Programa de Doctorado "Matemáticas"

PhD Dissertation

STATISTICAL ANALYSIS OF NEW MULTIVARIATE RISK MEASURES

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To my parents: Manuel and Carmen.

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Resumen

Como consecuencia de que los reguladores necesitan gestionar el riesgo en los distintos sectores, se está extendiendo de forma rápida una metodología basada en el riesgo. En las últimas décadas, este problema ha sido tratado en su mayoría en una versión univariante. Sin embargo, los riesgos envuelven normalmente varias variables aleatorias que son a menudo dependientes. Por tanto, es crucial trabajar en un marco multivariante. Por otro lado, los fenómenos están caracterizados frecuentemente por eventos extremos.

Esta tesis trata fundamentalmente dos problemas: la definición de medidas de riesgo en un marco multivariante y la estimatión de medidas de riesgo multivariantes teniendo en cuenta eventos extremos.

El Capítulo 1 es un capítulo introductorio. Presentamos el estado del arte para la noción de medidas de riesgo multivariantes. También, recordamos los principales resultados en Teoría de Cópulas, Teoría de Valores Extremos y Órdenes Estocásticos que son útiles en este trabajo .

Se introducen dos nuevas medidas de riesgo multivariantes en el Capítulo 2. Varias propiedades interesantes y, caracterizaciones bajo condiciones de cópulas Arquimedianas, se estudian para las medidas de riesgo propuestas. Además, se obtienen estimadores semiparamétricos para las nuevas medidas, y son ejemplificados considerando datos simulados y un conjunto de datos real de seguros.

El Capítulo 3 se centra en la estimación extrema no paramétrica de las medidas multivariantes propuestas en el Capítulo 2. Para este propósito, primero analizamos el comportamiento en la cola de las distribuciones condicionadas que definen dichas medidas. El principal resultado está constituido por el Teorema Central del Límite de los estimadores extremos. El rendimiento de los estimadores extremos se evalúa en datos simulados y para un conjunto de datos real de precipitaciones.

El estudio de la medida de riesgo multivariante asociada con *the Component-wise Excess(C.-E.) design realization* dada por Salvadori et al. (2011) se enmarca en el Capítulo 4. Se obtiene la expresión explícita de la medida para cópulas Arquimedianas. Asimismo, se proporciona un procedimiento de estimación extrema para la C.- E. design realization. Se estudia el comportamiento asintótico de los estimadores propuestos. Finalmente, los estimadores para la C.- E. design realization se aplican en datos simulados y para un conjunto de datos real de una presa.

Abstract

As a consequence of the need for regulators to manage risk in various sectors, a risk-based methodology is undergoing a fast expansion. Over recent decades, this problem has been mostly addressed via a univariate approach. However, risks usually involve several random variables that are often non-independent. Therefore, it is crucial to work in a multivariate setting. On the other hand, phenomena are frequently characterized by extreme events.

This thesis is fundamentally concerned with two problems: the definition of risk measures in a multivariate setting, and the estimation of multivariate risk measures by taking extreme events into account.

Chapter 1 is an introductory chapter. We present the state-of-art of the notion of multivariate risk measures. The main results in Copula Theory, Extreme Value Theory, and Stochastic Orders, which are useful in this work, are also provided.

Two new multivariate risk measures are introduced in Chapter 2. Several interesting properties and, characterizations under Archimedean copulas, are studied for the proposed risk measures. Furthermore, semi-parametric estimators for the new measures are obtained and are then exemplified considering simulated data and a real insurance data-set.

Chapter 3 deals with the non-parametric extreme estimation procedure of the multivariate measures proposed in Chapter 2. For this purpose, we first analyse the tail behaviour of the conditional distributions that define the aforementioned measures. The main result is given by the Central limit Theorem of the extreme estimators. The performance of the extreme estimators is evaluated in simulated data and for a real rainfall data-set.

The multivariate risk measure associated with the *Component-wise Excess (C.-E.)* design realization given by Salvadori et al. (2011) is outlined in Chapter 4. The explicit expression of the measure for Archimedean copulas is obtained. In addition, an extreme estimation procedure for the C.-E. design realization is provided and the asymptotic behaviour of the proposed estimators is studied. Finally, the estimators for the C.-E. design realization are applied to simulated data and a real dam data-set.

List of Symbols and Abbreviations

C.-E.	Component-wise Excess.
$VaR_\alpha(X)$	Value-at-Risk of X at level α.
$TVaR_\alpha(X)$	Tail Value-at-Risk of X at level α.
$CTE_\alpha(X)$	Conditional Tail Expectation of X at level α.
$CoVaR_\alpha^{j\|i}$	Conditional Value-at-Risk of X_j given X_i at level α.
$\partial\underline{L}(\alpha)$	Critical layer at level α given by $\{\mathbf{x} \in \mathbb{R}^d : F(\mathbf{x}) = \alpha\}$.
$\partial\underline{L}^<(\alpha)$	Critical layer at level α given by $\{\mathbf{x} \in \mathbb{R}^d : F(\mathbf{x}) < \alpha\}$.
$\partial\underline{L}^>(\alpha)$	Critical layer at level α given by $\{\mathbf{x} \in \mathbb{R}^d : F(\mathbf{x}) > \alpha\}$.
$\partial\overline{L}(\alpha)$	Critical layer at level α given by $\{\mathbf{x} \in \mathbb{R}^d : \overline{F}(\mathbf{x}) = 1-\alpha\}$.
$\delta_\omega(\alpha)$	Design realization with weight function ω defined over $\partial\underline{L}(\alpha)$.
EVT	Extreme Value Theory
POT Method	Peak Over Threshold Method.
GEV	Generalized Extreme Value.
MDA	Maximum Domain of Attraction.
$U_X(t)$	Left-continuous inverse of the distribution F of an rv X in $1 - 1/t$, $t > 1$.
RV_α	Regularly varying with index α.
GPD	Generalized Pareto Distribution.
$\leq_{st}$	Usual stochastic order.
$\leq_{sm}$	Supermodular order.
$PDR(Y\|X)$	Positive Regression Dependence of Y with respect to X.
$\leq_m$	Majorization order.
C	Copula function.
$W^d(\mathbf{u})$	Fréchet-Hoeffding lower bound.
$M^d(\mathbf{u})$	Fréchet-Hoeffding upper bound.
$\hat{C}$	Survival copula.
$\overline{C}$	Joint survival function for uniform distributions.
ϕ	Generator associated with an Archimedean copula C.
ϕ^{-1}	Inverse of ϕ.
φ	Generator associated with an Archimedean copula $\hat{C}$.
φ^{-1}	Inverse of φ.

$\delta_r(u)$	Self-nested diagonals of an Archimedean copula at each real order r.
λ_U	Upper tail dependence coefficient.
L_X	Weighted Loss function (WL) of an rv X.
$\underline{\mathrm{VaR}}_\alpha(\mathbf{X})$	Lower-Orthant Value-at-Risk of $\mathbf{X}$ at level α.
$\underline{\mathrm{VaR}}^i_\alpha(\mathbf{X})$	i-th component of Lower-Orthant Value-at-Risk of $\mathbf{X}$ at level α.
$\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$	Multivariate Lower-Orthant CoVaR at level α.
$\partial\underline{L}^{\geq}(\alpha)$	Critical layer at level α given by $\{\mathbf{x} \in \mathbb{R}^d_+ : F(\mathbf{x}) \geq \alpha\}$.
$\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$	Multivariate Upper-Orthant CoVaR at level α.
$\partial\overline{L}^{\leq}(\alpha)$	Critical layer at level α given by $\{\mathbf{x} \in \mathbb{R}^d_+ : \overline{F}(\mathbf{x}) \leq 1-\alpha\}$.
T_i	$[X_i \mid \mathbf{X} \in \partial\underline{L}(\alpha)]$.
T'_i	$[X_i \mid \mathbf{X} \in \partial\overline{L}(\alpha)]$.
$F_{T_i}(x\|\alpha)$	Distribution function of T_i.
$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$	i-th component of Lower-Orthant CoVaR at level α.
$\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$	i-th component of Upper-Orthant CoVaR at level α.
$\overline{\mathrm{VaR}}^i_\alpha(\mathbf{X})$	i-th component of Upper-Orthant Value-at-Risk of $\mathbf{X}$ at level α.
$\stackrel{d}{=}$	Equality in distribution.
MRV	Multivariate Regularly Varying.
$\Delta\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(X_j\|X_i)$	ΔCoVaR of X_j conditional on the distress of X_i.
$\Delta\underline{\mathrm{CoVaR}}^j_{\alpha,\boldsymbol{\omega}}(X_j\|\mathbf{X})$	ΔCoVaR of X_j conditional on the distress of the system.
RMSE	Relative Mean Square Error.
$\hat{\sigma}$	Empirical Standard Deviation.
ALAE	Allocated Loss Adjustment Expense.
ρ	Regularly varying index of generator ϕ.
γ^{T_i}	Tail index of T_i.
$\overline{F}_{T'_i}(x\|\alpha)$	Survival Distribution function of T'_i.
ρ'	Regularly varying index of generator φ.
$\gamma^{T'_i}$	Tail index of T'_i.
$\Lambda_U(x,y)$	Upper tail copula.
$2RV_{\gamma,\tau}$	Second Regularly Varying with first- and second-order indexes γ and τ.
$\xrightarrow{d}$	Convergence in distribution.
$\xrightarrow{\mathbb{P}}$	Convergence in probability.
$\delta_{CE}(\alpha)$	C.-E. design realization at level α.
ERV_γ	Extended Regularly Varying with index γ.
$2ERV_{\gamma,\tau}$	Second Extended Regularly Varying with first- and second-order indexes γ and τ.

Contents

Chapter 1

Introduction

1.1 Risk measures

Risk is a complex notion and can take on a variety of forms with diverse applications. Risk could be defined as the effect that lack of certainty produces on objectives (ISO (2009)). A risk-based approach for the supervision and regulation of different sectors is gaining ground in both emerging and industrialized countries. As part of this approach, regulators need to measure, monitor, and manage the risk. One of the most important areas in which this practice has been adopted is the insurance sector where the main challenge is to sell risk coverage. For instance, people rely on their savings to finance their old age. The study of the risk involved in the assessment of the profitable areas of business in finance is crucial. Furthermore, in order to prevent and to manage damages and losses due to a natural disaster, it is of prime importance to examine the risk in the environmental sector (The European Parliament and The Council (2007)).

As the recent financial crisis has shown, risks are generally difficult to measure and to manage. By concentrating on a framework to determine provisions and capital requirements in order to prevent insolvency, Denuit et al. (2005) introduce the following definition of risk measure.

Definition 1.1.1 (Definition 2.2.1 in Denuit et al. (2005))**.** *A risk measure is a functional ϱ mapping a risk X to a non-negative real number $\varrho(X)$, possibly infinite, representing the extra cash which has to be added to X to make it acceptable.*

Let Ω be the set of states of nature and assume that is finite. Let $\mathfrak{G}$ be the set of all real-valued functions on Ω. Artzner et al. (1999) define a risk measure as a mapping from $\mathfrak{G}$ into $\mathbb{R}$. According to Artzner et al. (1999), every risk measure should verify the following set of desirable properties, that is, the risk measure should be a coherent risk measure.

Definition 1.1.2 (Coherent Risk Measure)**.** *Let X and Y be two random variables such that $X, Y \in \mathfrak{G}$. A risk measure ϱ is coherent if it satisfies:*

- *Monotonicity:* $\mathbb{P}[X \leq Y] = 1 \Rightarrow \varrho[X] \leq \varrho[Y]$.
- *Subadditivity:* $\varrho[X+Y] \leq \varrho[X] + \varrho[Y]$.
- *Positive Homogeneity:* $\varrho[cX] = c\varrho[X]$, $c > 0$.
- *Translativity:* $\varrho[X+c] = \varrho[X] + c$, $c > 0$.

Over recent decades, the risk quantification problem has been mostly addressed via a univariate version. The classic univariate measure in financial sciences is that of the *Value-at-Risk* (VaR). This quantity represents the magnitude of a event that occurs at a given time and at a given site. More precisely, VaR_α, $\alpha \in (0,1)$, is a quantile which expresses the magnitude of the event that is exceeded with a probability $1-\alpha$. That is, for a random variable X with distribution function F_X, $\mathrm{VaR_\alpha(X) = inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}}$, $\alpha \in (0,1)$. If annual maxima observations are investigated, in the environmental sector, VaR is called *return level*, and $1/(1-\alpha)$ is called *return period* (RP). In general, the RP is traditionally defined as "the average time elapsing between two successive realizations of a prescribed event" (Salvadori et al. (2011)). The RP is given by $\mu/(1-\alpha)$ where μ is the average inter-arrival time of the realizations of X.

VaR is the most popular risk measure. For instance, Ahmed et al. (2016) use univariate return periods to construct seasonal drought maps for different climatic seasons in the Balochistan province (Pakistan). However, as we explain below, there are certain reasons for the rejection of VaR as an adequate measure of risk.

Firstly, the VaR measure fails to give any information about the thickness of the tail of the distribution function. That is, a regulator can know only the frequency of default but not the severity of default (Denuit et al. (2005)). In order to prevent the above shortcoming, *Tail Value-at-Risk* (TVaR) and *Conditional Tail Expectation* (CTE) risk measures are introduced. Let X be a random variable, *Tail Value-at-Risk* is defined as

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR_u(X)du}.$$

Furthermore, *Conditional Tail Expectation* is given by

$$CTE_\alpha(X) = E[X|X > VaR_\alpha(X)].$$

Secondly, VaR is not a coherent measure since it does not verify the subadditivity property in general. This result might imply that diversified portfolios are riskier than less diversified portfolios (Daníelsson et al. (2013), Elliott and Miao (2009), Peng (2013)). TVaR is a subadditive measure and CTE verifies the aforementioned property for continuous risks.

The risk allocation problem involves only internal risks associated with businesses in the subsidiaries. However, the solvability of financial institutions could also be affected by external risks whose sources cannot be controlled. These risks may also be heterogeneous in nature, difficult to diversify away, and frequently correlated. One can consider, for instance, contagion effects in a strongly interconnected system of financial companies or how a flood can be described by the volume, the peak and the duration (see Chebana and Ouarda (2011a,b)). For this reason, it is crucial to identify risks in a multivariate setting. The following consistent notion of multivariate RP is introduced in Salvadori et al. (2011).

Definition 1.1.3. *Let $\mathcal{X} = \{\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots\}$ be a sequence of independent and identically distributed d-dimensional random vectors, with $d > 1$: thus, each $\boldsymbol{X}_i$ has the same multivariate distribution of $\boldsymbol{X}$. Let $\mathcal{D}$ be a non-empty Borel set in $\mathbb{R}^d$ collecting all the values judged to be "dangerous" according to some suitable criterion. Let $\boldsymbol{X}$ be the vector that describes the phenomenon under investigation. The RP associated with the event $\{\boldsymbol{X} \in \mathcal{D}\}$ is given by $\mu/\mathbb{P}(\boldsymbol{X} \in \mathcal{D})$, where μ is the average inter-arrival time of the realizations in $\mathcal{X}$.*

In hydrology, the "value of the variable(s) characterizing the event associated with a given return period" is called *design quantile* or *multivariate return level* (Salvadori et al. (2013)). The design quantile coincides with return level in the univariate case.

The generalization of return level is not univalent (see e.g., Serfling (2002), Vandenberghe et al. (2012)) and several definitions can be found in the recent literature. Since different combinations of probabilities may produce the same return period, a multivariate return level is inherently ambiguous. Events that have equal probability of exceedance define iso-hyper-surfaces, otherwise known as *critical layers*. Salvadori et al. (2011) provide the following definition.

Definition 1.1.4. *Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a random risk vector with joint distribution function F. For $\alpha \in (0,1)$ and $d \geq 2$, the critical layer $\partial \underline{L}(\alpha)$ at level α is defined as $\partial \underline{L}(\alpha) = \{\boldsymbol{x} \in \mathbb{R}^d : F(\boldsymbol{x}) = \alpha\}$.*

Definition 1.1.4 provides a partition composed of three probability regions: $\partial \underline{L}^{<}(\alpha) = \{\boldsymbol{x} \in \mathbb{R}^d : F(\boldsymbol{x}) < \alpha\}$ (the sub-critical region); $\partial \underline{L}(\alpha)$ (the critical region where all the events have a constant F); and $\partial \underline{L}^{>}(\alpha) = \{\boldsymbol{x} \in \mathbb{R}^d : F(\boldsymbol{x}) > \alpha\}$ (the super-critical region).

In practice, at any occurrence of the phenomenon, only these three mutually exclusive events may occur (see Belzunce et al. (2007)). The multivariate return level can be defined with respect to one of the above three areas. For instance, in hydrology, the sub-critical region may be of interest if droughts are to be investigated, while the study of floods may require the use of super-critical regions (Salvadori et al. (2013)).

Furthermore, Definition 1.1.4 can be presented in terms of the joint survival distribution function of $\mathbf{X}$, $\overline{F}$. We denote $\partial \overline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}^d : \overline{F}(\mathbf{x}) = 1 - \alpha\}$.

