonal polynomials with respect to $\bar{\sigma}(x)\rho(x)$.

If instead of the uniform lattice $x(s) = s$ we consider a non uniform lattice of the form $x(s) = q^s$, for $0 < q < 1$, the operator (1.26) becomes, after some linear manipulations

$$D(\cdot) = \tilde{\sigma}(x) D_q D_{q^{-1}}(\cdot) + \tilde{\tau}(x) D_q(\cdot), \quad \tilde{\sigma}, \tilde{\tau} \in \mathbb{C}[x], \, \mathrm{dgr}(\tilde{\sigma}) \leq 2, \, \mathrm{dgr}(\tilde{\tau}) = 1, \tag{1.29}$$

where

$$D_q(p(x)) = \frac{p(qx) - p(x)}{(q-1)x} \tag{1.30}$$

is the q-derivative. The orthogonal polynomials satisfying such a q-difference equation are called q-classical polynomials (see [18], [90], [93]). One can also see that these orthogonal polynomials are characterized by different properties, for instance the q-weight $\rho(q^x)$ satisfies a Pearson equation

$$D_q(\tilde{\sigma}(q^x)\rho(q^x)) = \tilde{\tau}(q^x)\rho(q^x),$$

the q-classical polynomials satisfy a Rodrigues formula

$$p_n(q^x) = B_n D^n_{q^{-1}}\left(\rho(q^{x+n})\prod_{k=1}^{n}\tilde{\sigma}(q^{x+k})\right),$$

and the sequence $(D_q p_n(q^x))_n$ is a sequence of orthogonal polynomials with respect to $\sigma(q^{x+1})\rho(q^{x+1})$ In fact the q-classical polynomials are a wider class to that presented above, but for the purpose of this thesis we will just consider this subclass of q-classical polynomials, that used to be called q-classical polynomials of the Hahn class.

The aim of this section is to introduce some basic definition and facts needed to go through the study of matrix orthogonal polynomials being eigenfunctions of matrix analogues of the previous operators.

Due to the non-commutativity of the matrix product and the existence of singular matrices several considerations have to be done in the study of matrix operators. For instance the order in which the coefficients and eigenvalues multiply in the operator equation has to be carefully chosen ([28]). We also miss in general the equivalence of the properties satisfied by the scalar classical, discrete classical and q-classical polynomials (see for instance [12], [39], [46], [57]).

But if the non-commutativity and the existence of singular matrices bring a higher complexity to this theory, such a complexity also implies new interesting features appearing in the theory of matrix polynomials which are absent in the well known scalar theory of classical, classical discrete or q-classical polynomials. One of these phenomena is that the elements of a family of orthogonal matrix polynomials can be eigenfunctions of several linearly independent (second order differential or difference) operators ([14], [30], [35], [72], [94]). As a dual situation we can also find (differential) operators having several families of orthogonal polynomials as eigenfunctions ([36]). We will study this fact for difference operators in Chapter 2 below.

Before starting, we need to establish the sort of matrix operators we will consider.

Let $D : \mathbb{C}^{N\times N}[x] \to \mathbb{C}^{N\times N}[x]$ be a matrix operator taking matrix polynomials into matrix polynomials. As a matrix analogue of the situations discussed above we are going to focus the study in those matrix operators such that

$$D(P_n(x)) = \Lambda_n P_n(x), \quad \Lambda_n \in \mathbb{C}^{N\times N},$$

for some family of matrix orthogonal polynomials $(P_n)_n$. To do so we need to introduce the key concept of a *symmetric* operator with respect to a weight matrix.

Definition 1.3.2. *Let $D:\mathbb{C}^{N\times N}[x]\to\mathbb{C}^{N\times N}[x]$ be a matrix operator taking matrix polynomials into matrix polynomails. We say that D is symmetric with respect to a weight matrix W if it satisfies*

$$\langle D(P),Q\rangle=\int D(P)dWQ^*=\int PdWD(Q)^*=\langle P,D(Q)\rangle. \tag{1.31}$$

The following lemma establishes the relation between symmetric operators and matrix operators having matrix orthogonal polynomials as eigenfunctions.

Lemma 1.3.3. *[28] Let W be a weight matrix and $(P_n)_n$ a sequence of orthonormal polynomials with respect to W. Let D be a matrix operator satisfying that for all polynomial P, $D(P)$ is a polynomial with degree at most the degree of P. Then, D is symmetric with respect to W if and only if $D(P_n)=\Lambda_nP_n$ for certain sequence $(\Lambda_n)_n$ of Hermitian matrices.*

In this thesis we deal with three types of matrix operators

1. Matrix differential operators

$$D(P(x))=\sum_{k=0}^{k_0}\frac{d^k}{dx^k}(P(x))F_k(x),\quad F_k(x)\in\mathbb{C}^{N\times N}[x]. \tag{1.32}$$

2. Matrix difference operators:

$$D(P(x))=\sum_{k=s}^{r}\mathfrak{s}_k(P(x))F_k(x),\quad \mathfrak{s}_k(P(x))=P(x+k),\ F_k(x)\in\mathbb{C}^{N\times N}[x]. \tag{1.33}$$

3. Matrix q-difference operators, for $0<q<1$:

$$D(P(x))=\sum_{k=s}^{r}\mathfrak{E}_k(P(x))F_k(x),\ \mathfrak{E}_k(P(x))=P(q^kx),\ F_k(x)\in\mathbb{C}^{N\times N}[x^{-1}]. \tag{1.34}$$

Notice that all these three types of matrix operators have matrix coefficients multiplying on the right, whereas the eigenvalues appearing in Lemma (1.3.3) are multiplying on the left. There are several reasons why we focus on these kinds of matrix operators (see [28], [38]).

Notice that if $(P_n)_n$ is a sequence of orthogonal polynomials then, for any sequence of non-singular matrices $(\Lambda_n)_n$, the sequence $(\Lambda_nP_n)_n$ is also a sequence of orthogonal polynomials, but in general $(P_n\Lambda_n)_n$ is not a sequence of orthogonal polynomials. This is the reason why we multiply the eigenvalues on the left.

The reason for the order of multiplication of the coefficients in the case of continuous weights and second order differential operators is explained in [28, Theorem 3.2], a result that can be easily generalized for the case of difference and q-difference operators:

Theorem 1.3.4. *The following conditions are equivalent:*

(i) W is a continuous matrix weight (resp. discrete weight, q-weight) with a left-hand side second order differential operator (resp. matrix difference, q-difference operator)

(ii) A non-singular matrix S exists for which $W = S^* W_d S$ where $W_d = \rho I$, with ρ a classical scalar weight (resp. classical discrete weight, q-classical weight).

In the following sections we investigate those matrix operators of the form (1.32), (1.33), (1.34) not raising the degree of matrix polynomials and being symmetric with respect to some weight.

1.3.1 Matrix differential operators

In this section we study matrix-valued differential operators having a sequence of orthogonal polynomials as eigenfunctions. Such differential operators are related with a discrete-continuous non-commutative bispectral problem (for more information on bispectral problem we refer to [24], and to [69] for the non-commutative version). Recall that the orthogonality of a sequence is equivalent to a second order difference operator given by the three term recurrence relation (1.21). Then, if $(P_n)_n$ is a sequence of orthogonal polynomials being eigenfunctions of a differential operator D, then they are common eigenfunctions of two operators

$$D(P_n) = \sum_{k=0}^{k_0} \frac{d^k}{dx^k}(P_n(x))F_k(x) = \Lambda_n P_n(x),$$
$$\Theta P_n(x) = A_n P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x) = x P_n(x).$$

If D is a differential operator with a sequence of orthogonal polynomials as eigenfunctions, then D is of the form [72]

$$D(\cdot) = \sum_{k=0}^{k_0} \frac{d^k}{dx^k}(\cdot)F_k(x), \quad F_k(x) \in \mathbb{C}_k^{N\times N}[x].$$

The non-commutativity of the product of matrices entail the existence of sequences of matrix orthogonal polynomials being eigenfunctions of several independent matrix differential operator as one can see, for instance, in [14], what brings into the picture the concept of algebra of operators associated to a weight matrix.

Let W be a weight matrix and $(P_n)_n$ a sequence of orthogonal polynomials with respect to it. We define the set $\mathfrak{D}(W)$ as

$$\mathfrak{D}(W) = \left\{ D = \sum_{k=0}^{k_0} \frac{d^k}{dx^k}(\cdot)F_k(x) \,\middle|\, D(P_n) = \Lambda_n P_n,\ F_k \in \mathbb{C}_k^{N\times N}[x],\ k_0 \in \mathbb{N} \right\}. \tag{1.35}$$

Notice that if $(Q_n)_n$ is another sequence of orthogonal polynomials with respect to W then, there exists a sequence of non singular matrices $(M_n)_n$ such that $Q_n = M_n P_n$ and if $D \in \mathfrak{D}(W)$ we have

$$D(Q_n) = D(M_n P_n) = M_n D(P_n) = M_n \Lambda_n M_n^{-1} Q_n,$$

so the definition of the algebra $\mathfrak{D}(W)$ does not depend on the sequence chosen, but only on the weight matrix.

It is worth observing that $\mathfrak{D}(W)$ is a subalgebra of the Weyl algebra

$$\mathbf{D}(W) = \left\{ D = \sum_{k=0}^{k_0} \frac{d^k}{dx^k}(\cdot)F_k(x) \,\middle|\, F_k \in \mathbb{C}^{N\times N}[x],\ k_0 \in \mathbb{N} \right\}.$$

Given a weight matrix W, and a sequence of orthogonal polynomials $(P_n)_n$ with respect to W, a differential operator $D \in \mathfrak{D}(W)$ is determined by the sequence of eigenvalues $(\Lambda_n)_n$.

Proposition 1.3.5. *[72] Given a sequence $(P_n)_n$ of orthogonal polynomials with respect to W, let us consider the algebra $\mathfrak{D}(W)$ defined in (1.35). Let $\Lambda(D,n) = \Lambda_n(D)$ be the eigenvalue associated to Q_n. Then, $D \mapsto \Lambda(D,n)$ is a representation of $\mathfrak{D}(W)$ into $\mathbb{C}^{N\times N}$. Moreover the sequence $(\Lambda_n)_n$ separates elements of $\mathfrak{D}(W)$.*

For $D \in \mathfrak{D}(W)$, $D = \sum_{k=0}^{k_0} \frac{d^k}{dx^k} F_k(x)$, with $F_k = \sum_{j=0}^{k} F_k^j x^j$. The eigenvalues Λ_n associated to the monic orthogonal polynomials $(P_n)_n$ form a sequence of polynomials in n of degree at most the order of D, given by

$$\Lambda_n = \sum_{k=0}^{k_0} \binom{n}{k} k! F_k^k. \tag{1.36}$$

One can define a $\star$-operation in the algebra of differential operators associated to a weight matrix via the adjoint of operators.

Theorem 1.3.6. *[72] For any $D \in \mathfrak{D}(W)$ there exists a unique differential operator $D^* \in \mathfrak{D}(W)$ such that $\langle D(P), Q\rangle = \langle P, D^*(Q)\rangle$, for all $P, Q \in \mathbb{C}^{N\times N}[x]$. We shall refer to D^* as the adjoint of D. The map $D \mapsto D^*$ is a $\star$-operation, and the orders of D and D^* coincide.*

By Lemma 1.3.3 it is clear that symmetric differential operators not raising the degree of polynomials are always in $\mathfrak{D}(W)$. If we denote by $\mathcal{S}(W) \subset \mathfrak{D}(W)$ the subset of symmetric differential operators, we have that $\mathcal{S}(W)$ is a real form of the space $\mathfrak{D}(W)$ (see [72]),

$$\mathfrak{D}(W) = \mathcal{S}(W) \oplus i\mathcal{S}(W).$$

In the rest of the section we focus on second order matrix differential operators

$$D(\cdot) = \frac{d^2}{dx^2}(\cdot)F_2(x) + \frac{d}{dx}(\cdot)F_1(x) + (\cdot)F_0 \tag{1.37}$$

The following theorem establishes criteria to assure the symmetry of a differential operator D of degree 2 with respect to a weight matrix having a differentiable density with respect to the Lebesgue measure $dW = W(x)dx$.

Theorem 1.3.7. *[38] Let D be a second order matrix differential operator of the form (1.37) and W a weight matrix with a differentiable density with respect to the Lebesgue measure $dW = W(x)dx$ and supported on an interval $\mathfrak{I} = (a,b)$ of the real line $\mathfrak{I} \subseteq \mathbb{R}$ (a and b finite or infinite). If for all $x \in \mathfrak{I}$, the coefficients of D and $W(x)$ satisfy the symmetry equations*

$$\begin{aligned} F_2(x)W(x) &= W(x)F_2(x)^* \\ 2\left(F_2(x)W(x)\right)' &= F_1(x)W(x) + W(x)F_1(x)^* \\ \left(F_2(x)W(x)\right)'' - \left(F_1(x)W(x)\right)' + F_0W(x) &= W(x)F_0^*, \end{aligned}$$

and the boundary conditions

$$\lim_{x\to\{a,b\}} x^n F_2(x)W(x) = 0, \quad \lim_{x\to\{a,b\}} x^n \big((F_2(x)W(x))' - F_1(x)W(x)\big) = 0, \quad \textit{for all } n \in \mathbb{N},$$

then, D is symmetric with respect to W.

The symmetry of a differential operator D, with respect to a weight matrix W, is also equivalent to a set of identities involving the moments of W and the coefficients of D.

Theorem 1.3.8. *[38] Let D be a second order matrix differential operator of the form (1.37) and W a weight matrix. Let us denote by $\mu_n = \int x^n dW(x)$ the moments associated to W, and we write*

$$F_k(x) = \sum_{i=0}^{k} F_k^i x^i, \quad F_k^i \in \mathbb{C}^{N\times N}, \quad k = 0, 1, 2.$$

for the coefficients of the differential operator (1.37). Then, the following properties are equivalent,

(i) D is symmetric with respect to W.

(ii) For $n = 2, 3, \cdots$,

$$F_2^2\mu_n + F_2^1\mu_{n-1} + F_2^0\mu_{n-2} = \mu_n \left(F_2^2\right)^* + \mu_{n-1}\left(F_2^1\right)^* + \mu_{n-2}\left(F_2^0\right)^*.$$

For $n = 1, 2, \cdots$,

$$2(1-n)\left(F_2^2\mu_n + F_2^1\mu_{n-1} + F_2^0\mu_{n-2}\right) = F_1^1\mu_n + F_1^0\mu_{n-1} + \mu_n\left(F_1^1\right)^* + \mu_{n-1}\left(F_1^0\right)^*.$$

And for $n = 0, 1, \cdots$,

$$n(1-n)\left(F_2^2\mu_n + F_2^1\mu_{n-1} + F_2^0\mu_{n-2}\right) - n\left(F_1^1\mu_n + F_1^0\mu_{n-1}\right) + F_0^0\mu_n = \mu_n\left(F_0^0\right)^*.$$

As we have already mentioned, in the matrix case we lose some of the equivalences among those properties that characterize the classical orthogonal polynomials. In general, for a family of matrix orthogonal polynomials $(P_n)_n$ that are eigenfunctions of a second order differential operator like (1.37), it is not true that the sequence formed by its derivatives, $(P_n')_n$, is a sequence of orthogonal polynomials with respect to some weight matrix. In [12] the authors established criteria to assure the orthogonality of $(P_n')_n$.

Theorem 1.3.9. *[12] Let W be a weight matrix supported in the interval $\mathfrak{I} = (a,b) \subseteq \mathbb{R}$, and $(P_n)_n$ the sequence of monic orthogonal polynomials with respect to W. Then, the following are equivalent,*

(i) There exists two matrix polynomials Φ, Ψ with $\operatorname{dgr}(\Phi) \le 2$, $\operatorname{dgr}(\Psi) \le 1$ satisfying $\det(\Phi(x)W(x)) \ne 0$ in $\mathfrak{I}$ and such that

$$\Phi(x)W(x) = W(x)\Phi(x)^*,$$

$$\frac{d}{dx}\left(\Phi(x)W(x)\right) = \Psi(x)W(x), \quad \textit{for all } x \in \mathfrak{I},$$

$$\lim_{x\to\{a,b\}} \Phi(x)W(x) = \lim_{x\to\{a,b\}} \Psi(x)W(x) = 0. \tag{1.38}$$

(ii) The sequence $(\frac{1}{n}P_n')_n$ is a sequence of monic orthogonal polynomials.

Moreover the sequence $(P_n')_n$ is orthogonal with respect to $\Phi(x)W(x)$.

The matrix-valued hypergeometric equation

For some particular cases of second order matrix differential operators, the polynomial eigenfunctions can be described in terms of matrix-valued hypergeometric functions. We present here two matrix-valued hypergeometric functions introduced by Tirao in [98]. These matrix-valued functions characterize, under certain restrictions on the coefficients, the vector-valued polynomial solutions around $x = 0$ of the matrix equation

$$x(1-x)P''(x) + P'(x)F_1(x) + P(x)F_0 = \Lambda P(x).$$

To do so we first make clear some notation that will be used.

Assume that $A, B, C \in \mathbb{C}^{N\times N}$ with $\sigma(C) \cap \{0, -1, -2, \cdots\} = \varnothing$. For $m \in \mathbb{N}$ we write $(A, B, C)_m$ for the matrix function

$$\begin{aligned}(A, B, C)_0 &= I,\\ (A, B, C)_{m+1} &= (A, B, C)_m\,(B + mI)(A + mI)(C + mI)^{-1},\end{aligned}$$

and $[A, B, C]_m$ for the matrix function

$$\begin{aligned}[A, B, C]_0 &= I,\\ [A, B, C]_{m+1} &= [A, B, C]_m \left(m^2 I + m(A - I) + B\right)(C + mI)^{-1}.\end{aligned}$$

Definition 1.3.10. *Let $A, B, C \in \mathbb{C}^{N\times N}$ such that $\sigma(C) \cap \{0, -1, -2, \cdots\} = \varnothing$, we define*

$$ {}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{z^n}{n!}(A, B, C)_n. \tag{1.39}$$

Definition 1.3.11. *Let $C, U, V \in \mathbb{C}^{N\times N}$ such that $\sigma(C) \cap \{0, -1, -2, \cdots\} = \varnothing$, we define*

$$ {}_2H_1\left[\begin{matrix}U, V\\ C\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{z^n}{n!}[U, V, C]_n. \tag{1.40}$$

The following theorem characterizes the vector-valued analytic solutions of certain second order matrix differential equations.

Theorem 1.3.12. *[98] If $\sigma(C) \cap \{0, -1, -2, \cdots\} = \varnothing$ then,*

1. *The function ${}_2H_1\left[\begin{smallmatrix}U,V\\ C\end{smallmatrix}; x\right]$ is analytic on $|x| < 1$ with values in $\mathbb{C}^{N\times N}$.*
2. *If $G_0 \in \mathbb{C}^N$, then ${}_2H_1\left[\begin{smallmatrix}U,V\\ C\end{smallmatrix}; x\right]G_0$ is a solution of the hypergeometric equation*

$$x(1-x)G'' + G'(C - xU) - GV = 0 \tag{1.41}$$

such that $G(0) = G_0$. Conversely any solution G at $x = 0$ is of this form.

In the particular case when there exist matrices A and B such that $U = I + A + B$ and $V = AB$, equation (1.41) can be solved by means of the matrix-hypergeometric function (1.39).

Theorem 1.3.13. *[98] If $\sigma(C) \cap \{0, -1, -2, \cdots\} = \varnothing$ then,*

1. *The function ${}_2F_1\left[\begin{smallmatrix} A,B \\ C \end{smallmatrix}; x\right]$ is analytic on $|x| < 1$ with values in $\mathbb{C}^{N\times N}$.*
2. *If $G_0 \in \mathbb{C}^N$, then ${}_2F_1\left[\begin{smallmatrix} A,B \\ C \end{smallmatrix}; x\right] G_0$ is a solution of the hypergeometric equation*

$$x(1-x)G'' + G'(C - x(I + A + B)) - GAB = 0$$

such that $G(0) = G_0$. Conversely any solution G at $x = 0$ is of this form.

Observation 1.3.14. *In general and because of the different order in which the coefficients and the eigenvalues are multiplied, the matrix equation*

$$x(1-x)P_n''(x) + P_n'(x)F_1(x) + P_n(x)F_0 = \Lambda_n P_n(x)$$

does not fit in the form of the previous ones. However, for a differential operator D such that there exists a family of orthogonal polynomials $(P_n)_n$ with $DP_n = \Lambda_n P_n$ and Λ_n is a diagonal matrix for all $n \geq 0$, then we can rewrite the previous equation as a set of N differential equations for the rows of the matrix orthogonal polynomials $P_{n,i}$, and these equations are of the form (1.41),

$$x(1-x)P_{n,i}''(x) + P_{n,i}'(x)(F_1(x) - \lambda_i I) + P_{n,i}(x)F_0 = 0.$$

Notice that, because of Lemma 1.3.3, for a symmetric differential operator D with respect to a weight matrix W we can always find a sequence of orthogonal polynomials $(P_n)_n$ such that $D(P_n) = \Lambda_n P_n$, and Λ_n are diagonal matrices.

1.3.2 Matrix difference operators

A finite difference operator can be expanded in terms of several basis. We can use powers of the difference operators Δ and ∇ defined by

$$\Delta(f(x)) = f(x+1) - f(x), \quad \nabla(f(x)) = f(x) - f(x-1).$$

Cross powers $\Delta^i \nabla^j$ are not needed since they are linear combinations of powers of Δ and ∇ (it is an easy consequence of the formula $\Delta\nabla = \Delta - \nabla$). We can also use the shift operators, $\mathfrak{s}_l(F)(x) = F(x+l)$ with $l \in \mathbb{Z}$. We can change from the basis Δ^k, ∇^k, $k \geq 0$, to the basis $\mathfrak{s}_l$, by using the formulas ($k \geq 0$)

$$\Delta^k = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \mathfrak{s}_l, \qquad \nabla^k = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \mathfrak{s}_{-l},$$

$$\mathfrak{s}_k = \sum_{l=0}^{k} \binom{k}{l} \Delta^l, \qquad \mathfrak{s}_{-k} = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \nabla^l.$$

Since $\mathfrak{s}_{-1} \circ \mathfrak{s}_1$ is the identity operator, all the shift operators are powers (positive or negative) of $\mathfrak{s}_1$. We will mostly use the shift operators $\mathfrak{s}_l$, $l \in \mathbb{Z}$ (see (1.33)) to define matrix difference operators, and eventually we will change to the Δ and ∇ operators.

Taking into account Lemma 1.3.3, if a weight matrix W has a symmetric second order difference operator D such that for any polynomial P, $D(P)$ is a polynomial with degree at most the degree of P, then their orthonormal polynomials $(P_n)_n$ (and then, any other sequence of orthogonal polynomials with respect to W) satisfy a second order difference equations of the form

$$P_n(x+1)F_1(x) + P_n(x)F_0(x) + P_n(x-1)F_{-1}(x) = \Lambda_n P_n(x). \tag{1.42}$$

The following lemma (Lemma 3.2 of [35]) characterizes the coefficients of finite difference operators, D such that for every polynomial P, $D(P)$ is a polynomial with degree at most the degree of P.

Lemma 1.3.15. *[35] Let D be a finite difference operator*

$$D(\cdot) = \sum_{l=s}^{r} \mathfrak{s}_l(\cdot)F_l. \tag{1.43}$$

The following conditions are equivalent

(i) For any matrix polynomial P, $D(P)$ is also a polynomial with degree at most the degree of P.

(ii) The functions F_l, $l = s, \cdots, r$, are polynomials of degree at most $r - s$ and $\operatorname{dgr}\left(\sum_{l=s}^{r} l^k F_l\right) \leq k$ for $k = 0, \cdots, r - s$.

Proceeding as we did for matrix differential operators with orthogonal polynomials as eigenfunctions, one can relate matrix orthogonal polynomials being eigenfunctions of difference operators with a discrete-discrete non-commutative bispectral problem.

In the discrete case one also finds families of orthogonal polynomials being eigenfunctions of various independent difference operator. Given a weight matrix W and a sequence of orthogonal polynomials with respect to it $(P_n)_n$, we define the algebra $\mathfrak{D}(W)$ by

$$\mathfrak{D}(W) = \left\{ D = \sum_{k=s}^{r} \mathfrak{s}_k F_k(x) \,\middle|\, D(P_n) = \Lambda_n P_n,\ s < r,\ n \geq 0 \right\}. \tag{1.44}$$

Although we use the same notation for the algebra of difference operators and differential operators, the context will make clear which we are dealing with. Because of the left linearity of such a difference operator, it is clear that the definition does not depend on the sequence of matrix orthogonal polynomials but only on the weight. We say that a difference operator D of the form (1.43) with F_s, $F_r \neq 0$ has order $r - s$ and genre (r, s).

Fixed a sequence of orthogonal polynomials with respect to a weight matrix W, each difference operator in $\mathfrak{D}$ is determined by the sequence of eigenvalues $(\Lambda_n(D))_n$.

Theorem 1.3.16. *[35] Let D be a difference operator in $\mathfrak{D}(W)$ with $D(P_n) = \Lambda_n(D)P_n$. Then, D is determined by the sequence $(\Lambda_n(D))_n$. Set $\Lambda(D,n) = \Lambda_n(D)$, for each $n \geq 0$, the application $D \to \Lambda(D,n)$ is a representation of $\mathfrak{D}(W)$ in $\mathbb{C}^{N\times N}$ and the sequence of representation $(\Lambda_n)_n$ separates elements of $\mathfrak{D}(W)$.*

If $D \in \mathfrak{D}(W)$, then the n-th eigenvalue associated to the sequence of monic orthogonal polynomials $(P_n)_n$ is a matrix polynomial in n with degree at most the order of D.

Theorem 1.3.17. *[35] Let* $D = \sum_{l=s}^{r} \mathfrak{s}_l F_l$ *a difference operator such that* $D(P_n) = \Lambda_n P_n$*, where* $(P_n)_n$ *is the sequence of monic orthogonal polynomials with respect to* W*. If* $F_l = \sum_{i=0}^{r-s} F_l^i x^i$*, then*

$$\Lambda_n = \sum_{k=0}^{r-s} \binom{n}{k} \sum_{l=s}^{r} l^k F_l^k.$$

Hence the matrix Λ_n *is a polynomial in* n *of degree at most the order* $r-s$ *of* D*.*

The algebra of difference operators associated to a weight matrix W can be endowed with a $\star$-operation.

Theorem 1.3.18. *[35] For any* $D \in \mathfrak{D}(W)$ *there is a unique difference operator* $D^* \in \mathfrak{D}(W)$ *such that* $\langle D(P), Q\rangle = \langle P, D^*(Q)\rangle$*. We shall refer to* D^* *as the adjoint of* D*. The map* $D \mapsto D^*$ *is then a* $\star$*-operation in the algebra* $\mathfrak{D}(W)$*. If the operator* D *has genre* (s, r) *then,* D^* *has genre* $(-r, -s)$*.*

By Lemma 1.3.3 it is clear that symmetric operators D not raising the degree of polynomials are always in $\mathfrak{D}(W)$ and can be characterized in terms of its eigenvalues with respect to a sequence of orthonormal polynomials. If we denote by $\mathcal{S}(W) \subset \mathfrak{D}(W)$ the subset of symmetric difference operator, then $\mathcal{S}(W)$ is a real form of the space $\mathcal{D}(W)$, [35],

$$\mathfrak{D}(W) = \mathcal{S}(W) \oplus i\mathcal{S}(W).$$

From theorem 1.3.18 it follows that a finite order difference operator D (1.43) symmetric with respect to a weight matrix and the degree of $D(P)$ is at most the degree of P for all matrix polynomials P has genre $(-r, r)$ and order $2r$. The symmetry of a finite order difference operator satisfying this condition with respect to a discrete weight matrix W can be guaranteed by a finite set of commuting and difference equations together with certain boundary conditions. In fact, the following theorem holds:

Theorem 1.3.19. *[35] For* $r \geq 0$*, let* D *be the finite difference operator*

$$D(\cdot) = \sum_{l=-r}^{r} \mathfrak{s}_l(\cdot) F_l(x), \tag{1.45}$$

where $F_l(x)$*,* $l = -r, \cdots, r$ *are matrix polynomials. Let* W *be the discrete weight matrix with support* $\mathfrak{S}$*, given by*

$$W = \sum_{x \in \mathfrak{S}} W(x)\delta_x. \tag{1.46}$$

Suppose that the coefficients F_l *and the weight matrix* W *satisfy the following equations*

$$F_l(x-l)W(x-l) = W(x)F_{-l}^*(x), \quad \text{for } x \in (l+\mathfrak{S}) \cap \mathfrak{S}, \text{ and } l = 0, 1, \cdots, r, \tag{1.47}$$

and the boundary conditions

$$F_l(x-l)W(x-l) = 0, \quad \text{for } x \in (l+\mathfrak{S}) \backslash \mathfrak{S}, \text{ and } l = 1, \cdots, r, \tag{1.48}$$

$$W(x)F_{-l}^*(x) = 0, \quad \text{for } x \in \mathfrak{S} \backslash (l+\mathfrak{S}), \text{ and } l = 1, \cdots, r. \tag{1.49}$$

Then, the difference operator (1.45) is symmetric with respect to W*.*

1.3.3 Matrix q-difference operators

In the whole thesis, q denotes a real number with $0 < q < 1$. A q-difference operator can be expressed in terms of q-differential basic operators (q-derivatives)

$$D_q(f(x)) = \frac{f(x) - f(qx)}{(1-q)x}, \quad D_{q^{-1}}(f(x)) = \frac{f(x) - f(q^{-1}x)}{(1-q^{-1})x}$$

or in terms of q-shift operators $\mathfrak{E}_l(f(x)) = f(q^l x)$. For a matter of simplicity in the result exhibited here and in the foregoing chapters, we chose to study matrix q-difference operators expressed in terms of q-shift operators. In particular, and because of Lemma 1.3.3 we focus our attention in those q-difference operators taking matrix polynomials into matrix polynomials and not raising the degree, i.e

$$D : \mathbb{C}_k^{N\times N}[x] \to \mathbb{C}_k^{N\times N}[x].$$

Then, we have to consider q-difference operators of the form

$$D(\cdot) = \sum_{l=s}^{r} \mathfrak{E}_l(\cdot) F_l(x), \quad F_l(x) \in \mathbb{C}^{N\times N}[x^{-1}],$$

where $F_l(x)$ satisfy some degree conditions, as it is shown in the following theorem, that is a straightforward adaptation of Theorem 3.2 [35]

Theorem 1.3.20. *Let*

$$D = \sum_{l=s}^{r} \mathfrak{E}_l(\cdot) F_l(x), \tag{1.50}$$

with r, s integers such that $s \le r$. The following conditions are equivalent:

(i) $D : \mathbb{C}_k^{N\times N}[x] \to \mathbb{C}_k^{N\times N}[x]$ *for all* $k \ge 0$.

(ii) $F_l(x) \in \mathbb{C}_{r-s}^{N\times N}[x^{-1}]$ *for* $l = s, \dots, r$ *and* $\sum_{l=s}^{r} q^{lk} F_l(x) \in \mathbb{C}_k^{N\times N}[x^{-1}]$ *for* $k = 0, \dots, r-s$.

We say that a q-difference operator of the form (1.50) has order $r - s$ and genre r, s). It happens that if a q-difference operator is symmetric with respect to a weight matrix, then $s = -r$. The following theorem establishes the symmetry equations and boundary condition that assure the symmetry of a q-difference operator with order $2r$ and genre $(r, -r)$, with respect to a q-weight.

Theorem 1.3.21. *For $r \ge 0$, let D be the q-difference operator*

$$D = \sum_{l=-r}^{r} \mathfrak{E}_l(\cdot) F_l(x) \tag{1.51}$$

where $F_l(x)$ are matrix polynomials in x^{-1}. Let W be a q-weight given by

$$W = \sum_{x=0}^{\infty} W(q^x) \delta_{q^x}.$$

Suppose that the coefficients F_l and the q-weight matrix W satisfy the following equations

(1.52) $$F_0(q^x)W(q^x) = W(q^x)F_0(q^x)^*, \quad x \in \mathbb{N},$$

(1.53) $$F_l(q^{x-l})W(q^{x-l}) = q^l W(q^x)F_{-l}(q^x)^*, \quad x \geq l$$

and the boundary conditions

(1.54) $$F_{-l}(q^x)W(q^x) = 0, \quad \text{for } x = 0, \cdots, l-1, \text{ and } l = 1, \cdots, r$$
$$\lim_{x\to\infty} q^{2x}F_l(q^x)W(q^x) \to 0, \quad \text{for } l = 1, \cdots, r$$
$$\lim_{x\to\infty} q^x\left(F_l(q^x)W(q^x) - W(q^x)F_l(q^x)^*\right) \to 0, \quad \text{for } l = 1, \cdots, r.$$

Then, the q-difference operator D is symmetric with respect to W.

Proof. Let D be a q-difference operator of the form (1.51) and W a q-weight such that the symmetry equations (1.52) and (1.53) as well as the boundary conditions (1.54) hold. For M large enough (much bigger that r) we consider the truncated inner product

$$\langle P, Q\rangle^M = \sum_{x=0}^{M} q^x P(q^x)W(q^x)Q(q^x)^*.$$

It is clear that for any two matrix polynomials P, $Q \in \mathbb{C}^{N\times N}[x]$

$$\lim_{M\to\infty} \langle P, Q\rangle^M = \langle P, Q\rangle.$$

We are going to prove that

$$\langle D(P), Q\rangle^M = \langle P, D(Q)\rangle^M + \Theta(M, x)$$

where $\Theta(M,x)$ is a matrix function such that $\lim_{M\to\infty}\Theta(M,x) = 0$.