By taking into account the notion of critical layers (see Definition 1.1.4), Salvadori et al. (2011) provide the following definition of design realization.

Definition 1.1.5 (Design realization)**.** *Let $\omega : \partial \underline{L}(\alpha) \to [0, \infty)$ be a weight function. The design realization $\delta_\omega \in \partial \underline{L}(\alpha)$ is defined as*

$$\delta_\omega(\alpha) = \underset{\boldsymbol{x} \in \partial \underline{L}(\alpha)}{\arg\max}\ \omega(\boldsymbol{x}) \tag{1.1}$$

where $\partial \underline{L}(\alpha) = \{\boldsymbol{x} \in \mathbb{R}^d : F(\boldsymbol{x}) = \alpha\}$ is the critical layer at level α with $\alpha \in (0, 1)$ and $d \geq 2$.

Definition 1.1.5 is based on the idea of introducing a suitable function that "weights" the realizations lying in the critical layer of interest. Salvadori et al. (2011) and Salvadori et al. (2014) provide practical guidelines for both terrestrial and coastal/offshore engineering, illustrating how to calculate suitable design realization by using several different weight functions.

In the last decade, much research has been devoted to risk measures and many multidimensional extensions have been investigated. On theoretical grounds, Jouini et al. (2004) propose a class of set-value coherent risk measures. Unsurprisingly, the main difficulty regarding multivariate generalizations of risk measures is the fact that vector preorders are, in general, partial preorders. In order to generalize the Value-at-Risk measure, Embrechts and Puccetti (2006), Nappo and Spizzichino (2009), and Prékopa (2012) use the notion of a quantile curve which is defined as the boundary of the upper-level set of a distribution function or the lower-level set of a survival function. Cousin and Di Bernardino (2013) introduce two alternative extensions of the classic univariate Value-at-Risk in a multivariate setting. The proposed measures are real-valued vectors with the same dimension as that of the considered portfolio of risks. This feature can be considered relevant from an operational point of view. Cousin and Di Bernardino (2014) propose two extensions of the classic univariate Conditional-Tail-Expectation (CTE) in a multivariate setting. The multivariate extensions in Cousin and Di Bernardino (2013) and Cousin and Di Bernardino (2014) are constructed from level sets of multivariate distribution functions and multivariate survival distribution functions, respectively. Such as level sets approach is also used in this thesis. Discussions concerning the meaning and use of return periods under non-stationary and multivariate conditions can be found in Serinaldi (2015b) and Serinaldi and Kilsby (2015). Serinaldi (2015a) points out several critical aspects that are often overlooked and should be carefully taken into account for a correct interpretation of return periods. Torres et al. (2015) provide a definition of design quantile based on rotated directional distribution functions.

On the other hand, Chebana and Ouarda (2011b) provide a parametric estimator for critical layers and use it to study a real rainfall data-set. An estimation for the bivariate critical layers $\partial \underline{L}(\alpha_n)$ by assuming $\alpha_n \to 1$, as $n \to \infty$, is presented in de Haan and Huang (1995). Fawcett and Walshaw (2016) briefly highlight the shortcomings of standard methods in the estimation of design quantiles and provide an estimation framework which substantially increases the precision of design quantile estimates. Multivariate frequency analysis is also used in spatial statistics in order to model the spatial variability of hydrological random variables (see e.g., De Paola and Ranucci (2012), De Paola et al. (2013)).

Another interesting recent risk measure is that of the CoVaR, which stands for *Conditional Value-at-Risk*. CoVaR is a systemic risk measure proposed by Adrian and Brunnermeier (2011) that measures a financial institution's contribution to systemic risk and its contribution to the risk of other financial institutions. In the original unidimensional model, the CoVaR (of a particular bank, portfolio of asset, etc.) indicates the Value-at-Risk for a financial institution which is conditional on a certain (stress) scenario. On assuming that X_j represents asset returns of the financial system (or bank j) and X_i represents the asset returns of bank i, the $\mathrm{CoVaR}_\alpha^{\mathrm{j|i}}$ can then be defined by:

$$P[X_j \leq \mathrm{CoVaR}_\alpha^{\mathrm{j|i}} \,|\, \mathrm{X_i} = \mathrm{VaR}_\alpha(\mathrm{X_i})] = \alpha, \quad \text{for } \alpha \in (0,1). \tag{1.2}$$

Equation (1.2) implicitly defines the CoVaR of the bank j which is conditional on bank i being at its α%-VaR level (see Adrian and Brunnermeier (2011)).

In the literature, several alternative definitions of CoVaR can be found (see Goodhart and Segoviano (2009), Girardi and Ergün (2013) and Bernardi et al. (2017)). Starting from (1.2), we can also consider the CoVaR given by

$$\mathrm{CoVaR}_\alpha^{\mathrm{j}}(\mathbf{X}) = \mathrm{VaR}_\alpha(\mathrm{L}|\mathrm{X_j} \geq \mathrm{VaR}_\alpha(\mathrm{X_j})),$$

where the financial system is represented via the total risk $L = X_1 + \ldots + X_d$, that is, the aggregated total risk of the firm network and the component j of the vector $\mathbf{X} = (X_1, \ldots, X_d)$ represents the risk exposure of the company j. Moreover, Adrian and Brunnermeier (2011) defined a systemic risk measure, called ΔCoVaR, as the difference between the VaR of the institution j (or financial system) conditional on the distress of a particular financial institution i (see (1.2)) and the VaR of the institution j. That is,

$$\Delta\mathrm{CoVaR}_{\alpha,\omega}(\mathrm{X_j}|\mathrm{X_i}) = \mathrm{VaR}_\omega(\mathrm{X_j}|\mathrm{X_i} = \mathrm{VaR}_\alpha(\mathrm{X_i})) - \mathrm{VaR}_\omega(\mathrm{X_j}), \tag{1.3}$$

for α and ω in $(0,1)$. It can be observed that, when j is the complete system, ΔCoVaR captures the marginal contribution of a particular institution to the overall systemic risk. An institution with low VaR but high ΔCoVaR is far riskier to the financial system than an institution with high VaR but low ΔCoVaR. The ΔCoVaR measure and other systemic risk measures are described in Mainik and Schaanning (2014).

1.2 Extreme Value Theory

When dealing with a data-set, it may be possible that a few observations overpower the remainder of the sample due to their large (or low) magnitude. That is, most of the data are concentrated in the "body" of the distribution, and rare observations found outside this range are called *extreme events* (or *outliers*). Since extreme events can exert very negative impacts, the quantification of the occurrence of extreme risks is gaining attention in several fields, such as insurance, finance, environmental sciences and Internet traffic (Longin (2016)). Salvadori et al. (2016) discuss the various international guidelines concerning risk assessment and state that these guidelines require the implementation of extreme event scenarios in order to manage the environmental risk.

One of the most important fundamental questions from a statistical point of view is how to model extreme events (McNeil et al. (2005)). An extreme event can be defined as a catastrophe that has not yet happened, for instance, an earthquake. In order to protect ourselves against this calamity, the quantile of this risk has to be considered at a sufficiently small level in order to measure the risk of occurrence (Einmahl et al. (2013)). However, standard statistical estimation may lead to severely biased results if it is used for the estimation of the behaviour of the tails (Hochrainer-Stigler and Pflug (2012)). Furthermore, by using classic theory, a specific probabilistic model would be fitted to the whole sample and that model would be used for the estimation of the tail probability (Longin (2016)). *Extreme Value Theory* (EVT) is concerned with the study of the asymptotic distribution of extreme events by considering sample extreme maximum (or minimum). The limit distribution in EVT is mainly characterized by the tail index. The tail index measures the "fatness" in the tail of the distribution.

In order to identify the extreme events in a data-set, the EVT provides two different approaches (Gilli and Këllezi (2006)):

- Block-Maxima Method: The maximum the variable takes in successive periods (for instance, months or years) is considered. These selected observations constitute the extreme events, also called *block-maxima*.

- Peak Over Threshold (POT) Method: This focuses on the realizations exceeding a given (high) threshold.

The most important results in EVT for the above two methods can now be presented.

The limit laws for the maximum of n independent and identically distributed (iid) random variables were derived by Fisher and Tippett (1928) and Gnedenko (1943).

Theorem 1.2.1 (Theorem 1.1.3 in de Haan and Ferreira (2006)). *Let $X_1, \ldots, X_n$ be a sequence of iid random variables with distribution F. If there exist a positive sequence $(a_n)_{n>0}$, a real sequence $(b_n)_{n>0}$, and some non-degenerate distribution function H_γ*

such that

$$\lim_{n\to+\infty} \mathbb{P}\left[\frac{\max\{X_1,\ldots,X_n\}-b_n}{a_n} \le x\right] = H_\gamma(x),\ x \in \mathbb{R}, \tag{1.4}$$

then H_γ *is an element of the* Generalized Extreme Value *(GEV) class:*

$$H_\gamma(x) = \begin{cases} e^{(-(1+\gamma x)^{-1/\gamma})}, & \gamma \neq 0; \\ e^{-e^{-x}}, & \gamma = 0, \end{cases}$$

with $1+\gamma x > 0$.

Definition 1.2.1 (Maximum Domain of Attraction (MDA))**.** *The random variable* X *with distribution* F *belongs to the maximum domain of attraction of* H_γ*, denoted by* $X \in MDA(\gamma)$ *or* $F \in MDA(\gamma)$*, if there exist positive* $(a_n)_{n>0}$ *and real* $(b_n)_{n>0}$ *such that* (1.4) *holds.*

If $\gamma > 0$, F belongs to the Fréchet distribution MDA and F is heavy-tailed . When $\gamma < 0$, F belongs to the Weibull distribution MDA and F is short-tailed. If $\gamma = 0$, F belongs to the Gumbel distribution MDA and F is light-tailed (see page 9 in de Haan and Ferreira (2006)).

The von Mises condition states a sufficient condition for a function to belong to a domain of attraction.

Definition 1.2.2 (*von Mises condition*)**.** *Let* F *be a distribution function and* x_F *its right endpoint. Let* F' *and* F'' *be the first and the second derivatives of* F*, respectively. Suppose* $F''(x)$ *exists and* $F'(x)$ *is positive for all* x *in some left neighborhood of* x_F*. The von Mises condition for* F *holds if, for some* $\gamma \in \mathbb{R}$*,*

$$\lim_{t\uparrow x_F} \frac{(1-F(t))F''(t)}{(F'(t))^2} = -\gamma - 1.$$

Theorem 1.2.2 (Theorem 1.1.8 in de Haan and Ferreira (2006))**.** *Let* F *be a distribution function and* x_F *its right endpoint. Let* F' *and* F'' *be the first and the second derivatives of* F*, respectively. Suppose* $F''(x)$ *exists and* $F'(x)$ *is positive for all* x *in some left neighborhood of* x_F*. If* F *verifies the von Mises condition in Definition 1.2.2, then* F *is in the domain of attraction of* H_γ*.*

Let X be a random variable with distribution function F. Henceforth, we denote

$$U_X(t) := F^{-1}(1-1/t),\ t > 1, \tag{1.5}$$

where F^{-1} is the left-continuous inverse of F.

From Theorem 1.2.1 and Corollary 1.2.10 in de Haan and Ferreira (2006), it is shown that $U \in RV_\gamma$ (see Appendix A for RV definition) iff F is in the Fréchet MDA with index $\gamma > 0$. Similarly, $x_F - U \in RV_\gamma$ iff F is in the Weibull MDA with index $\gamma < 0$ (see

Section 3.3.2 in Embrechts et al. (1997)). For further details, the interested reader is directed to Mao and Hu (2012). Although there is no direct link between Gumbel MDA and regular variation functions, certain extensions of regularly varying notion provide the characterization for the Gumbel MDA in Section 3.3.3 in Embrechts et al. (1997).

The POT method emerges from the next theorem (Balkema and de Haan (1974), Pickands III (1975)). This theorem proves that the conditional excess distribution function $F_u(x) = \mathbb{P}[X - u \leq x | X > u] = \frac{F(x+u)-F(u)}{1-F(u)}$ follows a Generalised Pareto Distribution (GPD).

Theorem 1.2.3 (Theorem 3.4.5 in Embrechts et al. (1997)). *Let x_F be the right end-point of F. For a large class of underlying distribution functions F the conditional excess distribution function $F_u(x)$, for u large, is well approximated by*

$$F_u(x) \approx V_{\gamma,\sigma}(x), \;\; u \to \infty,$$

where

$$V_{\gamma,\sigma}(x) := \begin{cases} 1 - (1 + \frac{\gamma}{\sigma}x)^{-1/\gamma}, & \gamma \neq 0, \sigma > 0; \\ 1 - e^{-\frac{x}{\sigma}}, & \gamma = 0, \sigma > 0 \end{cases} \tag{1.6}$$

is a GPD with $x \in [0, (x_F - u)]$ if $\gamma \geq 0$, and $x \in [0, -\sigma/\gamma]$ if $\gamma < 0$.

The link between block-maxima and POT methods is established in Theorem 3.4.13(b) in Embrechts et al. (1997).

For a in-depth mathematical analysis of EVT, the interested reader is directed to de Haan and Ferreira (2006) and Embrechts et al. (1997).

1.3 Stochastic Orders

In risk theory, when it is assumed that an individual prefers the option whit the greatest expected utility, or when we would like to measure which risk is the most dangerous, we use stochastic orders to find a formal way to establish these ideas. Stochastic orders enable comparison between the risks to be made. Denuit et al. (2005) (Chapter 3) explain the desirable properties for stochastic orderings.

Several useful definitions of stochastic orders are now recalled. Further details, equivalent definitions and applications may be found in Shaked and Shanthikumar (2007), Müller (1997), Joe (1997), and Kaas et al. (2001).

Definition 1.3.1 (Usual Stochastic Order). *Let X and Y be two random variables with distribution functions F_X and F_Y, respectively. X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{st} Y$, if*

$$F_X(x) \geq F_Y(x), \;\; \textit{for all } x \in \mathbb{R}.$$

Definition 1.3.2 (Supermodular function). *A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be supermodular if, for any $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^d$, it satisfies*

$$f(\boldsymbol{x}) + f(\boldsymbol{y}) \leq f(\boldsymbol{x} \wedge \boldsymbol{y}) + f(\boldsymbol{x} \vee \boldsymbol{y}),$$

where the operators $\wedge$ and $\vee$ denote coordinate-wise minimum and maximum, respectively.

Definition 1.3.3 (Supermodular Order). *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two $d-$dimensional random vectors. $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ with respect to the supermodular order (denoted by $\boldsymbol{X} \leq_{sm} \boldsymbol{Y}$) iff*

$$\mathbb{E}(f(\boldsymbol{X})) \leq \mathbb{E}(f(\boldsymbol{Y})),$$

for all supermodular functions $f : \mathbb{R}^d \to \mathbb{R}$, provided the expectations exist.

While the usual stochastic order compares risks according to their “magnitude”, the supermodular order constitutes a condition sufficient to obtain positive dependence between the risks.

Lehmann (1966) introduces the following dependence notion based on the increasing stochastic condition.

Definition 1.3.4 (Positive Regression Dependence). *A bivariate random vector (X, Y) is said to admit positive regression dependence with respect to X, $PRD(Y|X)$, if $[Y|X = x_1] \leq_{st} [Y|X = x_2]$, $\forall x_1 \leq x_2$.*

We now recall the notion of majorization ordering from Marshall et al. (2011).

Definition 1.3.5 (Majorization Ordering). *Let $\boldsymbol{a} = (a_1, \ldots a_d)$ and $\boldsymbol{b} = (b_1, \ldots b_d)$ be two points in $\mathbb{R}^d$ and denote by $a_{[1]}, \ldots, a_{[d]}$ and $b_{[1]}, \ldots, b_{[d]}$ the components of $\boldsymbol{a}$ and $\boldsymbol{b}$ rearranged in decreasing order. The point $\boldsymbol{a}$ is said to be* majorized *by the point $\boldsymbol{b}$ (written $\boldsymbol{a} \prec_m \boldsymbol{b}$) if $\sum_{j=1}^{d} a_{[j]} = \sum_{j=1}^{d} b_{[j]}$ and $\sum_{j=1}^{k} a_{[j]} \leq \sum_{j=1}^{k} b_{[j]}$ for $k = 1, \ldots, d-1$.*

1.4 Copula Theory

A *copula* is (the restriction of) a d-dimensional distribution in $[0, 1]^d$ whose marginal distributions are uniformly distributed. The copula establishes the link between the marginal distribution functions to generate the joint distribution function. Although the functions themselves appear in previous work (Fréchet (1951), Dall’Aglio (1972)), the word “copula” was first used in a mathematical sense by Sklar (1959). From Sklar’s theorem (Sklar (1959), Theorem 2.10.9 in Nelsen (2006)), for every joint distribution function F with marginal F_{X_i}, $i = 1, \ldots, d$, there exists a copula C such that for all $\mathbf{x} = (x_1, \ldots, x_d) \in [-\infty, +\infty]^d$,

$$F(x_1, \ldots, x_d) = C(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)).$$

If F_{X_i}, $i = 1, \ldots, d$, are all continuous, then C is unique. Conversely, if C is a copula and, F_{X_i}, $i = 1, \ldots, d$, are distribution functions, then the function F is a joint distribution function with margins F_{X_i}, $i = 1, \ldots, d$.