Let P, $Q \in \mathbb{C}^{N\times N}[x]$. Then,

(1.55)
$$\begin{aligned}\langle D(P), Q\rangle^M &= \sum_{x=0}^{M}\sum_{k=-r}^{r} P(q^{k+x})F_k(q^x)W(q^x)Q(q^x)^* q^x \\ &= \sum_{x=0}^{M}\sum_{k=1}^{r} P(q^{x-k})F_{-k}(q^x)W(q^x)Q(q^x)^* q^x + \sum_{x=0}^{M} P(q^x)F_0(q^x)W(q^x)Q(q^x)^* q^x \\ &\quad + \sum_{x=0}^{M}\sum_{k=1}^{r} P(q^{k+x})F_k(q^x)W(q^x)Q(q^x)^* q^x.\end{aligned}$$

Taking into account the symmetry equation (1.53) we can write the first factor on the previous sum as

$$\begin{aligned}\sum_{k=1}^{r}\sum_{x=0}^{M} P(q^{x-k})F_{-k}(q^x)W(q^x)Q(q^x)^* q^x &= \sum_{k=1}^{r}\sum_{x=0}^{M} P(q^{x-k})q^k F_{-k}(q^x)W(q^x)Q(q^x)^* q^{x-k} \\ &= \sum_{k=1}^{r}\sum_{x=k}^{M} P(q^{x-k})W(q^{x-k})F_k(q^{x-k})^* Q(q^x)^* q^{x-k} \\ &\quad + \sum_{k=1}^{r}\sum_{x=0}^{k-1} P(q^{x-k})F_{-k}(q^x)W(q^x)Q(q^x)^* q^x.\end{aligned}$$

By the first boundary condition $\sum_{x=0}^{k-1} P(q^{x-k})F_{-k}(q^x)W(q^x)Q(q^x)^*q^x = 0$ for $k = 1,\cdots,r$, then, we get

(1.56)

$$\sum_{k=1}^{r}\sum_{x=0}^{M} P(q^{x-k})F_{-k}(q^x)W(q^x)Q(q^x)^*q^x = \sum_{k=1}^{r}\sum_{x=0}^{M} P(q^x)W(q^x)F_k(q^x)^*Q(q^{x+k})^*q^x - \sum_{k=1}^{r}\sum_{x=M-k+1}^{M} P(q^x)W(q^x)F_k(q^x)^*Q(q^{x+k})^*q^x$$

By proceeding as above we obtain the following identity concerning the third factor in (1.55)

(1.57)

$$\sum_{x=0}^{M}\sum_{k=1}^{r} P(q^{k+x})F_k(q^x)W(q^x)Q(q^x)^*q^x = \sum_{k=1}^{r}\sum_{x=0}^{M} P(q^x)W(q^x)F_{-k}(q^x)^*Q(q^{x-k})q^x + \sum_{k=1}^{r}\sum_{x=M-k+1}^{M} P(q^{x+k})F_k(q^x)W(q^x)Q(q^x)q^x.$$

For the remaining factor in (1.55), $\sum_{x=0}^{M} P(q^x)F_0(q^x)W(q^x)Q(q^x)^*q^x$ it suffices to use symmetry equation (1.52) to get

$$\sum_{x=0}^{M} P(q^x)F_0(q^x)W(q^x)Q(q^x)^*q^x = \sum_{x=0}^{M} P(q^x)W(q^x)F_0(q^x)^*Q(q^x)^*q^x. \tag{1.58}$$

Taking into account (1.56), (1.57) and (1.58) we can write (1.55) as

$$\begin{aligned}\langle D(P),Q\rangle^M &= \sum_{x=0}^{M}\sum_{k=-r}^{r} P(q^x)W(q^x)F_k(q^x)^*Q(q^{x+k})^*q^x + \Theta(M,x)\\ &= \langle P,D(Q)\rangle^M + \Theta(M,x).\end{aligned} \tag{1.59}$$

where

(1.60)

$$\Theta(M,x) = \sum_{k=1}^{r}\left(\sum_{x=M-k+1}^{M} P(q^{x+k})F_k(q^x)W(q^x)Q(q^x)q^x - \sum_{x=M-k+1}^{M} P(q^x)W(q^x)F_k(q^x)^*Q(q^{x+k})^*q^{x-k}\right).$$

By taking limits in (1.59) we get

$$\langle D(P),Q\rangle = \langle P,D(Q)\rangle + \lim_{M\to\infty}\Theta(M,x)$$

Then, it just remains to see that $\Theta(M,x)\to 0$ when $M\to\infty$ to obtain the desired result. To see this write $P(x) = P_0 + x\bar{P}(x)$ and $Q = Q_0 + x\bar{Q}(x)$. Then, by expanding the expression in (1.60) and taking into account the second and third boundary conditions (1.54) we obtain

$$\lim_{M\to\infty}\Theta(M,x) = 0$$

and then, we can conclude that $\langle D(P),Q\rangle = \langle P,D(Q)\rangle$ for all P, $Q\in\mathbb{C}^{N\times N}[x]$. □

Chapter 2

Matrix polynomials satisfying difference equations

Recall that the discrete classical orthogonal polynomials are those families of orthogonal polynomials satisfying a second order difference equation of the form

$$\sigma(x)\Delta\nabla p(x) + \tau(x)\Delta p(x) = \lambda p(x), \quad \sigma, \tau \in \mathbb{C}[x], \quad \mathrm{dgr}(\sigma) \leq 2, \mathrm{dgr}(\tau) = 1.$$

This difference equation can also be written in terms of shift operators

$$f_{-1}(x)\mathfrak{s}_{-1}p(x) + f_0(x)\mathfrak{s}_0 p(x) + f_1(x)\mathfrak{s}_1 p(x) = \lambda p(x), \tag{2.1}$$

where $f_1 = \sigma + \tau$, $f_0 = -2\sigma - \tau$, $f_{-1} = \sigma$.

The discrete classical orthogonal polynomials can be classified in four big families as one can see, for instance, in [62], [77], [83], [93]. These families are

1. **Charlier polynomials**. For $a > 0$,

$$c_n^a(x) = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix} \;\middle|\; -\frac{1}{a}\right) = \sum_{k=0}^{n}(-1)^k a^{-k}\binom{n}{k}\binom{x}{k}k!. \tag{2.2}$$

   They satisfy the orthogonality relation

$$\sum_{x=0}^{\infty}\frac{a^x}{x!}c_n^a(x)c_m^a(x) = a^{-n}e^a n!\delta_{mn}.$$

2. **Meixner polynomials**. For $c > 0$, $0 < a < 1$,

$$m_n^{c,a}(x) = {}_2F_1\left(\begin{matrix} -n - x \\ c \end{matrix} \;\middle|\; 1 - \frac{1}{mu}\right).$$

   They satisfy the orthogonality relation

$$\sum_{x=0}^{\infty}\frac{a^x\Gamma(x+c)}{\Gamma(c)x!}m_n^{c,a}(x)m_r^{c,a}(x) = \frac{a^{-n}n!}{(c)_n(1-a)^c}\delta_{nr}.$$

3. **Krawtchouk polynomials**. For $\kappa \in \mathbb{N}$ and $0 < p < 1$, the Krawchuck polynomials are defined by
$$k_n(x) = {}_2F_1\left(\begin{matrix} -n, -x \\ -\kappa \end{matrix} \,\middle|\, \frac{1}{p}\right).$$
They satisfy the orthogonality relation
$$\sum_{x=0}^{\kappa} \frac{(\kappa - x + 1)_x p^x (1-p)^{\kappa - x}}{x!} k_n(x) k_m(x) = \frac{(-1)^n n!}{(-\kappa)_n} \left(\frac{1-p}{p}\right)^n \delta_{nm}.$$

4. **Hahn polynomials**. For $\kappa \in \mathbb{N}$, $a > -1$ and $b > -1$ the Hahn polynomials are defined by
$$h_n(x) = {}_3F_2\left(\begin{matrix} -n, -n+a+b+1, -x \\ a+1, -\kappa \end{matrix} \,\middle|\, 1\right).$$
They satisfy the orthogonality relation
$$\sum_{x=0}^{\kappa} \frac{(a+1)_x (b+1)_{\kappa - x}}{x!(\kappa - x)!} h_n(x) h_m(x) = \frac{(-1)^n (n+a+b+1)_{\kappa+1} (b+1)_n n!}{(2n+a+b+1)(a+1)_n (-\kappa)_n (\kappa)!} \delta_{nm}.$$

For more details on the classical discrete orthogonal polynomials and some interesting relations among them we refer to [93], [76], [77].

We consider the following matrix analogue to (2.1)

$$\mathfrak{s}_{-1}(\cdot) F_1(x) + \mathfrak{s}_0(\cdot) F_0(x) + \mathfrak{s}_{-1}(\cdot) F_{-1}(x), \tag{2.3}$$

where F_i are matrix polynomials satisfying the conditions given in Lemma 1.3.15 that, in this particular case, take the form

$$\mathrm{dgr}(F_i) \le 2, \quad \mathrm{dgr}(F_1 - F_{-1}) \le 1, \quad \mathrm{dgr}(F_{-1} + F_0 + F_1) = 0.$$

These conditions imply that the degree of $D(P)$ is, at most, the degree of P. The symmetry equations (1.47) for such a difference equations are

$$F_1(x-1) W(x-1) = W(x) F_{-1}^*(x), \quad \text{for } x = 1, 2, \cdots, \kappa, \tag{2.4}$$
$$F_0(x) W(x) = W(x) F_0^*(x), \quad \text{for } x = 0, 1, \cdots, \kappa. \tag{2.5}$$

and the boundary conditions (1.48) are given by

$$W(0) F_{-1}^*(0) = 0, \tag{2.6}$$
$$F_1(\kappa) W(\kappa) = 0, \quad \text{if } \kappa \text{ is a positive integer.} \tag{2.7}$$

In this chapter we present a method to solve the symmetry equations (2.4). Using this method we will construct explicit examples of matrix difference operators and discrete weights satisfying both (2.4) and (2.5) as well as the boundary conditions (2.6), (2.7). That is, we construct pairs of difference operators and discrete weights (D, W) such that D is symmetric with respect to W.

Along the rest of this chapter, W is a discrete weight matrix

$$W = \sum_{x=0}^{\kappa} W(x) \delta_x, \quad \kappa \in \mathbb{N} \text{ or } \kappa = \infty,$$

and D is a difference operator of the form (2.3)

2.1 A method for solving the difference equations for the weight matrix

We now describe a method for constructing discrete weight matrices W and matrix polynomials F_1 and F_{-1} such that the first order difference equation (2.4) holds. The method is based on the following assumption on the coefficients F_1 and F_{-1}: there exists a scalar function s such that for $x \in \{1,2,\cdots,\kappa\}$, $s(x) \neq 0$ and

$$F_1(x-1)F_{-1}(x) = |s(x)|^2 I.$$

We now look for a weight matrix W factorized in the form

$$W = \sum_{x=0}^{\kappa} T(x)T^*(x)\delta_x,$$

where T is the matrix function satisfying $T(0) = I$ and the first order difference equation

$$T(x-1) = \frac{F_{-1}(x)}{s(x)}T(x), \quad \text{for } x \in \{1,2,\cdots,\kappa\}.$$

With this choice of W we have

$$\begin{aligned} F_1(x-1)W(x-1) &= F_1(x-1)T(x-1)T^*(x-1) \\ &= F_1(x-1)\frac{F_{-1}(x)}{s(x)}T(x)T^*(x)\frac{F_{-1}^*(x)}{\overline{s(x)}} \\ &= \frac{F_1(x-1)F_{-1}(x)}{s(x)\overline{s(x)}}W(x)F_{-1}^*(x) \\ &= W(x)F_{-1}^*(x). \end{aligned}$$

So equation (2.4) holds.

We have thus proved the following theorem.

Theorem 2.1.1. *Let κ be either a positive integer or infinite, and F_1 and F_{-1} matrix polynomials. Assume that there exists a scalar function $s(x)$ such that for $x = 1,\cdots,\kappa$, $s(x) \neq 0$ and*

$$F_1(x-1)F_{-1}(x) = |s(x)|^2 I, \quad x \in \{1,\cdots,\kappa\}.$$

Write T for the solution of the first order difference equation

$$T(x-1) = \frac{F_{-1}(x)}{s(x)}T(x), \quad \text{for } x \in \{1,\cdots,\kappa\}, \quad T(0) = I.$$

Then, the weight matrix

$$W = \sum_{x=0}^{\kappa} T(x)T^*(x)\delta_x,$$

satisfies the difference equation (2.4).

2.2 Four families of illustrative examples

In this section we show four families of orthogonal polynomials satisfying second order difference equations of the form

$$P_n(x-1)F_{-1}(x) + P_n(x)F_0(x) + P_n(x+1)F_1(x) = \Lambda_n P_n(x), \tag{2.8}$$

that we have constructed with our method. These families are classified in accordance with the degree of the coefficient F_1.

We recall that the matrices A and J are given by

$$A = \sum_{i=1}^{N-1} v_i \mathcal{E}_{i,i+1}, \qquad J = \sum_{i=1}^{N} (N-i)\mathcal{E}_{i,i},$$

where $v_1, v_2, \cdots, v_{N-1} \in \mathbb{C}$.

2.2.1 Example with $\mathrm{dgr}(F_1) = 0$

In our first example the coefficient F_1 of its associated second order difference operator does not depend on x. The example can be considered as a matrix relative of the Charlier scalar weight.

Theorem 2.2.1. *Let a be a positive real number. The second order difference operator*

$$D(\cdot) = a\mathfrak{s}_1(\cdot)\,(I+A) + \mathfrak{s}_0(\cdot)\left(-J - (I+A)^{-1}x\right) + \mathfrak{s}_{-1}(\cdot)\,(I+A)^{-1}x \tag{2.9}$$

is symmetric with respect to the weight matrix defined by

$$W = \sum_{x\in\mathbb{N}} \frac{a^x}{x!}(I+A)^x\,(I+A^*)^x\,\delta_x. \tag{2.10}$$

Moreover the monic orthogonal polynomials with respect to W are common eigenfunctions of D with eigenvalues given by

$$\Lambda_n = a(I+A) - J - n(I+A)^{-1}.$$

Before we prove Theorem 2.2.1 we announce an prove a Lemma that will be used.

Lemma 2.2.2. *Let A and J be the matrices given by (1.25), and let f be an analytic function at 0. Write $S = f(A)$. Then, $[S,J] = -Af'(A)$. In particular, for $S = \log(I+A)$ we have $[S,J] = -I + (I+A)^{-1}$.*

Proof. Write $f(z) = \sum_{k=0}^{\infty} a_k z^k$. It is easy to check that for $k \geq 0$ $\left[A^k, J\right] = -kA^k$. Then,

$$[S,J] = \sum_{k=0}^{\infty} a_k\left[A^k, J\right] = -\sum_{k=0}^{\infty} k a_k A^k = -Af'(A).$$

□

Proof. For this example we have that κ is infinite and

$$F_{-1} = (I + A)^{-1} x, \tag{2.11}$$

$$F_0 = -J - (I + A)^{-1} x, \tag{2.12}$$

$$F_1 = a(I + A). \tag{2.13}$$

We now use Theorem 1.3.19 for $r = 1$ to prove that the operator D is symmetric. We have to check the boundary condition $W(0)F_{-1}^*(0) = 0$ and the equations

$$F_1(x-1)W(x-1) = W(x)F_{-1}^*(x), \quad \text{for } x = 1, 2, \cdots$$
$$F_0(x)W(x) = W(x)F_0^*(x), \quad \text{for } x = 0, 1, \cdots.$$

We proceed in three steps.

First Step. Boundary condition $W(0)F_{-1}^*(0) = 0$.
Since $F_{-1}(x) = (I + A)^{-1} x$, we have $F_{-1}(0) = 0$ and the boundary condition follows straightforwardly.

Second Step. $F_1(x-1)W(x-1) = W(x)F_{-1}^*(x)$ *for* $x = 1, 2, \cdots$.
We use Theorem 2.1.1 to prove that F_1, F_{-1} and W satisfy this first order difference equation.

We first check that for $s(x) = \sqrt{ax}$ the matrix polynomials F_{-1} and F_1 satisfy

$$F_1(x-1)F_{-1}(x) = s^2(x)I, \quad x \geq 1.$$

But this is straightforward from the definition of F_{-1} and F_1 (see (2.11) and (2.13)).

We then factorize the matrix weight W in the form $W = \sum_{x=0}^{\infty} T(x)T^*(x)\delta_x$, where

$$T(x) = \sqrt{\frac{a^x}{x!}}(I + A)^x.$$

Since $T(0) = I$, we only have to check that for $x \geq 1$, $T(x-1) = \dfrac{F_{-1}(x)}{s(x)}T(x)$. Indeed, we have

$$\frac{F_{-1}(x)}{s(x)}T(x) = \frac{(I+A)^{-1}x}{\sqrt{ax}}\sqrt{\frac{a^x}{x!}}(I+A)^x = \sqrt{\frac{x}{a}}\sqrt{\frac{a^x}{x!}}(I+A)^{-1}(I+A)^x$$
$$= \sqrt{\frac{a^{x-1}}{(x-1)!}}(I+A)^{x-1} = T(x-1).$$

Theorem 2.1.1 gives now that F_1, F_{-1} and W satisfy $F_1(x-1)W(x-1) = W(x)F_{-1}^*(x)$ for $x = 1, 2, \cdots$.

Third Step. $F_0(x)W(x) = W(x)F_0^*(x)$ *for* $x \in \mathbb{N}$.
Since $W(x) = T(x)T^*(x)$ the equation $F_0W = WF_0^*$ is straightforwardly equivalent to the Hermiticity of the function

$$\chi(x) = T^{-1}(x)F_0(x)T(x), \quad x \in \mathbb{N}.$$

We now explicitly compute the function χ to see that actually it is a real diagonal matrix function. To do that, we expand the function $\chi(x)$ in a power series. By writing $S = \log(I+A)$ we get $(I+A)^x = e^{Sx}$. Taking into account that A and S commute, and using formula (1.24), we have

$$\begin{aligned}\chi(x) &= T^{-1}(x)F_0T(x) = e^{-Sx}\left(-J-(I+A)^{-1}x\right)e^{Sx}\\ &= \sum_{i=0}^{\infty}\frac{(-1)^{i+1}}{i!}\operatorname{ad}_S^i Jx^i-(I+A)^{-1}x.\end{aligned}$$

To compute $\operatorname{ad}_S^i J$, $i \geq 1$, we use Lemma 2.2.2 and we get

$$\operatorname{ad}_S J = [S,J] = (I+A)^{-1}-I.$$

For $\operatorname{ad}_S^i J$, $i \geq 2$, since S commutes with A, we have

$$\operatorname{ad}_S^i J = \operatorname{ad}_S^{i-1}(\operatorname{ad}_S J) = \operatorname{ad}_S^{i-1}\left((I+A)^{-1}-I\right) = 0.$$

Hence, we get

$$\chi(x) = -J - xI.$$

That is, the matrix function χ is real diagonal, and then Hermitian.

Thus, Theorem 1.3.19 for $r = 1$ shows that the second order operator D (3.5) is symmetric with respect to W.

We now use Lemma 1.3.3 to prove that the orthogonal polynomials are common eigenfunctions of the operator D. Since we have already proved that D is symmetric with respect to W, it is enough to check that for any polynomial P the degree of $D(P)$ is at most the degree of P. But that is a consequence of Lemma 1.3.15 taking into account that

$$\begin{aligned}&\operatorname{dgr}(F_i) \leq 2, \quad \text{for } i = 0,1,-1,\\ &\operatorname{dgr}(F_1-F_{-1}) = \operatorname{dgr}\left(a(I+A)-(I+A)^{-1}x\right) = 1,\\ &\operatorname{dgr}(F_0+F_1+F_{-1}) = \operatorname{dgr}(-J+a(I+A)) = 0.\end{aligned}$$

The expression for the eigenvalues follows by Theorem 1.3.17. This completes the proof. □

2.2.2 Example with $\operatorname{dgr}(F_1) = 1$ and unbounded support.

Our second example can be considered as a matrix relative of the Meixner scalar weight. It has unbounded support and the coefficient F_1 of its associated second order difference operator is a polynomials of degree 1.

Theorem 2.2.3. *Let A and J be the $N\times N$ nilpotent and diagonal matrices, respectively, given by (1.25). Given two positive real numbers a and c, with $0 < a < 1$, we consider the matrix*

$$R_{A,a} = (I-A)(I-aA)^{-1}, \tag{2.14}$$

and the weight matrix defined by

$$W = \frac{1}{\Gamma(c)}\sum_{x=0}^{\infty}\frac{a^x\Gamma(x+c)}{x!}R_{A,a}^x\left(R_{A,a}^*\right)^x\delta_x. \tag{2.15}$$

Then, the second order difference operator

$$D(\cdot) = \mathfrak{s}_1(\cdot)F_1(x) + \mathfrak{s}_0(\cdot)F_0(x) + \mathfrak{s}_{-1}(\cdot)F_{-1}(x), \tag{2.16}$$

where

$$F_{-1}(x) = R_{A,a}^{-1}x, \tag{2.17}$$

$$F_0(x) = (a-1)J - x(aR_{A,a} + R_{A,a}^{-1}), \tag{2.18}$$

$$F_1(x) = a(x+c)R_{A,a}, \tag{2.19}$$

is symmetric with respect to the weight matrix W. Moreover, the monic orthogonal polynomials with respect to W are common eigenfunctions of D, with eigenvalues given by

$$\Lambda_n = (a-1)J + acR_{A,a} + n\left(aR_{A,a} - R_{A,a}^{-1}\right).$$

Proof. We use again Theorem 1.3.19 for $r = 1$ to prove that the operator D is symmetric. We have to check the boundary condition $W(0)F_{-1}^*(0) = 0$ and the equations

$$F_1(x-1)W(x-1) = W(x)F_{-1}(x), \quad \text{for } x = 1, 2, \cdots, \tag{2.20}$$

$$F_0(x)W(x) = W(x)F_0^*(x), \quad \text{for } x = 0, 1, \cdots. \tag{2.21}$$

We omit the proofs of the boundary condition and the equation (2.20) because they are similar to those in Steps 1 and 2 in Theorem 2.2.1 (just taking here $s(x) = \sqrt{a(x+c-1)x}$). To prove $F_0(x)W(x) = W(x)F_0^*(x)$ for $x \in \mathbb{N}$, we proceed as follows. We first write $W(x) = T(x)T^*(x)$, where

$$T(x) = \sqrt{\frac{a^x\Gamma(x+c)}{\Gamma(c)x!}}R_{A,a}^x.$$

As we have already pointed out, the equation $F_0(x)W(x) = W(x)F_0^*(x)$ is then equivalent to the Hermiticity of the matrix function $\chi(x) = T^{-1}(x)F_0(x)T(x)$. We are going to show that this is, in fact, a real diagonal matrix.

Consider now the analytic function at 0

$$f(z) = \log\left(\frac{1-z}{1-az}\right),$$

and write $S = f(A)$. This gives $e^S = R_{A,a}$ and $e^{-S} = R_{A,a}^{-1}$, and hence

$$\begin{aligned}\chi(x) &= T^{-1}(x)F_0(x)T(x) \\ &= e^{-Sx}\left((a-1)J - ae^Sx - e^{-S}x\right)e^{Sx} \\ &= \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}(a-1)\,\mathrm{ad}_S^i\,Jx^i - ae^Sx - e^{-S}x.\end{aligned}$$

We now compute explicitly the function χ

Using Lemma 2.2.2, we see that $-(a-1)[S,J] - ae^S - e^{-S}$ is, in fact, a multiple of the identity matrix. Indeed, since

$$f'(z) = \frac{a}{1-az} - \frac{1}{1-z},$$

according to this lemma, we have

$$[S,J] = A(1-A)^{-1} - aA(1-aA)^{-1}.$$

On the other hand, a simple computation gives

$$-ae^S - e^{-S} = -(a+1)I - a(a-1)A(1-aA)^{-1} + (a-1)A(I-A)^{-1}.$$

Then, we get

$$-(a-1)[S,J] - ae^S - e^{-S} = -(1+a)I.$$

For the rest of the coefficients, we proceed as in Theorem 2.2.1 to get $\operatorname{ad}_R^i J = 0$ for $i \geq 2$.

We have then that

$$\chi(x) = (a-1)J - (1+a)Ix.$$

This is a real diagonal matrix for all $x \geq 0$, and so Hermitian. Then, $F_0(x)W(x) = W(x)F_0^*(x)$ holds for $x \in \mathbb{N}$.

Using Theorem 1.3.19 for $r = 1$ we can conclude that the second order difference operator (2.16) is symmetric with respect to the weight matrix W (2.15).

We now use Lemma 1.3.3 to see that the matrix orthogonal polynomials with respect to W are common eigenfunctions of the difference operator (2.16). Since we have already proved the symmetry of D, we have just to see that for any polynomial P, $D(P)$ has degree at most the degree of P. This is a consequence of Lemma 1.3.15 since

$$\begin{aligned}
&\operatorname{dgr}(F_i) \leq 2, \quad \text{for } i = 0, 1, -1,\\
&\operatorname{dgr}(F_1 - F_{-1}) = \operatorname{dgr}\left(a(x+c)R_{A,a} - R_{A,a}^{-1}x\right) = 1,\\
&\operatorname{dgr}(F_0 + F_1 + F_{-1}) = \operatorname{dgr}((a-1)J + acR_{A,a}) = 0.
\end{aligned}$$

The expression for the eigenvalues follows by Theorem 1.3.17.

The proof of the theorem is now complete.

□

2.2.3 Example with $\operatorname{dgr}(F_1) = 1$ and finite support

Our third example can be considered as a matrix relative of the Krawtchouk scalar weight. It has then finite support and the coefficient F_1 of its associated second order difference operator is a polynomial of degree 1 (the proof will be omitted because is similar to that of theorems 2.2.1 and 2.2.3).

Theorem 2.2.4. *Let A and J be the $N \times N$ matrices given by (1.25). Given a positive integer κ and a positive real number a, we consider the matrix*

$$R_{A,a} = (I+A)(I-aA)^{-1}, \tag{2.22}$$

and the matrix weight defined by

$$W = \sum_{x=0}^{\kappa} \frac{a^x \Gamma(\kappa+1)}{\Gamma(\kappa+1-x)x!} R_{A,a}^x \left(R_{A,a}^*\right)^x \delta_x. \tag{2.23}$$

Then, the second order difference operator

$$D(\cdot) = \mathfrak{s}_1(\cdot)F_1(x) + \mathfrak{s}_0(\cdot)F_0(x) + \mathfrak{s}_{-1}(\cdot)F_{-1}(x), \tag{2.24}$$

where

$$\begin{aligned} F_{-1}(x) &= R_{A,a}^{-1}x, \\ F_0(x) &= -(a+1)J + aR_{A,a}x - R_{A,a}^{-1}x, \\ F_1(x) &= -aR_{A,a}x + a\kappa R_{A,a}, \end{aligned}$$

is symmetric with respect to W. Moreover, the monic orthogonal polynomials with respect to W are common eigenfunctions of D and its eigenvalues are given by

$$\Lambda_n = -(a+1)J + a\kappa R_{A,a} - n\left(aR_{A,a} + R_{A,a}^{-1}\right).$$

2.2.4 Example with $\mathrm{dgr}(F_1) = 2$

Our third example seems not to have any scalar relative. The weight matrix has bounded support and the difference coefficient F_1 has degree two.

For a positive integer κ and complex numbers $v_i \in \mathbb{C}$, $1 \leq i \leq N-1$, we define the nilpotent matrices (with order of nilpotency 2 and $N-1$ respectively)

$$B = \frac{-2v_1}{\kappa+1}\mathcal{E}_{1,2}, \tag{2.25}$$

$$C = v_2\mathcal{E}_{1,3} + \sum_{\substack{i=1 \\ i\neq 2}}^{N-1} v_i\mathcal{E}_{i,i+1}. \tag{2.26}$$

We also define the diagonal matrix

$$L = \mathcal{E}_{1,1} + 5\mathcal{E}_{2,2} + \sum_{i=3}^{N}(2i-3)\mathcal{E}_{i,i}. \tag{2.27}$$

It is a matter of computation to check that these matrices satisfy the following equations

$$B^2 = 0, \tag{2.28}$$

$$BC = CB = 0, \tag{2.29}$$

$$\mathrm{ad}_C\, L = 2C - (\kappa+1)B, \tag{2.30}$$

$$\mathrm{ad}_{C^k}\, L = 2kC^k, \quad \text{for } k \geq 2, \tag{2.31}$$

$$\mathrm{ad}_B\, L = 4B. \tag{2.32}$$

We are now ready to introduce our last example.

Theorem 2.2.5. *Let B, C and L be the matrices defined by (2.25), (2.26) and (2.27) respectively. We consider the matrix*

$$R_C = \left(I - \frac{1}{2}C\right)\left(I + \frac{1}{2}C\right)^{-1} \tag{2.33}$$

and the weight matrix defined by

$$W = \sum_{x=0}^{\kappa} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-x)x!} \left(R_C^x - B\frac{x(x+1)}{2} \right) \left((R_C^*)^x - B^* \frac{x(x+1)}{2} \right) \delta_x. \tag{2.34}$$

Then, the second order difference operator

$$D(\cdot) = \mathfrak{s}_1(\cdot)F_1(x) + \mathfrak{s}_0(\cdot)F_0(x) + \mathfrak{s}_{-1}(\cdot)F_{-1}(x), \tag{2.35}$$

where

$$F_1 = Bx^2 + ((1-\kappa)B - R_C)\,x + \kappa\,(R_C - B)\,, \tag{2.36}$$

$$F_0 = -2Bx^2 - \left(R_C^{-1} + (1-\kappa)B - R_C\right)x + L, \tag{2.37}$$

$$F_{-1} = Bx^2 + R_C^{-1}x, \tag{2.38}$$

is symmetric with respect to the weight matrix W (2.34). Moreover, the monic orthogonal polynomials with respect to W are common eigenfunctions of D and its eigenvalues are given by

$$\Lambda_n = L + \kappa\,(R_C - B) + n\left((1-\kappa)B - R_C - R_C^{-1}\right) + n(n-1)B.$$

Proof. We use Theorem 1.3.19 for $r = 1$ to prove the symmetry of D with respect to W. For this we have to check the boundary conditions, $F_1(\kappa)W(\kappa) = 0$ and $W(0)F_{-1}^*(0) = 0$ and the equations

$$F_1(x-1)W(x-1) = W(x)F_{-1}^*(x), \quad \text{for } x = 1, \cdots, \kappa,$$
$$F_0(x)W(x) = W(x)F_0^*(x), \quad \text{for } x = 0, 1, \cdots, \kappa.$$

We skip over the proof of the boundary conditions and the first symmetry equation since it is similar to that for Theorem 2.2.1 (taking now $s(x) = \sqrt{x(\kappa+1-x)}$).

We write $W = \sum_{x=0}^{\kappa} T(x)T^*(x)\delta_x$ for

$$T(x) = \sqrt{\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-x)x!}} \left(R_C^x - B\frac{x(x+1)}{2} \right).$$

Then, the equation $F_0(x)W(x) = W(x)F_0^*(x)$ is equivalent to the Hermiticity of the matrix function

$$\chi(x) = T^{-1}(x)F_0(x)T(x).$$

We again compute explicitly the function χ to see that it is in fact a real diagonal matrix. Consider the analytic function at 0

$$f(z) = \log\left(\frac{1+\frac{z}{2}}{1-\frac{z}{2}} \right), \tag{2.39}$$

and write $S = f(C)$. This gives $R_C^{-1} = e^S$ and $R_C = e^{-S}$. The function $f(C)$ only contains odd powers of C, then by the property (2.29) we get that $SB = BS = 0$, and so

$$e^{Sx}B = Be^{Sx} = B = e^{-Sx}B = Be^{-Sx}.$$

Taking into account this remark as well as the property (2.28) we can write

$$\begin{aligned} T^{-1}(x) &= \sqrt{\frac{\Gamma(\kappa+1-x)x!}{\Gamma(\kappa+1)}}\left(e^{-Sx} - B\frac{x(x+1)}{2}\right)^{-1} \\ &= \sqrt{\frac{\Gamma(\kappa+1-x)x!}{\Gamma(\kappa+1)}}\left(\left(I - B\frac{x(x+1)}{2}\right)e^{-Sx}\right)^{-1} \\ &= \sqrt{\frac{\Gamma(\kappa+1-x)x!}{\Gamma(\kappa+1)}}\left(I + B\frac{x(x+1)}{2}\right)e^{Sx}. \end{aligned}$$

Because of (2.28) and (2.32), we have $\operatorname{ad}_B^i L = 0$ for $i \geq 2$. Taking into account all these properties we can write $\chi(x)$ as the power series

$$\begin{aligned} \chi(x) = L &+ \left((\kappa-1)B + e^{-S} - e^{S} + \operatorname{ad}_S L + \frac{1}{2}\operatorname{ad}_B L\right)x \\ &+ \left(-2B + \frac{1}{2}\operatorname{ad}_B\left(L + \operatorname{ad}_S L\right) + \frac{1}{2}\operatorname{ad}_S^2 L\right)x^2 \\ &+ \sum_{i=3}^{\infty}\left(\frac{1}{2}\operatorname{ad}_B\left(\frac{\operatorname{ad}_S^{i-1} L}{(i-1)!} + \frac{\operatorname{ad}_S^{i-2} L}{(i-2)!}\right) + \frac{\operatorname{ad}_S^{i} L}{i!}\right)x^i. \end{aligned} \tag{2.40}$$

We now prove that, except the first one, all the coefficients above vanish. To do that, we use the following lemma.

Lemma 2.2.6. *Let B, C and L be the matrices defined by (2.25), (2.26) and (2.27). For an analytic function f at 0 the following equations hold*

$$((\kappa+1)f'(0) - 2)B + \operatorname{ad}_{f(C)} L + \frac{1}{2}\operatorname{ad}_B L = 2Cf'(C), \tag{2.41}$$

$$\operatorname{ad}_{f(C)}^{i+1} L = 0, \quad \text{for } i \geq 1, \tag{2.42}$$

$$\operatorname{ad}_B\left(\operatorname{ad}_{f(C)} L\right) = 0. \tag{2.43}$$

Proof. Write $f(z) = \sum_{k\geq 0} a_k z^k$. Taking into account equations (2.30) and (2.31), we get

$$\operatorname{ad}_{f(C)} L = \sum_{k\geq 0} a_k\left[C^k, L\right] = 2\sum_{k\geq 0} a_k k C^k - (\kappa+1)a_1 B = 2Cf'(C) - (\kappa+1)f'(0)B,$$

and then, using equation (2.32) we get

$$\begin{aligned} ((\kappa+1)f'(0) - 2)B + \operatorname{ad}_{f(C)} L + \frac{1}{2}\operatorname{ad}_B L =& ((\kappa+1)f'(0) - 2)B + 2Cf'(C) \\ & - (\kappa+1)f'(0)B + 2B = 2Cf'(C). \end{aligned}$$

Since $BC = CB = 0$, any power of C commute with B then, we have

$$\begin{aligned} \operatorname{ad}_{f(C)}^{i+1} L &= \operatorname{ad}_{f(C)}^{i}\left(2Cf'(C) - (\kappa+1)a_1 B\right) = 0, \quad \text{for } i \geq 1, \\ \operatorname{ad}_B\left(\operatorname{ad}_{f(C)} L\right) &= \operatorname{ad}_B\left(2Cf'(C) - (\kappa+1)a_1 B\right) = 0. \end{aligned}$$

□

Using now equation (2.41) of Lemma 2.2.6 for the function f in (2.39) (for which $f'(0)=1$) and $S=f(C)$, we get

$$
\begin{aligned}
(\kappa-1)B+e^{-S}-e^{S}+\operatorname{ad}_S L+\frac{1}{2}\operatorname{ad}_B L &= e^{-S}-e^{S}+2Cf'(C)\\
&= e^{-S}-e^{S}+\left(\left(I+\frac{C}{2}\right)^{-1}+\left(I-\frac{C}{2}\right)^{-1}\right)\\
&= 0.
\end{aligned}
\tag{2.44}
$$

On the other hand, using equations (2.42) and (2.43) of Lemma 2.2.6, we obtain

$$-2B+\frac{1}{2}\operatorname{ad}_B(L+\operatorname{ad}_S L)+\frac{1}{2}\operatorname{ad}_S^2 L=0, \tag{2.45}$$

$$\frac{1}{2}\operatorname{ad}_B\left(\frac{\operatorname{ad}_S^{i-1}L}{(i-1)!}+\frac{\operatorname{ad}_S^{i-2}L}{(i-2)!}\right)+\frac{\operatorname{ad}_S^{i}L}{i!}=0,\ \ i\geq 3, \tag{2.46}$$

where we have also used the equation (2.32).

Equations (2.40), (2.44), (2.45) and (2.46) then give that $\chi(x)=L$, which it is a real diagonal matrix. So $F_0(x)W(x)=W(x)F_0(x)$ for $x=0,1,\cdots,\kappa$.
By Theorem 1.3.19 for $r=1$, we conclude that D is symmetric with respect to W.

To see that the orthogonal polynomials with respect to W are common eigenfunctions of D, we use Theorem 1.3.3. Since we have already proved that D is symmetric with respect to W, it remains to see that for any polynomial P, $D(P)$ is a polynomial with degree at most the degree of P. This is a consequence of Lemma 1.3.15, since

$$
\begin{aligned}
\operatorname{dgr}(F_i)&\leq 2 \quad \text{for } i=-1,0,1,\\
\operatorname{dgr}(F_1-F_{-1})&=\operatorname{dgr}\left(\left((1-\kappa)B-e^{-S}-e^{S}\right)x+\kappa\left(e^{-S}-B\right)\right)=1,\\
\operatorname{dgr}(F_0+F_1+F_{-1})&=\operatorname{dgr}(L+\kappa\left(e^{-S}-B\right))=0.
\end{aligned}
$$

Finally to obtain the expression for the eigenvalues we apply Theorem 1.3.17

□

Chapter 3

The convex cone associated to a difference operator

It has already been mentioned that in the matrix case the family of orthogonal polynomials can be eigenfunctions of several matrix difference operators, and the set of all these different operators form an algebra, $\mathcal{D}(W)$. In this chapter we show what one can call the dual situation of this fact. For a fixed difference operator D of the form

$$D(\cdot) = \sum_{k=-r}^{r} \mathfrak{s}_k(\cdot) F_k(x), \quad F_k(x) \in \mathbb{C}_k^{N\times N}[x], \tag{3.1}$$

we define the set of weight matrices

$$\Upsilon(D) = \{W : D \text{ is symmetric with respect to } W\}.$$

One straightforwardly has that, if $\Upsilon(D) \neq \varnothing$ then, it is a convex cone i.e., if $W_1, W_2 \in \Upsilon(D)$ and $\gamma, \zeta \geq 0$ (one of them non null) then, $\gamma W_1 + \zeta W_2 \in \Upsilon(D)$.