The notion of copulas rose in popularity at the end of nineties. The main reason for this was the explosive development of quantitative risk management methodology within finance and insurance (Durante and Sempi (2015)).

For every copula C and $\mathbf{u} = (u_1, \ldots, u_d) \in [0,1]^d$, it holds that

$$\max\{u_1 + \ldots + u_d - d + 1, 0\} \leq C(\mathbf{u}) \leq \min\{u_1, \ldots, u_d\}.$$

$W^d(\mathbf{u}) := \max\{u_1 + \ldots + u_d - d + 1, 0\}$ and $M^d(\mathbf{u}) := \min\{u_1, \ldots, u_d\}$ are the *Fréchet-Hoeffding* lower and upper bound respectively. When a vector follows a W^d copula ($d \leq 2$), the random variables are countermonotonic. If a vector follows a M^d copula, then the random variables are comonotonic. Furthermore, from Theorem 2.10.14 in Nelsen (2006), d continuous random variables $X_1, \ldots X_d$ are independent if and only if the copula associated with $X_1, \ldots X_d$ is $\Pi^d(\mathbf{u}) = u_1 \cdots u_d$.

On the other hand, let $(U_1, \ldots, U_d)$ be a random vector associated with C, the joint distribution function of $(1 - U_1, \ldots, 1 - U_d)$ is a copula $\hat{C}$ which is called the *survival copula* associated with C. Nelsen (2006) notices that $\hat{C}$ couples the joint survival function to its univariate survival margins in a manner completely analogous to the way in which a copula connects the joint distribution function to its margins. It should be borne in mind that one has be careful not to confuse the survival copula $\hat{C}$ with the joint survival function $\overline{C}$ for uniform distributions whose joint distribution function is the copula C. As presented in Joe (2015), the joint survival function $\overline{C}$ is given by

$$\overline{C}(u_1, \ldots, u_d) = \mathbb{P}[U_1 > u_1, \ldots, U_d > u_d] = 1 - \sum_{j=1}^{d} u_j + \sum_{S \subset \{1,\ldots,d\}, |S| \geq 2} (-1)^{|S|} C_S(u_i, i \in S)$$

where C_S is the copula of all the components in S and $|S|$ is the cardinality of S. It is verified that $\hat{C}(u_1, \ldots, u_d) = \overline{C}(1 - u_1, \ldots, 1 - u_d)$ (see Theorem 2 in Georges et al. (2001)).

The Archimedean copula class constitutes an important family of copulas, and has frequently been used in environmental sciences (see e.g., Salvadori et al. (2007) and Pappadà et al. (2016b)). For instance, Saad et al. (2015) propose a multivariate flood-risk model based on nested Archimedean Frank and Clayton copulas in a hydro-meteorological context in order to determine the 2011 Richelieu River flood-causing meteorological factors. Using the Gumbel copula, Zhang et al. (2016) obtain useful information about of reservoir regulation on drought evolution under precipitation variations in two cascade reservoirs located in China.

Note that a d-dimensional Archimedean copula with generator ϕ and its inverse ϕ^{-1}

is defined by

$$C(\mathbf{u}) = \phi^{-1}(\phi(u_1) + \ldots + \phi(u_d)), \text{ for all } \mathbf{u} = (u_1, \ldots, u_d) \in [0,1]^d.$$

The generator ϕ is a continuous, convex and strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\phi(1) = 0$. If $\phi(0) = +\infty$, then ϕ is called a strict generator. Furthermore, the generator of an Archimedean copula also satisfies several additional $d-$monotony properties (for further details, see Theorem 2.2 in McNeil and Nešlehová (2009)). In Table 1.1, we recall the generators of the Archimedean copulas that we are going to use in this thesis.

Copula	Parameter θ	Generator ϕ	Inverse Generator ϕ^{-1}
Ali-Mikhail-Haq	$[-1,1)$	$\log\left[\frac{1-\theta(1-t)}{t}\right]$	$\frac{1-\theta}{\exp(t)-\theta}$
Clayton	$[-1,\infty)\backslash\{0\}$	$\frac{1}{\theta}(t^{-\theta}-1)$	$(1+\theta t)^{-1/\theta}$
Frank	$\mathbb{R}\backslash\{0\}$	$-\log\left(\frac{\exp(-\theta t)-1}{\exp(-\theta)-1}\right)$	$-\frac{1}{\theta}\log(1+\exp(-t)(\exp(-\theta)-1))$
Gumbel	$[1,\infty)$	$(-\log(t))^{\theta}$	$\exp(-t^{1/\theta})$
Joe	$[1,\infty)$	$-\log(1-(1-t)^{\theta})$	$1-(1-\exp(-t))^{1/\theta}$

Table 1.1: The Archimedean copulas used in this thesis, the domain of the dependence parameter, the generators and the inverse generators.

McNeil and Nešlehová (2009) obtained an important stochastic representation of Archimedean copulas, recalled in Proposition 1.4.1 below.

Proposition 1.4.1 (McNeil and Nešlehová (2009))**.** *Let $\boldsymbol{U} = (U_1, \ldots, U_d)$ be distributed according to a d-dimensional Archimedean copula with generator ϕ, hence*

$$(\phi(U_1), \ldots, \phi(U_d)) \stackrel{d}{=} R\boldsymbol{S},$$

where $\boldsymbol{S} = (S_1, \ldots, S_d)$ is uniformly distributed on the unit simplex $\left\{\boldsymbol{x} \geq 0 | \sum_{k=1}^d x_k = 1\right\}$ and R is an independent non-negative scalar random variable which can be interpreted as the radial part of $(\phi(U_1), \ldots, \phi(U_d))$ since $\sum_{k=1}^d S_k = 1$. The random vector $\boldsymbol{S}$ follows a symmetric Dirichlet distribution, whereas the distribution of $R \stackrel{d}{=} \sum_{k=1}^d \phi(U_k)$ is directly related to the generator ϕ through the inverse Williamson transform of ϕ^{-1}.

As a result, any random vector $\mathbf{U} = (U_1, \ldots, U_d)$ which follows an Archimedean copula with generator ϕ can be represented as a deterministic function of $C(\mathbf{U})$ and an independent random vector $\mathbf{S} = (S_1, \ldots, S_d)$ uniformly distributed on the unit simplex, that is,

$$(U_1, \ldots, U_d) \stackrel{d}{=} (\phi^{-1}(S_1\phi(C(\mathbf{U}))), \ldots, \phi^{-1}(S_d\phi(C(\mathbf{U})))). \tag{1.7}$$

We now consider that $\mathbf{X} = (X_1, \dots, X_d)$ is distributed as $(\overline{F}_{X_1}^{-1}(V_1), \dots, \overline{F}_{X_d}^{-1}(V_d))$ where $\overline{F}_{X_i}$ denotes the i-th survival margin distribution of $\mathbf{X}$, $i = 1, \dots, d$, and $\mathbf{V} = (V_1, \dots, V_d)$ follows a survival Archimedean copula $\hat{C}$ with generator φ. Relation (1.7) therefore also holds for $\mathbf{V}$ and $\hat{C}$, that is,

$$(V_1, \dots, V_d) \stackrel{d}{=} (\varphi^{-1}(S_1\varphi(\hat{C}(\mathbf{V}))), \dots, \varphi^{-1}(S_d\varphi(\hat{C}(\mathbf{V})))). \tag{1.8}$$

The copula sections could be employed in the construction of copulas and could provide interpretations of certain dependence properties. The *diagonal section* of a d-dimensional copula C is given by $\delta_1(u) = C(u, \dots, u)$, $u \in [0,1]$, and δ_{-1} is the inverse function of δ_1, such that $\delta_1 \circ \delta_{-1}$ is the identity function. From Lemma 3.4 in Di Bernardino and Rullière (2013), one can write the family of self-nested diagonals of an Archimedean copula C of order $r \in \mathbb{R}$ as:

$$\delta_r(u) = \phi^{-1}(d^r\phi(u)), \quad \text{for } u \in (0,1),\ r \in \mathbb{R}. \tag{1.9}$$

The dependence tail properties for a copula are crucial to study the extreme estimators in Chapters 3 and 4.

Let $\mathbf{X} = (X_1, \dots, X_d)$ be a random vector with margin distributions F_{X_i}, $i = 1, \dots, d$. For the subsets $I,\ J \in \{1, \dots, d\}$, $I \cap J = \emptyset$, if the following limit exists everywhere on $\overline{\mathbb{R}}_+^d = [0, \infty]^d \setminus (\infty, \dots, \infty)$

$$\Lambda_U^{I,J}(\mathbf{x}) := \lim_{t\to\infty} P\left[X_i > F_{X_i}^{-1}(1 - x_i/t),\ \forall i \in I \,|\, X_j > F_{X_j}^{-1}(1 - x_j/t),\ \forall j \in J\right],$$

then the function $\Lambda_U^{I,J} : \overline{\mathbb{R}}_+^d \to \mathbb{R}$ is called an *upper tail copula* associated with F with respect to I, J (see Schmidt and Stadtmüller (2006)).

Let (X_i, X_j), $i \neq j$, be a bivariate random vector with marginal distribution functions F_{X_i} and F_{X_j}. It is said to be *upper tail dependent* if $\Lambda_U(1,1)$ exists and

$$\lambda_U := \Lambda_U(1,1) = \lim_{v\to 1^-} P[X_i > F_{X_i}^{-1}(v) \,|\, X_j > F_{X_j}^{-1}(v)] > 0. \tag{1.10}$$

Conversely, if $\lambda_U = 0$, then (X_i, X_j) is called *upper tail independent*. Furthermore, λ_U is referred to as the *upper tail dependence coefficient*. In Schmidt and Stadtmüller (2006), the following non-parametric rank-based estimator of λ_U is introduced. We now assume that (X_i, X_j), $(X_i^{(1)}, X_j^{(1)})$, ..., $(X_i^{(n)}, X_j^{(n)})$, $i \neq j$, are iid bivariate random vectors with distribution function F of marginal distribution functions F_{X_i} and F_{X_j}. The estimator of λ_U in Schmidt and Stadtmüller (2006) is given by

$$\widehat{\lambda}_U = \widehat{\Lambda}_{U,n}(1,1), \tag{1.11}$$

where

$$\widehat{\Lambda}_{U,n}(x,y) := \frac{1}{k_2}\sum_{w=1}^{n} 1_{\{R_i^{(w)}>n-k_2\,x \text{ and } R_j^{(w)}>n-k_2\,y\}},$$

with $k_2 = k_2(n) \to \infty$, $k_2/n \to 0$, as $n \to \infty$, and where $R_i^{(w)} = \sum_{h=1}^{n} 1_{\{X_i^{(h)} \leq X_i^{(w)}\}}$ (respectively $R_j^{(w)} = \sum_{h=1}^{n} 1_{\{X_j^{(h)} \leq X_j^{(w)}\}}$) is the rank of $X_i^{(w)}$ in $X_i^{(1)}, \ldots, X_i^{(n)}$ (is the rank of $X_j^{(w)}$ in $X_j^{(1)}, \ldots, X_j^{(n)}$, respectively), for $w = 1, \ldots, n$. The upper tail coefficients λ_U for the main Archimedean copulas can be found on page 215 in Nelsen (2006).

The Schur-concavity property for copulas constitutes a useful notion in Chapter 3 of this thesis. Firstly, we recall the definition of Schur-concave function from Marshall et al. (2011) (Definition A.1., page 80).

Definition 1.4.1 (Schur-concave function). *Let $\boldsymbol{a} = (a_1, \ldots a_d)$ and $\boldsymbol{b} = (b_1, \ldots b_d)$ be two points. A real valued function $g : \mathbb{A} \subseteq \mathbb{R}^n \to \mathbb{R}$, is Schur-concave (Schur-convex, respectively) on $\mathbb{A}$ if, for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{A}$, $\boldsymbol{a} \prec_m \boldsymbol{b}$ implies $g(\boldsymbol{a}) \geq g(\boldsymbol{b})$ ($g(\boldsymbol{a}) \leq g(\boldsymbol{b})$, respectively), where $\prec_m$ is defined in Definition 1.3.5.*

In the bivariate case, Durante (2006) provides the following result on Schur-concave copulas.

Proposition 1.4.2 (Proposition 10.1.7 in Durante (2006)). *A copula $C(u, v)$ is Schur-concave if, and only if, $\hat{C}(u, v)$ associated with $C(u, v)$ is Schur-concave.*

The next characterization shows the Schur-concavity property for the Archimedean copulas class.

Proposition 1.4.3 (Proposition 4.11 in Dolati and Dehgan Nezhad (2014)). *Every $d-$dimensional Archimedean copula is Schur-concave.*

For a thorough theoretical review of copulas, see Nelsen (2006), McNeil and Nešlehová (2009), Jaworski (2013) and Durante and Sempi (2015).

1.5 Main Goals and Contributions

The general goal of this work is to define, study and estimate new multivariate risk measures as well as to characterize and to estimate others existing multivariate measures.

In the context of trading firms, managing risk has been traditionally achieved by the introduction of Value-at-Risk (VaR) thresholds on the portfolio risk accumulated by traders. Over recent decades, this problem has been handled mostly in a univariate setting. However, the solvability of financial institutions could also be affected by external risks whose sources cannot be controlled. These risks may also be strongly heterogeneous in nature and difficult to diversify away. One can think, for instance, of systemic risk or contagion effects in a strongly interconnected system of financial companies. Therefore, the necessity of considering a multivariate framework to measure the risk emerges (for further details see Section 1.1). Unsurprisingly, the main difficulty regarding multivariate generalizations of risk measures is the fact that vector preorders are, in general, partial preorders.

Adrian and Brunnermeier (2011) introduce the systemic risk measure CoVaR given in Equation (1.2). CoVaR measures a financial institution's contribution to systemic risk and its contribution to the risk of other financial institutions. Today, CoVaR represents one of the major topics in the current regulatory and scientific discussion of systemic risks. In Chapter 2 (Definitions 2.2.1 and 2.2.2), we propose two new multivariate generalizations of CoVaR based on the multivariate quantile settings of Embrechts and Puccetti (2006), Cousin and Di Bernardino (2013), and Cousin and Di Bernardino (2014). The two generalizations are based on quantile functions of the conditional random variables $T_i := [X_i \,|\, \mathbf{X} \in \partial \underline{L}(\alpha)]$ and $T_i' := [X_i \,|\, \mathbf{X} \in \partial \overline{L}(\alpha)]$ (see Equation (2.6)). These proposed CoVaR measures can be useful in the analysis of multiple financial institutions taken all together in the systemic context. In addition, these new measures verify the *elicitability property*, which provides a natural methodology to perform backtesting (see Section 2.3.5). Since the two proposed measures are based on quantile functions, then they are more robust to extreme values than any other central tendency measures.

Artzner et al. (1999) justify that every risk measure should verify the set of desirable properties in Definition 1.1.2. Several properties have been obtained for our proposed risk measures. In particular, the positive homogeneity and translation property in Definition 1.1.2 are shown in Proposition 2.3.2.

In order to see how conservative are our proposed measures, we analyse in Section 2.3.2 how they behave with respect to the univariate VaR of margins and to the multivariate VaR in Cousin and Di Bernardino (2013). We also study how these new measures are influenced by considering comonotonic dependence in the respective vector (Propositions 2.3.5 and 2.3.6), by a change in risk level (Proposition 2.3.7 and Corollary 2.5.3) and by a change in dependence structure (Corollary 2.5.4).

Denuit et al. (2005) explain the importance of establishing a certain level of ordering between risks. In Section 2.4, the behaviour of multivariate CoVaR risk measures is provided under different stochastic ordering conditions. A future research could be the straight characterization of our proposed CoVaR with a stochastic ordering.

Obviously, an univariate risk approach does not let us consider several risks and, therefore, we misrepresent the relationship between them, i.e., the dependence structure between them. Archimedean copulas play a central role in the understanding of dependencies of multivariate random vector (see Nelsen (2006), McNeil and Nešlehová (2009), Durante and Salvadori (2010)). As Nelsen (2006) remarks, Archimedean copulas find a wide range of applications for a number of reasons: the ease with which they can be constructed, the great variety of families of copulas which belong to this class and the many desirable properties possessed by the members of this class.

In Section 2.5, we characterize the new multivariate CoVaR in the Archimedean copula class. In this framework, we tried to obtain the conditions under which the subadditivity is verified for our proposed multivariate CoVaR. Theorem 2.5.1 presents a weak subadditivity inequality for one of our generalization of CoVaR under regular variation conditions. Adrian and Brunnermeier (2011) also define a systemic risk measure ΔCoVaR as the difference between univariate CoVaR and VaR. In Section 2.5.3, we propose a general ΔCoVaR definition by using the multivariate CoVaRs. In a future perspective, a deep study of the subadditivity property for the two multivariate CoVaRs and of our proposed ΔCoVaR measure could be done. For instance, it is of great interest to develop an estimation procedure for ΔCoVaR.

A semi-parametric estimation procedure for the new multivariate CoVaR is provided in Section 2.6. However, as we point out in that same section, consistency and normal asymptotic properties of these estimators need a supplementary study that constitutes a future line to develop.

In order to study the accuracy of the proposed estimators in this thesis, we asses the performance of our estimators by using simulated data and we compare them with others competitor estimators (see Sections 2.7, 3.5 and 4.6).

As we mention in Section 1.2, events are usually described by distributions that contain extreme values. Although semi-parametric estimators achieve a good performance in terms of the bias and the variance, the aforementioned semi-parametric estimation (see Definitions 2.6.1 and 2.6.2) perform well only if the threshold is sufficiently low. This method cannot handle extreme events, that is, when we consider the risk level sufficiently smaller than $1/n$ where n is the sample size. It should be borne in mind that extreme events are specifically required for hydrological and environmental risk measures. Additionally, the random variable T_i (T_i', respectively) relies on $Z := F(X_1, \ldots, X_d)$ ($Z' := \overline{F}(X_1, \ldots, X_d)$, respectively), which is not observed. Therefore, in order to apply a quantile estimation procedure for our CoVaRs, Z (Z', respectively) has to be previously estimated. This type of plug-in procedure increases the variance of the final estimation and introduces statistical difficulties. In order to obtain an estimation procedure for multivariate CoVaR without the above two mentioned problems, we propose a non-parametric extreme estimation procedure for multivariate CoVaR by considering extreme events (see (3.10)) in Chapter 3.

By employing Archimedean copulas, tail index and the distribution of T_i (T_i', respectively) can be easily obtained, and we can avoid to estimate the latent random variables Z (Z', respectively) as we show in Chapter 3. The tail behaviour of conditional random variable T_i (T_i', respectively) are given in Proposition 3.2.1 (Remark 3.2.2, respectively). In Section 3.3, an extrapolation method is developed under the Archimedean copula assumption for the dependence structure of $\mathbf{X}$ and the von Mises condition for marginal

X_i. The main result in Chapter 3 is the Central Limit Theorem for our estimator which is provided in Theorem 3.4.3. Since our extreme estimator in (3.10) included an intermediate sequence that is unknown, an adaptive version of Theorem 3.4.3 is given in Section 3.6.

Salvadori et al. (2011) define their new design realization as the vector that maximizes a weight function given that the risk vector belongs to a given critical layer of its joint multivariate distribution function (Definition 1.1.5). Furthermore, Salvadori et al. (2011) propose to consider the survival joint distribution as the weight function (Definition 4.1). The aforementioned risk measure is called Component-wise Excess (C.-E.) design realization. Salvadori et al. (2011) present the design realization as a general definition based on a maximization problem without specifying directly the closed form.

In Chapter 4 (Proposition 4.2.1), we provide the closed-form expression of the C.-E. design realization under an Archimedean copula framework. At the same time, this measure is non-parametrically estimated by using Extreme Value Theory techniques and the asymptotic normality of its proposed estimator is obtained in Theorem 4.5.1. To this end, we firstly analyse in Section 4.3 the tail behaviour of the random variable $Y := \max\{V_1, \dots, V_d\}$ with V_i the i-th margin distribution with $i \in \{1, \dots, d\}$.

The proposed estimators for C.-E. design realization are contrasted with that of Salvadori et al. (2011) in the same data-set. It can be concluded that there is no significant statistical difference between our extreme estimator and that of Salvadori et al. (2011). Moreover, conversely to the estimator of Salvadori et al. (2011) based on a Gumbel model, we propose a non-parametric estimation procedure for the risk measure in this work. In our setting, only a general Archimedean copula framework and the heavy tailed behaviour of the margins are assumed in order to apply the proposed estimator.

Notice that, since Y can not be observed in real applications, in our extreme estimation procedure in Chapter 4, we disregard the uncertainty induced by the margins. That is, Theorem 4.5.1 is only valid under full knowledge of the margins (for more details see Remark 4.4.1). The improvement of Theorem 4.5.1 by using uncertainty induced by the margins could be developed in a future study.

It is of great importance to show to practitioners how they can apply the different risk measures in their data-sets. For this reason, it should be borne in mind that the proposed estimators in this thesis are also illustrated in real data-sets. The semi-parametric and non-parametric extreme estimators for multivariate CoVaRs are applied in an insurance and a rainfall data-set respectively (see Sections 2.8 and 3.7). The non-parametric extreme estimator for C.-E design realization is exemplified using a dam data-set (see Section 4.7).

Chapter 2

Multivariate Extensions of Conditional Value-at-Risk

2.1 Introduction

Let $\mathcal{L}_1(\Omega, \mathcal{A}, P)$ be the set of all random variables with finite expectations. Assuming that X is a random variable of $\mathcal{L}_1$ with distribution function F_X, the Weighted Loss (WL) function is defined by

$$L_X(x;\omega) = \omega\,\mathbb{E}[(X-x)^+] + (1-\omega)\,\mathbb{E}[(X-x)^-] \quad \text{for all } x \in \mathbb{R} \text{ and } \omega \in [0,1], \tag{2.1}$$

where $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. Note that if X is a non-negative random variable, then $L_X(x;\omega) = \omega\mathbb{E}[X]$ for all $x < 0$. This function plays a key role in an actuarial context. Indeed, it represents the expected cost for the reinsurance company, called *net premium*, where X denotes the risk for the insurance company. If the insurance company prefers not to bear all the risk, it passes on parts of the risk to a reinsurance company. The part retained by the original insurance company is usually called the *retention*. A *stop-loss* contract establishes a fixed retention x (see Section 8.3 in Müller and Stoyan (2002)). This means that the maximum risk for the insurance company is x. Thus, if $X > x$ then the reinsurance company will take over $X - x$. This class of contracts is useful to protect companies from insolvency due to excessive claims. In an actuarial context, the threshold x is often called the *deductible* or *priority* (see Section 1.7.1 in Denuit et al. (2005)).

Certain interesting properties of the WL function in (2.1) are now recalled. The properties (P1)-(P6) are trivially obtained by the same arguments as those used by Muñoz-Pérez and Sánchez-Gómez (1990) to prove the properties of the dispersion function.

(P1) It holds that

$$L_X(x;\omega) = \omega \int_x^{+\infty} \overline{F}_X(t)\,\mathrm{dt} + (1-\omega)\int_{-\infty}^{\mathrm{x}} \mathrm{F_X(t)}\,\mathrm{dt}.$$

(P2) Let C_F denote the set of continuity points of F_X and $X \in \mathcal{L}_1$. Therefore

$$F_X(x) = L_X^{'}(x;\omega) + \omega, \quad \forall\, x \in C_F \text{ and } x \geq 0$$

where L'_X is the derivative of L_X with respect to x.

(P3) The WL function is differentiable and its derivative has, at most, a countable number of discontinuity points.

(P4) $L_X(x;\omega)$ is a convex function on $\mathbb{R}_+$.

(P5) $\lim_{x\to+\infty} L_X^{'}(x;\omega) = 1-\omega$; and $\lim_{x\to-\infty} L_X^{'}(x;\omega) = 0$.

(P6) $\lim_{x\to+\infty}[L_X(x;\omega) - (1-\omega)x] = -(1-\omega)\,\mathbb{E}[X]$.

(P7) Finally, we can define the *Value-at-Risk* as

$$\mathrm{VaR}_\omega(\mathrm{X}) = \underset{x\in\mathbb{R}^+}{\arg\min}\ \mathrm{L_X}(x;\omega), \text{ for } \omega \in [0,1],$$

with $\mathrm{VaR}_0(\mathrm{X}) = x_{\mathrm{F}^-}$ and $\mathrm{VaR}_1(\mathrm{X}) = x_{\mathrm{F}^+}$, where x_{F^+} and x_{F^-} are, respectively, the right and left endpoints of F_X, such that $x_{F^+} = \sup\{x \in \mathbb{R} : F_X(x) < 1\}$ and $x_{F^-} = \inf\{x \in \mathbb{R} : F_X(x) > 0\}$.

It is easy to see that Properties (P1)-(P7) uniquely characterize a WL function, that is, if $L_X(x;\omega)$ is a function that satisfies Properties (P1)-(P7) above, then there exits a unique distribution function which has $L_X(x;\omega)$ as its WL function. Therefore, it uniquely determines a probability measure P_F on $\mathcal{B}$ (the σ-field of Borel set on $\mathbb{R}$).

An interesting interpretation of the WL function is that $2\,L_X(x;1/2)$ is the L_1-distance between F_X and the distribution function of the degenerate random variable at the point $x \in \mathbb{R}$ (Muñoz-Pérez and Sánchez-Gómez (1990)). It is also worth mentioning that $L_X(x;1)$ is the well-known *stop-loss* function of X, and that $L_X(x;0)$ could be interpreted as the *stop-gain* function of X. Consequently, the WL function is a weighting of both functions in terms of x. Now, let $\mathbf{X} = (X_1,\ldots,X_d)$ be a non-negative d-dimensional random vector[1]. Cousin and Di Bernardino (2013) defined, under certain regularity conditions, the multivariate Lower-Orthant Value-at-Risk at probability level α as the d-dimensional vector

$$\underline{\mathrm{VaR}}_\alpha(\mathbf{X}) = \mathbb{E}[\mathbf{X} \,|\, F(\mathbf{X}) = \alpha], \;\text{ for }\; \alpha \in (0,1),$$

[1] We restrict ourselves to $\mathbb{R}_+^d$ since, in our applications, components of $d-$dimensional vectors correspond to random losses and are then valued in $\mathbb{R}_+$.

where F is the distribution function of $\mathbf{X}$. In particular, the i-th component of this vector trivially verifies

$$\underline{\mathrm{VaR}}^i_\alpha(\mathbf{X}) = L_{X_i|F(\mathbf{X})=\alpha}(0;1).$$

Using Property (P7), our purpose is now to outline a new multivariate approach to the classic Conditional Value-at-Risk model (see CoVaR in (1.2) which, as introduced previously, is defined as the VaR of a financial institution, conditional on a certain scenario (see Adrian and Brunnermeier (2011))). In this case, the approach is based on the conditional scenario as a restriction for both financial institutions. Thus, in general, no relationship exists between the two CoVaRs.

From now on, assume that $\mathbf{X} = (X_1, \ldots, X_d)$ is a non-negative absolutely-continuous random vector (with respect to Lebesgue measure λ on $\mathbb{R}^d$) with distribution function F and survival function $\overline{F}$. Furthermore, the multivariate distribution function F is assumed to be partially strictly-increasing[2] such that $E(X_i) < \infty$ for $i = 1, \ldots, d$. Such F is said to verify the *regularity conditions*. Note that if $\overline{F}$ is the survival function of $\mathbf{X}$, and F verifies the regularity conditions, then $\overline{F}$ is a partially strictly-decreasing function.

The chapter is organized as follows. In Section 2.2, we introduce new multivariate extensions of CoVaR. In Section 2.3, interesting properties for the proposed multivariate CoVaR are shown. Furthermore, we analyse how these multivariate measures behave when the marginal risks or the copula structures increase with respect to stochastic orders (see Section 2.4). Illustrations and properties for the Archimedean copula class are presented in Section 2.5. In Section 2.6, an estimation procedure for the multivariate CoVaRs proposed is provided. In Sections 2.7 and 2.8, our proposed estimators are illustrated in simulated studies and for a real insurance data-set. The conclusion discusses possible directions for future work (see Section 2.9).

2.2 Definitions of the multivariate CoVaR

Two multivariate generalizations of the univariate CoVaR measure in (1.2) are now introduced in Definitions 2.2.1 and 2.2.2. Whereas Definition 2.2.1 is based on the level-sets of the joint distribution function, Definition 2.2.2 is constructed according to the level-sets of the joint survival distribution function.

Definition 2.2.1 (Multivariate Lower-Orthant CoVaR). *Consider a random vector $\boldsymbol{X}$ which satisfies the regularity conditions. For $\alpha \in (0,1)$, we define the multivariate lower-orthant* CoVaR *at probability level α by*

[2] A function $F(x_1, \ldots, x_n)$ is partially strictly-increasing on $\mathbb{R}^d_+\backslash\mathbf{0}$ if, for all $j \in \{1, \ldots, d\}$, the function of one variable $g_j(\cdot) = F(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_d)$ is strictly-increasing.

$$\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) = \mathrm{VaR}_{\boldsymbol{\omega}}(\boldsymbol{X}|\boldsymbol{X} \in \partial\underline{\mathrm{L}}^{\geq}(\alpha)) = \begin{pmatrix} \mathrm{VaR}_{\omega_1}(\mathrm{X}_1|\boldsymbol{X} \in \partial\underline{\mathrm{L}}^{\geq}(\alpha)) \\ \vdots \\ \mathrm{VaR}_{\omega_\mathrm{d}}(\mathrm{X}_\mathrm{d}|\boldsymbol{X} \in \partial\underline{\mathrm{L}}^{\geq}(\alpha)) \end{pmatrix}, \tag{2.2}$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)$ *is a marginal risk vector with* $\omega_i \in [0,1]$, *for* $i = 1, \ldots, d$, *and* $\partial\underline{L}^{\geq}(\alpha)$ *is the boundary of the set* $\underline{L}^{\geq}(\alpha) := \{\boldsymbol{x} \in \mathbb{R}^d_+ : F(\boldsymbol{x}) \geq \alpha\}$. *Since the* regularity conditions *are satisfied,* $\partial\underline{L}^{\geq}(\alpha)$ *is the* α*-level set of* F *denoted by* $\partial\underline{L}(\alpha)$, *therefore,*

$$\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) = \begin{pmatrix} \mathrm{VaR}_{\omega_1}(\mathrm{X}_1|\mathrm{F}(\boldsymbol{X}) = \alpha) \\ \vdots \\ \mathrm{VaR}_{\omega_\mathrm{d}}(\mathrm{X}_\mathrm{d}|\mathrm{F}(\boldsymbol{X}) = \alpha) \end{pmatrix}. \tag{2.3}$$

In a similar way, the multivariate upper-orthant CoVaR can be defined.

Definition 2.2.2 (Multivariate Upper-Orthant CoVaR)**.** *Consider a random vector* $\boldsymbol{X}$ *which satisfies the regularity conditions. For* $\alpha \in (0,1)$, *we define the multivariate upper-orthant* CoVaR *at probability level* α *by*

$$\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) = \mathrm{VaR}_{\boldsymbol{\omega}}(\boldsymbol{X}|\boldsymbol{X} \in \partial\overline{\mathrm{L}}^{\leq}(\alpha)) = \begin{pmatrix} \mathrm{VaR}_{\omega_1}(\mathrm{X}_1|\boldsymbol{X} \in \partial\overline{\mathrm{L}}^{\leq}(\alpha)) \\ \vdots \\ \mathrm{VaR}_{\omega_\mathrm{d}}(\mathrm{X}_\mathrm{d}|\boldsymbol{X} \in \partial\overline{\mathrm{L}}^{\leq}(\alpha)) \end{pmatrix}, \tag{2.4}$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)$ *is a marginal risk vector with* $\omega_i \in [0,1]$, *for* $i = 1, \ldots, d$, *and* $\partial\overline{L}^{\leq}(\alpha)$ *is the boundary of the set* $\overline{L}^{\leq}(\alpha) := \{\boldsymbol{x} \in \mathbb{R}^d_+ : \overline{F}(\boldsymbol{x}) \leq 1 - \alpha\}$. *Since the* regularity conditions *are satisfied,* $\partial\overline{L}^{\leq}(\alpha)$ *is the* $(1-\alpha)$*-level set of* $\overline{F}$ *denoted by* $\partial\overline{L}(\alpha)$, *therefore,*

$$\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) = \begin{pmatrix} \mathrm{VaR}_{\omega_1}(\mathrm{X}_1|\overline{\mathrm{F}}(\boldsymbol{X}) = 1 - \alpha) \\ \vdots \\ \mathrm{VaR}_{\omega_\mathrm{d}}(\mathrm{X}_\mathrm{d}|\overline{\mathrm{F}}(\boldsymbol{X}) = 1 - \alpha) \end{pmatrix}. \tag{2.5}$$

Remark 2.2.1. *Using the same notation and framework of Definitions 2.2.1 and 2.2.2, we can also consider a modified version of the multivariate upper-orthant and lower-orthant* CoVaR *proposed in Equations (2.2) and (2.4). Indeed, consider a financial institution* X_i *and the firm network without* X_i, *i.e.,* $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d) := \boldsymbol{X}_{d-1}$. *The following modified version of the lower-orthant* CoVaR *in Definition 2.2.1 can therefore be proposed:*

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) = \mathrm{VaR}_{\omega_\mathrm{i}}(\mathrm{X}_\mathrm{i}|\mathrm{F}(\boldsymbol{X}_{\mathrm{d}-1}) = \alpha),$$

where F_{d-1} is the $(d-1)$-dimensional distribution function associated with the vector $\boldsymbol{X}_{d-1}$. Analogously, a modified version of the upper-orthant CoVaR *in Definition 2.2.2 can be:*

$$\overline{\mathrm{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) = \mathrm{VaR}_{\omega_i}(\mathrm{X_i}|\overline{\mathrm{F}}(\boldsymbol{X}_{\mathrm{d}-1}) = 1-\alpha),$$

where $\overline{F}_{d-1}$ is the survival $(d-1)$-dimensional distribution function associated with the vector $\boldsymbol{X}_{d-1}$. It should be borne in mind that, using these modified versions, when $d=2$ and $\omega_i = \alpha$, then $\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X})$ and $\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X})$ become the classic CoVaR *in (1.2).*

The following interpretation of our measures can be considered. The i-th component of multivariate lower-orthant CoVaR of $\mathbf{X}$ (multivariate upper-orthant CoVaR of $\mathbf{X}$, respectively) corresponds to the point x^* that minimizes the WL function of the associated i-th marginal, given that $\mathbf{X}$ lies on the α-level curve of its multivariate distribution function (multivariate survival distribution function, respectively).