When $\Upsilon(D) \neq \varnothing$, it contains, at least, a half line: γW, $\gamma > 0$. In the scalar case, the convex cone of positive measures associated to a second order difference operator always reduces to the empty set except for those operators associated to the classical discrete measures in which case the convex cone is the half line defined by the classical measure itself. The situation is again rather different in the matrix orthogonality setup. The purpose of this chapter is to show examples of second order difference operators D for which $\Upsilon(D)$ is at least a two dimensional convex cone.

We provide two methods to find such examples (Section 3.1) and show a collection of instructive examples (Section 3.2). We remark that the convex cones generated by both methods have a completely different structure.

For similar results for the convex cone of weight matrices associated to a differential operator see [36]

3.1 Some necessary and/or sufficient conditions for symmetry

In this section we prove some necessary and/or sufficient conditions for the symmetry of a difference operator with respect to a weight matrix. These conditions will allow us to develop the two methods for studying the convex cone of weight matrices associated to a difference operator reported in the previous paragraph.

The first method is based on the method introduced in Chapter 2 to construct pairs of weights W and difference operators D satisfying the symmetry equations (1.47).

Lemma 3.1.1. *Let κ be either a positive integer or infinite, and consider the weight matrix*

$$W = \sum_{x=0}^{\kappa} T(x)T^*(x)\delta_x,$$

where T is certain matrix function satisfying that $T(0) = I$. Let D be the second order difference operator

$$D = \mathfrak{s}_{-1}F_{-1}(x) + \mathfrak{s}_0 F_0(x) + \mathfrak{s}_1 F_1(x),$$

where F_{-1} and F_1 satisfy that $F_{-1}(0) = 0$ and if κ is finite, $F_1(\kappa) = 0$. Assume that there exists a scalar function $s(x)$ such that for $x = 1, \cdots, \kappa$, $s(x) \neq 0$ and

$$F_1(x-1)F_{-1}(x) = |s(x)|^2 I, \quad x \in \{1, \cdots, \kappa\},$$

$$T(x-1) = \frac{F_{-1}(x)}{s(x)}T(x), \quad \text{for } x \in \{1, \cdots, \kappa\},$$

$$T^{-1}(x)F_0(x)T(x) \quad \textit{is diagonal for all } x = 0, \cdots, \kappa.$$

Then, for any diagonal matrix of numbers S with positive entries, the difference operator D is symmetric with respect to the matrix weight

$$W_S = \sum_{x=0}^{\kappa} T(x)ST^*(x)\delta_x.$$

Proof. In order to prove the symmetry of D with respect to W_S, we use Theorem 1.3.19 for $r = 1$. It is clear that the boundary conditions (1.48) and (1.49) hold since $F_{-1}(0) = 0$ and if κ is finite $F_1(\kappa) = 0$. By Theorem 2.1.1 it is clear that the first symmetry equation holds.

Now for the weight matrix $W_S = \sum_{x=0}^{\kappa} T(x)ST^*(x)\delta_x$, the symmetry equation (1.47) for $l = 0$ holds if and only if the matrix $T^{-1}(x)F_0(x)T(x)S$ is Hermitian for all x in the support. But this is a diagonal matrix, since by hypothesis $T^{-1}(x)F_0(x)T(x)$ is diagonal. We have then that for any diagonal matrix S with positive entries, the second order difference operator D is symmetric with respect to W_S. □

The assumptions in the previous Lemma are not so restrictive as at a first glance appeared to be. In fact, all the examples of weight matrices constructed in Chapter 2 satisfy those assumptions.

The next lemma provides a second method to construct convex cones associated to a difference operator. Given a difference operator D, we show how to choose a real number x_0 and a positive semidefinite matrix $M(x_0)$ such that the difference operator D is also symmetric with respect to all the weight matrices of the form $\gamma W + \zeta M(x_0)\delta_{x_0}$, $\gamma > 0, \zeta \geq 0$.

In the next Section, we will use this method to study the convex hull of two second order difference operators. In one case the real number x_0 belongs to the support of W (Theorem 3.2.5) and in the other it does not (Theorem 3.2.4).

Lemma 3.1.2. *Let D be a difference operator like (3.1) with $s = -r$. Assume that for $x_0 \in \mathbb{R}$, there exists an Hermitian matrix M_{x_0} such that*

$$F_l(x_0)M_{x_0} = 0, \quad l \neq 0, -r \leq l \leq r, \qquad F_0(x_0)M_{x_0} = M^*_{x_0}F^*_0(x_0). \tag{3.2}$$

If D is symmetric with respect to W then, for all real numbers $\gamma > 0, \xi \geq 0$, D is also symmetric with respect to the weight matrix $\gamma W + \xi M_{x_0}\delta_{x_0}$.

Proof. Notice that (3.2) easily implies that

$$D(P(x_0))M_{x_0}Q^*(x_0) = P(x_0)M_{x_0}D(Q(x_0))^*.$$

Hence, the inner product defined by $\widehat{W} = \gamma W + \xi M_{x_0}\delta_{x_0}$ satisfies that

$$\begin{aligned}\langle D(P), Q\rangle_{\widehat{W}} &= \gamma\langle D(P), Q\rangle_W + \xi D(P(x_0))M_{x_0}Q^*(x_0)\\ &= \gamma\langle P, D(Q)\rangle_W + \xi P(x_0)M_{x_0}D(Q(x_0))^* = \langle P, D(Q)\rangle_{\widehat{W}},\end{aligned}$$

where we have used the symmetry of D with respect to W. This shows that D is also symmetric with respect to $\widehat{W}$. □

The above Lemmas in this Section provide two methods for constructing weight matrices in the convex cone $\Upsilon(D)$ of a second order difference operator D. In some of the examples in the next Section we will prove that both methods produce all the weight matrices in the convex cone $\Upsilon(D)$. The key to prove that is the next lemma where we state necessary conditions for the symmetry of a second order difference operator with respect to a weight matrix in terms of the (generalized) moments of the weight matrix.

Lemma 3.1.3. *Let W be a weight matrix and write μ_n for its generalized moments $\mu_n = \int \binom{x}{n} dW(x)$, $n \geq 0$. Consider the second order difference operator $D = \mathfrak{s}_{-1}F_{-1} + \mathfrak{s}_0F_0 + \mathfrak{s}_1F_1$. Write*

$$\begin{aligned}F_{-1}(x) &= F_{-1,2}\binom{x}{2} + F_{-1,1}\binom{x}{1} + F_{-1,0}, \quad F_1(x) = F_{1,2}\binom{x}{2} + F_{1,1}\binom{x}{1} + F_{1,0},\\ G_0 &= F_{-1} + F_0 + F_1.\end{aligned}$$

If D is symmetric with respect to W and the degree of $D(P)$ is at most the degree of P then,

$$G_0\mu_0 = \mu_0G^*_0, \tag{3.3}$$

and for $n \geq 1$

$$\begin{aligned}&\left[\binom{n}{2}F_{1,2} + \binom{n}{1}F_{1,1} - \binom{n}{1}F_{-1,1} + G_0\right]\mu_n\\ &+\left[\binom{n-1}{2}F_{1,2} + \binom{n-1}{1}F_{1,1} + F_{1,0}\right]\mu_{n-1} - F_{-1,0}\sum_{j=0}^{n-1}(-1)^{j+n-1}\mu_j = \mu_nG^*_0.\end{aligned} \tag{3.4}$$

Proof. Using the basic identities $\binom{x}{n} = \binom{x-1}{n} + \binom{x-1}{n-1}$, and

$$\binom{x-j}{l}\binom{x}{j} = \binom{j+l}{l}\binom{x}{j+l}, \quad \binom{x-1}{n-1} = \sum_{j=0}^{n-1}(-1)^{j+n-1}\binom{x}{j},$$

we get

$$\begin{aligned} D\binom{x}{n} = &\binom{n+1}{2}[F_{1,2} - F_{-1,2}]\binom{x}{n+1} \\ &+ \left[2\binom{n}{2}F_{1,2} + \binom{n}{1}F_{1,1} - \binom{n}{2}F_{-1,2} - \binom{n}{1}F_{-1,1} + G_0\right]\binom{x}{n} \\ &+ \left[\binom{n-1}{2}F_{1,2} + \binom{n-1}{1}F_{1,1} + F_{1,0}\right]\binom{x}{n-1} - F_{-1,0}\sum_{j=0}^{n-1}(-1)^{j+n-1}\binom{x}{j}. \end{aligned}$$

Since the degree of $D(P)$ is at most the degree of P, we have that $F_{-1} - F_1$ and G_0 are polynomials of degree at most 1 and 0 respectively (see Lemma 1.3.15). Hence $F_{1,2} - F_{-1,2} = 0$ and $D(1) = G_0$. Then,

$$\begin{aligned} D\binom{x}{n} = &\left[\binom{n}{2}F_{1,2} + \binom{n}{1}F_{1,1} - \binom{n}{1}F_{-1,1} + G_0\right]\binom{x}{n} \\ &+ \left[\binom{n-1}{2}F_{1,2} + \binom{n-1}{1}F_{1,1} + F_{1,0}\right]\binom{x}{n-1} - F_{-1,0}\sum_{j=0}^{n-1}(-1)^{j+n-1}\binom{x}{j}. \end{aligned}$$

Since D is symmetric, we have $\langle D\binom{x}{n}, 1\rangle = \langle \binom{x}{n}, D(1)\rangle$, $n \geq 0$, from where (3.3) and (3.4) are deduced. ☐

3.2 Examples

Using the methods explained in the previous Section, we now study some illustrative examples of convex cones of weight matrices associated to some second order difference operators.

3.2.1 Examples constructed by using the first method

This Section will be devoted to a couple of examples of convex cones constructed by using the first method mentioned in the Introduction (for more details see Lemma 2.1.1).

We first consider the second order difference operator

$$D(\cdot) = \mathfrak{s}_1(\cdot)\begin{pmatrix} -a & -ab \\ 0 & -a \end{pmatrix} + \mathfrak{s}_0(\cdot)\begin{pmatrix} x+1 & -bx \\ 0 & x \end{pmatrix} + \mathfrak{s}_{-1}\begin{pmatrix} -x & xb \\ 0 & -x \end{pmatrix}. \tag{3.5}$$

We will show that its convex cone $\Upsilon(D)$ is formed by the weight matrices γW_ξ, $\gamma, \xi > 0$, where

$$W_\xi = \sum_{x\in\mathbb{N}} \frac{a^x}{x!}\begin{pmatrix} 1 & bx \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ bx & 1 \end{pmatrix}\delta_x.$$

First of all, we stress that the perturbation produced by the matrix $\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}$ has a nonlinear effect on the orthogonal polynomials with respect W_ξ (this is due to the non commutativity of the matrix product). To evaluate this effect we next display an explicit expression of the orthogonal polynomials with respect to W_ξ.

Lemma 3.2.1. *For $a > 0$, let $(c_n^a)_n$ be the Charlier polynomials given by (2.2). Consider the matrices $Y_1(n, \xi)$, $Y_2(n)$ and $Y_3(n)$ defined by*

$$Y_1(n,\xi) = \begin{pmatrix} 0 & b^2an + \xi \\ 1 & -(n+a)b \end{pmatrix}, \quad Y_2(n) = \begin{pmatrix} nab & -nb^2a(a+n-1) \\ 0 & nab \end{pmatrix}, \tag{3.6}$$

$$Y_3(n) = \begin{pmatrix} 0 & n(n-1)a^2b^2 \\ 0 & 0 \end{pmatrix}. \tag{3.7}$$

Then, for $\xi > 0$, the polynomials $(P_{n,\xi})_n$ defined by

$$P_{n,\xi}(x) = c_n^a(x)Y_1(n,\xi) + c_{n-1}^a(x)Y_2(n) + c_{n-2}^a(x)Y_3(n),$$

are orthogonal with respect to the weight matrix W_ξ.

Proof. Notice first that the leading coefficient of $P_{n,\xi}$ is $(-1)^n Y_1(n,\xi)$ which is a non singular matrix.

We then have to check that the polynomials $P_{n,\xi}$, $n \geq 0$, satisfy the orthogonality condition $\langle P_{n,\xi}, x^m \rangle_{W_\xi} = 0$, for $m < n$. To do that, we rewrite the weight matrix W_ξ as $W_\xi(x) = \frac{a^x}{x!} R(x,\xi)$ with

$$R(x,\xi) = \begin{pmatrix} \xi + b^2x^2 & bx \\ bx & 1 \end{pmatrix} = x^2R_2 + xR_1 + R_0(\xi),$$

where

$$R_2 = \begin{pmatrix} b^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad R_0(\xi) = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} \tag{3.8}$$

Accordingly, the inner product $\langle P_{n,\xi}, x^m \rangle_{W_\xi}$ can be rewritten as

$$\begin{aligned} \langle P_{n,\xi}, x^m \rangle_{W_\xi} &= \sum_{k=0}^{\infty} \frac{a^x}{x!} P_{n,\xi}(x) R(x,\xi) x^m \\ &= \sum_{k=0}^{\infty} \frac{a^x}{x!} P_{n,\xi}(x) \left(x^2R_2 + xR_1 + R_0(\xi) \right) x^m. \end{aligned} \tag{3.9}$$

Using the three term recurrence relation for the Charlier polynomials [77], pp. 247-249, we get

$$\begin{aligned} x^2c_n^a(x) &= c_{n+1}^a(x) - (2n+2a+1)c_{n+1}^a(x) + ((n+1)a + (n+a)^2 + na)c_n^a(x) \\ &\quad - ((n+a)na + na(n+a-1))c_{n-1}^a(x) + na^2(n-1)c_{n-1}^a(x). \end{aligned}$$

By carrying a careful computation, using the previous recurrence relation for the Charlier polynomials, as well as the expression of $P_{n,\xi}$ in terms of (3.6), (3.7) and (3.8), one can see that

$$\begin{pmatrix} 1/\xi & 0 \\ 0 & 1 \end{pmatrix} P_{n,\xi}(x)(x^2R_2 + xR_1 + R_0(\xi))$$
$$= \begin{pmatrix} b(-c^a_{n+1}(x) + (n+a)c^a_n(x)) & c^a_n(x) \\ b^2c^a_{n+2}(x) - (b^2 + nb^2 + b^2a)c^a_{n+1}(x) + (nb^2a + \xi + b^2a)c^a_n(x) & -bc^a_{n+1}(x) \end{pmatrix}.$$

Since the Charlier polynomials c^a_n, $n \geq 0$, are orthogonal with respect to the measure $\rho = \sum_{x=0}^{\infty} \frac{a^x}{x!}\delta_x$, we get from (3.9) that $\langle P_{n,\xi}, x^m\rangle_{W_\xi} = 0$ for $m < n$. Hence the polynomials $P_{n,\xi}$, $n \geq 0$, are orthogonal with respect to W_ξ. □

We now prove that the family of weight matrices γW_ξ, $\gamma, \xi > 0$, form the convex cone of the difference operator D (3.5).

Theorem 3.2.2. *For real numbers a and b with $a > 0$ and $b \neq 0$, let D be the second order difference operator defined by (3.5). Then,*

$$\Upsilon(D) = \{\gamma W_\xi : \gamma, \xi > 0\}.$$

Proof. Write $T(x) = \sqrt{\frac{a^x}{x!}} \begin{pmatrix} 1 & bx \\ 0 & 1 \end{pmatrix}$. An easy calculation shows that

$$T(x-1) = \frac{-1}{\sqrt{ax}} F_{-1}(x)T(x), \quad T^{-1}(x)F_0(x)T(x) = \begin{pmatrix} x+1 & 0 \\ 0 & x \end{pmatrix}, \quad F_1(x-1)F_{-1}(x) = axI,$$

where F_{-1}, F_0 and F_1 are the coefficients of $\mathfrak{s}_{-1}$, $\mathfrak{s}_0$ and $\mathfrak{s}_1$ in the second order difference operator D (3.5). Lemma 3.1.1 gives then that $\gamma W_\xi \in \Upsilon(D)$, $\gamma, \xi > 0$.

We now prove that actually the family γW_ξ, $\gamma, \xi > 0$, provides all the weight matrices in the convex cone $\Upsilon(D)$, that is

$$\Upsilon(D) = \{\gamma W_\xi : \gamma, \xi > 0\}.$$

Lemma 3.1.3 gives that the generalized moments $\mu_n = \int \binom{x}{n} dU(x)$, $n \geq 0$, of each weight matrix $U \in \Upsilon(D)$ have to satisfy the moment equations (3.3) and (3.4). If we write

$$F_{-1}(x) = F_{-1,2}\binom{x}{2} + F_{-1,1}\binom{x}{1}, \quad F_1(x) = F_{1,2}\binom{x}{2} + F_{1,1}\binom{x}{1} + F_{1,0},$$
$$G_0 = F_{-1} + F_0 + F_1,$$

these moment equations are (take into account that $F_{-1,0} = 0$), for $n = 0$,

$$G_0\mu_0 = \mu_0 G_0^*, \tag{3.10}$$

and for $n \geq 1$

$$\left[\binom{n}{2}F_{1,2} + nF_{1,1} - nF_{-1,1} + G_0\right]\mu_n + \left[\binom{n-1}{2}F_{1,2} + (n-1)F_{1,1} + F_{1,0}\right]\mu_{n-1} = \mu_n G_0^*. \tag{3.11}$$

In our example we have that

$$\binom{n}{2}F_{1,2}+nF_{1,1}-nF_{-1,1}+G_0=\begin{pmatrix} n+1-a & -nb-ab \\ 0 & n-a\end{pmatrix},\quad G_0^*=\begin{pmatrix} 1-a & -ab \\ 0 & -a\end{pmatrix}.$$

For a fixed $n\geq 2$, the matrices $\binom{n}{2}F_{1,2}+nF_{1,1}-nF_{-1,1}+G_0$ and G_0^* do not share any eigenvalue. This implies that the equation (3.11) defines μ_n, $n\geq 2$, in an unique way from μ_0 and μ_1 (see [61], p. 225). But μ_1 has to be Hermitian, since U is a weight matrix, hence it is easy to see that equation (3.11) for $n=1$ also defines μ_1 in an unique way from μ_0.

It is just a matter of calculation to see that the set of solutions μ_0 of the equation (3.10) is formed by the matrices

$$\begin{pmatrix} \eta & ab\gamma \\ ab\gamma & \gamma\end{pmatrix},$$

where $\gamma>0$ and $\eta>\gamma a^2b^2$ (since we are assuming that μ_0 is the first moment of a weight matrix, and so it has to be a positive definite matrix). On the other hand, the first moment of γW_ξ, $\gamma>0,\xi\in\mathbb{R}$, is

$$\gamma e^a\begin{pmatrix} \xi+ab^2+a^2b^2 & ab \\ ab & 1\end{pmatrix}.$$

By choosing $\xi=\eta/\gamma-ab^2-a^2b^2$, we see that for each weight matrix $U\in\Upsilon(D)$ there exist $\gamma>0$ and $\xi\in\mathbb{R}$ such that U has the same first moment μ_0 as $\gamma e^{-a}W_\xi$. Hence, since the (generalized) moments μ_n, $n\geq 1$, of U and γW_ξ are uniquely defined from μ_0, we conclude that U and γW_ξ have the same generalized moments μ_n, $n\geq 0$. It is then clear that U and γW_ξ have also the same moments s_n, $n\geq 0$, where $s_n=\int x^n dU(x)$.

We now prove that actually $U=\gamma W_\xi$ and then, $\xi>0$ (since U is a weight matrix). This can be done by different approaches. One of them uses Fourier transform and it works as follows: consider the Fourier transform $\mathcal{F}(X)$ of a weight matrix X defined by $\mathcal{F}(X)(x)=\int_{\mathbb{R}} e^{itx}dX(t)$, $x\in\mathbb{R}$. $\mathcal{F}(X)$ is a matrix $\mathcal{C}^\infty$-function in $\mathbb{R}$ and

$$\left(\frac{d}{dx}\right)^n(\mathcal{F}(X)(x))=\int_{\mathbb{R}}(it)^n e^{itx}dX(t),\quad x\in\mathbb{R}.$$

This shows that the moments of X are just $(d/dx)^n(\mathcal{F}(X))(0)/i^n$, $n\geq 0$.

Since U and γW_ξ have the same moments s_n, $n\geq 0$, we have that the Fourier transforms $\mathcal{F}(U)$ of U and $\mathcal{F}(\gamma W_\xi)$ of γW_ξ have at 0 the same derivatives of any order. These derivatives are equal to $i^n s_n$, $n\geq 0$.

It is easy to see that the Fourier transform $\mathcal{F}(\gamma W_\xi)$ is actually a matrix entire function, and accordingly

$$\lim_{n\to\infty}|(s_n)_{i,j}|\frac{r^n}{n!}=0,$$

for all $r>0$, where $Y_{i,j}$ denotes the (i,j) entry of a matrix Y. Since s_n, $n\geq 0$, are the moments of the weight matrix U and the even moments s_{2n} of a weight matrix are positive definite matrices, we conclude that

$$\lim_{n\to\infty}\operatorname{Tr}(s_{2n})\frac{r^{2n}}{(2m)!}=0.$$

Since U and γW_ξ have the same moments, according to Lemma 2.1 of [36], there exists an entire function Φ such that $\Phi(x) = \mathcal{F}(U)(x)$, $x \in \mathbb{R}$. Hence, the corresponding Fourier transforms of U and γW_ξ are entire functions which have at 0 the same derivatives of any order. That is, the entire functions Φ and $\mathcal{F}(\gamma W_\xi)$ are equal and then, $\mathcal{F}(\gamma W_\xi)(x) = \mathcal{F}(U)(x)$, $x \in \mathbb{R}$. So $U = \gamma W_\xi$. □

As we wrote in the previous Section, the assumptions in the Lemma 3.1.1 are not so restrictive as at a first glance appeared to be. In fact, all the examples of weight matrices constructed in Chapter 2 satisfy that assumptions. For the sake of completeness, we display a second example (the proof is omitted), now in arbitrary size, using the same method and based again in an example from Chapter 2. In this case the convex cone $\Upsilon(D)$ is, at least, N-dimensional.

For this example we recover the matrices, A, J in (1.25) and introduce the $N \times N$ matrix $R_{A,a}$:

$$A = \sum_{i=1}^{N-1} v_i \mathcal{E}_{i,i+1}, \qquad J = \sum_{i=1}^{N} (N-i)\mathcal{E}_{i,i}, \qquad R_{A,a} = (I-A)(I-aA)^{-1}, \tag{3.12}$$

where symbol $\mathcal{E}_{i,j}$ stands for the $N \times N$ matrix with entry (i,j) equal to 1 and 0 otherwise, and $v_1, \cdots, v_{N-1}$ are complex numbers.

Theorem 3.2.3. *For real numbers a and c with $0 < a < 1$ and $c > 0$ let D be the second order difference operator defined by*

$$D(\cdot) = \mathfrak{s}_1(\cdot)F_1(x) + \mathfrak{s}_0(\cdot)F_0(x) + \mathfrak{s}_{-1}(\cdot)F_{-1}(x),$$

where

$$F_{-1}(x) = R_{A,a}^{-1}x, \qquad F_1(x) = a(x+c)R_{A,a},$$
$$F_0(x) = (a-1)(J-(N-1)I) - x(aR_{A,a} + R_{A,a}^{-1}).$$

Then, $\{\gamma W_{\xi_1,\cdots,\xi_{N-1}} : \gamma, \xi_i > 0, i = 1, \cdots, N-1\} \subseteq \Upsilon(D)$ where

$$W_{\xi_1,\cdots,\xi_{N-1}} = \frac{1}{\Gamma(c)} \sum_{x=0}^{\infty} \frac{a^x \Gamma(x+c)}{x!} R_{A,a}^x \begin{pmatrix} \xi_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \xi_{N-1} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \left(R_{A,a}^*\right)^x \delta_x.$$

3.2.2 Examples constructed by using the second method

We now use the second method to study the convex cone of some second order difference operators.

We first consider the second order difference operator

$$D(\cdot) = \mathfrak{s}_1(\cdot)F_1(x) + \mathfrak{s}_0(\cdot)F_0(x) + \mathfrak{s}_{-1}(\cdot)F_{-1}(x). \tag{3.13}$$

where F_{-1}, F_0, F_1 are the matrix polynomials given by

$$F_{-1}(x) = \begin{pmatrix} -b(x-1)(a-1) & ab^2x(x-1)(a-1) \\ -1 & xba \end{pmatrix},$$

$$F_0(x) = \begin{pmatrix} b(a(c+x)-x)-b^{-1} & -ax\left(b^2(c+2x)(a-1)-1\right) \\ 1 & -2\,xba \end{pmatrix},$$

$$F_1(x) = \begin{pmatrix} 0 & a(c+x)\left((x+1)b^2(a-1)-1\right) \\ 0 & b(c+x)a \end{pmatrix},$$

along with the weight matrix

$$W = \sum_{x=0}^{\infty} \frac{a^x \Gamma(x+c)}{x!} \begin{pmatrix} x+b^2(a-1)^2x^2 & b(a-1)x \\ b(a-1)x & 1 \end{pmatrix} \delta_x.$$

Theorem 3.2.4. *For real numbers a, b and c with $0 < a < 1, 0 < c$, $a \neq 1/2 + 1/(2b^2)$ and $b \neq 0$, let D be the second order difference operator defined by (3.13). Then,*

$$\Upsilon(D) = \{\gamma W + \xi \begin{pmatrix} a|b|c & -\operatorname{sign}(b) \\ -\operatorname{sign}(b) & 1/(a|b|c) \end{pmatrix} \delta_{-c} : \gamma > 0, \xi \geq 0\},$$

where $\operatorname{sign}(b)$ *stands for the sign of b.*

Proof. It is a matter of calculation to check that W and D satisfy the symmetry equations

$$F_1(x-l)W(x-l) = W(x)F_{-1}^*(x), \quad \text{for } x \geq 1$$
$$F_0(x)W(x) = W(x)F_0^*(x), \quad \text{for } x \geq 0,$$

as well as the boundary condition $W(0)F_{-1}^*(0) = 0$. By Theorem 1.3.19 we can conclude that $W \in \Upsilon(D)$.

To simplify the writing, we set

$$M = \begin{pmatrix} a|b|c & -\operatorname{sign}(b) \\ -\operatorname{sign}(b) & 1/(a|b|c) \end{pmatrix}.$$

For $x = -c$ we then, have

$$F_{-1}(-c)M = \begin{pmatrix} b(c+1)(a-1) & acb^2(c+1)(a-1) \\ 1 & -cba \end{pmatrix} \begin{pmatrix} a|b|c & -\operatorname{sign}(b) \\ -\operatorname{sign}(b) & 1/(a|b|c) \end{pmatrix} = 0,$$

$$F_1(-c)M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a|b|c & -\operatorname{sign}(b) \\ -\operatorname{sign}(b) & 1/(a|b|c) \end{pmatrix} = 0,$$

$$\begin{aligned} F_0(-c)M &= \begin{pmatrix} \frac{-1+cb^2}{b} & -ac\left(cb^2a-cb^2+1\right) \\ 1 & 2cba \end{pmatrix} \begin{pmatrix} a|b|c & -\operatorname{sign}(b) \\ -\operatorname{sign}(b) & 1/(a|b|c) \end{pmatrix} \\ &= \begin{pmatrix} \operatorname{sign}(b)c^2a^2b^2 & -a|b|c \\ -a|b|c & -\operatorname{sign}(b) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} MF_0^*(-c) &= \begin{pmatrix} a|b|c & -\operatorname{sign}(b) \\ -\operatorname{sign}(b) & 1/(a|b|c) \end{pmatrix} \begin{pmatrix} \frac{-1+cb^2}{b} & 1 \\ -ac\left(cb^2a-cb^2+1\right) & 2cba \end{pmatrix} \\ &= \begin{pmatrix} \operatorname{sign}(b)c^2a^2b^2 & -a|b|c \\ -a|b|c & -\operatorname{sign}(b) \end{pmatrix}. \end{aligned}$$

By Theorem 3.1.2, the weight matrix $\gamma W + \xi M\delta_{-c}$, $\gamma > 0, \xi \geq 0$, is also in the convex cone $\Upsilon(D)$ of D.

We now prove that actually this family of weight matrices provides all the weight matrices in the convex cone $\Upsilon(D)$, that is $\Upsilon(D) = \{\gamma W + \xi M\delta_{-c} : \gamma > 0, \xi \geq 0\}$. The proof is similar to that of Theorem 3.2.2, hence we only sketch it.

Lemma 3.1.3 gives that the generalized moments $\mu_n = \int \binom{x}{n} dU(x)$, $n \geq 0$, of each weight matrix $U \in \Upsilon(D)$ have to satisfy the moment equations (3.3) and (3.4). If we write

$$F_{-1}(x) = F_{-1,2}\binom{x}{2} + F_{-1,1}\binom{x}{1} + F_{-1,0}, \qquad F_1(x) = F_{1,2}\binom{x}{2} + F_{1,1}\binom{x}{1} + F_{1,0},$$
$$G_0 = F_{-1} + F_0 + F_1,$$

these moment equations are, for $n = 0$,

$$G_0\mu_0 = \mu_0 G_0^*, \tag{3.14}$$

and for $n \geq 1$

$$\left[\binom{n}{2}F_{1,2} + \binom{n}{1}F_{1,1} - \binom{n}{1}F_{-1,1} + G_0\right]\mu_n + \left[\binom{n-1}{2}F_{1,2} + \binom{n-1}{1}F_{1,1} + F_{1,0}\right]\mu_{n-1} - F_{-1,0}\sum_{j=0}^{n-1}(-1)^{j+n-1}\mu_j = \mu_n G_0^*. \tag{3.15}$$

In our example, $\binom{n}{2}F_{1,2} + nF_{1,1} - nF_{-1,1} + G_0$ and G_0 are upper triangular matrices with diagonal entries equal to

$$b(a-1)n + abc + ab - b - 1/b, \quad abc - ab, \quad abc + ab - b - 1/b, \quad abc,$$

respectively. Hence, for a fixed $n \geq 1$, and except for the values $a = 1/2 + 1/(2b^2)$ and $a = 1 + 1/((n+1)b^2)$, the matrices $\binom{n}{2}F_{1,2} + nF_{1,1} - nF_{-1,1} + G_0$ and G_0^* do not share any eigenvalue. We need not to consider these exceptional values: $a = 1 + 1/((n+1)b^2)$, since we assume $a < 1$, and $a \neq 1/2 + 1/(2b^2)$. Hence, equation (3.15) defines μ_n, $n \geq 1$, in an unique way from μ_0 (see [61], p. 225).

It is just a matter of calculation to see that the set of solutions μ_0 of the equation (3.14) is formed by the matrices

$$\begin{pmatrix} \eta & -abc\tau \\ -abc\tau & \tau \end{pmatrix},$$

where $\tau > 0$ and $\eta > \tau a^2b^2c^2$ (since we are assuming that μ_0 is the first moment of a weight matrix, and so it has to be a positive definite matrix). On the other hand, the first moment of $\gamma W + \xi M\delta_{-c}$, $\gamma, \xi \in \mathbb{R}$, is

$$\frac{\gamma\Gamma(c)}{(1-a)^c}\begin{pmatrix} \dfrac{ac(a^2b^2c - ab^2(c-1) - 1 - b^2)}{a-1} & -abc \\ -abc & 1 \end{pmatrix} + \xi M.$$

It is then easy to see that given τ and η there exist $\gamma, \xi \in \mathbb{R}$ such that U has the same first moment μ_0 as $\gamma W + \xi M\delta_{-c}$. Hence, since the generalized moments μ_n, $n \geq 1$, of U and

$\gamma W + \xi M\delta_{-c}$ are uniquely defined from μ_0, we conclude that U and γW_ξ have the same generalized moments μ_n, $n \geq 0$. It is then clear that U and $\gamma W + \xi M\delta_{-c}$ have also the same moments s_n, $n \geq 0$, where $s_n = \int x^n dU(x)$.

In order to prove that actually $U = \gamma W + \xi M\delta_{-c}$, we can use again the Fourier transform (as in the proof of Theorem 3.2.2), taking into account that $\mathcal{F}(U)$ and $\mathcal{F}(\gamma W + \xi M\delta_{-c})$ have at 0 the same derivatives of any order, that $\mathcal{F}(\gamma W + \xi M\delta_{-c})$ is an analytic function in the open half plane $\{z \in \mathbb{C} : \log a < \Im z\}$ and that 0 is contained in that half plane (since $0 < a < 1$). □

We finally display an example in arbitrary size $N \times N$.

Consider again the $N \times N$ matrices given by (1.25),

$$A = \sum_{i=1}^{N-1} v_i \mathcal{E}_{i,i+1}, \qquad J = \sum_{i=1}^{N} (N-i)\mathcal{E}_{i,i},$$

where the parameters v_i satisfy the following constrains: for $i = 1, \cdots, N-2$,

$$(N-i-1)a|v_i|^2|v_{N-1}|^2 + (N-1)|v_i|^2 - i(N-i)|v_{N-1}|^2 = 0. \tag{3.16}$$

Let W be the discrete weight matrix given by

$$W = \sum_{x=0}^{\infty} \frac{a^x}{x!}(I+A)^x(I+A^*)^x\delta_x. \tag{3.17}$$

It was shown in Theorem 2.2.1 that the difference operator

$$D_1(\cdot) = \mathfrak{s}_{-1}(\cdot)(I+A)^{-1}x + \mathfrak{s}_0(\cdot)(-J-(I+A)^{-1}x) + a\mathfrak{s}_1(\cdot)(I+A)$$

is symmetric with respect to the weight matrix (3.17). In [57, Theorem 3] it was proved that, under the constrains (3.16), the difference operator

$$D_2(\cdot) = \mathfrak{s}_{-1}(\cdot)H_{-1}(x) + \mathfrak{s}_0(\cdot)H_0(x) + \mathfrak{s}_1 H_1(x)$$

where

$$\begin{aligned} H_{-1}(x) &= \left[(I+A)^{-1} - I\right]x^2 + \left(\frac{N-1}{a|v_{N-1}|^2}I + J\right)x, \\ H_1(x) &= \left[(I+A)^{-1} - I\right]x^2 + \left(2J - NI - aA + (I+A)^{-1}\right)x \\ &\quad + a(I+A)\left(\frac{N-1}{a|v_{N-1}|^2}I + J\right)(I+A^*), \\ H_0(x) &= -H_{-1}(x) - H_1(x), \end{aligned}$$

is also symmetric with respect to W.

Theorem 3.2.5. *Let w be the non zero (row) vector in $\mathbb{C}^N$ whose entries satisfy*

$$w_{i+1} = \frac{(N-i)}{a\bar{v}_i}w_i, \quad w_1 = 1, \tag{3.18}$$

and write M_0 for the $N \times N$ positive semidefinite matrix $M_0 = w^* w$. For

$$\lambda = -\frac{(N-1)(1+a|v_{N-1}|^2)}{a|v_{N-1}|^2},$$

consider the operator $D_3 = \lambda D_1 + D_2$. Then,

$$\{\gamma W + \xi M_0 \delta_0 : \gamma > 0,\ \xi \geq 0\} \subseteq \Upsilon(D_3).$$

Proof. Since D_3 is a linear combination of two symmetric operators with respect to W, it is also symmetric with respect to W. Write

$$D_3(\cdot) = \mathfrak{s}_{-1}(\cdot)F_{-1}(x) + \mathfrak{s}_0(\cdot)F_0(x) + \mathfrak{s}_1 F_1(x).$$

By Lemma 3.1.2, if we prove that D_3 satisfies the conditions

$$F_1(0)M_0 = 0, \qquad M_0 F_{-1}^*(0) = 0, \qquad F_0(0)M_0 = M_0^* F_0^*(0), \tag{3.19}$$

then, we have that D_3 is symmetric with respect to the weight matrices $\gamma W + \xi M_0 \delta_0$, $\gamma > 0$ and $\xi \geq 0$, i.e. $\gamma W + \xi M_0 \delta_0 \in \Upsilon(D_3)$.