As we mention before, under regularity conditions, $\partial\underline{L}^{\geq}(\alpha)(\partial\overline{L}^{\leq}(\alpha)$, respectively) is the α-level curve ($(1-\alpha)$-level curve, respectively) of F ($\overline{F}$, respectively) (see for instance Di Bernardino et al. (2011), Cuevas et al. (2006)). This means that there is no plateau in the graph of F for each level α. Therefore, the *regularity conditions* guarantee that the minimizer x^* is unique for each component $i = 1, \ldots, d$.

The solvency of an insurance company depends on the frequency of large claims. One of the advantages of working with the quantile function is that this function is more robust to extreme values than other central tendency measures.

In order to clarify the expressions in the proofs, the following notation is henceforth considered. We denote the conditional random variable X_i on the critical layer $\partial\underline{L}(\alpha)$ and $\partial\overline{L}(\alpha)$ for $i = 1, \ldots, d$, as

$$T_i := [X_i \,|\, \mathbf{X} \in \partial\underline{L}(\alpha)], \text{ and } T_i' := [X_i \,|\, \mathbf{X} \in \partial\overline{L}(\alpha)] \text{ for } \alpha \in (0,1). \tag{2.6}$$

One can interpret the random variable T_i (T_i', respectively) as the contribution (or the responsibility) of the marginal risk X_i in the case where the whole risk vector $\mathbf{X}$ belongs to the multivariate stress scenario represented by the critical layer $\partial\underline{L}(\alpha)$ ($\partial\overline{L}(\alpha)$, respectively), for some suitable level $\alpha \in (0,1)$.

Lemma 2.2.1, introduced below, shows the expression of the distribution function for T_i. In particular, when the vector $\mathbf{X}$ follows an Archimedean copula, Lemma 2.2.1 can be also obtained by adapting Lemma 3.4 in Brechmann (2014) in the case of $j = 1$.

Lemma 2.2.1. *Let $(X_1, \ldots, X_d)$ be a random vector that follows an Archimedean copula C with generator ϕ. Let $F_{T_i}(x|\alpha) = \mathbb{P}[T_i \leq x]$. Therefore, for $i = 1, \ldots, d$,*

$$F_{T_i}(x|\alpha) = \begin{cases} \left(1 - \frac{\phi(F_{X_i}(x))}{\phi(\alpha)}\right)^{d-1}, & \textit{if } x > Q_i(\alpha); \\ 0, & \textit{if } x \leq Q_i(\alpha), \end{cases}$$

where F_{X_i} is the marginal distribution of X_i and $Q_i(\alpha)$ is the associated quantile function at level $\alpha \in (0,1)$.

2.3 Characteristics of the multivariate CoVaR

In this section, the aim is to analyse the lower-orthant and upper-orthant CoVaR introduced in Definitions 2.2.1 and 2.2.2 in terms of classic suitable properties of risk measures (Artzner et al. (1999), Denuit et al. (2005)).

We focus on invariance properties (see Section 2.3.1). Furthermore, in Section 2.3.2, the relationships between our CoVaR, the univariate VaR, and the multivariate VaR introduced in Cousin and Di Bernardino (2013) are analysed. In Section 2.3.3, several comonotonic dependence properties for our measures are investigated. The behaviours of multivariate CoVaRs with respect to the risk levels are studied in Section 2.3.4. The advantages that our proposed CoVaRs present for backtesting are explained in Section 2.3.5.

2.3.1 Invariance properties

The results in Proposition 2.3.1 and Corollary 2.3.1 will be central in proving invariance properties of our risk measures.

Proposition 2.3.1. *Let the function h be such that $h(x_1,\dots,x_d) = (h_1(x_1),\dots,h_d(x_d))$. Let $\boldsymbol{\omega}$ be a vector in $[0,1]^d$ and $\alpha \in (0,1)$.*

(1) If $h_1,\dots,h_d$ are non-decreasing functions, then, for $i = 1,\dots,d$,

$$\underline{\mathrm{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(h(\boldsymbol{X})) = \mathrm{VaR}_{\omega_i}(\mathrm{h_i(X_i)}|\mathrm{F}(\boldsymbol{X}) = \alpha).$$

(2) If $h_1,\dots,h_d$ are non-increasing functions, then, for $i = 1,\dots,d$,

$$\underline{\mathrm{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(h(\boldsymbol{X})) = \mathrm{VaR}_{\omega_i}(\mathrm{h_i(X_i)}|\overline{\mathrm{F}}(\boldsymbol{X}) = \alpha).$$

Proof. By Definition 2.2.1,

$$\begin{aligned}
&\underline{\mathrm{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(h(\mathbf{X})) \\
&= \mathrm{VaR}_{\omega_i}(\mathrm{h_i(T_i)}) \\
&= \underset{x\in[h_i(\mathrm{VaR}_\alpha(\mathrm{X_i})),+\infty)}{\arg\min} \left\{\omega_i\,\mathbb{E}[(h_i(T_i)-x)^+] + (1-\omega_i)\,\mathbb{E}[(h_i(T_i)-x)^-]\right\},
\end{aligned}$$

where $h_i(T_i) = [h_i(X_i)|F_{h(\mathbf{X})}(h(\mathbf{X})) = \alpha]$, for $i = 1,\dots,d$.

Since

$$F_{h(\mathbf{X})}(y_1,\ldots,y_d)$$

$$= \begin{cases} F(h_1^{-1}(y_1),\ldots,h_d^{-1}(y_d)) \text{ if } h_1,\ldots,h_d \text{ are non-decreasing functions,} \\ \overline{F}(h_1^{-1}(y_1),\ldots,h_d^{-1}(y_d)) \text{ if } h_1,\ldots,h_d \text{ are non-increasing functions,} \end{cases}$$

then

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(h(\mathbf{X}))$$

$$= \begin{cases} \mathrm{VaR}_{\omega_i}(\mathrm{h_i}(\mathrm{X_i})|\mathrm{F}(\mathbf{X})=\alpha) \text{ if } \mathrm{h}_1,\ldots,\mathrm{h_d} \text{ are non-decreasing functions,} \\ \mathrm{VaR}_{\omega_i}(\mathrm{h_i}(\mathrm{X_i})|\overline{\mathrm{F}}(\mathbf{X})=\alpha) \text{ if } \mathrm{h}_1,\ldots,\mathrm{h_d} \text{ are non-increasing functions.} \end{cases}$$

□

As in Proposition 2.3.1, a similar result can also be obtained for the multivariate upper-orthant CoVaR, by interchanging F with $\overline{F}$. From Proposition 2.3.1, one can trivially obtain the following property which links the multivariate upper-orthant CoVaR and lower-orthant CoVaR.

Corollary 2.3.1. *Let h be a function such that $h(x_1,\ldots,x_d) = (h_1(x_1),\ldots,h_d(x_d))$ and h_i is a linear function, for $i=1,\ldots,d$. Let $\boldsymbol{\omega}$ be a vector in $[0,1]^d$ and $\alpha \in (0,1)$.*

(1) If $h_1,\ldots,h_d$ are non-decreasing functions, then

$$\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(h(\boldsymbol{X})) = h(\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}))$$

and

$$\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(h(\boldsymbol{X})) = h(\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X})).$$

(2) If $h_1,\ldots,h_d$ are non-increasing functions, then

$$\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(h(\boldsymbol{X})) = h(\overline{\mathrm{CoVaR}}_{1-\alpha,1-\boldsymbol{\omega}}(\boldsymbol{X}))$$

and

$$\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(h(\boldsymbol{X})) = h(\underline{\mathrm{CoVaR}}_{1-\alpha,1-\boldsymbol{\omega}}(\boldsymbol{X})).$$

The following result proves the positive homogeneity and invariance translation properties for risk measures in Definitions 2.2.1 and 2.2.2.

Proposition 2.3.2. *Consider a random vector $\boldsymbol{X}$ with a distribution function F, which satisfies the regularity conditions. Let $\boldsymbol{\omega}$ be a vector in $[0,1]^d$ and $\alpha \in (0,1)$. The multivariate lower-orthant and upper-orthant* CoVaR *satisfy the following properties:*
Positive Homogeneity: $\forall\, \boldsymbol{c} = (c_1,\ldots,c_d) \in \mathbb{R}^d_+$,

$$\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{c}\,\boldsymbol{X}) = \boldsymbol{c}\,\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) \text{ and } \overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{c}\,\boldsymbol{X}) = \boldsymbol{c}\,\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}),$$

where $\mathbf{c}\,\mathbf{X} = (c_1\,X_1, \ldots, c_d\,X_d)$.
Translation Invariance: $\forall \boldsymbol{c} \in \mathbb{R}^d_+$,

$$\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{c}+\boldsymbol{X}) = \boldsymbol{c} + \underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) \text{ and } \overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{c}+\boldsymbol{X}) = \boldsymbol{c} + \overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}).$$

The proof is trivially obtained from Corollary 2.3.1.

2.3.2 Multivariate CoVaR and other risk measures

The relationships between the marginal components of multivariate lower-orthant CoVaR (multivariate upper-orthant CoVaR , respectively) and the univariate VaR are given in Proposition 2.3.3. Furthermore, Proposition 2.3.4 provides a comparison between the multivariate VaR in Cousin and Di Bernardino (2013) and our corresponding multivariate CoVaR.

Proposition 2.3.3. *Consider a random vector* $\boldsymbol{X}$ *with distribution function* F*, which satisfies the regularity conditions. Let* $\boldsymbol{\omega}$ *be a vector in* $[0,1]^d$ *and* $\alpha \in (0,1)$*. Therefore,*

$$\overline{\mathrm{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) \leq \mathrm{VaR}_{\alpha}(\mathrm{X_i}) \leq \underline{\mathrm{CoVaR}}^{\mathrm{i}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}), \text{ for } \mathrm{i} = 1, \ldots, \mathrm{d}.$$

Proof. From Definitions 2.2.1 and 2.2.2,

$$\begin{aligned}\underline{\mathrm{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) &= \mathrm{VaR}_{\omega_\mathrm{i}}(\mathrm{T_i}) \\ &= \underset{x \in [\mathrm{VaR}_\alpha(\mathrm{X_i}),\, +\infty)}{\arg\min} \left\{\omega_i\,\mathbb{E}[(T_i - x)^+] + (1-\omega_i)\,\mathbb{E}[(T_i - x)^-]\right\},\end{aligned}$$

and

$$\begin{aligned}\overline{\mathrm{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) &= \mathrm{VaR}_{\omega_\mathrm{i}}(\mathrm{T'_i}) \\ &= \underset{x \in (-\infty,\, \mathrm{VaR}_\alpha(\mathrm{X_i})]}{\arg\min} \left\{\omega_i\,\mathbb{E}[(T'_i - x)^+] + (1-\omega_i)\,\mathbb{E}[(T'_i - x)^-]\right\},\end{aligned}$$

for $i = 1, \ldots, d$. Hence, the result is trivially verified since $\mathrm{VaR}_\alpha(\mathrm{X_i})$ is the lower and upper bound of the domain for the corresponding WL function, respectively. □

Proposition 2.3.4. *Let* α *be a fixed risk level in* $(0,1)$*. Let us denote by* $\underline{\mathrm{VaR}}^i_\alpha(\boldsymbol{X})$ *and* $\overline{\mathrm{VaR}}^i_\alpha(\boldsymbol{X})$ *the multivariate lower and upper VaR defined in Cousin and Di Bernardino (2013). Given a level* $\boldsymbol{\omega}_* \in [0,1]^d$ *such that* $\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}_*(\boldsymbol{X})} = \underline{\mathrm{VaR}}^i_\alpha(\boldsymbol{X})$*, for any* $i \in \{1, \ldots, d\}$*, then*

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) \geq \underline{\mathrm{VaR}}^i_\alpha(\boldsymbol{X}), \text{ for all } \boldsymbol{\omega} \geq \boldsymbol{\omega}_*.$$

Given a level $\boldsymbol{\omega}^* \in [0,1]^d$ *such that* $\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}^*}(\boldsymbol{X}) = \underline{\mathrm{VaR}}^i_\alpha(\boldsymbol{X})$ *for any* $i \in \{1, \ldots, d\}$*, then*

$$\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) \geq \overline{\mathrm{VaR}}^i_\alpha(\boldsymbol{X}), \text{ for all } \boldsymbol{\omega} \geq \boldsymbol{\omega}^*.$$

The proof is based on the increasing property of the quantile function. An illustration of Proposition 2.3.4 in the Clayton copula case is given in Example 2.5.4. Proposition 2.3.4 shows that our multivariate lower-orthant CoVaR (upper-orthant CoVaR, respectively) provides a quantification of the risk greater than the mean value given by the multivariate lower-orthant VaR (upper-orthant VaR, respectively).

2.3.3 Comonotonic dependence properties

Recall that a non-negative random vector $\mathbf{X}$ is said to be a comonotonic random vector if there exists a random variable Z and d increasing functions $g_1, \ldots, g_d$ such that $\mathbf{X} \stackrel{d}{=} (g_1(Z), \ldots, g_d(Z))$ (Proposition 5.16 in McNeil et al. (2005)). The following property of the multivariate CoVaR of a comonotonic random vector can be shown.

Proposition 2.3.5. *Consider a comonotonic random vector $\boldsymbol{X}$ with distribution function F, which satisfies the regularity conditions. Let $\boldsymbol{\omega}$ be a vector in $[0,1]^d$ and $\alpha \in (0,1)$. Therefore,*

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) = \mathrm{VaR}_\alpha(\mathrm{X_i}) = \overline{\mathrm{CoVaR}}^{\mathrm{i}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}), \ \textit{for } \mathrm{i} = 1, \ldots, \mathrm{d}.$$

Proof. Let $\alpha \in (0,1)$. Therefore

$$\begin{aligned}
\mathbb{E}[(X_i - x)^+ | F(\mathbf{X}) = \alpha] &= \mathbb{E}[(X_i - x)^+ | \min\{g_1^{-1}(x_1), \ldots, g_d^{-1}(x_d)\} = \mathrm{VaR}_\alpha(\mathrm{Z})] \\
&= \mathbb{E}[(X_i - x)^+ | g_i^{-1}(x_i) = \mathrm{VaR}_\alpha(\mathrm{Z})] \\
&= \mathbb{E}[(\mathrm{VaR}_\alpha(\mathrm{X_i}) - \mathrm{x})^+], \ \text{for all x in the support of } \mathrm{X_i}.
\end{aligned}$$

In the same way, $\mathbb{E}[(X_i - x)^- | F(\mathbf{X}) = \alpha] = \mathbb{E}[(\mathrm{VaR}_\alpha(\mathrm{X_i}) - \mathrm{x})^-]$, for all x in the support of X_i.