The coefficients of D_3 are given by

$$\begin{aligned}
F_1(x) =& \left[(I+A)^{-1} - I\right]x^2 + \left[2J - NI - aA + (I+A)^{-1}\right]x \\
&+ a\,(I+A)\left[\lambda I + \left(\frac{N-1}{a|v_{N-1}|^2} I + J\right)(I+A^*)\right], \\
F_{-1}(x) =& \left[(I+A)^{-1} - I\right]x^2 + \left[\frac{N-1}{a|v_{N-1}|^2} I + J + \lambda\,(I+A)^{-1}\right]x, \\
F_0(x) =& -F_{-1}(x) - F_1(x) - \lambda(J - a(I+A)).
\end{aligned}$$

It is obvious that the second condition in (3.19) holds since $F_{-1}(0) = 0$. To see that the first condition in (3.19) holds, we point out that under the constrains (3.16) we have

$$\begin{aligned}
F_1(0) =& a\,(I+A)\left[\lambda I + \left(\frac{N-1}{a|v_{N-1}|^2} I + J\right)(I+A^*)\right] \\
=& a\,(I+A)\left[-(N-1)I + J + \frac{N-1}{a|v_{N-1}|^2} A^* + JA^*\right] \\
=& a\,(I+A)\left[\sum_{i=1}^{N}(1-i)\mathcal{E}_{i,i} + \sum_{i=2}^{N}\left(\frac{(N-1)\bar{v}_{i-1}}{a|v_{N-1}|^2} + (N-i)\bar{v}_{i-1}\right)\mathcal{E}_{i,i-1}\right] \\
=& a\,(I+A)\left[\sum_{i=1}^{N}(1-i)\mathcal{E}_{i,i} + \sum_{i=2}^{N}\frac{(i-1)(N-i+1)}{a v_{i-1}}\mathcal{E}_{i,i-1}\right],
\end{aligned}$$

where the last equality holds because of the constrains on v_i, (3.16). Now it is straightforward to see that, because of (3.18), w^* is in the kernel of $F_1(0)$, and hence $F_1(0)M_0 = 0$.

For the third condition in (3.19), notice that

$$F_0(0)M_0 = (-F_1(0) - F_{-1}(0) - \lambda\,(J - a(I+A)))\,w^* w = -\lambda\,(J + \lambda a\,(I+A))\,w^* w.$$

And also $M_0F_0^*(0) = -\lambda w^*w\,(J - a\,(I + A^*))$. Then, to see that $F_0(0)M_0 = M_0^*F_0^*(0)$ it suffices to check the following equality, $(-J + aA)\,w^*w = w^*w\,(-J + aA^*)$. We write $w = \sum_{i=0}^{N} w_ie_i$, where $(e_i)_{i=1}^N$ is the canonical base of $\mathbb{C}^N$. Then, we have that

$$
\begin{aligned}
(-J + aA)M_0 &= \left(\sum_{i=1}^{N}(i - N)\mathcal{E}_{i,i} + a\sum_{i=1}^{N} v_i\mathcal{E}_{i,i+1}\right) w^*w \\
&= \left(\sum_{i=1}^{N}((i - N)\bar{w}_i + av_i\bar{w}_{i+1})e_i\right) w \\
&= \sum_{i,j=1}^{N} \left((i - N)\bar{w}_iw_j + av_i\bar{w}_{i+1}w_j\right)\mathcal{E}_{i,j}, \\
M_0(-J + aA^*) &= \sum_{i,j=1}^{N} \left((j - N)\bar{w}_iw_j + a\bar{v}_j\bar{w}_iw_{j+1}\right)\mathcal{E}_{i,j}.
\end{aligned}
$$

So we have that $(-J + aA)M_0 = M_0(-J + aA^*)$ if and only if

$$(i - N)\bar{w}_iw_j + av_i\bar{w}_{i+1}w_j = (j - N)\bar{w}_iw_j + a\bar{v}_j\bar{w}_iw_{j+1}.$$

By dividing both sides of the previous equality by $\bar{w}_iw_j$ we get that $(-J + aA)M_0 = M_0(-J + aA^*)$ if and only if

$$(i - N) + av_i\frac{\bar{w}_{i+1}}{\bar{w}_i} = (j - N) + a\bar{v}_j\frac{w_{j+1}}{w_j}.$$

But this is true since by (3.18) $\frac{\bar{w}_{i+1}}{\bar{w}_i} = \frac{(N - i)}{av_i}$, and then both sides of the previous equality are, indeed, 0. With this last equality it is proved the third condition in (3.19). □

Chapter 4

Matrix polynomials satisfying q-difference equations.

In the theory of scalar orthogonal polynomials, those families of orthogonal polynomials being solutions of a q-differential equation of the form (1.29)

$$\sigma(x)D_{q^{-1}}\left(D_q(p(x))\right) + \tau(x)D_q(p(x)) = \lambda p(x),$$

where σ and τ are polynomials and D_q is the q-differential operator (1.30), are called q-classical polynomials of the Hahn class (we will refer to them simply as q-classical polynomials).

The previous q-differential equation can be written in terms of q-shift operators, $\mathfrak{E}_l(f(x)) = f(q^l x)$ for $l \in \mathbb{Z}$, by

$$f_{-1}(x)\mathfrak{E}_{-1}p(x) + f_0(x)\mathfrak{E}_0 p(x) + f_1(x)\mathfrak{E}_1 p(x) = \lambda p(x), \tag{4.1}$$

where

$$f_{-1}(x) = -\frac{\sigma(x)}{(1-q^2)x^2} \quad f_0(x) = \frac{(1+q^{-1})\sigma(x)}{(1-q)(1-q^{-1})x^2} + \frac{\tau(x)}{(1-q)x},$$
$$f_1(x) = -\frac{\sigma(x)}{(1-q)(1-q^{-1})x^2} - \frac{\tau(x)}{(1-q)x}.$$

We consider here a matrix analogue of the q-difference operator, defined by

$$\mathfrak{E}_{-1}(\cdot)F_{-1}(x) + \mathfrak{E}_0(\cdot)F_0(x) + \mathfrak{E}_1(\cdot)F_1(x),$$

where $F_i(x)$ are matrix polynomials in x^{-1} such that $\operatorname{dgr}(F_i) \le 2$, $\operatorname{dgr}(qF_1 + F_0 + q^{-1}F_{-1}) \le 1$ and $\operatorname{dgr}(F_1 + F_0 + F_{-1}) = 0$.

Such a q-difference operator is symmetric with respect to a q-weight if it satisfies the symmetry equations

$$F_0(q^x)W(q^x) = W(q^x)F_0(q^x)^*, \quad x \in \mathbb{N}, \tag{4.2}$$

$$F_1(q^{x-1})W(q^{x-1}) = qW(q^x)F_{-1}(q^x)^*, \quad x \in \mathbb{N}\setminus\{0\}, \tag{4.3}$$

and the boundary conditions

$$W(1)F_{-1}(1)^* = 0, \tag{4.4}$$

$$q^{2x}F_1(q^x)W(q^x) \to 0, \quad \text{as } x \to \infty, \tag{4.5}$$

$$q^x\left(F_1(q^x)W(q^x) - W(q^x)F_1(q^x)^*\right) \to 0, \quad \text{as } x \to \infty. \tag{4.6}$$

In this chapter we introduce a method to solve the symmetry equation (4.3) for a second-order q-difference operator with respect to a q-weight

$$W = \sum_{x=0}^{\infty} q^x W(q^x)\delta_{q^x}. \tag{4.7}$$

We then construct a matrix relative of the little q-Jacobi polynomials and we explore some of their properties.

The following lemma give us a method to solve equation (1.53) in Theorem 1.3.21. We omit the proof for being completely analogous to that of Theorem 2.1.1.

Lemma 4.0.6. *Let $s(x)$ be a scalar function satisfying $s(q^x) \neq 0$ for $x \in \mathbb{N}\backslash\{0\}$. Assume that F_1 and F_{-1} are matrix-valued polynomials such that*

$$F_1(q^{x-1})F_{-1}(q^x) = q|s(q^x)|^2 I, \quad \forall x \in \mathbb{N}\backslash\{0\}. \tag{4.8}$$

Let T be a solution of the q-difference equation

$$T(q^{x-1}) = s(q^x)^{-1}F_{-1}(q^x)T(q^x), \quad x \in \mathbb{N}\backslash\{0\}, \quad T(1) = I. \tag{4.9}$$

Then, the q-weight defined by $W(q^x) = T(q^x)T(q^x)^*$ satisfies the symmetry equation

$$F_1(q^{x-1})W(q^{x-1}) = qW(q^x)F_{-1}(q^x)^*, \quad x \in \mathbb{N}\backslash\{0\}.$$

4.1 A matrix-valued relative of the little q-Jacobi polynomials

In this section we construct a family of matrix orthogonal polynomials that can be considered as a matrix relative of the little q-Jacobi polynomials.

For $\alpha \in \mathbb{R}\backslash\{0\}$ consider the matrix

$$R = \alpha A - ae^{\log(q)(J+A)} - e^{-\log(q)(J+A)}, \tag{4.10}$$

where A and J are the matrices given by (1.25). Notice that, since $-ae^{\log(q)(J+A)} - e^{-\log(q)(J+A)}$ has different eigenvalues, so does R, therefore it can be diagonalized. Let S be a matrix such that $S^{-1}RS$ is diagonal,

$$S^{-1}\left(\alpha A - ae^{\log(q)(J+A)} - e^{-\log(q)(J+A)}\right)S = -ae^{\log(q)J} - e^{-\log(q)J} = -aq^J - q^{-J}. \tag{4.11}$$

We set the matrices

$$M = S^{-1}(J+A)S, \quad L = q^M = S^{-1}q^{J+A}S, \quad U = \alpha S^{-1}AS. \tag{4.12}$$

Theorem 4.1.1. *Let $\alpha \in \mathbb{R}\backslash\{0\}$, $0 < a < q^{-1}$ and $b < q^{-1}$, and consider the matrices M, L and U given by (4.12). The q-difference operator*

$$D(\cdot) = \mathfrak{E}_{-1}(\cdot)\left(x^{-1} - 1\right)L^{-1} + \mathfrak{E}_0(\cdot)\left(U - x^{-1}(aL + L^{-1})\right) + \mathfrak{E}_1(\cdot)(ax^{-1} - abq)L \tag{4.13}$$

is symmetric with respect to the weight matrix W supported on $q^{\mathbb{N}}$ and given by

$$W(q^x) = \frac{a^x(bq;q)_x}{(q;q)_x} L^x (L^*)^x, \quad x \in \mathbb{N}. \tag{4.14}$$

Proof. To prove that (4.13) is symmetric with respect to (4.14) we apply Theorem 1.3.21 for $r = 1$. Then, it suffices to show that the symmetry equations (4.2), (4.3) and the boundary conditions (4.4), (4.5) and (4.6) hold. We proceed in three steps.

First Step. The symmetry equation $F_1(q^{x-1})W(q^{x-1}) = qW(q^x)F_{-1}(q^x)^*$ holds for $x \in \mathbb{N}$.

We use Lemma 4.0.6 to see that this q-difference equation holds. To do so notice that the coefficients $F_1 = (aq^{-x+1} - abq)L$ and $F_{-1} = (q^{-x} - 1)L^{-1}$ satisfy

$$F_1(q^{x-1})F_{-1}(q^x) = (aq^{-x+1} - abq)\,(q^{-x} - 1)\,LL^{-1} = \left|s(q^x)\right|^2 I, \quad x \geq 1,$$

and $s(q^x) = \sqrt{(aq^{-x+1} - abq)\,(q^{-x} - 1)} \neq 0$ for $x \geq 1$. On the other hand, the q-weight (4.14) can be factorized as

$$W(q^x) = T(q^x)T(q^x)^*, \quad T(q^x) = \sqrt{\frac{a^x(bq;q)_x}{(q;q)_x}} L^x, \tag{4.15}$$

and T satisfies $T(q^{x-1}) = \dfrac{\sqrt{q}}{s(q^x)}F_{-1}(q^x)T(q^x)$:

$$\begin{aligned}
\frac{\sqrt{q}}{s(q^x)}F_{-1}(q^x)T(q^x) &= \frac{\sqrt{q}(q^{-x} - 1)}{\sqrt{(aq^{1-x} - abq)(q^{-x} - 1)}} \sqrt{\frac{a^x(bq;q)_x}{(q;q)_x}} L^{-1}L^x \\
&= \sqrt{\frac{q(1 - q^x)}{qa(1 - bq^x)} \frac{a^x(bq;q)_x}{(q;q)_x}} L^{x-1} = \sqrt{\frac{a^{x-1}(bq;q)_{x-1}}{(q;q)_{x-1}}} L^{x-1} \\
&= T(q^{x-1}).
\end{aligned}$$

Then, Lemma 4.0.6 applies, and the symmetry equation

$$F_1(q^{x-1})W(q^{x-1}) = qW(q^x)F_{-1}(q^x)^*$$

holds.

Second step. The symmetry equation $F_0(q^x)W(q^x) = W(q^x)F_0(q^x)^*$ holds for $x \in \mathbb{N}$.

By using the factorization $W(q^x) = T(q^x)T(q^x)^*$ this symmetry equation is equivalent to the Hermiticity of the matrix function

$$T(q^x)^{-1}\left(U - q^{-x}(aL + L^{-1})\right)T(q^x), \quad \text{for all } x \in \mathbb{N}. \tag{4.16}$$

We see that this matrix is actually a real diagonal matrix for all $x \in \mathbb{N}$, hence it is Hermitian. Taking into account formulas (1.22), (4.15) and the definition of the matrix L (4.12), we can write

$$\begin{aligned}
&T(q^x)^{-1}\left(U - q^{-x}(aL + L^{-1})\right)T(q^x)\\
&= q^{-Mx}\left(U - q^{-x}(aS^{-1}q^{J+A}S + S^{-1}q^{-(J+A)}S)\right)q^{Mx}\\
&= \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}x^i\log(q)^i\left(\mathrm{Adj}_M^i U - aS^{-1}q^{J+A}S - S^{-1}q^{-(J+A)}S\right).
\end{aligned}$$

Notice that the matrices U and M given by (4.12) satisfy

$$[M,U] = \alpha S^{-1}(J+A)SS^{-1}AS - \alpha S^{-1}ASS^{-1}(J+A)S = \alpha S^{-1}\left(JA - AJ\right)S = \alpha S^{-1}AS = U.$$

Therefore $\mathrm{Adj}_M^i U = U$ for all $i \geq 0$ and then,

$$\begin{aligned}
T(q^x)^{-1}F_0(q^x)T(q^x) &= \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}x^i\log(q)^i\left(U - aS^{-1}q^{J+A}S - S^{-1}q^{-(J+A)}S\right)\\
&= \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}x^i\log(q)^iS^{-1}\left(\alpha A - aq^{(J+A)} - q^{-(J+A)}\right)S.
\end{aligned}$$

By (4.11) we get

$$T(q^x)^{-1}F_0(q^x)T(q^x) = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}x^i\log(q)^i\left(-aq^J - q^{-J}\right),$$

since all the parameters involved are real, this is an Hermitian matrix for all $x \geq 0$, therefore the second symmetry equation (4.3) holds.

Third Step. Boundary Conditions.

It just remains to check the boundary conditions

$$W(1)F_{-1}(1)^* = 0 \tag{4.17}$$

$$\lim_{x\to\infty} q^{2x}(F_1(q^x)W(q^x)) = 0 \tag{4.18}$$

$$\lim_{x\to\infty} q^{x}(F_1(q^x)W(q^x) - W(q^x)F_1(q^x)^*) = 0. \tag{4.19}$$

It is clear that the boundary condition (4.17) hold since $F_{-1}(x) = (x^{-1} - 1)L^{-1}$. To see that (4.18) is satisfied, we use the factorization of W (4.15). Write $W = \rho(q^x)L^x(L^x)^*$ where $\rho(q^x) = \dfrac{a^x(bq;q)_x}{(q;q)_x}$ is the q-weight associated to the little q-Jacobi polynomials, then, $q^xW(q^x) \to 0$ when $x \to \infty$, and we get

$$\lim_{x\to\infty} q^{2x}(F_1(q^x)W(q^x)) = \lim_{x\to\infty} q^{2x}(aq^{-x} - abq)LW(q^x) = 0.$$

For the last boundary condition (4.19) we see that

$$q^x\left(F_1(q^x)W(q^x) - W(q^x)F_1(q^x)^*\right) = q^x\frac{a^x(bq;q)_x}{(q;q)_x}K(q^x) = q^x\rho(q^x)K(q^x),$$

where $K(q^x)$ is a matrix polynomial in q^x. Then, since $\rho(q^x)$ is the scalar weight for the little q-Jacobi polynomials, we have that

$$\lim_{x\to\infty} q^x \rho(q^x) K(q^x) = 0.$$

Notice that for the matrices J and A (1.25), $J + A$ can be diagonalized and so does $S^{-1}(J + A)S$. Let S_1 be the matrix such that $(S_1)^{-1}S^{-1}(J + A)SS_1 = J$, then, since $L = S^{-1}e^{log(q)(J+A)}S$ we have $(S_1)^{-1}LS_1 = q^J$ and

$$\begin{aligned} L^{x+1}(L^*)^x - L^x(L^*)^{x+1} &= S_1\left(q^{J(x+1)}(S_1)^{-1}(S_1^*)^{-1}q^{Jx} - q^{J(x+1)}(S_1)^{-1}(S_1^*)^{-1}q^{Jx}\right)S_1^* \\ &= S_1 q^{Jx}\left(q^J(S_1)^{-1}(S_1^*)^{-1} - (S_1)^{-1}(S_1^*)^{-1}q^J\right)q^{Jx}S_1^*. \end{aligned}$$

We write

$$q^{Jx} = \begin{pmatrix} q^{(N-1)x} & & & \\ & q^{(N-2)x} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \tilde{Q}(x) + \mathcal{E}_{NN}, \tag{4.20}$$

where we recall that $\mathcal{E}_{i,j}$ is the matrix with 1 in its (i, j) entry and zero otherwise. Then,

$$\begin{aligned} L^{x+1}(L^*)^x - L^x(L^*)^{x+1} &= S_1(\tilde{Q}(x) + \mathcal{E}_{NN})\left(q^J(S_1)^{-1}(S_1^*)^{-1} - (S_1)^{-1}(S_1^*)^{-1}q^J\right)(\tilde{Q}(x) + \mathcal{E}_{NN})S_1^* \\ &= S_1\tilde{Q}(x)\left(q^J(S_1)^{-1}(S_1^*)^{-1} - (S_1)^{-1}(S_1^*)^{-1}q^J\right)\tilde{Q}(x)S_1^* + \Theta(x) \\ &= q^{2x}K_1(x) + \Theta(q^x), \end{aligned} \tag{4.21}$$

where $K_1(q^x)$ is a matrix polynomial in q^x and

$$\begin{aligned} \Theta(x) = {} & S_1\mathcal{E}_{NN}\left(q^J(S_1)^{-1}(S_1^*)^{-1} - (S_1)^{-1}(S_1^*)^{-1}q^J\right)\tilde{Q}(x)S_1^* \\ & + S_1\tilde{Q}(x)\left(q^J(S_1)^{-1}(S_1^*)^{-1} - (S_1)^{-1}(S_1^*)^{-1}q^J\right)S_1^*\mathcal{E}_{NN} \\ & + S_1\mathcal{E}_{NN}\left(q^J(S_1)^{-1}(S_1^*)^{-1} - (S_1)^{-1}(S_1^*)^{-1}q^J\right)\mathcal{E}_{NN}S_1^*. \end{aligned}$$

Notice that $\mathcal{E}_{NN}\left(q^J(S_1)^{-1}(S_1^*)^{-1} - (S_1)^{-1}(S_1^*)^{-1}q^J\right)\mathcal{E}_{NN} = 0$, therefore

$$\begin{aligned} \Theta(x) = {} & S_1\mathcal{E}_{NN}\left(q^J(S_1)^{-1}(S_1^*)^{-1} - (S_1)^{-1}(S_1^*)^{-1}q^J\right)\tilde{Q}(x)S_1^* \\ & + S_1\tilde{Q}(x)\left(q^J(S_1)^{-1}(S_1^*)^{-1} - (S_1)^{-1}(S_1^*)^{-1}q^J\right)S_1^*\mathcal{E}_{NN} \\ = {} & q^x K_2(q^x), \end{aligned} \tag{4.22}$$

where $K_2(q^x)$ is a matrix polynomial in q^x. Then, it follows from (4.15), (4.21) and (4.22) that

$$\begin{aligned} & \lim_{x\to\infty} q^x\left(F_1(q^x)W(q^x) - W(q^x)F_1(q^x)\right) \\ & = \lim_{x\to\infty} q^x(aq^{-x} - abq)\frac{a^x(bq;q)_x}{(q;q)_x}\left(L^{x+1}(L^*)^x - L^x(L^*)^{x+1}\right) \\ & = \lim_{x\to\infty} q^x(aq^{-x} - abq)\frac{a^x(bq;q)_x}{(q;q)_x}\left(q^{2x}K_1(q^x) + q^xK_2(q^x)\right) \\ & = \lim_{x\to\infty} q^x\frac{a^x(bq;q)_x}{(q;q)_x}(a - abq^{x+1})\left(q^xK_1(q^x) + K_2(q^x)\right) = 0, \end{aligned}$$

so the third boundary condition (4.19) also holds, and we have seen that Theorem 1.3.21 applies. Then, we can conclude that the q-difference operator given by (4.13) is symmetric with respect to the weight matrix W given by (4.14). □

Since the coefficients F_1, F_0 and F_{-1} satisfy the degree conditions given in Theorem 1.3.20, the q-difference operator D takes polynomials into polynomials and it does not raise the degree of polynomials. Then, we have the following corollary.

Corollary 4.1.2. *Let $(P_n)_n$ be the sequence of monic orthogonal polynomials with respect to the q-weight (4.14), then*

$$D(P_n) = \Lambda_n P_n, \quad \Lambda_n = -q^{-1}L^{-1} + U - abq^{n+1}L.$$

Observation 4.1.3. *The weight matrix presented in the previous theorem cannot be reduced to scalars. To see that, notice that from the definition of A, (1.25), for $v_1, \cdots, v_{N-1} \in \mathbb{C}$ not all of them zero, the matrix A is not normal, which implies that $W(q^x)W(q^y) \neq W(q^y)W(q^x)$ for all $y \neq x$.*

4.2 A matrix valued q-hypergeometric function

As in the continuous case, for certain vector-valued q-differential equation one can describe its analytic solutions around $x = 0$ in terms of a matrix analogue of basic hypergeometric functions. This can be used to describe the rows of the eigenfunctions of a matrix q-difference operator with diagonal eigenvalues.

In this section we introduce a matrix-valued basic hypergeometric function and show how this matrix function defines the analytic solutions of a q-difference equation. This work is based in the matrix-valued hypergeometric functions introduced by Tirao [98] and presented in the preliminaries. It is worth to mention that the matrix function that we introduce here is different form that introduced in [17], where some extra factorization assumption are required.

We start by defining the matrix basic hypergeometric function ${}_2\eta_1$.

Definition 4.2.1. *Let $A, B, C \in \mathbb{C}^{N\times N}$ where $\sigma(C) \cap q^{-\mathbb{N}\setminus\{0\}} = \varnothing$. Define*

$$(A, B; C; q)_0 = I,$$
$$(A, B; C; q)_k = (A, B; C; q)_{k-1}\left(I - q^{k-1}A - q^{2k-2}B\right)\left(I - q^kC\right)^{-1}, \qquad k \geq 1.$$

Define the function ${}_2\eta_1$ by

$$ {}_2G_1\left[\begin{matrix} A, B \\ C \end{matrix}; q, x\right] = \sum_{n=0}^{\infty} (A, B; C; q)_n \frac{x^n}{(q;q)_n}. \tag{4.23}$$

Theorem 4.2.2. *Let $A, B, C \in \mathbb{C}^{N\times N}$ such that $\sigma(C) \cap q^{-\mathbb{N}\setminus\{0\}} = \varnothing$. Then, for $Q^0 \in \mathbb{C}^N$, $Q^0 \neq 0$*

$$Q(x) = Q^0\, {}_2G_1\left[\begin{matrix} A, B \\ C \end{matrix}; q, qx\right] \tag{4.24}$$

is a solution of the vector-valued q-difference equation

$$Q(q^{-1}x)(1-x) + Q(x)\left(-I - C + xA\right) + Q(qx)\left(C + xB\right) = 0, \tag{4.25}$$

with initial condition $Q(0) = Q^0$. Conversely, any vector-valued analytic solution $Q(x)$ around 0 of (4.25) with initial condition $Q(0) \neq 0$ is of the form (4.24).

Proof. We will prove that any analytic solution of (4.25) is of the form (4.24).

Let $Q = \sum_{k=0}^{\infty} Q^k x^k$ be an analytic solution of (4.25) with $Q^0 \neq 0$. Applying the Frobenius method we get the following recursions formulas for the coefficients of Q:

$$\begin{aligned} Q^0 &= Q^0, \\ 0 &= Q^k\left((q^{-k}-1)I + (q^k-1)C\right) + Q^{k-1}\left(-q^{-k+1}I + A + q^{k-1}B\right), \quad k \geq 1. \end{aligned}$$

Then, we get

$$\begin{aligned} Q^k &= -Q^{k-1}\left(-q^{-k+1}I + A + q^{k-1}B\right)\left((q^{-k}-1)I + (q^k-1)C\right)^{-1} \\ &= Q^{k-1}q^{-k+1}\left(I - q^{k-1}A - q^{2k-2}B\right)\left((q^{-k}-1)I + (q^k-1)C\right)^{-1} \\ &= Q^{k-1}q\left(I - q^{k-1}A - q^{2k-2}B\right)\left((1-q^k)I - q^k(1-q^k)C\right)^{-1} \\ &= \frac{q}{(1-q^k)}Q^{k-1}\left(I - q^{k-1}A - q^{2k-2}B\right)\left(I - q^kC\right)^{-1}. \end{aligned}$$

By iterating this process we obtain

$$Q^k = \frac{q^k}{(q:q)_k}Q^0(A,B;C;q)_n,$$

and therefore, Q is given by 4.24. □

4.3 The 2×2 case in depth.

In this section we study in depth a family of matrix valued orthogonal polynomials introduced in Section 4.1 for the size $N = 2$. First, by exploiting the factorization of the weight matrix we can write the polynomials in terms of scalar little q-Jacobi polynomials. Then, we apply the results presented in the previous sections to write the orthogonal polynomials as basic hypergeometric functions. We also provide a Rodrigues formula as well as a three term recurrence relation for this family of polynomials.

The next theorem is just Theorem 4.1.1 for $N = 2$ and $\alpha = (1-q)(q^{-1}-a)$.

Theorem 4.3.1. *Assume a and b satisfy $0 < a < q^{-1}$ and $b < q^{-1}$. For $v \in \mathbb{C}$ define matrices*

$$U = \begin{pmatrix} 0 & v(1-q)(q^{-1}-a) \\ 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & v \\ 0 & 0 \end{pmatrix}, \qquad L = e^{\log(q)M} = \begin{pmatrix} q & -v(1-q) \\ 0 & 1 \end{pmatrix}. \tag{4.26}$$

The q-difference operator given by

$$D = \mathfrak{E}_{-1}F_{-1}(x) + \mathfrak{E}_0F_0(x) + \mathfrak{E}_1F_1(x), \tag{4.27}$$

$$F_{-1}(x) = (x^{-1}-1)L^{-1}, \quad F_0(x) = U - x^{-1}(L^{-1} + aL), \quad F_1(x) = (ax^{-1} - abq)L,$$

satisfies $DP_n = \Lambda_n P_n$, *where* $(P_n)_{n\geq 0}$ *are the monic polynomials orthogonal with respect to a weight matrix of the form (4.7) with*

$$W(q^x) = a^x \frac{(bq;q)_x}{(q;q)_x} L^x (L^*)^x, \tag{4.28}$$

and the eigenvalues are

$$\Lambda_n = \begin{pmatrix} -q^{-n-1} - abq^{n+2} & v(1-q)(abq^{n+1} - q^{1-n} + q^{-1} - a) \\ 0 & -q^{-n} - abq^{n+1} \end{pmatrix}. \tag{4.29}$$

4.3.1 Expression in terms of little q-Jacobi polynomials.

For $N = 2$ and $\alpha = (1-q)(q^{-1} - a)$ the weight function (4.14) takes the form

$$W(q^x) = \frac{a^x (bq;q)_x}{(q;q)_x} \begin{pmatrix} q^{2x} + |v|^2 (1-q^x)^2 & -v(1-q^x) \\ -\bar{v}(1-q^x) & 1 \end{pmatrix},$$

and by Theorem (4.3.1), the sequence of monic orthogonal polynomials $(\tilde{P}_n)_n$ with respect to W satisfy

$$\tilde{P}_n(q^{-1}x)L^{-1} + \tilde{P}_n(x)\left(U - x^{-1}(aL + L^{-1})\right) + (ax^{-1} - abq)\tilde{P}_n(qx)L = \Lambda_n \tilde{P}_n(x), \tag{4.30}$$

where

$$\tilde{\Lambda} n = \begin{pmatrix} -q^{-n-1} - abq^{n+2} & v(1-q)(abq^{n+1} - q^{1-n} + q^{-1} - a) \\ 0 & -q^{-n} - abq^{n+1} \end{pmatrix}. \tag{4.31}$$

The following theorem gives the explicit expression of the matrix orthogonal polynomials in terms of the little q-Jacobi polynomials.

Theorem 4.3.2. *The monic matrix-valued orthogonal polynomials are of the form*

$$\tilde{P}_n(x) = M_n^{-1} \begin{pmatrix} \kappa_{11}^n p_n(x;aq^2,b;q) & \kappa_{12}^n p_{n+1}(x;a,b;q) + \kappa_{11}^n (1-x) v p_n(x;aq^2,b;q) \\ \kappa_{21}^n p_{n-1}(x;aq^2,b;q) & \kappa_{22}^n p_n(x;a,b;q) + \kappa_{21}^n (1-x) v p_{n-1}(x;aq^2,b;q) \end{pmatrix},$$

where

$$M_n = \begin{pmatrix} 1 & -\dfrac{\mu_n}{1-abq^{2n+2}} v \\ 0 & 1 \end{pmatrix}, \tag{4.32}$$

$p_n(x;a,b;q)$ *are the little* q*-Jacobi polynomials*

$$p_n(x;a,b|q) = {}_2\phi_1 \left[\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} ; q, qx \right], \tag{4.33}$$

and the coefficients $\kappa_{i,j}^n$ *are given by*

$$\kappa_{11}^n = (-1)^n q^{\binom{n}{2}} \frac{(aq^3;q)_n}{(abq^{n+3};q)_n}, \qquad \kappa_{12}^n = (-1)^{n+1} v q^{\binom{n+1}{2}} \frac{(aq;q)_{n+1}}{(abq^{n+2};q)_{n+1}}, \tag{4.34}$$

$$\kappa_{21}^n = \xi_n a q^{-n+2} \frac{(1-q^n)(1-bq^n)}{(1-aq)(1-aq^2)} \overline{\kappa_{12}^{n-1}}, \qquad \kappa_{22}^n = \xi_n \frac{\kappa_{12}^{n-1}}{v},$$

where

$$\xi_n = \left(1 + aq|v|^2 \frac{(1-q^n)(1-bq^n)}{(1-abq^{n+1})(1-aq^{n+1})}\right)^{-1}.$$

Proof. Let $(P_n)_n$ be the sequence of orthogonal polynomials given by $P_n = M_n\tilde{P}_n$. Then,

$$D(P_n)) = M_n^{-1}\tilde{\Lambda}nM_nP_n = \Lambda_nP_n.$$

A straightforward computation gives that for M_n and $\tilde{\Lambda}_n$ given by (4.31), the following holds

$$\Lambda_n = M_n^{-1}\tilde{\Lambda}nM_n = \mathrm{diag}(-q^{-n-1} - abq^{n+2}, -q^{-n} - abq^{n+1}).$$

We now write

$$P_n(q^x) = M_n\tilde{P}_n(q^x) = \begin{pmatrix} \tilde{p}_{11}^n(q^x) & \tilde{p}_{12}^n(q^x) \\ \tilde{p}_{21}^n(q^x) & \tilde{p}_{22}^n(q^x), \end{pmatrix},$$

$$Q_n(q^x) = P_n(q^x)L^x = \begin{pmatrix} q^x\tilde{p}_{11}^n(q^x) & \tilde{p}_{12}^n - (1-q^x)v\tilde{p}_{11}^n(q^x) \\ q^x\tilde{p}_{21}^n(q^x) & \tilde{p}_{22}^n - (1-q^x)v\tilde{p}_{21}^n(q^x) \end{pmatrix} = \begin{pmatrix} r_{11}^n(q^x) & r_{12}^n(q^x) \\ r_{21}^n(q^x) & r_{22}^n(q^x) \end{pmatrix}. \tag{4.35}$$

Taking into account that the sequence of monic orthogonal polynomials, $(\tilde{P}_n)_n$ satisfies (4.30) and that $L^{-x}F_0(q^x)L^x$ is diagonal (see Theorem 4.1.1) we obtain

$$\begin{aligned}
(DP_n)(q^x)L^x =& P_n(q^{x-1})(q^{-1}x)L^{-1}L^x + P_n(q^x)L^xL^{-x}\left(U - x^{-1}(aL + L^{-1})\right)L^x \\
&+ (ax^{-1} - abq)P_n(qx)LL^x \\
=& (q^{-x} - 1)Q_n(q^{x-1}) + Q_n(q^x)q^{-x}\begin{pmatrix} -(q^{-1} + aq) & 0 \\ 0 & -(1+a) \end{pmatrix} \\
&+ (aq^{-x} - abq)Q_n(q^{x+1}) \\
=& \mathrm{diag}(-q^{-n-1} - abq^{n+2}, -q^{-n} - abq^{n+1})Q_n(q^x).
\end{aligned} \tag{4.36}$$

Since the eigenvalues, as well as all the matrix coefficients involved, are diagonal, (4.36) gives four uncoupled scalar-valued q-difference equations. These equations can be matched with the q-difference equation for the little q-Jacobi polynomials, giving

$$p_{11}^n = \kappa_{11}^n p_n(x; aq^2, b; q), \qquad p_{12}^n = \kappa_{12}^n p_{n-1}(x; aq^2, b; q),$$
$$r_{21}^n = \kappa_{21}^n p_{n+1}(x; aq^2, b; q), \qquad r_{22}^n = \kappa_{22}^n p_n(x; a, b; q),$$

where $p_n(x; a, b; q)$ are little q-Jacobi polynomials, and $\kappa_{ij} \in \mathbb{C}$ are constants that need to be determined. Then, we can write

$$P_n(x) = \begin{pmatrix} \kappa_{11}^n p_n(x; aq^2, b; q) & \kappa_{12}^n p_{n+1}(x; a, b; q) + \kappa_{11}^n(1-x)vp_n(x; aq^2, b; q) \\ \kappa_{21}^n p_{n-1}(x; aq^2, b; q) & \kappa_{22}^n p_n(x; a, b; q) + \kappa_{21}^n(1-x)vp_{n-1}(x; aq^2, b; q) \end{pmatrix}.$$

From the expression of the leading coefficient of P_n, M_n, the coefficients κ_{11}^n and κ_{12}^n are determined and one can easily see that they are given by (4.34). The expression of M_n also gives the relation

$$\kappa_{22}^n = q^{-n}\frac{(q, aq; q)_n}{(q^{-n}, abq^{n+1}; q)_n} - \kappa_{21}^n vq^{n-1}\frac{(1-aq)(1-aq^2)}{(1-abq^{n+1})(1-aq^{n+1})}. \tag{4.37}$$

By using the orthogonality of P_n we can determine completely κ_{21}^n and κ_{22}^n and they are given by (4.34). This completes the proof of the theorem. □

Corollary 4.3.3. *For the matrix-valued polynomials* $(P_n)_{n\geq 0}$ *with diagonal eigenvalues we have*

$$\langle P_m, P_n\rangle_W = H_n\delta_{m,n},$$

where H_n *is the diagonal matrix*

(4.38)
$$H_n = \mathrm{diag}(|\kappa_{11}^n|^2 h_n(aq^2,b;q) + |\kappa_{12}^n|^2 h_n(a,b;q), |\kappa_{21}^n|^2 h_n(aq^2,b;q) + |\kappa_{22}^n|^2 h_n(a,b;q)),$$

and $h_n(a,b;q) = \|p_n(a,b;q)\|^2$.