In addition,

$$\begin{aligned}
\mathrm{VaR}_{\omega_\mathrm{i}}(\mathrm{X_i} | \mathrm{F}(\mathbf{X}) = \alpha) &= \underset{x \in [\mathrm{VaR}_\alpha(\mathrm{X_i}), +\infty)}{\arg\min} \left\{ \omega_i \, \mathbb{E}[(\mathrm{VaR}_\alpha(\mathrm{X_i}) - \mathrm{x})^+] \right. \\
&\quad \left. + (1 - \omega_i) \, \mathbb{E}[(\mathrm{VaR}_\alpha(\mathrm{X_i}) - \mathrm{x})^-] \right\} \\
&= \underset{x \in [\mathrm{VaR}_\alpha(\mathrm{X_i}), +\infty)}{\arg\min} (1 - \omega_i) \{x - \mathrm{VaR}_\alpha(\mathrm{X_i})\} \\
&= \mathrm{VaR}_\alpha(\mathrm{X_i}), \ \text{for } \mathrm{i} = 1, \ldots, \mathrm{d}.
\end{aligned}$$

By using similar arguments to the lower CoVaR and taking into account that $\overline{F}_{\mathbf{Z}}(u_1, \ldots, u_d) = \overline{F}_{\mathbf{Z}}(\max_{i=1,\ldots,d} u_i)$, then the result for the upper CoVaR is obtained. □

The additivity of the multivariate CoVaR for a π-comonotonic pair of random vectors is now proposed. From Puccetti and Scarsini (2010), a pair $(\mathbf{X}, \mathbf{Y})$ of d-dimensional random vectors is a π-comonotonic random vector if there exists a d-dimensional random vector $\mathbf{Z} = (Z_1, \ldots, Z_d)$ and non-decreasing functions $f_1, \ldots, f_d, g_1, \ldots, g_d$ such that

$$(\mathbf{X}, \mathbf{Y}) \stackrel{d}{=} ((f_1(Z_1), \ldots, f_d(Z_d)), (g_1(Z_1), \ldots, g_d(Z_d))).$$

Proposition 2.3.6. *Let $(\boldsymbol{X}, \boldsymbol{Y})$ be a π-comonotonic pair of random vectors. Therefore, for $\boldsymbol{\omega} \in [0,1]^d$ and $\alpha \in (0,1)$,*

$$\begin{aligned}
\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}+\boldsymbol{Y}) &= \underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) + \underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{Y}),\\
\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}+\boldsymbol{Y}) &= \overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) + \overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{Y}).
\end{aligned}$$

Proof. Let $\mathbf{X}$ and $\mathbf{Y}$ be two π-comonotonic random vectors. There exists a random vector $\mathbf{Z}$ such that, for any $i = 1, \ldots, d$, $X_i = f_i(Z_i)$ and $Y_i = g_i(Z_i)$, where f_i and g_i are non-decreasing functions. Let f be the function defined by $f(x_1, \ldots, x_d) = (f_1(x_1), \ldots, f_d(x_d))$, g be the function defined by $g(x_1, \ldots, x_d) = (g_1(x_1), \ldots, g_d(x_d))$, and h be the function defined by $h(x_1, \ldots, x_d) = (h_1(x_1), \ldots, h_d(x_d))$, where $h_i := f_i + g_i, i = 1, \ldots, d$. Since the function h_i, $i = 1, \ldots, d$ is a sum of non-decreasing functions, h_i is a non-decreasing function for $i = 1, \ldots, d$. Furthermore, $\mathbf{X} + \mathbf{Y} = h(\mathbf{Z})$. From Proposition 2.3.1, it follows that

$$\begin{aligned}
\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}+\mathbf{Y}) &= \mathrm{VaR}_{\omega_i}(\mathrm{h_i}(\mathrm{Z_i})|\mathrm{F}_{\mathbf{Z}}(\mathbf{Z}) = \alpha)\\
&= \mathrm{VaR}_{\omega_i}(\mathrm{f_i}(\mathrm{Z_i})|\mathrm{F}_{\mathbf{Z}}(\mathbf{Z}) = \alpha) + \mathrm{VaR}_{\omega_i}(\mathrm{g_i}(\mathrm{Z_i})|\mathrm{F}_{\mathbf{Z}}(\mathbf{Z}) = \alpha),
\end{aligned}$$

where $F_{\mathbf{Z}}$ denotes the distribution function of $\mathbf{Z}$. Consequently,

$$\mathrm{VaR}_{\omega_i}(\mathrm{f_i}(\mathrm{Z_i})|\mathrm{F}_{\mathbf{Z}}(\mathbf{Z}) = \alpha) = \mathrm{VaR}_{\omega_i}(\mathrm{f_i}(\mathrm{Z_i})|\mathrm{F}_{\mathrm{f}(\mathbf{Z})}(\mathrm{f}(\mathbf{Z})) = \alpha) = \underline{\mathrm{CoVaR}}^{\mathrm{i}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}),$$

and

$$\mathrm{VaR}_{\omega_i}(\mathrm{g_i}(\mathrm{Z_i})|\mathrm{F}_{\mathbf{Z}}(\mathbf{Z}) = \alpha) = \mathrm{VaR}_{\omega_i}(\mathrm{g_i}(\mathrm{Z_i})|\mathrm{F}_{\mathrm{g}(\mathbf{Z})}(\mathrm{g}(\mathbf{Z})) = \alpha) = \underline{\mathrm{CoVaR}}^{\mathrm{i}}_{\alpha,\boldsymbol{\omega}}(\mathbf{Y}),$$

which concludes the proof for the lower-orthant CoVaR. Similar arguments can be used for the upper-orthant CoVaR. □

2.3.4 Multivariate CoVaR in terms of risk levels

Trivially, due to the increasing property of the quantile function, the components of the multivariate risk measures $\underline{\mathrm{CoVaR}}$ and $\overline{\mathrm{CoVaR}}$ are increasing functions of the risk levels $\omega_i \in [0,1]$.

A property of the monotony of the CoVaR for the risk level α is now given. The increasing behaviour of CoVaR in terms of level α means that the measures increase with the level of danger of the stress scenarios considered. This monotony is based on the concept of *positive regression dependence, PRD,*(see Definition 1.3.4).

We denote $U_i = F_{X_i}(X_i)$, $\mathbf{U} = (U_1, \ldots, U_d)$, $V_i = \overline{F}_{X_i}(X_i)$, and $\mathbf{V} = (V_1, \ldots, V_d)$.

Proposition 2.3.7. *Consider a d-dimensional random vector $\boldsymbol{X}$, which satisfies the regularity conditions, with marginal distributions F_{X_i}, for $i = 1, \ldots, d$, copula C and survival copula $\hat{C}$.*

(1) *If* $(U_i, C(\boldsymbol{U}))$ *is* $PRD(U_i|C(\boldsymbol{U}))$ *then, for* $\boldsymbol{\omega} \in [0,1]^d$, $\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X})$ *is a non-decreasing function of* α.

(2) *If* $(V_i, \hat{C}(\boldsymbol{V}))$ *is* $PRD(V_i|\hat{C}(\boldsymbol{V}))$ *then, for* $\boldsymbol{\omega} \in [0,1]^d$, $\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X})$ *is a non-decreasing function of* α.

Proof. If $\alpha_1 \leq \alpha_2$, then $[U_i|C(\mathbf{U}) = \alpha_1] \leq_{st} [U_i|C(\mathbf{U}) = \alpha_2]$ and $[V_i|\hat{C}(\mathbf{V}) = 1-\alpha_2] \leq_{st} [V_i|\hat{C}(\mathbf{V}) = 1-\alpha_1]$ hold. By using Theorem 1.A.3.a from Shaked and Shanthikumar (2007), it is verified that

$$[F^{-1}_{X_i}(U_i)|C(\mathbf{U}) = \alpha_1] \leq_{st} [F^{-1}_{X_i}(U_i)|C(\mathbf{U}) = \alpha_2],$$

and

$$[\overline{F}^{-1}_{X_i}(V_i)|\hat{C}(\mathbf{V}) = 1-\alpha_2] \geq_{st} [\overline{F}^{-1}_{X_i}(V_i)|\hat{C}(\mathbf{V}) = 1-\alpha_1].$$

Thus, $\underline{\mathrm{CoVaR}}^i_{\alpha_1,\boldsymbol{\omega}}(\mathbf{X}) \leq \underline{\mathrm{CoVaR}}^i_{\alpha_2,\boldsymbol{\omega}}(\mathbf{X})$ and $\overline{\mathrm{CoVaR}}^i_{\alpha_1,\boldsymbol{\omega}}(\mathbf{X}) \leq \overline{\mathrm{CoVaR}}^i_{\alpha_2,\boldsymbol{\omega}}(\mathbf{X})$, for any $\alpha_1 \leq \alpha_2$ which proves that $\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ and $\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ are non-decreasing functions of α. □

Assumptions of Proposition 2.3.7 are automatically satisfied by the large class of Archimedean copulas. This result will be proved in Corollary 2.5.3.

2.3.5 Elicitability property

Functionals that are defined as the minimizers of a suitable expected loss are called *elicitable functions* in statistical decision theory (Gneiting (2011)). As shown in property (P7), CoVaRs verify the *elicitability* property. This property was studied by Gneiting (2011), while Bellini and Bignozzi (2013) suggested a slightly more restrictive definition. More recently, Embrechts and Hofert (2013) stated that elicitability is a very important property of a risk measure since it provides a natural methodology to perform backtesting. Ziegel (2014) has also studied the connections between elicitability and coherence properties of risk measures.

2.4 CoVaR relations by stochastic orders

The comparison of risks constitutes an important topic of actuarial sciences, especially in insurance business. The behaviour of multivariate CoVaR risk measures is studied under different stochastic ordering conditions. The results below compare the multivariate CoVaR risk measures for random vectors with the same copula by assuming that margins change according to some particular stochastic order.

Proposition 2.4.1. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two d-dimensional random vectors, that satisfy the regularity conditions and with the same copula C. If $X_i \leq_{st} Y_i$, then*

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) \leq \underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{Y}),$$

and

$$\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) \leq \overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{Y}),$$

for $\alpha \in (0,1)$ and $\boldsymbol{\omega} \in [0,1]^d$.

Proof. Let us denote the i-margins of $\mathbf{X}$ and $\mathbf{Y}$ by F_{X_i} and F_{Y_i} respectively. Since $X_i \leq_{st} Y_i$, then $F^{-1}_{X_i}(u) \leq F^{-1}_{Y_i}(u)$, $\forall\, u \in [0,1]$. Using Sklar's Theorem (see Section 1.4), the random variables $U_i \stackrel{d}{=} F_{X_i}(X_i)$, for $i = 1, \ldots, d$, are uniformly distributed and their joint distribution is equal to that of C. Similarly, the random variables $U'_i \stackrel{d}{=} F_{Y_i}(Y_i)$, for $i = 1, \ldots, d$. Therefore,

$$[X_i \,|\, C(\mathbf{U}) = \alpha] \stackrel{d}{=} [F^{-1}_{X_i}(U_i) \,|\, C(\mathbf{U}) = \alpha], \text{ and}$$
$$[Y_i \,|\, C(\mathbf{U}') = \alpha] \stackrel{d}{=} [F^{-1}_{Y_i}(U'_i)|C(\mathbf{U}') = \alpha],$$

for $i = 1, \ldots, d$. Observe that $[U_i \,|\, C(\mathbf{U}) = \alpha] \stackrel{d}{=} [U'_i \,|\, C(\mathbf{U}') = \alpha]$. From Theorem 1.A.2 in Shaked and Shanthikumar (2007), $[X_i \,|\, C(\mathbf{U}) = \alpha] \leq_{st} [Y_i \,|\, C(\mathbf{U}') = \alpha]$ holds. Hence, the statement for the lower-orthant CoVaR is verified. The proof of the second statement is also verified using the same arguments. □

The result in Proposition 2.4.1 will be illustrated in the Archimedean case in Example 2.5.5.

Corollary 2.4.1. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two d-dimensional random vectors satisfying the regularity conditions and with the same copula C. If $X_i \stackrel{d}{=} Y_i$, then, for $\alpha \in (0,1)$ and $\boldsymbol{\omega} \in [0,1]^d$,*

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) = \underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{Y}), \text{ and } \overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) = \overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{Y}).$$

Finally, some results are provided for the behaviour of our CoVaR measures with respect to a variation of the copula structure, with unchanged marginal distributions.

Proposition 2.4.2. *Let $\boldsymbol{X}$ and $\boldsymbol{X}^*$ be two d-dimensional continuous random vectors, which satisfy the regularity conditions with joint distribution functions F and G, and with the same margins F_{X_i} and $F_{X^*_i}$, for $i = 1, \ldots, d$. Let C (C^*, respectively) be the copula function associated with $\boldsymbol{X}$ ($\boldsymbol{X}^*$, respectively) and $\hat{C}$ ($\hat{C}^*$, respectively) be the survival copula function associated with $\boldsymbol{X}$ ($\boldsymbol{X}^*$, respectively).*

(1) *Let $U_i = F_{X_i}(X_i)$, $U^*_i = F_{X_i{}^*}(X^*_i)$, $\boldsymbol{U} = (U_1, \ldots, U_d)$ and $\boldsymbol{U}^* = (U^*_1, \ldots, U^*_d)$.*
If $[U_i \,|\, C(\boldsymbol{U}) = \alpha] \leq_{st} [U^*_i \,|\, C^*(\boldsymbol{U}^*) = \alpha]$, then

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) \leq \underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}^*) \text{ for } \alpha \in (0,1), \omega_i \in [0,1], i = 1, \ldots, d.$$

(2) *Let* $V_i = \overline{F}_{X_i}(X_i)$, $V_i^* = \overline{F}_{X_i^*}(X_i^*)$, $\boldsymbol{V} = (V_1, \ldots, V_d)$ *and* $\boldsymbol{V}^* = (V_1^*, \ldots, V_d^*)$. *If* $[V_i \,|\, \hat{C}(\boldsymbol{V}) = 1-\alpha] \leq_{st} [V_i^* \,|\, \hat{C}^*(\boldsymbol{V}^*) = 1-\alpha]$, *then*

$$\overline{\text{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) \geq \overline{\text{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}^*) \ \textit{for } \alpha \in (0,1), \omega_i \in [0,1], i = 1, \ldots, d.$$

Proof. By using (P2) and (P7), for $i = 1, \ldots, d$, trivially it holds that

$$\omega_i = F_{X_i^* | F_{\mathbf{X}^*}(\mathbf{X}^*)=\alpha}(\underline{\text{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}^*)) = F_{X_i | F_{\mathbf{X}}(\mathbf{X})=\alpha}(\underline{\text{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})). \tag{2.7}$$

On the other hand, since $F^{-1}_{X_i}(u)$ for $u \in [0,1]$ is a non-decreasing function, and since X_i and X_i^* have the same distribution, then from Theorem 1.A.3.a in Shaked and Shanthikumar (2007), it is verified that

$$F_{F^{-1}_{X_i}(U_i^*)\,|\,C^*(\mathbf{U}^*)=\alpha}(u) \leq F_{F^{-1}_{X_i}(U_i)\,|\,C(\mathbf{U})=\alpha}(u), \ \forall u \in [0,1]. \tag{2.8}$$

Therefore, from (2.7) and (2.8), $\underline{\text{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \leq \underline{\text{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}^*)$.

Following the above development for $X_i \,|\, \overline{F}_{\mathbf{X}}(\mathbf{X}) = 1-\alpha$ and $X_i^* \,|\, \overline{F}_{\mathbf{X}^*}(\mathbf{X}^*) = 1-\alpha$, and by using the survival quantile function $\overline{F}^{-1}_{X_i}(u)$ for $u \in [0,1]$, the result for upper-orthant CoVaR holds. □

An application of Proposition 2.4.2 in the case of Archimedean copulas is given in Corollary 2.5.4.

2.5 Multivariate CoVaR for the class of Archimedean copulas

Interestingly enough, one can readily show that when the random vector $\mathbf{X}$ follows an Archimedean copula then the analytical expression for the CoVaR can be easily computed, in a similar way to that used in Cousin and Di Bernardino (2013) to compute their multivariate Value-at-Risk. Indeed, Archimedean copulas have useful relationships between their generator and the probability associated with their level curves $\overline{L}^{\leq}(\alpha)$ and $\underline{L}^{\geq}(\alpha)$ (see the notion of multivariate probability integral transformation in Genest and Rivest (2001), Barbe et al. (1996) and references therein). Furthermore, the results and properties, which were previously proved in this chapter, can easily be applied in the large class of Archimedean copulas.

Corollary 2.5.1. *Let* $\boldsymbol{X}$ *be a* d*-dimensional random vector with an Archimedean copula with generator* ϕ *and* $\alpha \in (0,1)$. *Therefore,*

$$\underline{\text{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) = \text{VaR}_{\omega_\text{i}}\left[\text{F}^{-1}_{\text{X}_\text{i}}(\phi^{-1}(\text{S}_\text{i}\phi(\alpha)))\right], \ \textit{for } \text{i} = 1, \ldots, \text{d}, \tag{2.9}$$

where $\boldsymbol{\omega} \in [0,1]^d$ *and* S_i *is a random variable with* $Beta(1, d-1)$ *distribution.*

Proof. Note that $\mathbf{X}$ is distributed as $(F_{X_1}^{-1}(U_1), \ldots, F_{X_d}^{-1}(U_d))$, where $\mathbf{U} = (U_1, \ldots, U_d)$ follows an Archimedean copula C with generator ϕ. Consequently, each component $i = 1, \ldots, d$ of the multivariate risk measure introduced in Definition 2.2.1 can be expressed as

$$\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \underset{x \in [\mathrm{VaR}_\alpha(\mathrm{X_i}), +\infty)}{\arg\min} \left\{ \omega_i \, \mathbb{E}[(T_i - x)^+] + (1 - \omega_i) \, \mathbb{E}[(T_i - x)^-] \right\},$$

where $T_i = [F_{X_i}^{-1}(U_i) | C(\mathbf{U}) = \alpha]$. Moreover, from representation (1.7), the following relation is verified

$$[\mathbf{U} | C(\mathbf{U}) = \alpha] \stackrel{d}{=} (\phi^{-1}(S_1 \phi(\alpha)), \ldots, \phi^{-1}(S_d \phi(\alpha))), \tag{2.10}$$

since $\mathbf{S}$ and $C(\mathbf{U})$ are stochastically independent. The result comes from the fact that the random vector $\mathbf{S}$ follows a symmetric Dirichlet distribution. □

Corollary 2.5.2. *Let $\boldsymbol{X}$ be a d-dimensional random vector with an Archimedean survival copula with generator φ and $\alpha \in (0, 1)$. Therefore,*

$$\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) = \mathrm{VaR}_{\omega_\mathrm{i}} \left[\overline{\mathrm{F}}_{\mathrm{X_i}}^{-1}(\varphi^{-1}(\mathrm{S_i} \varphi(1 - \alpha))) \right], \ \textit{for } \mathrm{i} = 1, \ldots, \mathrm{d}, \tag{2.11}$$

where $\boldsymbol{\omega} \in [0, 1]^d$ and S_i is a random variable with $Beta(1, d - 1)$ distribution.