4.3.2 Expression as a matrix-valued q-hypergeometric function

Let $P_{n,i}$ be the i-th row of the matrix-valued polynomial P_i. Since the matrix polynomials P_n satisfy $D(P_n) = \Lambda_n P_n$ where D is given by (4.27) and Λ_n are diagonal matrices, this q-difference equation can be written as two decoupled row equations

$$DP_{n,i}(x) = P_{n,i}(q^{-1}x)F_{-1}(x) + P_{n,i}(x)F_0(x) + P_{n,i}(qx)F_1(x) = \lambda_{n,i}P_{n,i}, \tag{4.39}$$

where $i = 1,2$, $\lambda_{n,1} = -q^{-n-1} - abq^{n+2}$, $\lambda_{n,2} = -q^{-n} - abq^{n+1}$ and $P_{n,i}$ are the rows of the matrix polynomials P_n.

We rewrite (4.39) by multiplying on the right by xL, where L is given by (4.26), we get fro $i = 1,2$,

(4.40)
$$P_{n,i}(q^{-1}x)\,(1-x) + P_{n,i}(x)\left(x\,(U-\lambda_{n,i}I)\,L - \left(I + aL^2\right)\right) + P(qx)\left((a-abqx)L^2\right) = 0.$$

Proposition 4.3.4. *The vector-valued orthogonal polynomial solution of (4.40) is given by*

$$P_{n,i}(x) = P_{n,i}(0)\,{}_2G_1\left[\begin{matrix} UL - \lambda_{n,i}L, -abqL^2 \\ aL^2\end{matrix}; q, qx\right]. \tag{4.41}$$

Proof. Since $\sigma(aL^2)\cap q^{-\mathbb{N}\setminus\{0\}} = \{1,q^2\}\cap q^{-\mathbb{N}\setminus\{0\}} = \varnothing$, we apply Theorem 4.2.2 on (4.40) to get (4.41).

Notice that the initial values of the terminating series (4.41) can be determined by Theorem 4.3.2, and are given by

$$P_n(0) = \begin{pmatrix} \kappa_{11}^n & \kappa_{12}^n + \kappa_{11}^n v \\ \kappa_{21}^n & \kappa_{22}^n + \kappa_{21}^n v\end{pmatrix}.$$

□

4.3.3 The three term recurrence relation and the Rodrigues formula

In this section we compute the coefficients of the three term recurrence relation for $(P_n)_n$ and we show a Rodrigues formula for them.

Theorem 4.3.5. *The sequence of orthogonal polynomials $(P_n)_n$ satisfies the three term recurrence relation*

$$xP_n(x) = A_nP_{n+1}(x) + B_nP_n(x) + C_nP_{n-1}, \tag{4.42}$$

where the matrices A_n, B_n and C_n are given by

$$A_n = M_nM_{n+1}^{-1} \quad C_n = A_nH_nH_{n-1}^{-1},$$
$$B_n = -A_nP_{n+1}(0)(P_n(0))^{-1} - C_nP_{n-1}(0)(P_n(0))^{-1},$$

and M_n and H_n are given by (4.32) and (4.38) respectively.

Proof. Consider the recurrence relation

$$xP_n(x) = A_nP_{n+1}(x) + B_nP_n(x) + C_nP_{n-1}. \tag{4.43}$$

By comparing the leading coefficients of (4.43) we get

$$A_n = M_nM_{n+1}^{-1} = \begin{pmatrix} 1 & -\dfrac{q^n(1-q)(1-aq)(1+abq^{2n+3})}{(abq^{2n+2};q^2)_2}v \\ 0 & 1 \end{pmatrix}.$$

From (4.43) we can compute C_n as

$$C_n = A_n\langle P_n, P_n\rangle\langle P_{n-1}, P_{n-1}\rangle^{-1}.$$

Therefore by Corollary 4.3.3 we can write $C_n = A_nH_nH_{n-1}^{-1}$. To find B_n we first recall that, by Theorem (4.3.2),

$$P_n(0) = \begin{pmatrix} \kappa_{11}^n & \kappa_{12}^n + \kappa_{11}^n v \\ \kappa_{21}^n & \kappa_{22}^n + \kappa_{21}^n v \end{pmatrix}$$

and $\det(P_n(0)) = \kappa_{11}^n\kappa_{22}^n - \kappa_{21}^n\kappa_{12}^n > 0$, by Theorem 4.3.2. If we plug-in $x = 0$ in (4.43) we find

$$B_n = -A_nP_{n+1}(0)(P_n(0))^{-1} - C_nP_{n-1}(0)(P_n(0))^{-1}.$$

□

The following theorem gives a Rodrigues formula for $(P_n)_n$.

Theorem 4.3.6. *Let $R(n)$ be the matrix function given by*

$$R(n) = \begin{pmatrix} \dfrac{(1-aq^{n+2})(1-abq^{n+3}) + av^2q^2(1-q^n)(1-bq^{n+1})}{1-abq^{2n+3}} & 0 \\ -(1-q^n)avq^2 & 1-aq^{n+2} \end{pmatrix}.$$

Then, the expression

$$\bar{P}_n(q^x) = q^{-x}D_q^n\left(\frac{a^xq^{(n+1)x}(bq;q)_x}{(q;q)_{x-n}}L^xR(n)(L^x)^*\right)W(q^x)^{-1}, \tag{4.44}$$

defines a sequence of matrix polynomials orthogonal with respect to (4.28)

Before we prove Theorem 4.3.6 we formulate a Lemma that will be proved later.

Lemma 4.3.7. *For* $1 < k < n$,

$$D_q^{n-k}\left(\frac{a^{x+k-1}q^{(n+1)(x+k-1)}(bq;q)_{x+k-1}}{(q;q)_{x+k-n-1}}T(q^{x+k-1})R(n)T(q^{x+k-1})^*\right)D_q^k(q^{xm})\Big|_{x=0} = 0, \tag{4.45}$$

and

$$D_q^{n-k}\left(\frac{a^{x+k-1}q^{(n+1)(x+k-1)}(bq;q)_{x+k-1}}{(q;q)_{x+k-n-1}}T(q^{x+k-1})R(n)T(q^{x+k-1})^*\right)D_q^k(q^{xm})\Big|_{x=\infty} = 0. \tag{4.46}$$

We will also make use of the q-Leibniz rule

$$D_q^n\left(f(q^x)g(q^x)\right) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}D_q^{n-k}f(q^{k+x})D_q^k g(q^x), \tag{4.47}$$

and the following formula

$$D_q^n\left(f(q^x)\right) = \frac{1}{(1-q)^n q^{\binom{n}{2}}q^{nx}}\sum_{j=0}^{n}(-1)^{n-j}\begin{bmatrix}n\\j\end{bmatrix}q^{\binom{n-j}{2}}f(q^{j+x}). \tag{4.48}$$

Proof of Theorem 4.3.6. We need to prove that for all $n \geq 0$, $\bar{P}_n$ is a matrix-valued polynomial of degree n with non-singular coefficient and they satisfy an orthogonality relation with respect to (4.28).

First step. $\bar{P}_n$ is a matrix polynomial of degree n with non-singular coefficient.

Let us write $q^x W(q^x) = \rho(q^x)L^x(L^x)^*$, where the matrix L is given by 4.26 and $\rho(q^x) = q^x\dfrac{a^x(bq;q)_x}{(q;q)_x}$ is the weight associated to the scalar little q-Jacobi polynomials with parameters a and b. Taking into account that $L^xR(n)(L^x)^*$ is a matrix-valued polynomial of degree 2 in q^x and the q-Leibniz rule (4.47), we can write $\bar{P}_n$ as

$$\begin{aligned}
\bar{P}_n(q^x) =& D_q^n\left(\frac{a^xq^{(n+1)x}(bq;q)_x}{(q;q)_{x-n}}L^xR(n)(L^x)^*\right)W(q^x)^{-1}\\
=& D_q^n\left(\frac{a^xq^{(n+1)x}(bq;q)_x}{(q;q)_{x-n}}\right)(\rho(x))^{-1}L^xR(n)(L^x)^*\left(L^x(L^x)^*\right)^{-1}\\
&+\begin{bmatrix}n\\1\end{bmatrix}_q D_q^{n-1}\left(\frac{a^{x+1}q^{(n+1)(x+1)}(bq;q)_{x+1}}{(q;q)_{x-n+1}}\right)(\rho(x))^{-1}D_q\left(L^xR(n)(L^x)^*\right)\left(L^x(L^x)^*\right)^{-1}\\
&+\begin{bmatrix}n\\2\end{bmatrix}_q D_q^{n-2}\left(\frac{a^{x+2}q^{(n+1)(x+2)}(bq;q)_{x+2}}{(q;q)_{x-n+2}}\right)(\rho(x))^{-1}D_q^2\left(L^xR(n)(L^x)^*\right)\left(L^x(L^x)^*\right)^{-1}.
\end{aligned} \tag{4.49}$$

Let us first deal with the scalar part of (4.49). By making use of the formula (4.48) we can write

$$D_q^n\left(\frac{a^x q^{(n+1)x}(bq;q)_x}{(q;q)_{x-n}}\right)\rho(q^x)^{-1} = \frac{1}{(1-q)^n q^{\binom{n}{2}}}\sum_{j=0}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j\end{bmatrix}_q q^{\binom{n-j}{2}} q^{(n+1)j} a^j \times (bq^{x+1};q)_j (q^{x-n+j+1};q)_{n-j} = t_n(q^x), \tag{4.50}$$

$$D_q^{n-1}\left(\frac{a^{x+1} q^{(n+1)(x+1)}(bq;q)_{x+1}}{(q;q)_{x+1-n}}\right)\rho(q^x)^{-1} = \frac{q^x}{(1-q)^{n-1} q^{\binom{n-1}{2}}}\sum_{j=0}^{n-1}(-1)^{n-j-1}\begin{bmatrix} n-1 \\ j\end{bmatrix}_q q^{\binom{n-j-1}{2}} \times q^{(n+1)(j+1)} a^{j+1} (bq^{x+1};q)_{j+1} (q^{x-n+j+2};q)_{n-j-1} = q^x r_n(q^x) \tag{4.51}$$

$$D_q^{n-2}\left(\frac{a^{x+2} q^{(n+1)(x+2)}(bq;q)_{x+2}}{(q;q)_{x+2-n}}\right)\rho(q^x)^{-1} = \frac{q^{2x}}{(1-q)^{n-2} q^{\binom{n-2}{2}}}\sum_{j=0}^{n-2}(-1)^{n-j-2}\begin{bmatrix} n-2 \\ j\end{bmatrix}_q q^{\binom{n-j-2}{2}} a^{j+2} \times q^{(n+1)(j+2)} (bq^{x+1};q)_{j+2} (q^{x-n+j+3};q)_{n-j-2} = q^{2x} s_n(q^x), \tag{4.52}$$

where t_n, r_n and s_n are polynomials of degree n. Moreover, one can directly see that these polynomials have leading coefficients given by

$$t_n^n = \frac{1}{(1-q)^n q^{\binom{n}{2}}}\sum_{j=0}^{n}(-1)^{n-j}\begin{bmatrix} n \\ j\end{bmatrix}_q q^{\binom{j+1}{2}} q^{(n+1)j} a^j b^j, \tag{4.53}$$

$$r_n^n = \frac{1}{(1-q)^{n-1} q^{\binom{n-1}{2}}}\sum_{j=0}^{n}(-1)^{n-j-1}\begin{bmatrix} n-1 \\ j\end{bmatrix}_q q^{\binom{j+2}{2}} q^{(n+1)(j+1)} a^{j+1} b^{j+1}, \tag{4.54}$$

$$s_n^n = \frac{1}{(1-q)^{n-2} q^{\binom{n-2}{2}}}\sum_{j=0}^{n}(-1)^{n-j-2}\begin{bmatrix} n-2 \\ j\end{bmatrix}_q q^{\binom{j+3}{2}} q^{(n+1)(j+2)} a^{j+2} b^{j+2}. \tag{4.55}$$

By applying the q-Leibniz rule (4.47) to the matrix part of (4.49) we can write

$$D_q(L^x R(n)(L^x)^*)(L^{-x})^* L^{-x} = \frac{1}{(1-q)q^x} L^x R_1(n) L^{-x}, \tag{4.56}$$

$$D_q^2(L^x R(n)(L^x)^*)(L^{-x})^* L^{-x} = \frac{1}{(1-q)^2 q^{2x}} L^x R_2(n) L^{-x},$$

where

$$R_1(n) = R(n) - LR(n)L^*, \quad R_2(n) = R(n) - (1+q^{-1})LR(n)L^* + q^{-1}L^2 R(n)(L^*)^2.$$

Taking into account (4.50), (4.51), (4.52) and (4.56), we can write (4.49) as

$$\bar{P}_n(q^x) = t_n(q^x) L^x R(n) L^{-x} + \begin{bmatrix} n \\ 1\end{bmatrix}_q \frac{q^x r_n(q^x)}{(1-q)q^x} L^x R_1(n) L^{-x} + \begin{bmatrix} n \\ 2\end{bmatrix}_q \frac{q^{2x} s_n(q^x)}{(1-q)^2 q^{2x}} L^x R_2(n) L^{-x}. \tag{4.57}$$

By plugging in the explicit expressions of the polynomials t_n, r_n and s_n and of the matrices $R(n)$, $R_1(n)$, $R_2(n)$ and L^x and carrying direct computations we get that (4.57) is a polynomial of degree n with non-singular leading coefficient.

Second step. The orthogonality relation.
To prove that the sequence of polynomials given by (4.44) is orthogonal, we see that for $n \geq 1$ and $0 \leq m < n$, $\langle P_n, x^m I\rangle_W = 0$ holds.

By using the q-Leibniz rule (4.47), the formal identity given in (4.48) Lemma 4.3.7, we get

$$\begin{aligned}\langle P_n, x^m\rangle_W =& \frac{1}{q-1}\sum_{x=0}^{\infty} D_q^n\left(\frac{a^x q^{(n+1)x}(bq;q)_x}{(q;q)_{x-n}}L^x R(n)(L^x)^*\right)q^{xm}\\ =& D_q^{n-1}\left(\frac{a^x q^{(n+1)x}(bq;q)_x}{(q;q)_{x-n}}L^x R(n)(L^x)^*\right)D_q(q^{xm})\Big|_{x=0}^{\infty}\\ &+\frac{1}{q-1}\sum_{x=0}^{\infty} D_q^{n-1}\left(\frac{a^{x+1} q^{(n+1)(x+1)}(bq;q)_{x+1}}{(q;q)_{x+1-n}}L^x R(n)(L^x)^*\right)D_q(q^{xm})\\ =& \frac{1}{q-1}\sum_{x=0}^{\infty} D_q^{n-1}\left(\frac{a^{x+1} q^{(n+1)(x+1)}(bq;q)_{x+1}}{(q;q)_{x+1-n}}L^x R(n)(L^x)^*\right)D_q(q^{xm}),\end{aligned}$$

By repeating this process we obtain

$$\begin{aligned}\langle P_n, x^m\rangle =& \frac{1}{q-1}\sum_{x=0}^{\infty} D_q^{n-m-1}\left(\frac{a^{x+m+1} q^{n(x+m+2)}(bq;q)_{x+m+1}}{(q;q)_{x+m+1-n}}L^x R(n)(L^x)^*\right)D_q^{m+1}(q^{xm})q^x\\ =& 0\end{aligned}$$

because $D_q^{m+1}(q^{xm}) = 0$. This gives the desired result. □

We now prove Lemma 4.3.7.

Proof of Lemma 4.3.7. To see that the first boundary condition (4.45) holds, we use the expression

$$\frac{1}{(q;q)_{x-n}} = \frac{(q^{x-n+1};q)_n}{(q;q)_x},$$

that vanishes at $x = 0$. Any other quantity involved in (4.45) is bounded in $x = 0$, then, we have that (4.45) holds.

For (4.46), use that $0 < a < q^{-1}$ and so $a^{x+k-1}q^{(n+1)(x+k-1)}$ goes to 0 when x goes to ∞. Since all the other quantities remain bounded when x goes to ∞, we obtain the desired result. □

Chapter 5

A family of matrix polynomials satisfying second order differential equations.

In this chapter we introduce a family of continuous weight matrices and their sequences of orthogonal polynomials. We start by presenting the weights via an LDU-decomposition (see [75]) and we give an alternative expression of them in terms of scalar Gegenbauer polynomials. We then show that the sequences of orthogonal polynomials are eigenfunctions of two independent differential operators. This property can be exploited to obtain a matrix Pearson equation for the weight matrices in the sense of [12] and a compact Rodrigues formula for the polynomials. We also show how the families of polynomials introduced here can be expressed in terms of matrix hypergeometric functions introduced in Chapter 1, and we give the expression of these matrix polynomials in terms of scalar Racah polynomials and Gegenbauer polynomials.

Along the whole chapter we will make use of several summation formulas regarding hypergeometric functions. Such formulas are listed in Section (1.1).

For a matter of simplicity in the formulas exhibited in this chapter, we perform a tinny change in the notation used for the matrices, by shifting the index summation range. From now, a matrix $M \in \mathbb{C}^{(N+1)\times(N+1)}$ will be written as

$$M = \sum_{i=0}^{N}\sum_{j=0}^{N} M_{ij}\mathcal{E}_{i,j},$$

and $\mathcal{E}_{i,j}$ is the matrix with a 1 in its $(i+1, j+1)$ entry, and zero othewise.

5.1 The weight matrix

We define a family of weights matrices $(W^{(\nu)})_{\nu>0}$ absolutely continuous with respect to the Lebesgue measure on $(-1,1)$. Then their densities (that we also denote by $W^{(\nu)}$) are positive semi-definite matrices on $(-1,1)$ so they can be expressed in terms of an LDU decomposition of the form $L(x)D(x)L^{*}(x)$ where $L(x)$ is a triangular unipotent matrix and $D(x)$ is a positive semi-definite diagonal matrix for all $x \in (-1,1)$. In the rest of the chapter we will use the term weight matrix to refer both to the weight matrix itself and to the matrix functions that defines it, $dW^{(\nu)}(x) = W^{(\nu)}(x)dx$.

To do so we will use the Gegenabuer polynomials defined by (see, for instance, [76], [59])

$$C_n^{(\nu)}(x) = \frac{(2\nu)_n}{n!} \, {}_2F_1\left(\begin{matrix} -n, n+2\nu \\ \frac{1}{2}+\nu \end{matrix} ; \frac{(1-x)}{2} \right). \tag{5.1}$$

The Gegenbauer polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} C_n^{(\nu)}(x) C_m^{(\nu)}(x)(1-x^2)^{\nu-1/2} dx = \frac{(2\nu)_n \sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{n!(n+\nu)\Gamma(\nu)} \delta_{m,n}. \tag{5.2}$$

For $\nu > 0$ and $\ell \in \frac{1}{2}\mathbb{N}$, $N = 2\ell+1$, we define the matrix polynomials $L^{(\nu)}\colon [-1,1] \to \mathbb{C}^{N\times N}$ by

$$\left(L^{(\nu)}(x)\right)_{m,k} = \begin{cases} 0 & \text{if } m < k \\ \dfrac{m!}{(2\nu+2k)_{m-k}\, k!} C_{m-k}^{\nu+k}(x) & \text{if } m \geq k, \end{cases} \tag{5.3}$$

for $k,\ m = 0, \cdots, 2\ell$. Notice that they are unipotent lower triangular matrices.

We also introduce the diagonal matrix polynomial

$$T^{(\nu)}\colon [-1,1] \to \mathbb{C}^{N\times N}$$

$$\begin{aligned} \left(T^{(\nu)}(x)\right)_{k,k} &= t_k^{(\nu)} (1-x^2)^k, \quad k = 0, \cdots, 2\ell \\ t_k^{(\nu)} &= \frac{k!\,(\nu)_k}{(\nu+1/2)_k} \frac{(2\nu+2\ell)_k\,(2\ell+\nu)}{(2\ell-k+1)_k\,(2\nu+k-1)_k}. \end{aligned} \tag{5.4}$$

A direct check shows that $T^{(\nu)}(x)$ is a positive definite matrix for all $x \in (-1,1)$ and for all $\nu > 0$.

Definition 5.1.1. *Let $\nu > 0$ and $\ell \in \frac{1}{2}\mathbb{N}$, set $N = 2\ell+1$. Let $W^{(\nu)}$ be the continuous weight matrix supported on $(-1,1)$ and given by $dW^{(\nu)} = W^{(\nu)}(x)dx$, where*

$$W^{(\nu)}(x) = L^{(\nu)}(x) T^{(\nu)}(x) \left(L^{(\nu)}(x)\right)^*. \tag{5.5}$$

Observation 5.1.2. *Notice that*

$$\left(L^{(\nu)}(1)\right)_{i,j} = \binom{i}{j}, \quad \left(L^{(\nu)}(-1)\right)_{i,j} = (-1)^{i-j}\binom{i}{j}, \quad \text{for } i \geq j,$$

it is also clear that $\left(T^{(\nu)}(\pm 1)\right)_{k,k} = t_0^{(\nu)}\delta_{k,0}$. If we write $W^{(\nu)} = (1-x^2)^{\nu-1/2} W_{\text{pol}}^{(\nu)}(x)$ then we get

$$\left(W_{\text{pol}}^{(\nu)}(1)\right)_{i,j} = t_0^{(\nu)} = 2\ell+\nu, \quad \left(W_{\text{pol}}^{(\nu)}(-1)\right)_{i,j} = (-1)^{i-j}(2\ell+\nu). \tag{5.6}$$

Observation 5.1.3. *The matrix $L(x)$ is invertible, and it is remarkable that the inverse is again completely described in terms of hypergeometric series. Explicitly, see Cagliero and Koornwinder [11, (4.7), Thm. 4.1],*

$$\left(\left(L^{(\nu)}(x)\right)^{-1}\right)_{k,n} = \frac{k!}{n!\,(2\nu+k+n-1)_{k-n}} C_{k-n}^{(1-\nu-k)}(x), \qquad k \geq n, \tag{5.7}$$

where we follow the convention of [11] for $C_n^{(-p)}(x)$ for $p \in \mathbb{N}$,

$$C_n^{-p} = \frac{(-2p)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n-2p \\ \frac{1}{2}-p \end{matrix}; \frac{(1-x)}{2}\right).$$

The next theorem gives an alternative expression for the weight matrices $W^{(\nu)}$ in terms of scalar Gegenbauer polynomials.

Theorem 5.1.4. *For $\ell \in \frac{1}{2}\mathbb{N}$ and $\nu > 0$, the weight function $W^{(\nu)}$ can be expressed as*

$$\left(W^{(\nu)}(x)\right)_{i,j} = (1-x^2)^{\nu-1/2} \sum_{t=\max(0,i+j-2\ell)}^{i} \alpha_t^{(\nu)}(i,j) C_{i+j-2t}^{(\nu)}(x), \tag{5.8}$$

$$\begin{aligned}\alpha_t^{(\nu)}(i,j) = (-1)^i \frac{j!\,i!\,(i+j-2t)!}{t!\,(2\nu)_{i+j-2t}\,(\nu)_{i+j-t}} \frac{(\nu)_{j-t}(\nu)_{i-t}}{(j-t)!\,(i-t)!} \frac{(i+j-2t+\nu)}{(i+j-t+\nu)} \\ \times (2\ell-i)!(j-2\ell)_{i-t}(-2\ell-\nu)_t \frac{(2\ell+\nu)}{(2\ell)!},\end{aligned} \tag{5.9}$$

where $C_n^{(\nu)}$ stands for the Gegenbauer polynomials (5.1), $i, j \in \{0, 1, \cdots, 2\ell\}$ and $j \geq i$, and $\left(W^{(\nu)}(x)\right)_{i,j} = \left(W^{(\nu)}(x)\right)_{j,i}$ for $j < i$.

Before we prove Theorem 5.1.4 we announce a lemma that is needed and will be proved afterwards.

Lemma 5.1.5. *For $\nu > -1/2$, $i \leq j$, and $t, k \in \{0, 1, \cdots, i\}$ the following holds*

$$\begin{aligned}\alpha_t^{(\nu)}(i,j) \frac{(2\nu)_{i+j-2t}\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{(i+j-2t)!(i+j-2t+\nu)\Gamma(\nu)} = \sum_{k=0}^{i} \frac{i!\,j!\,t_k^{(\nu)}}{k!\,k!\,(2\nu+2k)_{i-k}(2\nu+2k)_{j-k}} \\ \times \int_{-1}^{1} C_{i-k}^{(\nu+k)}(x)\, C_{j-k}^{(\nu+k)}(x) C_{i+j-2t}^{(\nu)}(x)\,(1-x^2)^{k+\nu-1/2}\,dx.\end{aligned} \tag{5.10}$$

We can now prove Theorem 5.1.4.

Proof of Theorem 5.1.4. Because of the symmetries of the matrices involved it suffices to see that for $i \leq j$ the (i,j)-entry of (5.5) is equal to the right hand side of (5.8), that is

$$\sum_{t=0}^{i} \alpha_t^{(\nu)}(i,j)\, C_{i+j-2t}^{(\nu)}(x) = \sum_{k=0}^{i} \frac{i!\,j!\,t_k^{(\nu)}\,(1-x^2)^k}{k!\,k!\,(2\nu+2k)_{i-k}(2\nu+2k)_{j-k}} C_{i-k}^{(\nu+k)}(x)\, C_{j-k}^{(\nu+k)}(x), \tag{5.11}$$

By the orthogonality relation for the Gegenbauer polynomials (5.2), (5.11) is equivalent to see that for $r \neq i+j-2t$, $t = 0, \cdots, i$,

$$\sum_{k=0}^{i} \frac{i!\,j!\,t_k^{(\nu)}}{k!\,k!\,(2\nu+2k)_{i-k}(2\nu+2k)_{j-k}} \int_{-1}^{1} C_{i-k}^{(\nu+k)}(x)\, C_{j-k}^{(\nu+k)}(x) C_r^{(\nu)}(x)(1-x^2)^{k+\nu-1/2} = 0, \tag{5.12}$$

and for $r = i + j - 2t$ with $t = 0, \cdots, i$,

$$(5.13)\quad \alpha_t^{(\nu)}(i,j)\|C_{i+j-2t}^{(\nu)}\|_\nu^2 = \sum_{k=0}^{i} \frac{i!\,j!\,t_k^{(\nu)} \int_{-1}^{1} C_{i-k}^{(\nu+k)}(x) C_{j-k}^{(\nu+k)}(x) C_{j+i-2t}^{(\nu)}(x)(1-x^2)^{k+\nu-1/2}}{k!k!(2\nu+2k)_{i-k}(2\nu+2k)_{j-k}}.$$

To see that (5.12) holds, we apply the connection formula of $C_r^{(\nu)}$ and the linearization formula for the product $C_{i-k}^{(\nu+k)} C_{j-k}^{(\nu+k)}$ (see for instance [76]),

$$(5.14)\quad C_r^{(\gamma)} = \sum_{j=0}^{[\frac{r}{2}]} \frac{(\gamma-\beta)_j(\gamma)_{r-j}}{j!(\beta+1)_{r-j}} \frac{\beta+r-2j}{\beta} C_{r-2j}^{(\beta)},$$

$$(5.15)\quad C_n^{(\gamma)} C_m^{(\gamma)} = \sum_{j=0}^{\min(m,n)} \frac{(n+m-2j+\gamma)(n+m-2j)!(\gamma)_j}{(n+m-j+\gamma)j!(n-j)!(m-j)!} \times \frac{(\alpha)_{n-j}(\alpha)_{m-j}(2\alpha)_{m+n-j}}{(\alpha)_{m+n-j}(2\alpha)_{m+n-2j}} C_{n+m-2j}^{(\alpha)},$$

and we obtain

$$\int_{-1}^{1} C_{i-k}^{(\nu+k)}(x)\, C_{j-k}^{(\nu+k)}(x) C_r^{(\nu)}(x)(1-x^2)^{k+\nu-1/2} = 0$$

for $i + j - r$ odd and $r < j + i$ or $r > i + j$. That is $r \neq i + j - 2t$ for $t = 0, \cdots, i$. So (5.12) is, indeed, equal to zero. For (5.13), we apply Lemma 5.1.5 and we have the result since by (5.2),

$$\|C_{i+j-2t}^{(\nu)}\|_\nu^2 = \frac{(2\nu)_{i+j-2t}\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{(i+j-2t)!(i+j-2t+\nu)\Gamma(\nu)}.$$

□

We now prove Lemma 5.1.5.

Proof of Lemma 5.1.5. To see that (5.10) holds we start by evaluating the integral on the right hand side

$$(5.16)\quad \int_{-1}^{1} C_{i-k}^{(\nu+k)}(x)\, C_{j-k}^{(\nu+k)}(x) C_{i+j-2t}^{(\nu)}(x)\,(1-x^2)^{k+\nu-1/2}\,dx.$$

By applying the the connection formula, (5.14) and the linearization formula, (5.15) for the Gegenbauer polynomials, we get that (5.16) can be written as

$$\frac{(\nu+k)_{i-k}(2\nu+2k)_{j-k}(-i)_{i-t}(\nu)_{j-t}\Gamma(\nu+k+1/2)}{(i-k)!\,(j-k)!\,(i-t)!\,\Gamma(\nu+i+j-t+1)}\sqrt{\pi} \times {}_4F_3\left(\begin{matrix} k-i, -i-k-2\nu+1, t-i, \nu+1+j-t \\ -i, -i-\nu+1, j-i+1 \end{matrix}; 1\right),$$

where we have also used the orthogonality relations for the Gegenbauer polynomials (5.2). Using Whipple's transformation, (see 1.6), twice (once with (n, a, d) of as $(i-k, -i-2\nu-k+1, -i)$ and the second time as $(t, -i-j-\nu+t, -i)$) the ${}_4F_3$-series can be rewritten as

$$(5.17)\quad \frac{(1-j-i-2\nu)_{i-k}(-j)_t(\nu)_t}{(1+j-i)_{i-k}(1-\nu-i)_t(1-i-j-2\nu)_t}\, {}_4F_3\left(\begin{matrix} -k, k+2\nu-1, -t, t-i-j-\nu \\ -i,\ -j,\ \nu \end{matrix}; 1\right).$$

Then, the integral (5.16) remains

$$
\begin{aligned}
&\int_{-1}^{1} C_{i-k}^{(\nu+k)}(x)\, C_{j-k}^{(\nu+k)}(x) C_{i+j-2t}^{(\nu)}(x)\,(1-x^2)^{k+\nu-1/2}\,dx = \\
&\sqrt{\pi}\frac{(\nu+k)_{i-k}(2\nu+2k)_{j-k}(-i)_{i-t}(\nu)_{j-t}}{(i-k)!\,(j-k)!\,(i-t)!}\frac{\Gamma(\nu+k+1/2)}{\Gamma(\nu+i+j-t+1)} \\
&\times \frac{(1-j-i-2\nu)_{i-k}(-j)_t(\nu)_t}{(1+j-i)_{i-k}(1-\nu-i)_t(1-i-j-2\nu)_t} \\
&\times \sum_{m=0}^{\min(k,t)} \frac{(-k)_m(k+2\nu-1)_m(-t)_m(t-i-j-\nu)_m}{(-i)_m(-j)_m(\nu)_m m!}.
\end{aligned}
\tag{5.18}
$$

We now plug (5.18) in (5.10) We next interchange the summations over k and m, and in the summation $\sum_{k=m}^{i}$ we replace $k=p+m$. This gives for the inner sum (the k-dependent part)

$$
\begin{aligned}
&\frac{(2\nu+2\ell)_m(1-j-i-2\nu)_{i-m}(2\nu+m)_m}{(-2\ell)_m(2\nu+m)_i(i-m)!\,(1+j-i)_{i-m}} \\
&\qquad\times {}_5F_4\left(\begin{matrix} 2\nu+2m-1, \nu+m+\frac{1}{2}, m-j, 2\nu+2l+m, m-i \\ \nu+m-\frac{1}{2}, 2\nu+m+j, -2\ell+m, 2\nu+i+m \end{matrix};1\right) \\
&= \frac{(1-j-i-2\nu)_i(j-2\ell)_i}{(-2\ell)_i i!\,(1+j-i)_i(2\nu+j)_i}\frac{(2\nu+2\ell)_m(-i)_m(-j)_m}{(1-j-i+2\ell)_m}
\end{aligned}
$$

where we have used the Dougall summation formula for a very-well-poised ${}_5F_4$-series, (1.7).

This shows that the right hand side of (5.10) can be written as a single sum; explicitly

$$
\begin{aligned}
&\frac{(-i)_{i-t}(\nu)_{j-t}\sqrt{\pi}(-j)_t(\nu)_t j!\,(2\ell+\nu)(\nu)_i\Gamma(\nu+\frac{1}{2})(1-i-j-2\nu)_i(j-2\ell)_i}{(i-j)!\,(i-t)!\,\Gamma(\nu+i+j-t+1)(1-\nu-i)_t(1-i-j-2\nu)_t(-2\ell)_i} \\
&\times \frac{1}{(1+j-i)_i(2\nu+j)_i}\, {}_3F_2\left(\begin{matrix} -t, t-i-j-\nu, 2\nu+2\ell \\ \nu, 1-i-j+2\ell \end{matrix};1\right).
\end{aligned}
$$

The balanced ${}_3F_2$-series is summable to $\frac{(1-i-j-2\nu)_t(-2\ell-\nu)_t}{(\nu)_t(2\ell+1-i-j)_t}$ by the Pfaff-Saalschütz formula, (1.5). Next, a straightforward verification using the expression of $\alpha_t^{(\nu)}(i,j)$ in (5.9) shows that this is equal to the left hand side of (5.10), which proves Lemma 5.1.5.

□

As it was mentioned in the preliminaries, given a weight matrix one can always construct a sequence of orthogonal polynomials with respect to it. We denote by $P_n^{(\nu)}$ the corresponding monic matrix orthogonal polynomial of degree n, thus

$$
\begin{aligned}
&\int_{-1}^{1} P_n^{(\nu)}(x)\, W^{(\nu)}(x) \big(P_m^{(\nu)}(x)\big)^*\,dx = \delta_{n,m} H_n^{(\nu)}, \\
&P_n^{(\nu)}(x) = x^n I + x^{n-1}P_{n,n-1}^{(\nu)} + \cdots + xP_{n,1}^{(\nu)} + P_{n,0}^{(\nu)}, \qquad P_{n,n}^{(\nu)} = I, P_{n,i}^{(\nu)} \in \mathbb{C}^{2\ell+1\times 2\ell+1},
\end{aligned}
\tag{5.19}
$$

where $H_n^{(\nu)}$ is a positive definite matrix. We denote the matrix inner product by

$$
\langle P, Q\rangle^{(\nu)} = \int_{-1}^{1} P(x)\, W^{(\nu)}(x)\,(Q(x))^*\,dx, \qquad \nu > 0.
\tag{5.20}
$$

In particular, the case $n = 0$ in (5.19) can be evaluated explicitly using the orthogonality relations for the Gegenbauer polynomials (5.2) and (5.5), to see that $H_0^{(\nu)}$ is a diagonal matrix with entries

(5.21)
$$\left(H_0^{(\nu)}\right)_{k,k} = \alpha_k^{(\nu)}(k,k)\int_{-1}^{1}(1-x^2)^{\nu-1/2}\,dx = (2\ell+\nu)\sqrt{\pi}\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)}\frac{k!\,(2\ell-k)!\,(\nu+1)_{2\ell}}{(2\ell)!\,(\nu+1)_k(\nu+1)_{2\ell-k}}.$$

Before we study the polynomials closely, and in order to be able to establish a result concerning the decomposition of the weight matrix, we determine the commutant of the weight, that is the algebra of matrices that commute with the weight.