The proof is similar to Corollary 2.5.1 and is therefore omitted here.

From (2.9) and (2.11), analytical expressions of the lower-orthant and the upper-orthant CoVaR for a vector $\mathbf{X} = (X_1, \ldots, X_d)$ with a particular Archimedean copula are now derived. Assume that X_i is uniformly-distributed on $[0, 1]$, for $i = 1, \ldots, d$. Since Archimedean copulas are exchangeable, the components of $\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ ($\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$, respectively) are equal in the case where $\omega_1 = \ldots = \omega_d$. Furthermore, it is also possible to obtain expressions for the upper-orthant $\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}$ for $\tilde{\mathbf{X}} = (1 - X_1, \ldots, 1 - X_d)$ since, by using Corollary 2.3.1:

$$\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\tilde{\mathbf{X}}) = 1 - \underline{\mathrm{CoVaR}}^i_{1-\alpha,1-\boldsymbol{\omega}}(\mathbf{X}).$$

In the following, Corollary 2.5.1 is illustrated for some commonly used Archimedean copula families (see Examples 2.5.1 - 2.5.3).

Example 2.5.1 (Bivariate Clayton family). *In Table 2.1 (left), the bivariate random vector (X, Y) is considered with uniform marginal distributions and a Clayton copula with parameter $\theta \geq -1$ is considered. One can readily show that*

$$\frac{\partial \underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}}{\partial \theta} \leq 0 \ \textit{ and } \ \frac{\partial \overline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}}{\partial \theta} \geq 0, \ \textit{for } \ \theta \geq -1, \ \ \alpha \in (0, 1) \ \textit{ and } \ \omega \in [0, 1].$$

Hence, the components of the multivariate $\underline{\mathrm{CoVaR}}$ *(*$\overline{\mathrm{CoVaR}}$*, respectively) are decreasing (increasing, respectively) functions of the dependence parameter θ. Interestingly enough,* *in the comonotonic case, both multivariate risk measures* $\underline{\mathrm{CoVaR}}$ *and* $\overline{\mathrm{CoVaR}}$ *correspond to the vector composed of the univariate* VaR *at level* α *associated with each component. These properties are illustrated in Figure 2.1 where upper and lower* CoVaR *are plotted as functions of the risk level* $\boldsymbol{\omega}$ *for different values of the dependence parameter* θ *and for a fixed level* α*. Note that, when the parameter* θ *increases, the lower* CoVaR *tends to decrease. Conversely, the upper bound for the upper* CoVaR *is represented by the perfect positive dependence case. The latter empirical behaviours will be formally confirmed in the following (see Corollary 2.5.4).*

θ	$\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega},\theta}(X,Y)$
$(-1,\infty)$	$\left(1+\left(\frac{1}{\alpha^\theta}-1\right)(1-\omega_1)\right)^{-1/\theta}$
-1	$1-(1-\omega_1)(1-\alpha)$
0	$\alpha^{1-\omega_1}$
1	$\frac{\alpha}{(1-\alpha)(1-\omega_1)+\alpha}$
∞	α

θ	$\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega},\theta}(X,Y)$
$[-1,1)$	$\frac{1-\theta}{\left(\frac{1-\theta(1-\alpha)}{\alpha}\right)^{(1-\omega_1)}-\theta}$
0	$\alpha^{1-\omega_1}$

Table 2.1: $\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(X,Y)$, for a bivariate Clayton copula (left) and for a bivariate Ali-Mikhail-Haq copula (right).

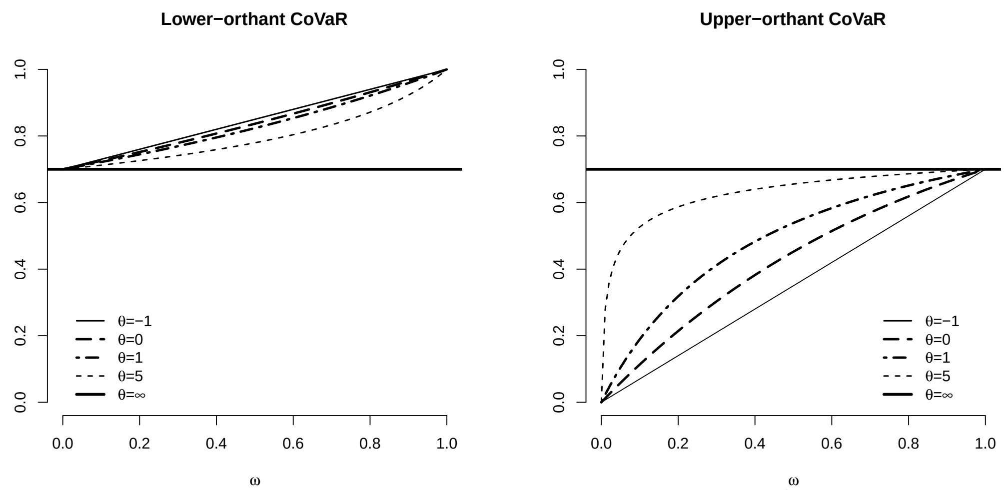

Figure 2.1: Behaviour of $\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(X,Y)$ (left panel) and $\overline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(1-X,1-Y)$ (right panel) with respect to the risk level ω for different values of dependence parameter θ and for $\alpha = 0.7$. Here, (X,Y) is a bivariate random vector with uniform marginal distributions and a Clayton copula with parameter $\theta \geq -1$.

Example 2.5.2 (Bivariate Ali-Mikhail-Haq family)**.** *Table 2.1 (right) illustrates the analytical expressions of* $\underline{\mathrm{CoVaR}}$ *for the first component of a bivariate random vector with uniform marginal distributions and an Ali-Mikhail-Haq copula, for* $\theta \in [-1,1)$*.*

Recall that bivariate Archimedean copulas can be extended to $d-$dimensional copulas, with $d > 2$, on the condition that the generator ϕ is a $d-$monotone function in $[0, \infty)$ (see McNeil and Nešlehová (2009)). The bivariate Gumbel family can be generalized in dimension d, for $\theta \geq 1$ (see Example 4.25 in Nelsen (2006)).

Example 2.5.3 (3-dimensional Gumbel family)**.** *In this case, analytical expressions of the first component of the lower-orthant* CoVaR *of a 3-dimensional random vector* (X_1, X_2, X_3) *with uniform marginal distributions and a Gumbel copula, for* $\theta \geq 1$*, are provided in Table 2.2.*

θ	$\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega},\theta}(X_1, X_2, X_3)$
$[1, \infty)$	$\alpha^{(1-\sqrt{\omega_1})^{1/\theta}}$
1	$\alpha^{(1-\sqrt{\omega_1})}$
∞	α

Table 2.2: $\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(X_1, X_2, X_3)$ for a 3-dimensional Gumbel copula.

2.5.1 Properties of multivariate CoVaR for Archimedean copulas

In the following, some theoretical properties presented in Sections 2.3 and 2.4 are illustrated in the large class of d-dimensional Archimedean copula. Firstly, using Corollary 2.5.1, an illustration of Proposition 2.3.4 in the Clayton copula case is provided.

Example 2.5.4. *Assume that* $\boldsymbol{X}$ *is a bivariate random vector with uniform marginal distributions and a Clayton copula. The distribution function of* $\boldsymbol{X}$ *is therefore given by:*

$$F(x_1, x_2) = \left[\max\{x_1^{-\theta} + x_2^{-\theta} - 1, 0\}\right]^{-1/\theta},$$

for $\theta \in [-1, \infty)\backslash\{0\}$ *and* $(x_1, x_2) \in [0, 1]^2$*. Therefore, by straightforward computation, one can obtain, for* $\alpha \in (0, 1)$ *and* $\omega_1 \in [0, 1]$*,*

$$\underline{\mathrm{VaR}}^1_{\alpha}(\boldsymbol{X}) = \tfrac{\theta}{\theta-1}\tfrac{\alpha^{\theta}-\alpha}{\alpha^{\theta}-1}, \quad \textit{and} \quad \underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X}) = \left[1 + \left(\tfrac{1}{\alpha^{\theta}} - 1\right)(1 - \omega_1)\right]^{-1/\theta},$$

where $\underline{\mathrm{VaR}}^1_{\alpha}(\boldsymbol{X})$ *is the first-component lower-orthant* VaR *proposed by Cousin and Di Bernardino (2013). Consequently, both measures coincide in*

$$\omega_* = \left(\alpha^{-\theta} - \left(\tfrac{\theta}{\theta-1}\tfrac{\alpha^{\theta}-\alpha}{\alpha^{\theta}-1}\right)^{-\theta}\right)[\alpha^{-\theta} - 1]^{-1}.$$

For a fixed $\alpha = 0.6$ *we obtain the results gathered in Figure 2.2. In Figure 2.2 (left panel), we gather* $\underline{\mathrm{VaR}}^1_{\alpha}(\boldsymbol{X})$ *and* $\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X})$ *in terms of* ω*. In Figure 2.2 (right panel), we gather the ratio between* $\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X})$ *and* $\underline{\mathrm{VaR}}^1_{\alpha}(\boldsymbol{X})$ *in terms of* ω*.* $\underline{\mathrm{VaR}}_{\alpha}(\boldsymbol{X})$

represents the case that the complete risk of the insurance company is reinsured by another company ($x = 0$) (see Cousin and Di Bernardino (2013)). The insurance company gives the total weight to the expected cost of the reinsurance company, that is, establishes $\omega = 1$. By contrast, CoVaR *defines the minimum retention of the insurance company given a weight $\omega \in [0, 1]$ for the expected cost of the reinsurance company. For instance, for $\theta = 2$, it can be observed in Figure 2.2 that $\underline{\mathrm{VaR}}^1_{0.6}(\boldsymbol{X}) = 0.75$ and the cut-off point is $\omega_* = 0.56$. Furthermore, we can easily observe in Figure 2.2 (right panel) that lower-orthant CoVaR is larger (smaller, respectively) than lower-orthant VaR for every $\omega < (>, respectively)\omega^*$ and, that both measures coincide in the respective ω^*. Similarly, analytical expressions can be obtained for multivariate upper-orthant* CoVaR *and comparisons with the associated $\overline{\mathrm{VaR}}_\alpha(\boldsymbol{X})$ (see Cousin and Di Bernardino (2013)).*

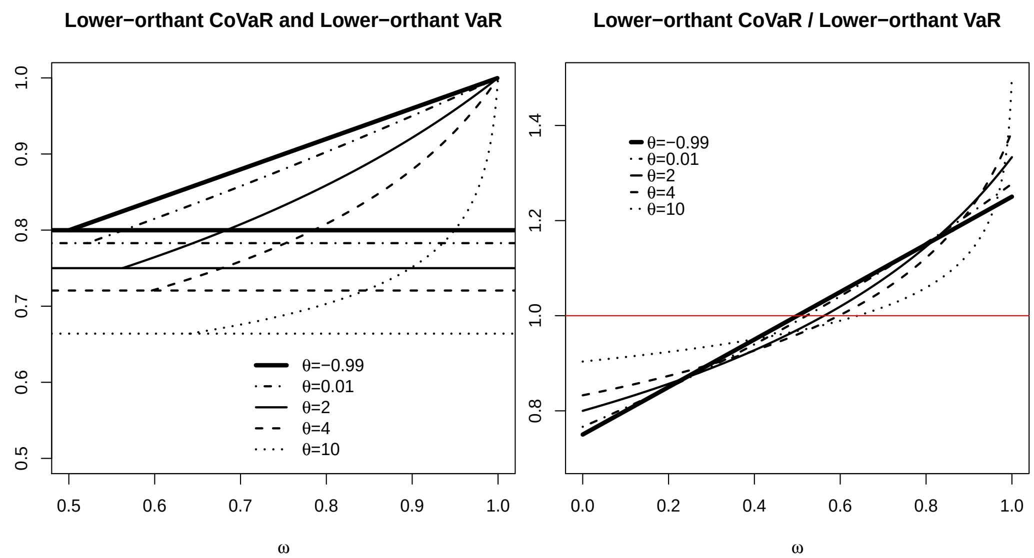

Figure 2.2: (X, Y) is a bivariate random vector with uniform marginal distributions and a Clayton copula with parameter $\theta \geq -1$, and $\alpha = 0.6$. $\underline{\mathrm{VaR}}^1_\alpha(\mathbf{X})$ and $\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ (left panel). $\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(\mathbf{X})/\underline{\mathrm{VaR}}^1_\alpha(\mathbf{X})$ (right panel).

Corollary 2.5.3 proves that assumptions of Proposition 2.3.7 are automatically satisfied in the large class of d-dimensional Archimedean copulas.

Corollary 2.5.3. *Consider a d-dimensional random vector $\boldsymbol{X}$, which satisfies the regularity conditions, with marginal distributions F_{X_i}, for $i = 1, \ldots, d$, copula C and survival copula $\hat{C}$.*

(1) If C is a d-dimensional Archimedean copula, then $\underline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X})$ is a non-decreasing function of α with $\boldsymbol{\omega} \in [0, 1]^d$.

(2) If $\hat{C}$ is a d-dimensional Archimedean copula, then $\overline{\mathrm{CoVaR}}^i_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X})$ is a non-decreasing function of α with $\boldsymbol{\omega} \in [0, 1]^d$.

Proof. Let $U_i = F_{X_i}(X_i)$, $\mathbf{U} = (U_1, \ldots, U_n)$, $V_i = \overline{F}_{X_i}(X_i)$ and $\mathbf{V} = (V_1, \ldots, V_n)$. Since C is the copula of $\mathbf{X}$, then $\mathbf{U}$ is distributed as C. If C is an Archimedean copula, then from Lemma 2.2.1, $P(U_i > u | C(\mathbf{U}) = \alpha)$ is a non-decreasing function of α. Similarly, $P(V_i > u | \hat{C}(\mathbf{V}) = 1 - \alpha)$ is a non-decreasing function of α. The results are therefore trivially derived from Proposition 2.3.7. □

In the following, an illustration of Proposition 2.4.1 is provided in the Archimedean case.

Example 2.5.5. *Three different random vectors (X, Y_i), for $i = 1, \ldots, 3$ are considered with the same bivariate Clayton copula with dependence parameter 2, such that*

$$X \sim Exp(1), \;\; Y_1 \sim Exp(2), \;\; Y_2 \sim Burr(5,1), \;\; Y_3 \sim Fréchet(4).$$

If $X \sim Burr(c,k)$, then the distribution function of X is given by $F(x) = 1-(1+x^c)^{-k}$, with $c > 0$ and $k > 0$. Furthermore, recall that if $X \sim Fréchet(\beta)$, then the distribution function of X is given by $F(x) = \exp\{-x^{-\beta}\}$, $\beta > 0$ (see Section 1.2). It should be borne in mind that the above three distributions are usually applied in studies of household income, insurance risk and reliability analysis. Since $Y_1 \leq_{st} Y_2 \leq_{st} Y_3$, from Proposition 2.4.1, then

$$\underline{\mathrm{CoVaR}}^2_{\alpha,\boldsymbol{\omega}}(X, Y_1) \leq \underline{\mathrm{CoVaR}}^2_{\alpha,\boldsymbol{\omega}}(X, Y_2) \leq \underline{\mathrm{CoVaR}}^2_{\alpha,\boldsymbol{\omega}}(X, Y_3),$$

for any $\boldsymbol{\omega} \in [0,1]^2$ and $\alpha \in (0,1)$. The results are collected in Figure 2.3. It should also be emphasized that, by Corollary 2.4.1, the first components of the multivariate lower-orthant CoVaR and upper-orthant CoVaR for the four vectors coincide.