Proposition 5.1.1. *Let $\nu > 0$, $N = 2\ell+1$. The commutant algebra, $A^{(\nu)} = \{T \in \mathbb{C}^{N\times N} \mid [T, W^{(\nu)}(x)] = 0,\ \forall x \in (-1,1)\}$, is generated by $\mathfrak{J}$, where $\mathfrak{J} \in \mathbb{C}^{N\times N}$ is the involution defined by $e_j \mapsto e_{2\ell-j}$.*

Proof. We first show that $\mathfrak{J} \in A^{(\nu)}$. Notice that $W^{(\nu)}$ commutes with $\mathfrak{J}$ if and only if

$$\left(W^{(\nu)}(x)\right)_{j,2\ell-i} = \left(W^{(\nu)}(x)\right)_{2\ell-j,i}.$$

From the expression of $W^{(\nu)}$ in Theorem 5.1.4 and comparing the coefficients of the Gegenbauer polynomials we need to prove

$$\alpha_t^{(\nu)}\big(\min(2\ell-m,n),\max(2\ell-m,n)\big) = \alpha_{t+m-n}^{(\nu)}\big(\min(2\ell-n,m),\max(2\ell-n,m)\big).$$

This is straightforwardly verified from the expression of $\alpha_t^{(\nu)}$ (5.9). Notice that the summation ranges in (5.8) also match.

We now prove that if S is in the commutant then it is a linear combination of $\mathfrak{J}$ and I. Write $W^{(\nu)}(x) = (1-x^2)^{\nu-1/2}W_{\text{pol}}^{(\nu)}(x)$, and $W_{\text{pol}}^{(\nu)}(x) = \sum_{k=0}^{2\ell} W_k x^k$ where $W_k \in \mathbb{C}^{N\times N}$ are Hermitian matrices. Then $S \in A^{(\nu)}$ if and only if $[S, W_k] = 0$ for all $k = 0,\cdots,2\ell$.

Take $k = 2\ell$. From (5.8) and (5.9) we observe that $(W_{2\ell})_{ij} \neq 0$ only for $j = 2\ell - i$, then $(SW_{2\ell})_{ij} = (W_{2\ell}S)_{ij}$ if and only if $(S)_{i,2\ell-j}\,(W_{2\ell})_{2\ell-j,j} = (W_{2\ell})_{i,2\ell-i}\,(S)_{2\ell-i,j}$, then for $i,j = 0,\cdots,2\ell$

(5.22)
$$(S)_{i,2\ell-j} = \frac{(W_{2\ell})_{i,2\ell-i}}{(W_{2\ell})_{2\ell-j,j}}\,(S)_{2\ell-i,j} = \frac{(W_{2\ell})_{i,2\ell-i}\,(W_{2\ell})_{2\ell-i,i}}{(W_{2\ell})_{2\ell-j,j}\,(W_{2\ell})_{j,2\ell-j}}\,(S)_{i,2\ell-j}$$

From the expression in (5.8) and since $\nu > 0$, we have that $(W_{2\ell})_{j,2\ell-j} = (W_{2\ell})_{i,2\ell-i}$ if and only if $j = i$ or $j = 2\ell - i$. Therefore, from (5.22) we get that if $S \in A^{(\nu)}$ then $S_{i,j} = 0$ unless $j = i$ or $j = 2\ell - i$. Moreover

(5.23)
$$S_{i,2\ell-i} = S_{2\ell-i,i}, \qquad S_{i,i} = S_{2\ell-i,2\ell-i}.$$

Now take $k = 2\ell-1$. From (5.8) and taking into account that the Gegenbauer polynomials of even order are even and of odd order are odd we get that $(W_{2\ell-1})_{i,j}$ vanishes except for $|i + j - 2\ell| = 1$. Then, we have that $S \in A^{(\nu)}$ if and only if

(5.24)
$$\begin{aligned} S_{i,2\ell-j+1}\,(W_{2\ell-1})_{2\ell-j+1,j} + S_{i,2\ell-j-1}\,(W_{2\ell-1})_{2\ell-j-1,j} &= \\ S_{2\ell-i+1,j}\,(W_{2\ell-1})_{i,2\ell-i+1} + S_{2\ell-i-1,j}\,(W_{2\ell-1})_{i,2\ell-i-1}&. \end{aligned}$$

Taking into account that $S_{i,j} = 0$ unless $j = i$ or $j = 2\ell - i$, we have that (5.24) is trivial except for $i = 2\ell - j + 1,\ j + 1,\ 2\ell - j - 1,\ j - 1$. By plugging in each of the values for which (5.24) is non-trivial and taking into account the symmetries of $W_{2\ell-1}$ we get that

$$S_{j,j} = S_{2\ell-j+1,2\ell-j+1},\ S_{j,2\ell-j} = S_{j-1,2\ell-j+1}, \quad \text{for } j = 1, \cdots, 2\ell,\ j \neq \ell,$$

and, in the case $j = \ell$, we also get $S_{\ell,\ell} = S_{\ell+1,\ell+1} + S_{\ell+1,\ell-1}$. This, together with (5.23), gives that S is a linear combination of $\mathfrak{J}$ and I. So, we can conclude that $A^{(\nu)}$ is spanned by I and $\mathfrak{J}$ for all $\nu > -1/2$. □

Observation 5.1.6. *It follows that $A^{(\nu)}$ is a two-dimensional algebra, and that the invariant subspaces of $\mathbb{C}^{2\ell+1}$ for $W^{(\nu)}(x)$ are the ±1-eigenspaces of $\mathfrak{J}$. Then, the restrictions of $W^{(\nu)}(x)$ to the ±1-eigenspaces are irreducible. Explicitly,*

$$W^{(\nu)}(x) = Y_\ell^t \begin{pmatrix} W_+^{(\nu)}(x) & 0 \\ 0 & W_-^{(\nu)}(x) \end{pmatrix} Y_\ell, \qquad Y_\ell \in \mathrm{SO}(2\ell+1),$$

$$Y_{p+\frac{1}{2}} = \frac{1}{2}\sqrt{2}\begin{pmatrix} I_{p+1} & \mathfrak{J}_{p+1} \\ -\mathfrak{J}_{p+1} & I_{p+1} \end{pmatrix}, \quad Y_p = \frac{1}{2}\sqrt{2}\begin{pmatrix} I_p & 0 & \mathfrak{J}_p \\ 0 & \sqrt{2} & 0 \\ -\mathfrak{J}_p & 0 & I_p \end{pmatrix}, \quad p \in \mathbb{N}, \tag{5.25}$$

where I_p denotes the identity as $p \times p$-matrix and $\mathfrak{J}_p$ the antidiagonal matrix in $\mathbb{C}^{p\times p}$,

$$\mathfrak{J}_p = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ & & \iddots & & \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

That means that the weight matrix $W^{(\nu)}$ can be reduced to lower size, (5.25) and it admits no more reduction.

Note that $W_+^{(\nu)}(x) \in \mathbb{C}^{r_1\times r_1}$ and $W_-^{(\nu)}(x) \in \mathbb{C}^{r_2\times r_2}$ with $r_1 = \lceil \ell + \frac{1}{2} \rceil$ and $r_2 = \lfloor \ell + \frac{1}{2} \rfloor$, where $\lceil\ \rceil$ and $\lfloor\ \rfloor$ are the ceiling and floor functions,

$$\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}, \quad \lfloor x \rfloor \max\{m \in \mathbb{Z} \mid m \leq x\}.$$

The corresponding monic orthogonal polynomials are denoted $P_{\pm,n}^{(\nu)}$.

Since $C_n^{(\nu)}(-x) = (-1)^n C_n^{(\nu)}(x)$, the involution $\mathfrak{F} \in \mathbb{C}^{2\ell+1\times 2\ell+1}$, $e_j \mapsto (-1)^j e_j$ satisfies $W^{(\nu)}(x)\mathfrak{F} = \mathfrak{F}W^{(\nu)}(-x)$. Since $\mathfrak{F}J = (-1)^{2\ell}J\mathfrak{F}$, we find, in the case when the dimension $2\ell+1$ is even that the weights $W_+^{(\nu)}(-x) = \mathfrak{F}JW_-^{(\nu)}(x)J\mathfrak{F}$ (where $\mathfrak{F}$ and $\mathfrak{J}$ are now the corresponding diagonal and antidiagonal operators in $\mathbb{C}^{\frac{2\ell+1}{2}\times\frac{2\ell+1}{2}}$.) For odd $2\ell+1$ there is no link between the weights $W_\pm^{(\nu)}$.

5.2 Polynomial eigenfunctions of matrix differential operators

In this section we present, for each $\nu > 0$ a first order and a second order matrix differential operators which are symmetric with respect to the weight function $W^{(\nu)}$, and such that they do not raise the degree of polynomials. In particular (see Lemma 1.3.3) this implies that the corresponding monic matrix orthogonal polynomials are eigenfunctions for these differential operators. These differential operators can be combined to obtain matrix functions that connect to the results by Cantero, Moral and Velázquez in [12]. This leads to forward shift and backward shift operators which can be used to give an explicit evaluation of the squared norm and to establish a Rodrigues formula.

Theorem 5.2.1. *For $\nu > 0$, let $D^{(\nu)}$ and $E^{(\nu)}$ be the matrix differential operators*

$$D^{(\nu)}(\cdot) = (1-x^2)\frac{d^2}{dx^2}(\cdot) + \frac{d}{dx}(\cdot)(C^{(\nu)} - xU^{(\nu)}) - V^{(\nu)}, \tag{5.26}$$

$$E^{(\nu)}(\cdot) = \frac{d}{dx}(\cdot)(xB_1^{(\nu)} + B_0^{(\nu)}) + A_0^{(\nu)}, \tag{5.27}$$

where the matrices $C^{(\nu)}$, $U^{(\nu)}$, $V^{(\nu)}$, $B_0^{(\nu)}$, $B_1^{(\nu)}$ and $A_0^{(\nu)}$ are given by

$$C^{(\nu)} = \sum_{i=0}^{2\ell}(2\ell-i)\mathcal{E}_{i,i+1} + \sum_{i=0}^{2\ell} i\mathcal{E}_{i,i-1}, \qquad U^{(\nu)} = (2\ell+2\nu+1)I,$$

$$V^{(\nu)} = -\sum_{i=0}^{2\ell} i(2\ell-i)\mathcal{E}_{i,i} + (\nu-1)(2\ell+\nu+1)I,$$

$$B_0^{(\nu)} = \sum_{i=0}^{2\ell}\frac{(2\ell-i)}{2\ell}\mathcal{E}_{i,i+1} - \sum_{i=0}^{2\ell}\frac{i}{2\ell}\mathcal{E}_{i,i-1}, \qquad B_1^{(\nu)} = -\sum_{i=0}^{2\ell}\frac{(\ell-i)}{\ell}\mathcal{E}_{i,i},$$

$$A_0^{(\nu)} = \sum_{i=0}^{2\ell}\left(\frac{(2\ell+2)(i-2\ell)}{2\ell} - (\nu-1)\frac{(\ell-i)}{\ell}\right)\mathcal{E}_{i,i}.$$

Then, $D^{(\nu)}$ and $E^{(\nu)}$ are symmetric with respect to the weight $W^{(\nu)}$, and $D^{(\nu)}$ and $E^{(\nu)}$ commute. Moreover, for every integer $n \geq 0$, the monic orthogonal polynomials with respect to $W^{(\nu)}$, $(P_n^{(\nu)})_n$ satisfy:

(5.28)

$$D^{(\nu)}(P_n^{(\nu)}) = \Lambda_n(D^{(\nu)})P_n^{(\nu)}, \quad \Lambda_n(D^{(\nu)}) = \sum_{i=0}^{2\ell}\big(i(2\ell-i) - (n+\nu-1)(2\ell+\nu+n+1)\big)\mathcal{E}_{i,i},$$

(5.29)

$$E^{(\nu)}(P_n^{(\nu)}) = \Lambda_n(E^{(\nu)})P_n^{(\nu)}, \; \Lambda_n(E^{(\nu)}) = \sum_{i=0}^{2\ell}\frac{(\ell+1)(i-2\ell) - n(\ell-i) - (\nu-1)(\ell-i)}{\ell}\mathcal{E}_{i,i}.$$

Proof. We make use of Theorem 1.3.7 to see that the differential operators $E^{(\nu)}$ and $D^{(\nu)}$ are symmetric with respect to $W^{(\nu)}$.

The symmetry of $E^{(\nu)}$ with respect to $W^{(\nu)}$. By Theorem 1.3.7 and writing $W^{(\nu)}(x) = \rho(x)W_{\text{pol}}^{(\nu)}(x)$ with $\rho(x) = (1-x^2)^{\nu-1/2}$, we have that the symmetry equations for $E^{(\nu)}$ and $W^{(\nu)}$ are given by

$$0 = W_{\text{pol}}^{(\nu)}(x)(xB_1^{(\nu)} + B_0^{(\nu)})^* + (xB_1^{(\nu)} + B_0^{(\nu)})W_{\text{pol}}^{(\nu)}(x), \tag{5.30}$$

$$\begin{aligned} 0 = {} & -A_1'(x)W_{\text{pol}}^{(\nu)}(x) - \frac{\rho'(x)}{\rho(x)}(xB_1^{(\nu)} + B_0^{(\nu)})W_{\text{pol}}^{(\nu)}(x) \\ & - (xB_1^{(\nu)} + B_0^{(\nu)})W_{\text{pol}}^{(\nu)}(x)' + A_0W_{\text{pol}}^{(\nu)}(x) - W_{\text{pol}}^{(\nu)}(x)A_0^{(\nu)}. \end{aligned} \tag{5.31}$$

And the boundary condition is

$$\lim_{x\to\pm 1} (xB_1 + B_0)\, W^{(\nu)}(x) = 0. \tag{5.32}$$

We start by proving the boundary condition (5.32). Because of Observation 5.6 we have that for all $i, j = 0, \cdots 2\ell$

$$\begin{aligned}\left(\lim_{x\to 1} (xB_1 + B_0)\, W^{(\nu)}(x)\right)_{i,j} &= \left(\lim_{x\to 1}(1-x^2)^{(\nu-1/2)}\,(xB_1 + B_0)\, W_{\mathrm{pol}}(1)\right)_{i,j} \\ &= \lim_{x\to 1}(1-x^2)^{(\nu-1/2)}\,(x(B_1)_{i,i} + (B_0)_{i,i+1} + (B_0)_{i,i-1}) \\ &= \lim_{x\to 1}(1-x^2)^{(\nu-1/2)}(1-x)(2\ell+\nu)\frac{\ell-i}{i} = 0,\end{aligned}$$

since $\nu > 0$. Similarly, we can see that

$$\left(\lim_{x\to -1} (xB_1 + B_0)\, W^{(\nu)}(x)\right)_{i,j} = 0, \quad \text{for } i, j = 0, \cdots, 2\ell.$$

For the symmetry equations, we check entry per entry the matrix equations (5.30) and (5.31). We do it for $i \le j-1$, the other cases can be done similarly. We start with the proof of (5.30). By preforming the matrix product we get that the (i,j)-component of (5.30) is given by

$$\begin{aligned}&\left(W^{(\nu)}_{\mathrm{pol}}(x)(xB^{(\nu)}_1 + B^{(\nu)}_0)^* + (xB^{(\nu)}_1 + B^{(\nu)}_0)W^{(\nu)}_{\mathrm{pol}}(x)\right)_{i,j} = \\ &-\frac{i}{2\ell}\sum_{t=0}^{i-1}\alpha^{(\nu)}_t(i-1,j)C^{(\nu)}_{i+j-2t-1}(x) - \frac{\ell-i}{\ell}\sum_{t=0}^{i} x\alpha^{(\nu)}_t(i,j)C^{(\nu)}_{i+j-2t}(x) \\ &+\frac{2\ell-i}{2\ell}\sum_{t=0}^{i+1}\alpha^{(\nu)}_t(i+1,j)C^{(\nu)}_{i+j-2t+1}(x) - \frac{j}{2\ell}\sum_{t=0}^{i}\alpha^{(\nu)}_t(i,j-1)C^{(\nu)}_{i+j-2t-1}(x) \\ &-\frac{\ell-j}{\ell}\sum_{t=0}^{i} x\alpha^{(\nu)}_t(i,j)C^{(\nu)}_{i+j-2t}(x) + \frac{2\ell-j}{2\ell}\sum_{t=0}^{i}\alpha^{(\nu)}_t(i,j+1)C^{(\nu)}_{i+j-2t+1}(x).\end{aligned} \tag{5.33}$$

where we have used the explicit expressions of $W^{(\nu)}$ from Theorem 5.1.4 and that $xB_1^{(\nu)} + B_0^{(\nu)}$ is tridiagonal. Next, we use the three-recurrence relation for Gegenbauer polynomials, see e.g. [76], [77],

$$xC^{(\nu)}_r(x) = \frac{(r+1)}{2(r+\nu)}C^{(\nu)}_{r+1}(x) + \frac{(r+2\nu-1)}{2(r+\nu)}C^{(\nu)}_{r-1},$$

to get rid of the multiplication by x. Notice that the coefficient of $C^{(\nu)}_{m+n+1}$ is equal to

$$\begin{aligned}&\alpha^{(\nu)}_0(i,j)\frac{i+j+1}{2(i+j+\nu)}\frac{j-\ell}{\ell} + \alpha^{(\nu)}_0(i,j+1)\frac{2\ell-j}{2\ell} \\ &\quad+\alpha^{(\nu)}_0(i,j)\frac{i+j+1}{2(i+j+\nu)}\frac{i-\ell}{\ell} + \alpha^{(\nu)}_0(i+1,j)\frac{2\ell-i}{2\ell} = 0,\end{aligned}$$

what follows from the expression of $\alpha_t^{(\nu)}(i,j)$ (5.9). Regrouping and using the explicit expressions for $\alpha_t^{(\nu)}(m,n)$ we get that the right hand side of (5.33) equals

$$\sum_{t=0}^{i}\alpha_t^{(\nu)}(i,j)\frac{(i+j-2t+2\nu-1)}{2\ell(i+j-2t+\nu)}C_{i+j-2t-1}^{(\nu)}(x)$$
$$\times\Bigg[\frac{(i+j-2t+\nu-1)(i+j+\nu-t)(2\ell-i+1)(i-t)}{(2\ell-i-j+t+1)(i+j-2t)(i+\nu-t-1)}$$
$$+\frac{(i+j-2t+\nu-1)(i+j+\nu-t)(2\ell-j+1)(j-t)}{2\ell(2\ell-i-j+t+1)(i+j-2t)(j+\nu-t-1)}$$
$$-(2\ell-i-j)+\frac{(i+j-1-2t+\nu)(2\ell+\nu-t)(i+1)(j-t)}{(i+j-2t)(t+1)(j+\nu-t-1)}$$
$$+\frac{(i+j-2t+\nu-1)(2\ell+\nu-t)(j+1)(i-t)}{(i+j-2t)(t+1)(i+\nu-t-1)}$$
$$-\frac{(2\ell-i-j)(i+j-2t+2\nu-2)(i-t)(i+j+\nu-t)(j-t)(2\ell+\nu-t)}{(j+\nu-t-1)(i+j-2t)(\nu+i-t-1)(2\ell-i-j+t+1)(t+1)}\Bigg],$$

and it is a straightforward check that the coefficient in square brackets is equal to 0.

For the proof of the second symmetry equation we proceed in the same way. For the (i,j)-entry of (5.31)

$$(2\nu-1)x((xB_1^{(\nu)}+B_0^{(\nu)})W_{\mathrm{pol}}^{(\nu)}(x))_{i,j}-(1-x^2)((xB_1^{(\nu)}+B_0^{(\nu)})W_{\mathrm{pol}}^{(\nu)}(x)')_{i,j}$$
$$+(1-x^2)[(A_0^{(\nu)})_{i,i}-(A_0^{(\nu)})_{j,j}-(B_1^{(\nu)})_{i,i}](W_{\mathrm{pol}}^{(\nu)}(x))_{i,j},$$

we use the three term recurrence for the Gegenbauer polynomials to get rid of the multiplication by x and by x^2. We also have to use

$$(1-x^2)\frac{dC_r^{(\nu)}}{dx}(x)=\frac{(r+2\nu-1)(r+2\nu)}{2(r+\nu)}C_{r-1}^{(\nu)}(x)-\frac{r(r+1)}{2(r+\nu)}C_{r+1}^{(\nu)}(x), \tag{5.34}$$

see e.g. [76, (4.5.7)] with the convention $C_{-1}^{(\nu)}(x)=0$. So this gives again a sum only involving Gegenbauer polynomials as in the check of the (5.30). Insert the explicit expression of the coefficients $\alpha_t^{(\nu)}(i,j)$ (5.9) to complete the proof by a straightforward but cubersome computation to see that each coefficient in the expansion of Gegenbauer polynomials vanishes.

This completes the proof of the symmetry of $E^{(\nu)}$ with respect to $W^{(\nu)}$.

The symmetry of $D^{(\nu)}$ with respect to $W^{(\nu)}$. The symmetry equations for $D^{(\nu)}$ and $W^{(\nu)}$ are

(5.35)
$$2\frac{d}{dx}\Big((1-x^2)W^{(\nu)}(x)\Big)-(C^{(\nu)}-xU^{(\nu)})W^{(\nu)}(x)=W^{(\nu)}(x)(C^{(\nu)}-xU^{(\nu)})^*,$$
(5.36)
$$\frac{d^2}{dx^2}\Big((1-x^2)W^{(\nu)}(x)\Big)-\frac{d}{dx}\Big((C^{(\nu)}-xU^{(\nu)})W^{(\nu)}(x)\Big)+V^{(\nu)}W^{(\nu)}(x)=W^{(\nu)}(x)(V^{(\nu)})^*,$$

and the boundary condition are

$$\text{(5.37)}\quad \lim_{x\to\pm1}(1-x^2)W^{(\nu)}(x)=0, \qquad \lim_{x\to\pm1}\big((1-x^2)W^{(\nu)}(x)\big)'-(C^{(\nu)}-xU^{(\nu)})W^{(\nu)}(x)=0.$$

We start by proving the symmetry equations.

We skip the proof of (5.35) because it is completely similar to that of (5.30).

For the symmetry equation (5.36) by making use of (5.35) we see that it suffices to prove

$$\text{(5.38)}\quad \big(W^{(\nu)}(x)(C^{(\nu)}-xU^{(\nu)})^*-(C^{(\nu)}-xU^{(\nu)})W^{(\nu)}(x)\big)'=2(W^{(\nu)}(x)V^{(\nu)}-V^{(\nu)}W^{(\nu)}(x)).$$

We prove instead that

$$\text{(5.39)}\quad \begin{aligned} &W^{(\nu)}(x)(C^{(\nu)}-xU^{(\nu)})^*-(C^{(\nu)}-xU^{(\nu)})W^{(\nu)}(x)\\ &\quad-2\Big(\int W^{(\nu)}(x)dx\Big)V^{(\nu)}+2V^{(\nu)}\Big(\int W^{(\nu)}(x)dx\Big)=0,\end{aligned}$$

which is obtained by integrating (5.38) with respect to x. Then, (5.38) follows by taking the derivative with respect to x.

We prove that the (i,j) component for $i<j$ is zero. The other cases can be proved similarly. By performing the matrix product we see that the (i,j) component of $\big(W(x)(C^{(\nu)}-xU^{(\nu)})^*-(C^{(\nu)}-xU^{(\nu)})W(x)\big)$ is given by

$$\text{(5.40)}\quad \begin{aligned}&\rho(x)\sum_{t=0}^{i}C^{(\nu)}_{i+j-2t-1}\Big[j\alpha_t^{(\nu)}(i,j-1)+(2\ell-j)\alpha_{t+1}^{(\nu)}(i,j+1)-i\alpha_{t+1}^{(\nu)}(i-1,j)-(2\ell-i)\alpha_{t+1}^{(\nu)}(i+1,j)\Big]\\ &=-\sum_{t=0}^{i}\rho(x)C^{(\nu)}_{i+j-2t-1}[(\nu-1)(i+j-2t\ell-2t-2\ell-1+i)+2(i-t)(j-t)(\ell+1)]\\ &\quad\times\frac{(2\ell-i-j)(i-j)(i+j-1-2t+\nu)(i+j+2\nu-1-2t)}{(i+j-2t+\nu)(i+j-2t)(i+\nu-t-1)(2\ell-i-j+t+1)(j+\nu-t-1)}.\end{aligned}$$

In order to compute the part of (5.39) involving the integrals, we use the following formula for the Gegenbauer polynomials:

$$\text{(5.41)}\quad \int\rho(x)C_r^{(\nu)}(x)dx=\rho(x)\left(\frac{(r+1)C_{r+1}^{(\nu)}(x)}{2(\nu+r)(2\nu+r)}-\frac{(2\nu+r-1)C_{r-1}^{(\nu)}(x)}{2r(\nu+r)}\right),$$

which follows from the Rodrigues formulas for the Gegenbauer polynomials [76, (4.5.11)], [76, (4.5.5)] and (5.34). Using (5.41) for the integrals in (5.39) we obtain

$$\begin{aligned}&2\Big(\Big(\int W^{(\nu)}(x)dx\Big)V^{(\nu)}-V^{(\nu)}\Big(\int W^{(\nu)}(x)dx\Big)\Big)_{i,j}\\ &\quad=(i-j)(2\ell-i-j)\rho(x)\sum_{t=0}^{i}\alpha_t^{(\nu)}(i,j)\Big[-\frac{(2\nu+i+j-2t-1)}{(i+j-2t)(\nu+i+j-2t)}\\ &\qquad\times\frac{\alpha_t^{(\nu)}(i,j)}{\alpha_{t+1}^{(\nu)}(i,j)}\frac{(i+j-2t-1}{(\nu+i+j-2t-2)(2\nu+i+j-2t-2)}\Big]C^{(\nu)}_{i+j-2t-1}(x),\end{aligned}$$

which is (5.40).

We now prove the boundary condition (5.37). The first one is direct since $W^{(\nu)}(x) = (1-x^2)^{\nu-1/2}W^{(\nu)}_{\text{pol}}(x)$, $\nu > 0$ and $W^{(\nu)}_{\text{pol}}(x)$ is a matrix polynomial. For the second boundary condition we use the first symmetry equation (5.35) to see that it is equivalent to prove that

$$\lim_{x\to\pm 1}\Big(\big(W^{(\nu)}(x)(xU^{(\nu)}-C^{(\nu)})^* - (xU^{(\nu)}-C^{(\nu)})W^{(\nu)}(x)\big)\Big)_{i,j} = 0, \quad \text{for all } i,\, j = 0,\cdots,2\ell.$$

Since $U^{(\nu)}$ is a multiple of the identity matrix we get that the second boundary condition for $D^{(\nu)}$ holds if and only if $\lim_{x\to\pm1}\big(W^{(\nu)}(x)(C^{(\nu)})^* - C^{(\nu)}W^{(\nu)}(x)\big)_{i,j} = 0$ for all $i,\, j = 0,\cdots,2\ell$. By performing the matrix product we get

(5.42)
$$\begin{aligned}\lim_{x\to\pm1}\Big(W^{(\nu)}(x)(C^{(\nu)})^* - C^{(\nu)}W^{(\nu)}(x)\Big)_{i,j} = \lim_{x\to\pm1}\Big(&(W^{(\nu)})_{i,j-1}(x)C^{(\nu)}_{j,j-1} + (W^{(\nu)})_{i,j+1}(x)C^{(\nu)}_{j,j+1}\\ &-(C^{(\nu)})_{i,i-1}(W^{(\nu)}(x))_{i-1,j} - (C^{(\nu)})_{i,i+1}(W^{(\nu)}(x))_{i+1,j}\Big).\end{aligned}$$

From the expression of $W^{(\nu)}$ in Definition 5.1.1 it follows that

$$\lim_{x\to\pm1}(W^{(\nu)}(x))_{i,j} = t_0^{(\nu)}\lim_{x\to\pm1}(1-x^2)^{\nu-1/2}L^{(\nu)}_{i,0}(x)L^{(\nu)}_{j,0}(x).$$

Therefore (5.42) becomes

$$\begin{aligned}\lim_{x\to\pm1}(1-x^2)^{\nu-1/2}\big(&j\,L^{(\nu)}_{i,0}(x)L^{(\nu)}_{j-1,0}(x) + (2\ell-j)\,L^{(\nu)}_{i,0}(x)L^{(\nu)}_{j+1,0}(x)\\ &-i\,L^{(\nu)}_{j,0}(x)L^{(\nu)}_{i-1,0}(x) - (2\ell-i)\,L^{(\nu)}_{j,0}(x)L^{(\nu)}_{i+1,0}(x)\big).\end{aligned}$$

By introducing the explicit values from the definition of the matrix functions $L^{(\nu)}(x)$ and taking into account the expression of the Gegenbauer polynomials (5.1) and the symmetry relation

$$C^{(\nu)}_n(x) = (-1)^n C^{(\nu)}_n(-x) = (-1)^n\frac{(2\nu)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+2\nu\\ \nu+\frac{1}{2}\end{matrix};\frac{1+x}{2}\right),$$

we get that

$$\begin{aligned}&\big(j\,L^{(\nu)}_{i,0}(x)L^{(\nu)}_{j-1,0}(x) + (2\ell-j)\,L^{(\nu)}_{i,0}(x)L^{(\nu)}_{j+1,0}(x)\\ &\quad -i\,L^{(\nu)}_{j,0}(x)L^{(\nu)}_{i-1,0}(x) - (2\ell-i)\,L^{(\nu)}_{j,0}(x)L^{(\nu)}_{i+1,0}(x)\big)\\ &=(1-x)K_1(x) = (1+x)K_2(x),\end{aligned}$$

where $K_1(x)$ and $K_2(x)$ are matrix polynomials. Therefore, since $\nu > 0$ we have

$$\lim_{x\to\pm1}\Big(W^{(\nu)}(x)(C^{(\nu)})^* - C^{(\nu)}W^{(\nu)}(x)\Big)_{i,j} = 0, \quad \text{for all } i,\, j = 0,\cdots,2\ell.$$

That is, the second boundary condition in (5.37) holds. So, by Theorem 1.3.7, $D^{(\nu)}$ is symmetric with respect to $W^{(\nu)}$.

It just remains to see that the monic orthogonal polynomials satisfy (5.28) and (5.29). To do so we first notice that $E^{(\nu)}$ and $D^{(\nu)}$ do not raise the degree of polynomials, so Lemma 1.3.3 applies and we get that the monic orthogonal polynomials are eigenfunctions of $D^{(\nu)}$ and $E^{(\nu)}$. The expression of the eigenvalues in (5.28) and (5.29) follows directly by equating the corresponding coefficients.

□

For matrix orthogonal polynomials, Cantero, Moral and Velázquez [12] established criteria for the derivatives of matrix orthogonal polynomials to be again matrix orthogonal polynomials. A result that is recovered in Theorem 1.3.9.

In order to establish the connection with Theorem 1.3.9, we consider the following second order differential operator for $\nu > 0$[1],

$$D^{(\nu)}_{(\Phi,\Psi)} = (E^{(\nu)})^2 + (2\ell+2)E^{(\nu)} + \left(\frac{(\ell+\nu)^2}{\ell^2}\right)D^{(\nu)} + \frac{\nu\,(\nu-1)\,(2\,\ell+\nu+1)\,(\nu+2\,\ell)}{\ell^2}I. \tag{5.43}$$

By Theorem 5.2.1, $D^{(\nu)}_{(\Phi,\Psi)}$ is symmetric with respect to the weight $W^{(\nu)}$. Moreover, it commutes with $\mathfrak{J}$.

Proposition 5.2.1. *The matrix differential operator $D^{(\nu)}_{(\Phi,\Psi)}$ commutes with $\mathfrak{J}$.*

Proof. A straightforward calculation shows that $D^{(\nu)}$ commutes with $\mathfrak{J}$, and that the differential operator $E^{(\nu)}$ satisfies

$$\mathfrak{J}E^{(\nu)}\mathfrak{J} = -\left(E^{(\nu)} + (\ell+1)I\right).$$

Then, by the definition of $D^{(\nu)}_{(\Phi,\Psi)}$, it is clear that

$$\begin{aligned}\mathfrak{J}D^{(\nu)}_{(\Phi,\Psi)}\mathfrak{J} &= \mathfrak{J}\left((E^{(\nu)})^2 + (2\ell+2)E^{(\nu)} + \left(\frac{(\ell+\nu)^2}{\ell^2}\right)D^{(\nu)} + \frac{\nu\,(\nu-1)\,(2\,\ell+\nu+1)\,(\nu+2\,\ell)}{\ell^2}I\right)\mathfrak{J}\\ &= (E^{(\nu)})^2 + (2\ell+2)E^{(\nu)} + \left(\frac{(\ell+\nu)^2}{\ell^2}\right)D^{(\nu)} + \frac{\nu\,(\nu-1)\,(2\,\ell+\nu+1)\,(\nu+2\,\ell)}{\ell^2}I.\end{aligned}$$

□

A straightforward calculation shows that $D^{(\nu)}_{(\Phi,\Psi)}$ can be written as

$$D^{(\nu)}_{(\Phi,\Psi)} = \frac{d^2}{dx^2}\Phi^{(\nu)}(x)^* + \frac{d}{dx}\Psi^{(\nu)}(x)^*, \tag{5.44}$$

[1]The notation of this operator is motivated by equation (5.44)

where $\Phi^{(\nu)}$ and $\Psi^{(\nu)}$ are the explicit matrix polynomials of degree 2 and 1,

$$(5.45)\qquad \begin{aligned}\Phi^{(\nu)}(x) =&x^2\sum_{i=0}^{2\ell}\frac{(\ell-i)^2-(\ell+\nu)^2}{\ell^2}\mathcal{E}_{i,i}+x\sum_{i=1}^{2\ell}\frac{(i-1-2\ell)(2\ell-2i+1)}{2\ell^2}\mathcal{E}_{i,i-1}\\ &+x\sum_{i=0}^{2\ell-1}\frac{(i+1)(2\ell-2i-1)}{2\ell^2}\mathcal{E}_{i,i+1}+\sum_{i=2}^{2\ell}\frac{(2\ell-i+2)(2\ell-i+1)}{4\ell^2}\mathcal{E}_{i,i-2}\\ &+\sum_{i=0}^{2\ell}\frac{-i(2\ell-i+1)-(2\ell-i)(i+1)+4(\ell+\nu)^2}{4\ell^2}\mathcal{E}_{i,i}\\ &+\sum_{i=0}^{2\ell-2}\frac{(i+2)(i+1)}{4\ell^2}\mathcal{E}_{i,i+2},\end{aligned}$$

$$(5.46)\qquad \begin{aligned}\Psi^{(\nu)}(x) =& -x\sum_{i=0}^{2\ell}\frac{(2\ell+2\nu+1)\,(\nu+i)\,(\nu+2\,\ell-i)}{\ell^2}\mathcal{E}_{i,i}\\ &-(\ell+\nu+\frac{1}{2})\left(\sum_{i=1}^{2\ell}\frac{(i-1-2\,\ell)\,(\nu+i-1)}{\ell^2}\mathcal{E}_{i,i-1}+\sum_{i=0}^{2\ell-1}\frac{(i+1)\,(\nu+2\,\ell-i-1)}{\ell^2}\mathcal{E}_{i,i+1}\right).\end{aligned}$$

Proposition 5.2.2. *Let $\nu>0$. The weight matrix $W^{(\nu)}(x)$ and the matrix polynomials $\Phi^{(\nu)}(x)$ and $\Psi^{(\nu)}(x)$ satisfy $\left(W^{(\nu)}(x)\Phi^{(\nu)}(x)\right)'=W^{(\nu)}(x)\Psi^{(\nu)}(x)$.*

Proof. Since $D^{(\nu)}_{(\Phi,\Psi)}$ is a combination of symmetric differential operators it is symmetric, so it satisfies the symmetry equations

$$(5.47)\qquad (\Phi^{(\nu)}(x)^*W^{(\nu)}(x))''-(\Psi^{(\nu)}(x)^*W^{(\nu)}(x))'=0,$$

and the boundary condition

$$(5.48)\qquad \lim_{x\to-1}\left(\Phi^{(\nu)}(x)^*W^{(\nu)}(x)\right)'-\Psi^{(\nu)}(x)^*W^{(\nu)}(x)=0$$

holds. Integrating (5.47) in $(-1,x)$ and taking into account (5.48) we get

$$(\Phi^{(\nu)}(x)^*W^{(\nu)}(x))'=\Psi^{(\nu)}(x)^*W^{(\nu)}(x).$$

The result follows by taking adjoints. □

Theorem 5.2.3. *Let $\nu>0$, $x\in(-1,1)$ then, $W^{(\nu+1)}(x)=c^{(\nu)}W^{(\nu)}(x)\Phi^{(\nu)}(x)$, where* $c^{(\nu)}=\frac{(2\nu+1)(2\ell+\nu+1)\ell^2}{\nu(2\nu+2\ell+1)(2\ell+\nu)(\ell+\nu)}$.

Proof. From Proposition 5.2.2 and taking into account that the boundary condition

$$\lim_{x\to-1}\Phi^{(\nu)}(x)^*W^{(\nu)}(x)=0$$

holds, it is equivalent to prove

$$(5.49)\qquad c^{(\nu)}W^{(\nu)}(x)\Psi^{(\nu)}(x)=\left(W^{(\nu+1)}(x)\right)'.$$

The proof of this theorem uses the same ingredients as the proof of Theorem 5.2.1. Let us calculate the (i,j) entry of both sides in (5.49). We assume $i<j$, the other situations can be treated in an analogue way. Now $\big(W^{(\nu)}(x)\Psi^{(\nu)}(x)\big)_{i,j}$ can be written explicitly using Theorem 5.1.4 and (5.46). This gives explicit terms involving Gegenbauer polynomials multiplied by x. Then, using the three term recurrence relation for Gegenbauer polynomials and regrouping we can write $\big(W^{(\nu)}(x)\Psi^{(\nu)}(x)\big)_{i,j}$ as the following expansion in terms of Gegenbauer polynomials:

$$\frac{(2\nu+1)(2\ell+\nu+1)}{\nu(2\ell+\nu)(\ell+\nu)}\sum_{t=0}^{m}\eta_t^{(\nu)}(i,j)C^{(\nu)}_{i+j-2t+1}(x), \tag{5.50}$$

where

$$\begin{aligned}
\eta_t^{(\nu)}(i,j) =& \frac{-(\nu+j)(\nu+2\ell-j)(i+j-2t+1)}{i+j-2t+\nu}\alpha_t^{(\nu)}(i,j)-(j-2\ell)(\nu+j)\alpha_t^{(\nu)}(i,j+1)\\
&+j(\nu+2\ell-j)\alpha_t^{(\nu)}(i,j-1)-\frac{(\nu+j)(\nu+2\ell-j)(2\nu+i+j-2t+1)}{i+j-2t+\nu+2}\alpha_{t-1}^{(\nu)}(i,j),\\
\eta_0^{(\nu)}(i,j) =& \frac{-(\nu+j)(\nu+2\ell-j)(i+j+1)}{(i+j+\nu)}\alpha_0^{(\nu)}(i,j)-(j-2\ell)(\nu+j)\alpha_0^{(\nu)}(i,j+1),\\
\eta_{i+1}^{(\nu)}(i,j) =& j(\nu+2\ell-j)\alpha_i^{(\nu)}(i,j-1)-\frac{(\nu+j)(\nu+2\ell-j)}{(j-i+\nu)}\alpha_i^{(\nu)}(i,j),
\end{aligned}$$

for $t\in\{1,\cdots,i\}$. On the other hand, by (5.34) we can expand $\big(W^{(\nu+1)}(x)\big)'_{i,j}$ in the same basis leading to

$$(1-x^2)^{\nu-1/2}\sum_{t=0}^{i}\frac{-(i+j-2t+1)(2\nu+i+j-2t+1)}{2\nu}\alpha_t^{(\nu+1)}(i,j)C^{(\nu)}_{i+j-2t+1}(x). \tag{5.51}$$

To obtain (5.51) we calculate

$$\int_{-1}^{1}C_k^{(\nu)}(x)\left(W_{i,j}^{(\nu+1)}(x)\right)'\,dx=-\int_{-1}^{1}\frac{dC_k^{(\nu)}}{dx}(x)(1-x^2)^{\nu+\frac12}\sum_{t=0}^{i}\alpha_t^{(\nu+1)}(i,j)C^{(\nu+1)}_{i+j-2t}(x)\,dx.$$

Since $\frac{dC_k^{(\nu)}}{dx}(x)=2\nu C_{k-1}^{(\nu+1)}$ by [76, (4.5.5)], the integral on the right hand side of the previous equation can be calculated using the orthogonality relations for the Gegenbauer polynomials (5.2), leading to (5.51).

By a straightforward computation we check that (5.50) and (5.51) are the same, up to the constant, as given in Theorem 5.2.3.

□

By the previous theorems and Theorem 1.3.9 we have the following corollary.

Corollary 5.2.4. *For $\nu>0$, the sequence $(\frac{dP_n^{(\nu)}}{dx})_n$ is a sequence of matrix orthogonal polynomials with respect to weight function $W^{(\nu+1)}$ on $[-1,1]$. In particular, $\frac{dP_n^{\nu}}{dx}(x)=nP_{n-1}^{(\nu+1)}(x)$, $n\geq 1$.*

Then, we get that $\frac{d}{dx}$ connects the sequence $(P_n^{(\nu)})_n$ with $(P_n^{(\nu+1)})_n$ and we say that $\frac{d}{dx}$ is a forward shift operator. We now show how to go form $(P_n^{(\nu+1)})_n$ to $(P_n^{(\nu)})_n$ via a differential operator.

Lemma 5.2.5. *Let $\nu > 0$. Define the first order matrix differential operator, $T^{(\nu)}$ by*

$$\big(T^{(\nu)}Q\big)(x) = \frac{dQ}{dx}(x)\big(\Phi^{(\nu)}(x)\big)^* + Q(x)\big(\Psi^{(\nu)}(x)\big)^*, \tag{5.52}$$

then, $\big\langle \frac{dP}{dx}, Q\big\rangle^{(\nu+1)} = -c^{(\nu)}\big\langle P, T^{(\nu)}Q\big\rangle^{(\nu)}$, for matrix polynomials P and Q.

Proof. For $\nu > 0$ and P, $Q \in \mathbb{C}^{N\times N}$ we have

$$\begin{aligned}\Big\langle \frac{dP}{dx}, Q\Big\rangle^{(\nu+1)} &= \int_{-1}^{1} \frac{d}{dx}P(x)W^{(\nu+1)}Q(x)\\ &= -\int_{-1}^{1} P(x)\left(\frac{d}{dx}\left(W^{(\nu+1)}\right)Q(x) + W^{(\nu+1)}\frac{d}{dx}Q(x)\right)dx,\end{aligned}$$

where we have used the boundary condition for $W^{(\nu+1)}$. Now by applying Lemma 5.2.3 and Proposition 5.2.2 we obtain

$$\begin{aligned}\Big\langle \frac{dP}{dx}, Q\Big\rangle^{(\nu+1)} &= -c^{(\nu)}\int_{-1}^{1} P(x)\left(W^{(\nu)}\Psi^{(\nu)}(x)Q(x) + W^{(\nu)}\Phi^{(\nu)}(x)\frac{d}{dx}Q(x)\right)dx\\ &= -c^{(\nu)}\big\langle P, T^{(\nu)}Q\big\rangle^{(\nu)}.\end{aligned}$$

□

We can now exploit Lemma 5.2.5 in order to obtain information about the monic matrix orthogonal polynomials $(P_n^{(\nu)})_n$.

Theorem 5.2.6. *Let $(P_n^{(\nu)})_n$ be the sequence of monic orthogonal polynomials with respect to $W^{(\nu)}$. Then*

(i) the squared norm of $P_n^{(\nu)}$, $H_n^{(\nu)}$ is a diagonal matrix whose entries are given by

$$\begin{aligned}\big(H_n^{(\nu)}\big)_{k,k} =& \sqrt{\pi}\,\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)}\,\frac{\nu(2\ell+\nu+n)}{\nu+n}\,\frac{n!\,(\ell+\frac{1}{2}+\nu)_n(2\ell+\nu)_n(\ell+\nu)_n}{(2\ell+\nu+1)_n(\nu+k)_n(2\ell+2\nu+n)_n(2\ell+\nu-k)_n}\\ &\times\frac{k!\,(2\ell-k)!\,(n+\nu+1)_{2\ell}}{(2\ell)!\,(n+\nu+1)_k(n+\nu+1)_{2\ell-k}}.\end{aligned} \tag{5.53}$$

(ii) The following Rodrigues formula holds:

$$\begin{aligned}P_n^{(\nu)}(x) =& G_n^{(\nu)}\frac{d^n}{dx^n}\big(W^{(\nu+n)}(x)\big)W^{(\nu)}(x)^{-1}\\ \big(G_n^{(\nu)}\big)_{j,k} =& \delta_{j,k}\frac{(-1)^n(\nu)_n(\ell+\nu+\frac{1}{2})_n(\ell+\nu)_n(2\ell+\nu)_n}{(\nu+\frac{1}{2})_n(\nu+k)_n(2\ell+\nu+1)_n(2\ell+2\nu+n)_n(2\ell+\nu-k)_n}.\end{aligned} \tag{5.54}$$

Proof. To see that (i) holds we first notice that $D^{(\nu)}_{\Phi,\Psi} = T^{(\nu)} \circ \frac{d}{dx}$ then,

$$T^{(\nu)}(P^{(\nu+1)}_{n-1}(x)) = \frac{1}{n} T^{(\nu)} \frac{d}{dx} P^{(\nu)}_n(x) = \frac{1}{n} D^{(\nu)}_{\Phi,\Psi} P^{(\nu)}_n(x) = \frac{1}{n} K^{(\nu)}_n P^{(\nu)}_n(x),$$

where $K^{(\nu)}_n$ can be determined from the leading coefficients of $\Phi^{(\nu)}$ and $\Psi^{(\nu)}$. In particular, $T^{(\nu)}$ is a backward shift operator. Explicitly, $K^{(\nu)}_n$ is a diagonal invertible matrix for all $\nu > 0$,

$$\left(K^{(\nu)}_n\right)_{k,k} = -\frac{(\nu+k)(2\ell+2\nu+n)}{\ell^2}(2\ell+\nu-k).$$

Then, we can write

$$\begin{aligned} H^{(\nu)}_n &= \langle P^{(\nu)}_n, P^{(\nu)}_n \rangle^{(\nu)} = \left(K^{(\nu)}_n\right)^{-1} \langle D_{\Phi,\Psi} P^{(\nu)}_n, P^{(\nu)}_n \rangle^{(\nu)} \\ &= \left(K^{(\nu)}_n c^{(\nu)}_n\right)^{-1} \langle \frac{d}{dx} P^{(\nu)}_n, \frac{d}{dx} P^{(\nu)}_n \rangle^{(\nu+1)} = \left(K^{(\nu)}_n\right)^{-1} \frac{n^2}{c^{(\nu)}_n} \langle P^{(\nu+1)}_{n-1}, P^{(\nu+1)}_{n-1} \rangle^{(\nu+1)}. \end{aligned}$$

Iterating, we get

$$H^{(\nu)}_n = \prod_{i=0}^{n-1} (n-i)^2 \left(c^{(\nu+i)} K^{(\nu+i)}_{n-i}\right)^{-1} H^{(\nu+n)}_0,$$

where the order in the product does not matter since all the $K^{(\nu+i)}_{n-i}$ are diagonal. By performing the product using the expression (5.21) of $H^{(\nu+n)}_0$, we get the result.

For (ii), use Theorem 5.2.3 and (5.49) to see that $T^{(\nu)}$ can be written as

$$T^{(\nu)} Q(x) = (c^{(\nu)})^{-1} \frac{d}{dx} \left(Q(x) W^{(\nu+1)}\right) \left(W^{(\nu)}\right)^{-1}, \quad x \in (-1,1), \tag{5.55}$$

where $Q \in \mathbb{C}^{N\times N}[x]$ and $\mathrm{dgr}(T^{(\nu)} Q) = \mathrm{dgr}(Q) + 1$. Iterating leads to

$$\left(\prod_{i=0}^{n-1} c^{(\nu+i)}\right) \left(((Q T^{(\nu+n-1)}) \cdots T^{(\nu+1)}) T^{(\nu)}\right)(x) = \frac{d^n}{dx^n}\left(Q(x) W^{(\nu+n)}(x)\right) W^{(\nu)}(x)^{-1}.$$

Now take $Q(x) = P^{(\nu+n)}_0(x) = 1$, so the previous equality becomes

$$\prod_{i=0}^{n-1} c^{(\nu+i)} P^{(\nu)}_n(x) = \left(\prod_{i=0}^{n-1} K^{(\nu+i)}_{n-i}\right)^{-1} \frac{d^n}{dx^n}\left(W^{(\nu+n)}(x)\right) W^{(\nu)}(x)^{-1}.$$

A direct check shows that

$$G_n = \left(\prod_{i=0}^{n-1} c^{(\nu+i)} K^{(\nu+i)}_{n-i}\right)^{-1},$$

and this proves the result. □

5.3 Expression in terms of matrix hypergeometric functions

In this section we link the matrix orthogonal polynomials with respect to $W^{(\nu)}$ to Tirao's matrix hypergeometric series ${}_2H_1$ (see defintion 1.40). In order to show this link we switch from the interval $[-1,1]$ to $[0,1]$ using $x = 1-2u$. Set

$$R_n^{(\nu)}(u) = (-1)^n 2^{-n} P_n^{(\nu)}(1-2u), \qquad Z^{(\nu)}(u) = W_{\mathrm{pol}}^{(\nu)}(1-2u). \tag{5.56}$$

Hence, the rescaled monic matrix orthogonal polynomials $R_n^{(\nu)}$ satisfy

$$\int_0^1 R_n^{(\nu)}(u) Z^{(\nu)}(u) R_m^{(\nu)}(u)^* \left(u(1-u)\right)^{\nu-1/2} du = \delta_{n,m} 2^{-2n-2\nu} H_n^{(\nu)}. \tag{5.57}$$

In this setting Theorem 5.2.1 gives the following corollary.

Corollary 5.3.1. *Let $\widetilde{D}^{(\nu)}$ and $\widetilde{E}^{(\nu)}$ be the matrix differential operators*

$$\widetilde{D}^{(\nu)} = u(1-u)\frac{d^2}{du^2} + \frac{d}{dx}(\widetilde{C}^{(\nu)} - u\widetilde{U}^{(\nu)}) - \widetilde{V}^{(\nu)}, \quad \widetilde{E}^{(\nu)} = \frac{d}{dx}(u\widetilde{B}_1^{(\nu)} + \widetilde{B}_0^{(\nu)}) + \widetilde{A}_0^{(\nu)},$$

where $\widetilde{C}^{(\nu)}$, $\widetilde{U}^{(\nu)}$, $\widetilde{V}^{(\nu)}$, $\widetilde{B}_0^{(\nu)}$, $\widetilde{B}_1^{(\nu)}$ and $\widetilde{A}_0^{(\nu)}$ are given by

$$\widetilde{C}^{(\nu)} = -\sum_{i=0}^{2\ell}\frac{(2\ell-i)}{2}\mathcal{E}_{i,i+1} + \sum_{i=0}^{2\ell}\frac{(2\ell+2\nu+1)}{2}\mathcal{E}_{i,i} - \sum_{i=0}^{2\ell}\frac{i}{2}\mathcal{E}_{i,i-1}, \quad \widetilde{U}^{(\nu)} = (2\ell+2\nu+1)I,$$

$$\widetilde{V}^{(\nu)} = -\sum_{i=0}^{2\ell} i(2\ell-i)\mathcal{E}_{i,i} + (\nu-1)(2\ell+\nu+1)I,$$

$$\widetilde{B}_0^{(\nu)} = -\sum_{i=0}^{2\ell}\frac{(2\ell-i)}{4\ell}\mathcal{E}_{i,i+1} + \sum_{i=0}^{2\ell}\frac{(\ell-i)}{2\ell}\mathcal{E}_{i,i} + \sum_{i=0}^{2\ell}\frac{i}{4\ell}\mathcal{E}_{i,i-1}, \qquad \widetilde{B}_1^{(\nu)} = -\sum_{i=0}^{2\ell}\frac{(\ell-i)}{\ell}\mathcal{E}_{i,i},$$

$$\widetilde{A}_0^{(\nu)} = \sum_{i=0}^{2\ell}\left(\frac{(\ell+1)(i-2\ell)}{\ell} - (\nu-1)\frac{(\ell-i)}{\ell}\right)\mathcal{E}_{i,i}.$$

Then, $\widetilde{D}^{(\nu)}$ and $\widetilde{E}^{(\nu)}$ are symmetric with respect to the weight $\big(u(1-u)\big)^{\nu-1/2}Z^{(\nu)}(u)$, and $\widetilde{D}^{(\nu)}$ and $\widetilde{E}^{(\nu)}$ commute. Moreover, for every integer $n \geq 0$, $R_n^{(\nu)}\widetilde{D}^{(\nu)} = \Lambda_n(\widetilde{D}^{(\nu)})R_n^{(\nu)}$, and $R_n^{(\nu)}\widetilde{E}^{(\nu)} = \Lambda_n(\widetilde{E}^{(\nu)})R_n^{(\nu)}$ with

$$\Lambda_n(\widetilde{D}^{(\nu)}) = ((\nu-1)(2\ell+\nu+1) - n(2\ell+2\nu+n))I - \sum_{i=0}^{2\ell} i(2\ell-i)\mathcal{E}_{i,i},$$

$$\Lambda_n(\widetilde{E}^{(\nu)}) = -(n+\nu+2\ell+1)I - (n+\nu+\ell)\sum_{i=0}^{2\ell}\frac{i}{\ell}\mathcal{E}_{i,i}.$$

We can describe the rows of $R_n^{(\nu)}$ in terms of matrix hypergeometric functions (see [98]). We first need to study the $\mathbb{C}^{2\ell+1}$-valued polynomial solutions of $P\widetilde{D}^{(\nu)} = \lambda P$, $\lambda \in \mathbb{C}$. In order to avoid technical problems we consider

$$D_\alpha^{(\nu)} = \widetilde{D}^{(\nu)} + \alpha\widetilde{E}^{(\nu)} = u(1-u)\frac{d^2}{du^2} + \frac{d}{du}(C_\alpha^{(\nu)} - uU_\alpha^{(\nu)}) - V_\alpha^{(\nu)}, \qquad \alpha \in \mathbb{R}, \tag{5.58}$$

where $C_\alpha^{(\nu)} = \widetilde{C}^{(\nu)} + \alpha\widetilde{B}_0^{(\nu)}$, $U_\alpha^{(\nu)} = \widetilde{U}^{(\nu)} - \alpha\widetilde{B}_1^{(\nu)}$ and $V_\alpha^{(\nu)} = \widetilde{V}^{(\nu)} - \alpha\widetilde{A}_0^{(\nu)}$. It follows from Corollary 5.3.1 that

$$R_n^{(\nu)} D_\alpha^{(\nu)} = \Lambda_n(D_\alpha^{(\nu)}) R_n^{(\nu)}, \quad \Lambda_n(D_\alpha^{(\nu)}) = -n^2 - n(\widetilde{U}_\alpha^{(\nu)} - 1) - \widetilde{V}_\alpha^{(\nu)}, \qquad \text{for all } n \in \mathbb{N}_0. \tag{5.59}$$

We denote by $\lambda_n^\alpha(j)$ the j-th diagonal entry of $\Lambda_n(D_\alpha^{(\nu)})$, i.e.

$$\begin{aligned}\lambda_n^\alpha(j) = &- n^2 - n\,\frac{(2\ell(\ell+\nu) + \alpha(\ell - j) - \ell)}{\ell} \\ &- \frac{(2\ell - j)(\alpha(\ell+1) - \ell j)}{\ell} + (\nu - 1)\frac{\ell(2\ell + \nu + 1) - \alpha(\ell - j)}{\ell}.\end{aligned}$$

It follows from (5.59) that the i-th row of $R_n^{(\nu)}$ is a solution of

$$u(1-u)F''(u) + F'(u)(C_\alpha^{(\nu)} - uU_\alpha^{(\nu)}) - F(u)(V_\alpha^{(\nu)} + \lambda) = 0, \quad \lambda = (\Lambda_n(D_\alpha^{(\nu)}))_{i,i}, \tag{5.60}$$

which is an instance of the matrix hypergeometric equation [98]. In order to be able to apply Tirao's approach [98], and to have the rows of $R_n^{(\nu)}$ defined by this solution, we need the following lemma, whose proof is skipped since it is completely analogue to that in [79, Lem. 4.3].

Lemma 5.3.2. *(i) The eigenvalues of* $C_\alpha^{(\nu)}$ *are* $(2j+2\nu+1)/2$, $j \in \{0, \dots, 2\ell\}$. *In particular,* $\sigma(C_\alpha^{(\nu)}) \cap \{-\mathbb{N}\} = \varnothing$.
(ii) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. *Then,* $(j,n) = (i,m) \in \{0, 1 \cdots, 2\ell\} \times \mathbb{N}$ *if and only if* $\lambda_n^\alpha(j) = \lambda_m^\alpha(i)$.

Since the eigenvalues of $C_\alpha^{(\nu)}$ are not in $-\mathbb{N}$, for any $F_0 \in \mathbb{C}^{2\ell+1}$, a (row-)vector-valued solution to (5.60) is given by

$$F(u) = \left({}_2H_1\left(\begin{matrix} (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda \\ (C_\alpha^{(\nu)})^t \end{matrix} ; u \right) F_0 \right)^t, \tag{5.61}$$

where the matrix hypergeometric function ${}_2H_1$, defined as the power series

$$\begin{gathered} {}_2H_1\left(\begin{matrix} U, V \\ C \end{matrix} ; z \right) = \sum_{i=0}^{\infty} \frac{z^i}{i!} [C, U, V]_i, \\ [C, U, V]_0 = I, \quad [C, U, V]_{i+1} = (C + i)^{-1}\big(i^2 + i(U - 1) + V\big)[C, U, V]_i, \end{gathered} \tag{5.62}$$

converges for $|z| < 1$ in $\mathbb{C}^{2\ell+1 \times 2\ell+1}$.

So (5.61) is valid and this gives a series representation for the rows of the monic polynomial $R_n^{(\nu)}$. Since each row is a polynomial, the series has to terminate and there exists $n \in \mathbb{N}$ so that $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1}$ is singular and $0 \neq P_0 \in \operatorname{Ker}\Big([(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1}\Big)$.

Suppose that n is the least integer for which $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1}$ is singular, i.e. $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_i$ is regular for all $i \leq n$. Since

$$\begin{aligned} &[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1} \\ &\quad = ((C_\alpha^{(\nu)})^t + n)^{-1}\Big(n^2 + n((U_\alpha^{(\nu)})^t - 1) + (V_\alpha^{(\nu)})^t + \lambda\Big)[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_n \end{aligned} \tag{5.63}$$

and since the matrix $(C_\alpha^{(\nu)}+n)$ is invertible by Lemma 5.3.2(i), $[C_\alpha^{(\nu)}, U_\alpha^{(\nu)}, V_\alpha^{(\nu)}+\lambda]_{n+1}$ is a singular matrix if and only if the diagonal matrix

$$M_n^\alpha(\lambda) = \Big(n^2 + n((U_\alpha^{(\nu)})^t - 1) + V_\alpha^t + \lambda\Big) = \Big(n^2 + n(U_\alpha^{(\nu)} - 1) + V_\alpha^{(\nu)} + \lambda\Big) = \lambda - \Lambda_n(D_\alpha^{(\nu)}) \tag{5.64}$$

is singular. Note that the diagonal entries of $M_n^\alpha(\lambda)$ are of the form $\lambda - \lambda_n^\alpha(j)$, so that $M_n^\alpha(\lambda)$ is singular if and only if $\lambda = \lambda_n^\alpha(j)$ for some $j \in \{0, 1, \cdots, 2\ell\}$. Because of Lemma 5.3.2 (ii) the value of λ and the corresponding n (and j) is uniquely determined for α irrational.

Assume α irrational, so that $M_n^\alpha\big(\lambda_m^\alpha(i)\big)$ is singular if and only if $n = m$. Then, in the series (5.61) the matrix $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1}$ is singular and $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_i$ is non-singular for $0 \le i \le n$. Furthermore, the kernel of $[(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda]_{n+1}$ is one-dimensional if and only if $\lambda = \lambda_n^\alpha(i)$, $i \in \{0, 1, \cdots, 2\ell\}$. In case $\lambda = \lambda_n^\alpha(i)$ we see that (5.61) is a (row-)vector-valued polynomial for

$$P_0 = [(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda_n^\alpha(i)]_n^{-1} e_i$$

determined uniquely up to a scalar, where e_i, $i \in \{0, 1 \cdots, 2\ell\}$, is the standard orthonormal basis vector in $\mathbb{C}^{2\ell+1}$.

This leads to the main result of this section, expressing the monic polynomials $R_n^{(\nu)}$ as a matrix hypergeometric function.

Theorem 5.3.3. *The entries of the monic matrix orthogonal polynomials $R_n^{(\nu)}$ are given by*

$$\big(R_n^{(\nu)}(u)\big)_{i,j} = \left({}_2H_1\left(\begin{matrix} (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda_n^\alpha(i) \\ (C_\alpha^{(\nu)})^t \end{matrix} ; u \right) n!\, [(C_\alpha^{(\nu)})^t, (U_\alpha^{(\nu)})^t, (V_\alpha^{(\nu)})^t + \lambda_n^\alpha(i)]_n^{-1} e_i \right)_j^t,$$

for all $\alpha \in \mathbb{R}$.

In particular, the right hand side is independent of α.

Remark 5.3.4. *Let $F(u) = {}_2H_1\left(\begin{smallmatrix} U,V \\ C \end{smallmatrix}; u\right) F(0)$. Since $[C, U, V]_{n+1} = [C+1, U+2, V+U]_n [C, U, V]_1$ and $[C, U, V]_1 = C^{-1}V$, and $F'(0) = C^{-1}VF(0)$ then, $F'(u) = {}_2H_1\left(\begin{smallmatrix} U+2, V+U \\ C+1 \end{smallmatrix}; u\right) F'(0)$. So we can use this in Theorem 5.3.3 to calculate $\left(\frac{dR_n^{(\nu)}}{du}(u)\right)_{ij}$ in terms of matrix hypergeometric functions. On the other hand, using Corollary 5.2.4 and (5.56) we see that $\left(\frac{dR_n^{(\nu)}}{du}(u)\right)_{i,j} = n\big(R_{n-1}^{(\nu+1)}(u)\big)_{i,j}$.*

5.4 Three-term recurrence relation

Since the monic matrix polynomials $R_n^{(\nu)}$ given by (5.56), are orthogonal, they satisfy a three term recurrence relation of the form

$$uR_n^{(\nu)}(u) = R_{n+1}^{(\nu)}(u) + X_n^{(\nu)} R_n^{(\nu)}(u) + Y_n^{(\nu)} R_{n-1}^{(\nu)}(u), \quad n \ge 0,$$

where $R_{-1}^{(\nu)} = 0$ and $X_n^{(\nu)}, Y_n^{(\nu)} \in \mathbb{C}^{2\ell+1\times 2\ell+1}$ are matrices depending on n but not on u.

In order to obtain the coefficients $X_n^{(\nu)}$, $Y_n^{(\nu)}$ explicitly, we exploit the explicit expression of $R_n^{(\nu)}$ in terms of Tirao's matrix hypergeometric function (see Theorem 5.3.3), the expression of the norms of $R_n^{(\nu)}$, $2^{2n-2\nu}H_n^{(\nu)}$, and the explicit expression of $H_n^{(\nu)}$, (5.53).

Before we state the main result of this section we need the following Lemma.

Lemma 5.4.1. *Let $R_n^{(\nu)}$ be the matrix orthogonal polynomial appearing in Theorem 5.3.3 then,*

$$R_{n,n-1}^{(\nu)} = \sum_{j=0}^{2\ell} \frac{jn}{4(j+n+\nu-1)}\mathcal{E}_{j,j-1} - \sum_{j=0}^{2\ell}\frac{n}{2}\mathcal{E}_{j,j} + \sum_{j=0}^{2\ell}\frac{n(2\ell-j)}{4(2\ell+n+\nu-j-1)}\mathcal{E}_{j,j+1}. \tag{5.65}$$

Proof. We compute $R_{n,n-1}^{(\nu)}$ by considering the coefficient of u^{n-1} in Theorem 5.3.3. Using the recursive definition of $[C_\alpha^{(\nu)}, U_\alpha^{(\nu)}, V_\alpha^{(\nu)} + \lambda_n^\alpha(i)]_n$ we obtain that the matrix entries of $R_{n,n-1}^{(\nu)}$ are given by

$$(R_{n,n-1}^{(\nu)})_{i,j} = (e_i^t(C_\alpha^{(\nu)} + (n-1)I)M_{n-1}^\alpha(\lambda_n^\alpha(i))^{-1})_j.$$

Observe that this is well defined since the matrix $M_{n-1}^\alpha(\lambda_n^\alpha(i))$, (5.64), is invertible for any irrational α. Now the lemma follows by a straightforward computation from the expression of $C_\alpha^{(\nu)}$ (see 5.58) and $M_{n-1}^\alpha(\lambda_n^\alpha(i))^{-1})_j$, (5.64). □

We can now give the explicit three term recurrence relation for $R_n^{(\nu)}$.

Theorem 5.4.2. *For any $\ell \in \frac{1}{2}\mathbb{N}$ and $\nu > 0$, the monic orthogonal polynomials $R_n^{(\nu)}$ satisfy the following three term recurrence relation*

$$uR_n^{(\nu)}(u) = R_{n+1}^{(\nu)}(u) + X_n^{(\nu)}R_n^{(\nu)}(u) + Y_n^{(\nu)}R_{n-1}^{(\nu)}(u), \tag{5.66}$$

where the matrices $X_n^{(\nu)}$, $Y_n^{(\nu)}$ are given by

$$X_n^{(\nu)} = \sum_{j=0}^{2\ell}\Bigg[\frac{-j(j+\nu-1)}{4(j+n+\nu-1)(j+n+\nu)}\mathcal{E}_{j,j-1} + \frac{\mathcal{E}_{j,j}}{2} - \frac{(2\ell-j)(2\ell-j+\nu-1)}{4(2\ell-j+n+\nu-1)(2\ell+n-j+\nu)}\mathcal{E}_{j,j+1}\Bigg], \tag{5.67}$$

$$Y_n^{(\nu)} = \sum_{j=0}^{2\ell}\frac{n(n+\nu-1)(2\ell+n+\nu)(2\ell+n+2\nu-1)}{16(2\ell+n+\nu-j-1)(2\ell+n+\nu-j)(j+n+\nu-1)(j+n+\nu)}\mathcal{E}_{j,j}. \tag{5.68}$$

Proof. Since the matrix polynomials $R_n^{(\nu)}$ are monic, by equating coefficients in (5.66) we get that $X_n^{(\nu)} = R_{n,n-1}^{(\nu)} - R_{n+1,n}^{(\nu)}$. Then, (5.67) follows by plugging (5.65) in the previous equality.

To see that (5.68) holds, notice that from (5.66) we have

$$2^{-2n-2\nu}H_n^{(\nu)} = \langle uR_n, R_{n-1}\rangle = Y_n^{(\nu)}\langle R_{n-1}, R_{n-1}\rangle = 2^{-2n+2-2\nu}H_{n-1}^{(\nu)}.$$

Therefore $Y_n^{(\nu)} = \frac{1}{4}H_n^{(\nu)}\left(H_{n-1}^{(\nu)}\right)^{-1}$ and (5.68) follows from the expression of $H_n^{(\nu)}$, (5.53). □

Corollary 5.4.1. *The monic matrix polynomials $P_n^{(\nu)}$ defined in (5.19) satisfy*

$$xP_n^{(\nu)}(x) = P_{n+1}^{(\nu)}(x) + (1-2X_n^{(\nu)})P_n^{(\nu)}(x) + 4Y_n^{(\nu)}P_{n-1}^{(\nu)}(x), \tag{5.69}$$

where $X_n^{(\nu)}$ and $Y_n^{(\nu)}$ are given by (5.67) and (5.68), respectively.

Note that in the limit $n\to\infty$ the coefficients become constant; $\lim_{n\to\infty}(1-2X_n^{(\nu)}) = 0$, and $\lim_{n\to\infty} 4Y_n^{(\nu)} = \frac{1}{4}I$.

Taking derivatives and using Corollary 5.2.4 gives the following identity

$$P_n^{(\nu)}(x)+nxP_{n-1}^{(\nu+1)}(x) = (n+1)P_n^{(\nu+1)}(x)+n(1-2X_n^{(\nu)})P_{n-1}^{(\nu+1)}(x)+4(n-1)Y_n^{(\nu)}P_{n-2}^{(\nu+1)}(x), \tag{5.70}$$

for $n\geq 1$ and with the convention $P_{-1}^{(\nu+1)}(x) = 0$.

Combining (5.70) with the three term recurrence for $xP_{n-1}^{(\nu+1)}(x)$, (5.69), gives the following corollary.

Corollary 5.4.2. *The monic orthogonal polynomials $P_n^{(\nu)}$ satisfy the following connection formula,*

$$P_n^{(\nu)}(x) = P_n^{(\nu+1)}(x) + 2(X_{n-1}^{(\nu+1)} - X_n^{(\nu)})P_{n-1}^{(\nu+1)}(x) + 4((n-1)Y_n^{(\nu)} - nY_{n-1}^{(\nu+1)})P_{n-2}^{(\nu+1)}(x). \tag{5.71}$$

5.5 The matrix orthogonal polynomials related to Gegenbauer and Racah polynomials

The matrix entries of $P_n^{(\nu)}$ can be expressed in terms of Gegenbauer and Racah polynomials.

For doing that we first switch to $D_0^{(\nu)} = \widetilde{D}^{(\nu)} - 2\ell\widetilde{E}^{(\nu)}$, since this will lead to a diagonalisable operator after conjugation by an appropriate matrix function. In particular $D_0^{(\nu)}$ has the following form

$$\begin{aligned} D_0^{(\nu)} &= u(1-u)\frac{d^2}{du^2} + \left(\frac{d}{du}\right)(K_0^1 - uK_1^1) + K_0, \\ K_1^1 &= \sum_{i=0}^{2\ell}(2i+2\nu+1)\,\mathcal{E}_{i,i}, \quad K_0^1 = -\sum_{i=0}^{2\ell} i\,\mathcal{E}_{i,i-1} + \sum_{i=0}^{2\ell}\frac{(2i+2\nu+1)}{2}\,\mathcal{E}_{i,i}, \\ K_0 &= \sum_{i=0}^{2\ell}\big((2\ell-i)(2\ell+i+2) - (\nu-1)(\nu+2i+1)\big)\,\mathcal{E}_{i,i}. \end{aligned} \tag{5.72}$$

Notice that $D_0^{(\nu)}$ is a matrix hypergeometric differential operator in the sense of Tirao [98].

We conjugate this differential operator by $M^{(\nu)}(u) = L^{(\nu)}(1-2u)$, where $L^{(\nu)}$ is the matrix function (5.1). Going through to the calculations as in Section 6 in [79], we obtain Proposition 5.5.1. Recall that for a matrix G, G_L and G_R denote the left and right multiplication operators given by (1.23).

Proposition 5.5.1. *The differential operator* $\mathcal{D}^{(\nu)} = M_R^{(\nu)} \circ D_0^{(\nu)} \circ \big(M^{(\nu)}\big)_R^{-1}$ *is the diagonal differential operator*

$$\mathcal{D}^{(\nu)} = u(1-u)\frac{d^2}{du^2} + \left(\frac{d}{du}\right) T_1(u) + T_0,$$

where $T_1(u) = \frac{1}{2}T_1^1 - uT_1^1$,

$$T_1^1 = \sum_{k=0}^{2\ell}(2k+2\nu+1)\mathcal{E}_{k,k}, \qquad T_0 = \sum_{k=0}^{2\ell}(2\ell-k-\nu+1)(2\ell+k+\nu+1)\mathcal{E}_{k,k}.$$

Moreover, $\mathcal{R}_n^{(\nu)}(u) = R_n^{(\nu)}(u)M^{(\nu)}(u)$ *satisfies*

$$\mathcal{R}_n^{(\nu)}\,\mathcal{D}^{(\nu)} = \Lambda_n(\mathcal{D}^{(\nu)})\,\mathcal{R}_n^{(\nu)}, \qquad \Lambda_n(\mathcal{D}^{(\nu)}) = \Lambda_n(\widetilde{D}^{(\nu)}) - 2\ell\,\Lambda_n(\widetilde{E}^{(\nu)}).$$

Since $R_n^{(\nu)}$ and $M^{(\nu)}$ are matrix polynomials, Proposition 5.5.1 implies that for $j,\, k = 0\cdots 2\ell$, $\big(\mathcal{R}_n^{(\nu)}(u)\big)_{k,j}$ is a polynomial solution to the hypergeometric differential operator

$$u(1-u)f''(u)+((j+\nu+\frac{1}{2})-u(2j+2\nu+1))f'(u)-(j-k-n)(j+n+k+2\nu)f(u) = 0. \tag{5.73}$$

Since the polynomial solutions of (5.73) are unique up to a constant factor we find

$$\big(\mathcal{R}_n^{(\nu)}(u)\big)_{k,j} = c_{k,j}^{(\nu)}(n)\;{}_2F_1\left(\begin{matrix} j-k-n,\, n+k+j+2\nu \\ j+\frac{1}{2}+\nu\end{matrix};u\right). \tag{5.74}$$

Note that the ${}_2F_1$-series gives a Gegenbauer polynomial $C_{n+k-j}^{(\nu+j+1)}(1-2u)$ scaled to $[0,1]$.

We are now going to calculate the explicit values of the constants $c_{k,j}^{(\nu)}(n)$. To do so, let us consider the following differential operator, $\mathcal{E}^{(\nu)} = M_R^{(\nu)} \circ \widetilde{E}^{(\nu)} \circ \big((M^{(\nu)})^{-1}\big)_R$. It is clear that the operator $\mathcal{E}^{(\nu)}$ satifies

$$\mathcal{E}^{(\nu)}\,\mathcal{R}_n^{(\nu)} = \Lambda_n(\mathcal{E}^{(\nu)})\,\mathcal{R}_n^{(\nu)}, \quad \Lambda_n(\mathcal{E}^{(\nu)}) = \Lambda_n(\widetilde{E}^{(\nu)}). \tag{5.75}$$

Moreover, by Corollary 5.3.1, the matrix differential operators $\widetilde{D}^{(\nu)}$ and $\widetilde{E}^{(\nu)}$ commute then, $\mathcal{E}^{(\nu)}$ and $\mathcal{D}^{(\nu)}$ also commute, i.e. $\mathcal{E}^{(\nu)} \circ \mathcal{D}^{(\nu)} = \mathcal{D}^{(\nu)} \circ \mathcal{E}^{(\nu)}$.

An explicit calculation gives

$$\begin{aligned} \mathcal{E}^{(\nu)} &= \frac{d}{du}S_1(u) + S_0(u), \\ S_1(u) &= u(1-u)\sum_{i=0}^{2\ell}\frac{i(i+2\nu-2)(2\nu+i+2\ell-1)}{\ell(2\nu+2i-1)(2\nu+2i-3)}\mathcal{E}_{i,i-1} + \sum_{i=0}^{2\ell}\frac{(2\ell-i)}{4\ell}\mathcal{E}_{i,i+1}, \\ S_0(u) &= (1-2u)\sum_{i=0}^{2\ell}\frac{i(2\nu+i-2)(2\nu+i+2\ell-1)}{2\ell(2\nu+2i-3)}\mathcal{E}_{i,i-1} \\ &\qquad + \sum_{i=0}^{2\ell}\frac{i(2\nu+i-1)-4\ell(\ell+1)-2\ell(\nu-1)}{2\ell}\mathcal{E}_{i,i}. \end{aligned} \tag{5.76}$$

Let $F(\lambda) \subseteq \mathbb{C}^{2\ell+1}$ be the eigenspace of $\mathcal{D}^{(\nu)}$ for an eigenvalue λ, $F(\lambda) = \{f \in \mathbb{C}^{2\ell+1} \,|\, \mathcal{D}^{(\nu)} f = \lambda f\}$. By Tirao, [98], the evaluation at zero $ev(0)$

$$ev(0) : F(\lambda) \to \mathbb{C}^{2\ell+1}$$
$$Q^\lambda \mapsto Q^\lambda(0),$$

is an isomorphism between $F(\lambda)$ and $\mathbb{C}^{2\ell+1\times 2\ell+1}$. Since $\mathcal{E}^{(\nu)}$ commutes with $\mathcal{D}^{(\nu)}$, $\mathcal{E}^{(\nu)}$ preserves the eigenspaces then, there exists a linear map $N(\lambda)$ such that the following diagram commutes

$$\begin{array}{ccc} F(\lambda) & \xrightarrow{\mathcal{E}^{(\nu)}} & F(\lambda) \\ \downarrow{\scriptstyle ev(0)} & & \downarrow{\scriptstyle ev(0)} \\ \mathbb{C}^{2\ell+1} & \xrightarrow[N(\lambda)]{} & \mathbb{C}^{2\ell+1} \end{array} \tag{5.77}$$

Let us show how the function $N(\lambda)$ looks like. Let $Q^\lambda \in \mathbb{C}^{N\times N}$ be a polynomial eigenfunction of $D^{(\nu)}$ for an eigenvalue λ. Then, by (5.61) (or see [98]), the rows of Q^λ, Q^λ_j can be written as

$$Q^\lambda_j(u) = \left({}_2H_1 \left(\begin{matrix} T^1_1, \lambda - T_0 \\ \frac{1}{2}T^1_1 \end{matrix} ; u \right) Q^\lambda(0)^t \right)^t_j,$$

so that $\frac{dQ^\lambda_j}{du}(0) = Q^\lambda_j(0)(\lambda - T_0)(\frac{1}{2}T^1_1)^{-1}$ (see Observation 5.3.4). Now (5.76) and (5.77) shows that

$$N(\lambda)(Q^\lambda_j(0)) = (\mathcal{E}^{(\nu)}Q^\lambda)_j(0) = Q^\lambda_j(0)(\lambda - T_0)\left(\frac{1}{2}T^1_1\right)^{-1} S_1(0) + Q^\lambda_j(0)S_0(0).$$

Therefore

$$N(\lambda) = (\lambda - T_0)\left(\frac{1}{2}T^1_1\right)^{-1} S_1(0) + S_0(0)$$

acting from the right on row-vectors from $\mathbb{C}^{2\ell+1}$.

With the explicit expression of the function $N(\lambda)$, we can now calculate the value of the constants $c^{(\nu)}_{k,j}(n)$ in (5.74).

The k-th row $\left((\mathcal{R}^{(\nu)}_n)_{k,j}\right)^{2\ell}_{j=0}$ is an eigenfunction of $\mathcal{D}^{(\nu)}$ for the eigenvalue $\lambda_n(k) = \Lambda_n(\mathcal{D}^{(\nu)})_{k,k}$ (see Proposition 5.5.1). On the other hand, by (5.75) the k-th row of $\mathcal{R}^{(\nu)}_n$ is an eigenfunction of $\mathcal{E}^{(\nu)}$ for the eigenvalue $\mu_n(k) = \Lambda_n(\mathcal{E}^{(\nu)})_{k,k}$. Since $\left((\mathcal{R}^{(\nu)}_n(0))_{k,j}\right)^{2\ell}_{j=0} = (c^{(\nu)}_{k,j}(n))^{2\ell}_{j=0}$, the row-vector $c^{(\nu)}_k = (c^{(\nu)}_{k,j}(n))^{2\ell}_{j=0}$ satisfies $c^{(\nu)}_k N(\lambda_n(k)) = \mu_n(k)\, c^{(\nu)}_k$. Explicitly,

$$\begin{aligned} & - \frac{(j+k+n+2\nu-1)(j-k-n-1)(2\ell-j+1)}{(2j+2\nu-1)} c^{(\nu)}_{k,j-1}(n) \\ & + (j(j+2\nu-1) - 4\ell(\ell+1) - 2\ell(\nu-1))c^{(\nu)}_{k,j}(n) + \frac{(j+1)(j+2\nu-1)(2\ell+j+2\nu)}{(2j+2\nu-1)} c^{(\nu)}_{k,j+1}(n) \\ & = (-2n(\ell-k) + (2\ell+2)(k-2\ell) - 2(\nu-1)(\ell-k))c^{(\nu)}_{k,j}(n). \end{aligned} \tag{5.78}$$

This is a finite three term recurrence relation for the coefficient $c^{(\nu)}_{k,j}(n)$ in the subindex j, which can be solved explicitly in terms of Racah polynomials, (see [102], [77])

$$\begin{aligned} c^{(\nu)}_{k,j} &= c^{(\nu)}_{k,0}\,(-1)^j \frac{(-2\ell)_j\,(-n-k)_j}{j!\,(2\nu+2\ell)_j}\, R_k(\lambda(j);-2\ell-1,-n-k-\nu,\nu-1,\nu-1) \\ &= c^{(\nu)}_{k,0}\,(-1)^j \frac{(-2\ell)_j\,(-n-k)_j}{j!\,(2\nu+2\ell)_j}\, {}_4F_3\left(\begin{matrix}-j,j+2\nu-1,-k,-n-\nu-2\ell \\ \nu,-n-k,-2\ell\end{matrix};1\right). \end{aligned} \tag{5.79}$$

Switching to the variable x, we find

$$(\mathcal{P}^{(\nu)}_n)_{k,j} = (P^{(\nu)}_n(x)L^{(\nu)}(x))_{k,j} = (-2)^n c^{(\nu)}_{k,j}(n)\frac{(n+k-j)!}{(2j+2\nu)_{n+k-j}} C^{(\nu+j)}_{n+k-j}(x).$$

Then, the orthogonality relations (5.19) and the orthogonality relations for the (scalar-valued) Gegenbauer polynomials, see (5.2), imply

$$\delta_{n,m}(H_n)_{k,i} = \delta_{n+k,m+i}(-2)^{n+m}\sum_{j=0}^{2\ell\wedge(n+k)} c^{(\nu)}_{k,j}(n)\overline{c^{(\nu)}_{i,j}(m)}\,t^{(\nu)}_j\,\frac{(n+k-j)!}{(2\nu+2j)_{n+k-j}}\,\frac{\sqrt{\pi}\Gamma(\nu+j+1/2)}{(n+k+\nu)\Gamma(\nu+j)}.$$

It follows from (5.79) that

$$\begin{aligned} &\delta_{n,m}(H_n)_{k,k}(-2)^{-n-m}\frac{(n+k+\nu)\Gamma(\nu)\Gamma(n+k+2\nu)}{(2\ell+\nu)(n+k)!\Gamma(\nu+1/2)\sqrt{\pi}\Gamma(2\nu)} \\ &\qquad = |c_{k0}(n)|^2\sum_{j=0}^{2\ell\wedge(n+k)}\frac{(-2\ell)_j(-n-k)_j(2\nu-1)_j(\nu+1/2)_j}{j!(2\ell+2\nu)_j(n+k+2\nu)_j(\nu-1/2)_j} \\ &\times R_k(\lambda(j);-2\ell-1,-k-n-\nu,\nu-1,\nu-1)\,R_{k+n-m}(\lambda(j);-2\ell-1,-k-n-\nu,\nu-1,\nu-1), \end{aligned} \tag{5.80}$$

which corresponds to the orthogonality relations for the corresponding Racah polynomials (see [102], [77]). From this we find that the sum in (5.80) equals

$$\delta_{n,m}\,M\,\frac{(-2\ell-n-\nu)_k(-2\ell-n-k-2\nu+1)_k(-2\ell-\nu+1)_k(-n-k-\nu+1)_k k!}{(-2\ell-n-k-\nu+1)_{2k}(-2\ell)_k(-n-k)_k(\nu)_k},$$

where

$$M = \frac{(n+k+\nu)_{2\ell}(2\nu)_{2\ell}}{(n+k+2\nu)_{2\ell}(\nu)_{2\ell}} = \frac{(2\ell+\nu)_{n+k}(2\nu)_{n+k}}{(2\ell+2\nu)_{n+k}(\nu)_{n+k}}.$$

Hence,

$$|c^{(\nu)}_{k,0}(n)|^2 = \frac{4^{-2n}(\nu)_n^2(2\ell+2\nu)_n^2}{(k+\nu)_n^2(2\ell+\nu-k)_n^2}.$$

It just remains to see the sign of $c^{(\nu)}_{k,0}(n)$. If we take the $(k,0)$-entry of the three term recurrence relation for $\mathcal{R}^{(\nu)}_n(u)$ we obtain a polynomial identity in u. If we take the leading coefficient it gives

$$c^{(\nu)}_{k,0}(n+1) = -c^{(\nu)}_{k,0}(n)\frac{(n+k+2\nu)}{4(n+k+\nu)} + c^{(\nu)}_{k+1,0}(n)\frac{(2\ell-k)(2\ell-k+\nu-1)(n+k+\nu)}{4(2\ell-k+n+\nu-1)(2\ell+n-k+\nu)}.$$

If we plug $c_{k,0}^{(\nu)}(n) = \mathrm{sign}(c_{k,0}^{(\nu)}(n))\,|c_{k,0}^{(\nu)}(n)|$ in the equation above, we obtain

$$\begin{aligned}(\nu+n)(2\ell+2\nu+n)\,&\mathrm{sign}(c_{k,0}^{(\nu)}(n+1))\\ &= -(n+k+2\nu)(2\ell+n+\nu-k)\,\mathrm{sign}(c_{k,0}^{(\nu)}(n)) + (k+\nu)(2\ell-k)\,\mathrm{sign}(c_{k+1,0}^{(\nu)}(n)).\end{aligned}$$

It then follows that $\mathrm{sign}(c_{k,0}^{(\nu)}(n)) = \mathrm{sign}(c_{k+1,0}^{(\nu)}(n)) = -\mathrm{sign}(c_{k,0}^{(\nu)}(n+1))$. This implies that $\mathrm{sign}(c_{k,0}^{(\nu)}(n)) = (-1)^n$.

The previous discussion is summarized in the following theorem.

Theorem 5.5.2. *The polynomials $\mathcal{R}_n^{(\nu)}$ are given by*

$$\begin{aligned}\left(\mathcal{R}_n^{(\nu)}(u)\right)_{k,j} &= c_{k,0}^{(\nu)}(n)\,(-1)^j \frac{(-2\ell)_j\,(-n-k)_j}{j!\,(2\nu+2\ell)_j}\\ &\times {}_4F_3\left(\begin{matrix}-j, j+2\nu-1, -k, -n-\nu-2\ell\\ \nu, -n-k, -2\ell\end{matrix};1\right)\,{}_2F_1\left(\begin{matrix}j-k-n, n+k+j+2\nu\\ j+\frac{1}{2}+\nu\end{matrix};u\right).\end{aligned}$$

where the constant $c_{k,0}^{(\nu)}(n)$ is given by

$$c_{k,0}^{(\nu)}(n) = \frac{(-1)^n 4^{-n}(\nu)_n(2\ell+2\nu)_n}{(k+\nu)_n(2\ell+\nu-k)_n}.$$

Note that Theorem 5.5.2 extends [79, Thm. 6.2]. Switching back to $P_n^{(\nu)}$ and using the inverse of $L^{(\nu)}(x)$ in (5.7) due Cagliero and Koornwinder [11], we get the following corollary.

Corollary 5.5.3. *Using the notation of Theorem 5.5.2 the monic matrix polynomials have the explicit expansion*

$$\begin{aligned}(P_n^{(\nu)}(x))_{k,i} &= \frac{(-2)^n}{i!}\,c_{k,0}^{(\nu)}(n)\sum_{j=i}^{2\ell}\frac{(n+k-j)!(-2\ell)_j(-1)^j(-n-k)_j}{(2\nu+2j)_{n+k-j}(2\nu+j+i-1)_{j-i}(2\nu+2\ell)_j}\\ &\times R_k(\lambda(j);-2\ell-1,-n-k-\nu,\nu-1,\nu-1)\,C_{n+k-j}^{(\nu+j)}(x)\,C_{j-i}^{(1-\nu-j)}(x).\end{aligned}\tag{5.81}$$

Observation 5.5.1. *It isn't obvious that the right hand side of (5.81) is a polynomial of degree at most n in case $k>i$. In particular, the coefficients of x^p with $p>n$ are zero. Notice that for $k>i$ the leading coefficient of the right hand side is zero, and this gives the following non-trivial identity regarding the Racah polynomials.*

$$\begin{aligned}&\sum_{j=i}^{2\ell}\frac{(\nu+j)_{n+k-j}(-2\ell)_j(-1)^j(-n-k)_j}{(2\nu+2j)_{n+k-j}(2\nu+j+i-1)_{j-i}(2\nu+2\ell)_j}\\ &\times\frac{(1-\nu-j)_{j-i}}{(j-i)!}R_k(\lambda(j);-2\ell-1,-n-k-\nu,\nu-1,\nu-1)=0.\end{aligned}$$

Bibliography

[1] N. Aldenhoven, E. Koelink, A. M. de los Ríos. Matrix-Valued Little q-Jacobi Polynomials. *Preprint*. http://arxiv.org/abs/1308.2540. xvi

[2] R. Álvarez-Nodarse, A. J. Durán and A. M. de los Ríos. Orthogonal matrix polynomials satisfying second order difference equations. *J. Approx. Theory*, **169**, (2013), 40–55. xiii

[3] R. Álvarez-Nodarse, J. C. Medem. q-classical polynomials and the q-Askey and Nikiforov-Uvarov tableaus. *J. Comput. Appl. Math.*, **135** (2001), 157–196. vii

[4] G. E. Andrews and R. Askey. Classical orthogonal polynomials. In *Orthogonal polynomials and applications (Bar-le-Duc, 1984)*, Lecture Notes in Math.,Springer, Berlin (1985) 36–62. vii

[5] G. E. Andrews, R. Askey and R. Roy. *Special functions*, volume **71** of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999. 2

[6] W. N. Bailey. *Generalized hypergeometric series*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 32. Stechert-Hafner, Inc., New York, 1964. 1, 2

[7] H. Bavinck and R. Koekoek. On a difference equation for generalizations of Charlier polynomials. *J. Approx. Theory*, **81**(2) (1995), 195–206. xiv

[8] S. Böchner. Uber Sturm-Liouvillesche Polynomsysteme. *Math. Zeit.*, **29**(1) (1929), 730–736. v

[9] J. Borrego, M. Castro and A. J. Durán. Orthogonal matrix polynomials satisfying differential equations with recurrence coefficients having non-scalar limits. *Integral Transforms Spec. Funct.*, **23**(9) (2012), 685–700. viii

[10] J. M. Berezans′kiĭ. *Expansions in eigenfunctions of selfadjoint operators*. Translated from the Russian by R. Bolstein, J. M. Danskin, J. Rovnyak and L. Shulman. Translations of Mathematical Monographs, Vol. **17**. American Mathematical Society, Providence, R.I., 1968. viii

[11] L. Cagliero and T.H. Koornwinder. Explicit matrix inverses for lower triangular matrices with entries involving Jacobi polynomials. *J. Approx. Theory*, (2014) In press. 69, 92

[12] M. J. Cantero, L. Moral and L. Velázquez. Matrix orthogonal polynomials whose derivatives are also orthogonal. *J. Approx. Theory*, **146**(2) (2007), 174–211. viii, x, xvii, 10, 14, 67, 74, 79

[13] M. M. Castro and F. A. Grünbaum. Orthogonal matrix polynomials satisfying first order differential equations: a collection of instructive examples. *J. Nonlinear Math. Phys.*, **12** (2005), 63–76. viii

[14] M. M. Castro and F. A. Grünbaum. The algebra of differential operators associated to a family of matrix-valued orthogonal polynomials: five instructive examples. *Int. Math. Res. Not.*, pages Art. ID 47602, 33, 2006. viii, x, 10, 12

[15] M. M. Castro and F. A. Grünbaum. The noncommutative bispectral problem for operators of order one. *Constr. Approx.*, **27** (3) (2008), 329–347. viii

[16] T. S. Chihara. *An introduction to orthogonal polynomials.* Gordon and Breach Science Publishers, New York-London-Paris, 1978. Mathematics and its Applications, Vol. **13**. 9

[17] A. Conflitti and M. J. Schlosser. Noncommutative hypergeometric and basic hypergeometric equations. *J. Nonlinear Math. Phys.*, **17**(4) (2010), 429–443. 56

[18] R. S. Costas, F. Marcellán. q-Classical orthogonal polynomials: A general difference calculus approach. *Acta Appl. Math.* **111**(1) (2010), 107–128. 10

[19] D. Damanik, A. Pushnitski and B. Simon. The analytic theory of matrix orthogonal polynomials. *Surv. Approx. Theory*, **4** (2008), 1–85. viii, 6

[20] E. Daneri, A. J. Durán. Ratio asymptotics for orthogonal matrix polynomials with unbounded recurrence coefficients. *J. Approx. Theory* **110**(1) (2001), 1–17. viii

[21] E. Daneri, A. J. Durán. Weak convergence for orthogonal matrix polynomials. *Indag. Math. (N.S.)* **13**(1) (2002), 47–62. viii

[22] E. Defez, A. J. Durán. Orthogonal matrix polynomials and quadrature formulas. *Linear Algebra Appl.* **345** (2002), 71–84. viii

[23] P. Delsarte, Y. V. Genin and Y. G Kamp. Orthogonal polynomial matrices on the unit circle. *IEEE Trans. Circuits and Systems*, **CAS-25**(3) (1978), 149–160. 4

[24] J. J. Duistermaat and F. A. Grünbaum. Differential equations in the spectral parameter. *Comm. Math. Phys.*, **103**(2) (1986), 177–240. 12

[25] A. J. Durán. A generalization of Favard's theorem for polynomials satisfying a recurrence relation. *J. Approx. Theory*, **74**(1) (1993), 83–109. viii

[26] A. J. Durán. On orthogonal polynomials with respect to a positive definite matrix of measures. *Canad. J. Math.*, **47** (1) (1995), 1180–1195. viii, 6

[27] A. J. Durán. Markov's theorem for orthogonal matrix polynomials. *Canad. J. Math.*, **48**(6) (1996), 1180–1195. viii

[28] A. J. Durán. Matrix inner product having a matrix symmetric second order differential operator. *Rocky Mountain J. Math.*, **27**(2) (1997), 585–600. ix, 10, 11

[29] A. J. Durán. Ratio asymptotics for orthogonal matrix polynomials. *J. Approx. Theory*, **100**(2) (1999), 304–344. viii

[30] A. J. Durán. A method to find weight matrices having symmetric second-order differential operators with matrix leading coefficient. *Constr. Approx.*, **29**(2) (2009), 181–205. viii, ix, x, 10

[31] A. J. Durán. Generating orthogonal matrix polynomials satisfying second order differential equations from a trio of triangular matrices. *J. Approx. Theory.*, **161** (1) (2009), 88–113. viii

[32] A. J. Durán. Rodrigues' formulas for orthogonal matrix polynomials satisfying second-order differential equations. *Int. Math. Res. Not. IMRN*, **5** (2010), 824–855. viii

[33] A. J. Durán. Rodrigues's formulas for orthogonal matrix polynomials satisfying higher-order differential equations. *Exp. Math.*, **20** (1) (2011), 15–24. viii

[34] A. J. Durán. A miraculously commuting family of orthogonal matrix polynomials satisfying second order differential equations. *J. Approx. Theory*, **163** (12) (2011), 1815–1833. viii, x

[35] A. J. Durán. The algebra of difference operators associated to a family of orthogonal polynomials. *J. Approx. Theory*, **164**(5) (2012),586–610. x, xi, 10, 17, 18, 19

[36] A. J. Durán and M. D. de la Iglesia. Second-order differential operators having several families of orthogonal matrix polynomials as eigenfunctions. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 084, 24, 2008. viii, x, xiii, 10, 37, 44

[37] A. J. Durán and M. D. de la Iglesia. Some examples of orthogonal matrix polynomials satisfying odd order differential equations. *J. Approx. Theory*, **150**(2) (2008), 153–174. viii, x

[38] A. J. Durán and F. A. Grünbaum. Orthogonal matrix polynomials satisfying second-order differential equations. *Int. Math. Res. Not.*, **10** (2004),461–484. viii, ix, 8, 11, 13, 14

[39] A. J. Durán and F. A. Grünbaum. Structural formulas for orthogonal matrix polynomials satisfying second-order differential equations. I. *Constr. Approx.*, **22**(2) (2005), 255–271. viii, x, 10

[40] A. J. Durán and F. A. Grünbaum. A characterization for a class of weight matrices with orthogonal matrix polynomials satisfying second-order differential equations. *Int. Math. Res. Not.*, **23** (2005), 1371–1390. viii

[41] A. J. Durán and F. A. Grünbaum. Orthogonal matrix polynomials, scalar-type Rodrigues' formulas and Pearson equations. *J. Approx. Theory*, **134** (2) (2005), 267–280. viii

[42] A. J. Durán and F. A. Grünbaum. A survey on orthogonal matrix polynomials satisfying second order differential equations. *J. Comput. Appl. Math.*, **178** (1-2) (2005), 169–190. viii, x

[43] A. J. Durán and F. A. Grünbaum. P. A. M. Dirac meets M. G. Krein: matrix orthogonal polynomials and Dirac's equation. *J. Phys. A*, **39** (14) (2006), 3655-3662. viii

[44] A. J. Durán and F. A. Grünbaum. Matrix orthogonal polynomials satisfying second-order differential equations: coping without help from group representation theory. *J. Approx. Theory*, **148**(1) (2007), 35–48. viii

[45] A. J. Durán and F. A. Grünbaum. Matrix differential equations and scalar polynomials satisfying higher order recursions. *J. Math. Anal. Appl.*, **354**(1) (2009), 1–11. viii

[46] A. J. Durán and F. A. Grünbaum. Orthogonal matrix polynomials, scalar-type Rodrigues' formulas and Pearson equations. *J. Approx. Theory*, **134**(2) (2009), 267–280. x, 10

[47] A. J. Durán and P. López-Rodríguez. Orthogonal matrix polynomials: zeros and Blumenthal's theorem. *J. Approx. Theory*, **84**(1) (1996), 96–118. viii, 6

[48] A. J. Durán and P. López-Rodríguez. The $\mathcal{L}^p$ space of a positive definite matrix of measures and density of matrix polynomials in $\mathcal{L}^1$. *J. Approx. Theory*, **90** (2), 299–318 viii

[49] A. J. Durán and P. López-Rodríguez. Density questions for the truncated matrix moment problem. *Canad. J. Math.*, **49** (4) (1997), 708–721 viii

[50] A. J. Durán and P. López-Rodríguez. N-extremal matrices of measures for an indeterminate matrix moment problem. *J. Funct. Anal.*, **174** (2) (2000), 301–321 viii

[51] A. J. Durán and P. López-Rodríguez. Orthogonal matrix polynomials. In *Laredo Lectures on Orthogonal polynomials and Special Functions*, Adv. Theory Spec. Funct. Orthogonal polynomials, Nova Sci. Publ., Hauppauge, NY (2004) 13–44. viii

[52] A. J. Durán and P. López-Rodríguez. Structural formulas for orthogonal matrix polynomials satisfying second-order differential equations. II. *Constr. Approx.*, **26** (1) (2007), 29–47 viii, x

[53] A. J. Durán, P. López-Rodríguez and E. B. Saff. Zero asymptotic behaviour for orthogonal matrix polynomials. *J. Anal. Math.*, **78** (1999), 37–60 viii

[54] A. J. Durán and B. Polo. Gaussian quadrature formulae for matrix weights. *Linear Algebra Appl.*, **355** (2002), 119–146. viii

[55] A. J. Durán and B. Polo. Matrix Christoffel functions. *Constr. Approx.*, **20** (2004), 353–376. viii

[56] A. J. Durán and A.M. de los Ríos. The convex cone of weight matrices associated to a second-order matrix difference operator. *Integral Transforms Spec. Funct.*, **25** (8) (2014), 663–679. xi, xiv

[57] A. J. Durán and V. Sánchez-Canales. Orthogonal matrix polynomials whose differences are also orthogonal. *J. Approx. Theory*, **179** (2014), 112–127. x, xi, 10, 47

[58] A. J. Durán and W. Van Assche. Orthogonal matrix polynomials and higher-order recurrence relations. *Linear Algebra Appl.*, **219** (1995), 261–280. viii

[59] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi. *Higher transcendental functions. Vol. II.* Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman, Reprint of the 1953 original. 68

[60] R. Gangolli and V. S. Varadarajan. Harmonic analysis of spherical functions on real reductive groups. *R*esults in Mathematics and related Areas. Springer Verlag, Belin 1988. ix

[61] F. R. Gantmacher. *The theory of matrices. Vols. 1, 2.* Translated by K. A. Hirsch. Chelsea Publishing Co., New York, 1959. 43, 46

[62] A. G. García, F. Marcellán and L. Salto. A distributional study of discrete classical orthogonal polynomials. *J. Comput. Appl. Math.*, **57** (1995), 147–162. 23

[63] G. Gasper and M. Rahman. *Basic hypergeometric series*, volume 96 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, second edition, 2004. With a foreword by Richard Askey. 1, 3

[64] J. S. Geronimo. Matrix orthogonal polynomials on the unit circle. *J. Math. Phys.*, **22**(7) (1981), 1359–1365. 4

[65] J. S. Geronimo. Scattering theory and matrix orthogonal polynomials on the real line. *Circuits Systems Signal Process.*, **1**(3-4) (1982), 471–495. viii

[66] F. A. Grünbaum. Matrix valued Jacobi polynomials. *Bull. Sci. Math.*, **127**(3) (2003), 207–214. viii, ix

[67] F. A. Grünbaum and M. D. de la Iglesia. Matrix-valued orthogonal polynomials related to SU($N+1$), their algebras of differential operators and the corresponding curves. *Experiment. Math.*, **16**(2) (2007), 189–207. ix

[68] F. A. Grünbaum and M. D. de la Iglesia. Matrix valued orthogonal polynomials arising from group representation theory and a family of quasi-birth-and-death processes. *SIAM J. Matrix Anal. Appl.*, **30**(2) (2008), 741–761. viii

[69] F. A. Grünbaum, P. Iliev. A noncommutative version of the bispectral problem. *J. Comput. Appl. Math.*, **161**(2) (2003), 99–118. 12

[70] F. A. Grünbaum, I. Pacharoni and J. Tirao. Matrix valued spherical functions associated to the complex projective plane. *J. Funct. Anal.*, **188**(2) (2002), 350–441. viii, ix

[71] F. A. Grünbaum, I. Pacharoni and J. Tirao. Matrix valued orthogoonal polynomials of the Jacobi type *Indag. Math.*, **14**(3-4) (2003), 353–366. viii, ix

[72] F. A- Grünbaum and Juan Tirao. The algebra of differential operators associated to a weight matrix. *Integral Equations Operator Theory*, **58**(4) (2007),449–475. viii, x, 10, 12, 13

[73] W. Hahn. Über Orthogonalpolynome, die q-Differenzengleichungen genügen. *Math. Nachr.*, **2** (1949), 4–34. vi

[74] D. Holger and B. Reuther. Random block matrices and matrix orthogonal polynomials. *J. Theoret. Probab.*, **23**(2), (2010), 378–400. viii

[75] R. A. Horn and C. R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, second edition, 2013. 8, 67

[76] M. E. H. Ismail. *Classical and quantum orthogonal polynomials in one variable*, volume 98 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2009. With two chapters by Walter Van Assche, With a foreword by Richard A. Askey, Reprint of the 2005 original. 1, 24, 68, 70, 75, 76, 77, 81

[77] R. Koekoek, P. A. Lesky and R. F. Swarttouw. *Hypergeometric orthogonal polynomials and their q-analogues*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010. With a foreword by Tom H. Koornwinder. vii, 1, 23, 24, 41, 75, 91

[78] E. Koelink, M. Van Pruijssen and P. Román. Matrix-valued orthogonal polynomials related to $(SU(2) \times SU(2), diag)$. *Int. Math. Res. Not. IMRN*, **24** (2012),5673–5730. ix, xvi

[79] E. Koelink, M. Van Pruijssen and P. Román. Matrix-valued orthogonal polynomials related to $(\mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{diag})$, II. *Publ. Res. Inst. Math. Sci.*, **49**(2) (2013), 271–312. xvi, 85, 88, 92

[80] E. Koelink, A. M. de los Ríos and P. Román. Matrix-valued Gegenbauer polynomials. *Preprint*. `http://arxiv.org/abs/1403.2938`. xviii

[81] M. G. Kreĭn. Infinite J-matrices and a matrix-moment problem. *Doklady Akad. Nauk SSSR (N.S.)*, **69** (1949), 125–128. viii

[82] M. G. Kreĭn. The fundamental propositions of the theory of representations of Hermitian operators with deficiency index (m, m). *Ukrain. Mat. Žurnal*, **1**(2) (1949), 3–66. viii

[83] O. E. Lancaster. Orthogonal polynomials defined by difference equations. *Amer. J. Math.* **63** (1941), 185–207 vii, 23

[84] P. López-Rodríguez. Riesz's theorem for orthogonal matrix polynomials. *Constr. Approx.* **15** (1) (1999), 135–151. viii

[85] P. López-Rodríguez. The Nevanlinna parametrization for a matrix moment problem. *Math. Scand.* **89** (2) (2001), 245–267. viii

[86] P. Lesky. Über polynomsysteme, die Sturm-Liouvilleschen sifferenzengleichungen genügen. *Math. Zeit.* **78** (1962), 439–445. vii

[87] F. Marcellán, M. A. Piñar and H. O. Yakhlef. Relative asymptotics for orthogonal matrix polynomials with convergent recurrence coefficients. *J. Approx. Theory.*, **111** (2001), 1–30. viii

[88] F. Marcellán, M. A. Piñar and H. O. Yakhlef. Perturbations in the Nevai matrix class of orthogonal matrix polynomials. *Linear Algebra Appl.*, **336** (2001), 131–254. viii

[89] F. Marcellán and I. Rodríguez. A class of matrix orthogonal polynomials on the unit circle. *Linear Algebra Appl.*, **121** (1989), 233–241. 4

[90] J. C. Medem, R. Álvarez-Nodarse, F. Marcellán. On the q-polynomials: a distributional study. *J. Comput. Appl. Math.* **135** (2001), 157–196. vii, 10

[91] W. Miller. Symmetry Groups and their Applications. *Academic Press, New York*, (1972). 8

[92] Luisa Miranian. *Matrix valued orthogonal polynomials.* ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)–University of California, Berkeley. 6

[93] A. F. Nikiforov and U. B. Uvarov. *Special Functions of Mathematical Physics.* Birkhäuser Verlag, Basilea 1988. viii, 9, 10, 23, 24

[94] I. Pacharoni and P. Román. A sequence of matrix valued orthogonal polynomials associated to spherical functions. *Constr. Approx.*, **28**(2) (2008), 127–147. viii, ix, x, 10

[95] I. Pacharoni and J. Tirao. Matrix valued orthogonal polynomials arising from the complex projective space. *Constr. Approx.*, **25**(2) (2007), 177–192. viii, ix

[96] A. Sinap and W. Van Assche. Orthogonal matrix polynomials and applications. In *Proceedings of the Sixth International Congress on Computational and Applied Mathematics (Leuven, 1994)*, **66** (1996), 27–52. viii

[97] G. Szegö. Orthogonal polynomials. *Amer. Math. Soc. Coll. Pub*, **23** AMS, Providence, Rhode Island, 1974. v

[98] J. A. Tirao. The matrix-valued hypergeometric equation. *Proc. Natl. Acad. Sci. USA*, **100**(14) (2003),8138–8141. xv, xvii, 15, 16, 56, 84, 85, 88, 90

[99] J. A. Tirao. The algebra of differential operators associated to a weight matrix: a first example. In *Groups, algebras and applications*, Contem. Math, Amer. Math. Soc., Providence, RI. **537** (2011), 291–324. x

[100] N. Ja. Vilenkin. Special functions and the theory of group representation. *T*rans. of Math. Monographs. , **22**] AMS, Providence, Rhode Island, 1986. ix

[101] L. Vinet and A. Zhedanov. Representations of the Schrödinger group and matrix orthogonal polynomials. *J. Phys. A*, **44**(35) (2011), 355201, 28. ix

[102] J. A. Wilson. Some hypergeometric orthogonal polynomials. *SIAM J. Math. Anal.*, **11**(4) (1980), 690–701. 91