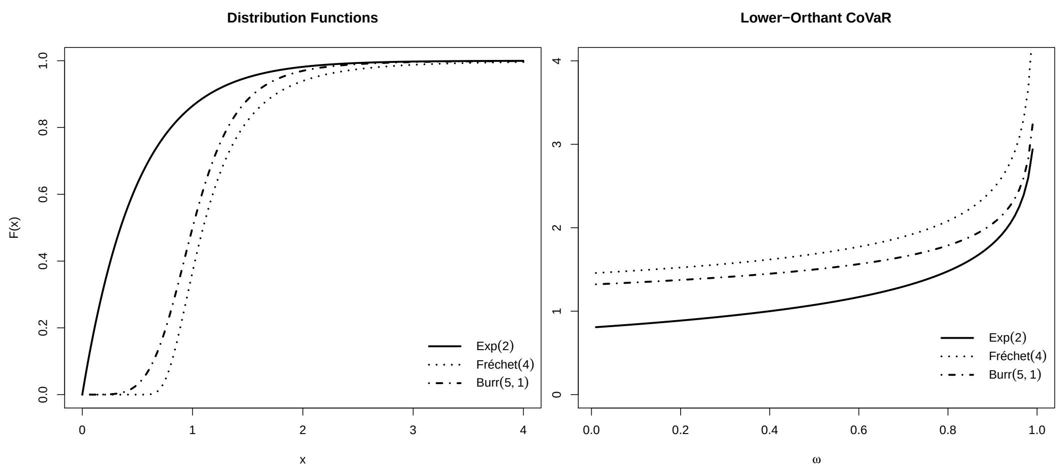

Figure 2.3: Distribution functions of random variables Y_i, for $i = 1, \ldots, 3$, with $Y_1 \sim Exp(2)$, $Y_2 \sim Burr(5,1)$ and $Y_3 \sim Fréchet(4)$ (left panel). $\underline{\mathrm{CoVaR}}^2_{\alpha,\boldsymbol{\omega}}(X, Y_i)$ for $i = 1, \ldots, 3$, with the same Clayton copula with parameter 2, $X \sim Exp(1)$, $Y_1 \sim Exp(2)$, $Y_2 \sim Burr(5,1)$, $Y_3 \sim Fréchet(4)$ and $\alpha = 0.8$ (right panel).

The following remark will be useful in Corollary 2.5.4.

Remark 2.5.1. *Let $\boldsymbol{U}$ and $\boldsymbol{U}^*$ be two random vectors with copula C and C^*, respectively, and with uniform marginal distributions. It is easy to prove that $\boldsymbol{U} \leq_{sm} \boldsymbol{U}^*$ implies $C(\boldsymbol{u}) \leq C^*(\boldsymbol{u})$, for $\boldsymbol{u} \in [0,1]^d$ (Section 6.3.3 in Denuit et al. (2005)). In addition, for Gumbel, Frank, Clayton, and Ali-Mikhail-Haq families, it can be shown that an increase of θ yields an increase of dependence in the sense of the supermodular order (see examples in Wei and Hu (2002), Joe (1997)). As a consequence, in these cases,*

$$\theta \leq \theta^* \Rightarrow C(\boldsymbol{u}) \leq C^*(\boldsymbol{u}), \quad for \quad \boldsymbol{u} \in [0,1]^d. \tag{2.12}$$

Corollary 2.5.4. *Let $\boldsymbol{X}$ be a d-dimensional random vector satisfying the regularity conditions with copula C and survival copula $\hat{C}$.*

If C is a d-dimensional Archimedean copula that satisfies Property (2.12) in Remark (2.5.1), then each component of $\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X})$ is a decreasing function of θ, with $\alpha \in (0,1)$ and $\boldsymbol{\omega} \in [0,1]^d$.

If $\hat{C}$ is a d-dimensional Archimedean copula that satisfies Property (2.12) in Remark (2.5.1), each component of $\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X})$ is a increasing function of θ, with $\alpha \in (0,1)$ and $\boldsymbol{\omega} \in [0,1]^d$.

Proof. We consider two Archimedean copulas of the same family, C_θ (associated with vector $\mathbf{U}$) and C_{θ^*} (associated with vector $\mathbf{U}^*$) with generator ϕ_θ and ϕ_{θ^*} such that $\theta \leq \theta^*$. By Proposition 2.4.2, we have to prove that $[U_i^* | C_{\theta^*}(\mathbf{U}^*) = \alpha] \leq_{st} [U_i | C_\theta(\mathbf{U}) = \alpha]$ holds for $i = 1, \ldots, d$. On the other hand, from Lemma 2.2.1, it is readily obtained that

$$[U_i^* | C_{\theta^*}(\mathbf{U}^*) = \alpha] \leq_{st} [U_i | C_\theta(\mathbf{U}) = \alpha] \text{ for any } \alpha \in (0,1) \Leftrightarrow \frac{\phi_{\theta^*}}{\phi_\theta} \text{ is a decreasing function.}$$

Finally, by taking Remark 2.5.1 into account, if C verifies Property (2.12), then the function $\frac{\phi_{\theta^*}}{\phi_\theta}$ is decreasing when $\theta \leq \theta^*$. Therefore, from Proposition 2.4.2, an increase of the parameter θ yields a decrease in each component of $\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$. The second statement is obtained trivially using the same arguments. □

It should be noted that if C ($\hat{C}$, respectively) belongs to the Gumbel, Frank, Clayton or Ali-Mikhail-Haq families, assumptions of Corollary 2.5.4 are satisfied. The reader is referred, for instance, to the behaviour of the lower-orthant and upper-orthant CoVaR with respect to the copula parameter θ presented in Figure 2.1.

2.5.2 A weak subadditivity tail property in the Archimedean case

The additivity of our CoVaR is provided in Section 2.3.3 in a comonotonic dependence vectorial case (see Proposition 2.3.6 for π-comonotonic vectors). In the following, the aim is to study the condition for a copula to obtain subadditivity inequalities for our lower-orthant CoVaR.

To this end, as in the univariate case (see Daníelsson et al. (2013)), we focus on the *tails* of the considered multivariate distribution. The notions of regularly varying function (RV) and of multivarite regularly varying vector (MRV) are recalled in Appendix A.

Theorem 2.5.1. *Let $\boldsymbol{X}$ be a bivariate random vector with distribution function F, Archimedean copula C with generator ϕ and marginals F_{X_i}, $i = 1, 2$. Assume that the marginals of $\boldsymbol{X}$ are identically distributed, that ϕ is twice differentiable, and that $(\phi \circ F_{X_1}) \in RV_{-\beta}$, $\beta > 0$. Therefore $\boldsymbol{T} := (T_1, T_2) \in MRV$.*

Proof. Firstly, the copula of random vector $\mathbf{T}$ is computed. Note that

$$F(x_1, x_2) = \phi^{-1}(\phi(F_{X_1}(x_1)) + \phi(F_{X_2}(x_2))).$$

For simplicity, the univariate random variable $F(X_1, X_2)$ is denoted by V. Similarly to Theorem 1 in Wang and Oakes (2008), we obtain

$$\mathbb{P}[V \le \alpha, X_1 \le x_1, X_2 \le x_2] = \begin{cases} \alpha - \frac{\phi(\alpha)}{\phi'(\alpha)} + \frac{\phi(F(x_1,x_2))}{\phi'(\alpha)}, & \text{if } 0 < \alpha \le F(x_1, x_2); \\ 0, & \text{if } \alpha > F(x_1, x_2). \end{cases} \tag{2.13}$$

By straightforward calculation, it can be shown that the distribution function of $\mathbf{T}$ is defined as

$$\begin{aligned} F_{\mathbf{T}}(x_1, x_2) &= \begin{cases} \frac{\mathbb{P}[V=\alpha, X_1 \le x_1, X_2 \le x_2]}{P(V=\alpha)}, & \text{if } 0 < \alpha \le F(x_1, x_2); \\ 0, & \text{if } \alpha > F(x_1, x_2), \end{cases} \\ &= \begin{cases} 1 - \frac{\phi(F(x_1,x_2))}{\phi(\alpha)}, & \text{if } 0 < \alpha \le F(x_1, x_2); \\ 0, & \text{if } \alpha > F(x_1, x_2), \end{cases} \end{aligned} \tag{2.14}$$

where $P(V = \alpha)$ is the density in α of random variable V.

On the other hand, for $i = 1, 2$, by using Lemma 2.2.1, we obtain that

$$F_{T_i}^{-1}(w_i) = \begin{cases} (\phi \circ F_{X_i})^{-1}(\phi(\alpha)(1 - w_i)), & \text{if } 0 < w_i \le 1; \\ 0, & \text{if } w_i = 0. \end{cases}$$

Therefore, the copula of the random vector $\mathbf{T}$ is

$$C_{\mathbf{T}}(u_1, u_2) = F_{\mathbf{T}}(F_{T_1}^{-1}(u_1), F_{T_2}^{-1}(u_2)) = \begin{cases} u_1 + u_2 - 1, & \text{if } u_1 + u_2 \ge 1; \\ 0, & \text{otherwise.} \end{cases}$$

It is now shown that $\mathbf{T} \in MRV$ by Theorem 3.2 in Weng and Zhang (2012). Therefore, conditions (C1) and (C2) of Theorem 3.2 in Weng and Zhang (2012) are proved. As a result of $(\phi \circ F_{X_1}) \in RV_{-\beta}$, $\beta > 0$, we trivially obtain $\overline{F}_{T_1} \in RV_{-\beta}$, $\beta > 0$ (C1).

In addition, since $\mathbf{X}$ has the same margins, then

$$\lim_{t \to \infty} \frac{\overline{F}_{T_2}(t)}{\overline{F}_{T_1}(t)} = 1,$$

that is, $\overline{F}_{T_1}$ and $\overline{F}_{T_2}$ have equivalent tails. (C2)

Finally, the lower tail dependence function of the survival copula of $\mathbf{T}$,

$$\lambda_2(u_1, u_2) = \lim_{t \to 0^+} \frac{\hat{C}_{\mathbf{T}}(tu_1, tu_2)}{t},$$

is equal to 0. Hence, considering (C1) and (C2), by Theorem 3.2 in Weng and Zhang (2012), $\mathbf{T} \in MRV$.

$\square$

Remark 2.5.2. *Note that, if $(\phi \circ F_{X_1}) \in RV_{-\beta}$, $\beta > 1$, by applying Theorem 2.5.1 and Proposition 1 in Daníelsson et al. (2013) for $\boldsymbol{T}$, then the* VaR *of $\boldsymbol{T}$ is sufficiently deeply subadditive*[3] *in the tail regions. In this case, a weak subadditivity of the proposed multivariate lower-orthant* CoVaR is obtained, that is, since $\mathrm{VaR}_\omega(\mathrm{T_i}) = \underline{\mathrm{CoVaR}}^{\mathrm{i}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$, then

$$\mathrm{VaR}_\omega(\mathrm{T}_1 + \mathrm{T}_2) < \underline{\mathrm{CoVaR}}^{1}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) + \underline{\mathrm{CoVaR}}^{2}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) \tag{2.15}$$

sufficiently deep in the tail regions.

An illustration of Remark 2.5.2 is now presented (see Figure 2.4 and Example 2.5.6 below).

Example 2.5.6. *In this example, a bivariate random vector, $\boldsymbol{X}$, with $X_1 \sim X_2 \sim$ Pareto(2) and a Gumbel copula with parameter $\theta = 2$, is considered. Analytical expressions of $\underline{\mathrm{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\boldsymbol{X})$, $i = 1, 2$ are obtained. In addition,* $\mathrm{VaR}_\omega(\mathrm{T}_1 + \mathrm{T}_2)$ *is calculated by numeric approximation. The obtained results are shown in Figure 2.4: for $\omega = \alpha \in (0, 1)$ (see Figure 2.4, left panel) and for $\alpha = 0.75$, $\omega \in (0, 1)$ (see Figure 2.4, right panel). It can be easily observed that (2.15) is verified for a large ω.*

One line of future research could entail the study of the condition for a survival copula to obtain subadditivity inequalities for our upper-orthant CoVaR.

[3] That is, VaR_ω of $\mathbf{T}$ is subadditive for a sufficiently small level ω.

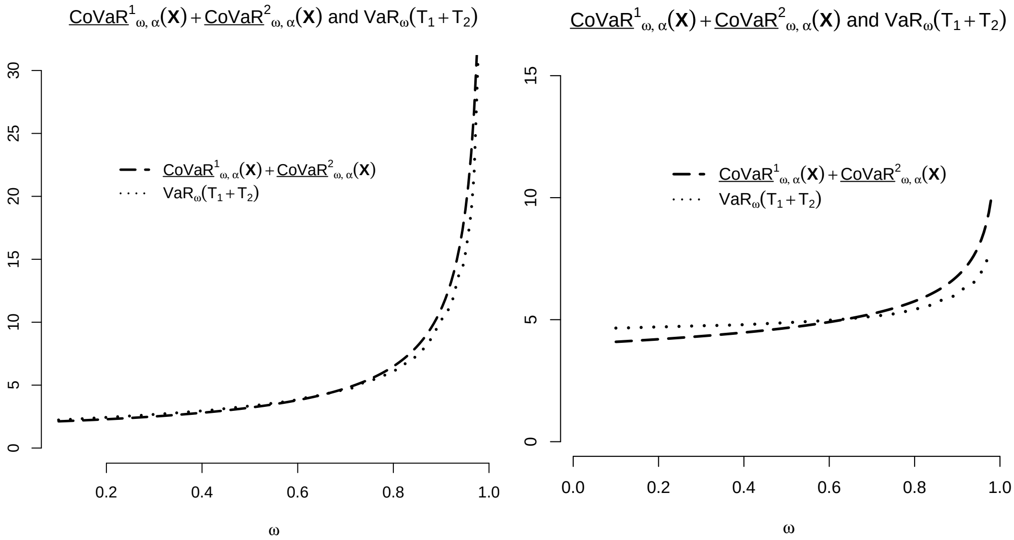

Figure 2.4: $\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) + \underline{\mathrm{CoVaR}}^2_{\alpha,\boldsymbol{\omega}}(\mathbf{X})$ and $\mathrm{VaR}_\omega(\mathrm{T}_1+\mathrm{T}_2)$ for $\mathbf{X}$ with $X_1 \sim X_2 \sim$ *Pareto*(2) and a Gumbel copula with $\theta = 2$, as in Example 2.5.6, for $\alpha = \omega$ (left panel) and for $\alpha = 0.75$ (right panel).

2.5.3 Measuring Systemic Risk

Systemic risk can be defined as the risk of collapse of an entire financial system, as opposed to risk associated with any one individual system component, that can be contained therein without harming the entire system. During financial crises, losses spread across financial institutions threatening the financial system as a whole and, as a consequence, systemic risk is brought about. A company that is highly interconnected with others is also a source of systemic risk.

VaR is the most common measure of risk used by financial institutions. However, VaR is focused on the risk of an individual institution in isolation and does not necessarily reflect the systemic risk. Adrian and Brunnermeier (2011) defined the systemic risk measure ΔCoVaR as the difference between the VaR of the institution j (or financial system) conditional on the distress of a particular financial institution i and the VaR of the institution j (see Equation (1.3)).

In the same way as in Adrian and Brunnermeier (2011), a new systemic risk measure can be considered using the proposed multivariate risk CoVaR in Definition 2.2.1. The contribution of institution j to the system is defined as

$$\Delta\underline{\mathrm{CoVaR}}^j_{\alpha,\boldsymbol{\omega}}(X_j|\mathbf{X}) = \underline{\mathrm{CoVaR}}^j_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) - \mathrm{VaR}_{\omega_\mathrm{j}}(\mathrm{X_j}) \tag{2.16}$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)$ is a marginal risk vector with $\omega_j \in [0,1]$, for $j = 1, \ldots, d$, and $\alpha \in (0,1)$. Analogously, using Definition 2.2.2, we can also propose the associated $\Delta\overline{\mathrm{CoVaR}}^j_{\alpha,\boldsymbol{\omega}}(X_j|\mathbf{X})$.

Remark 2.5.3. *From Proposition 2.3.5, if* $\boldsymbol{X}$ *is a comonotonic random vector and* $\alpha = \omega_j$*, then* $\Delta\underline{\mathrm{CoVaR}}^j_{\alpha,\boldsymbol{\omega}}(X_j|\boldsymbol{X}) = \Delta\overline{\mathrm{CoVaR}}^j_{\alpha,\boldsymbol{\omega}}(X_j|\boldsymbol{X}) = 0$. *Therefore, in this case, each institution* j *of the financial system protects itself using its associated univariate* VaR.

In Figure 2.5 (left panel), the lower $\Delta\underline{\mathrm{CoVaR}}$ is represented for a bivariate random vector (X_1, X_2) with $X_1 \sim X_2 \sim Pareto(2,1)$ and Clayton copula, with parameter $\theta > 0$. Similarly, the lower $\Delta\underline{\mathrm{CoVaR}}$ of bivariate random vector (X_1, X_2) with $X_1 \sim X_2 \sim Exp(2)$ and Gumbel copula, with parameter $\theta \geq 1$, is illustrated in Figure 2.5 (right panel).
Firstly, it can be observed that $\Delta\underline{\mathrm{CoVaR}}$ is lower when the risk level $\alpha = \omega$ decreases (see Corollary 2.5.3). Secondly, $\Delta\underline{\mathrm{CoVaR}}$ is also lower when the dependence parameter θ increases, since the multivariate lower CoVaR is decreasing with respect to θ, as proved in Corollary 2.5.4. In addition, when θ increases, that is, the vector exhibits more positive dependency, then the ΔCoVaR measure goes to 0. This behaviour is consistent with Proposition 2.3.5 and Corollary 2.5.4. Finally, in Figure 2.5, the evaluated $\Delta\underline{\mathrm{CoVaR}}$ are always larger than 0, i.e., the components of the financial system take part in the systemic risk.

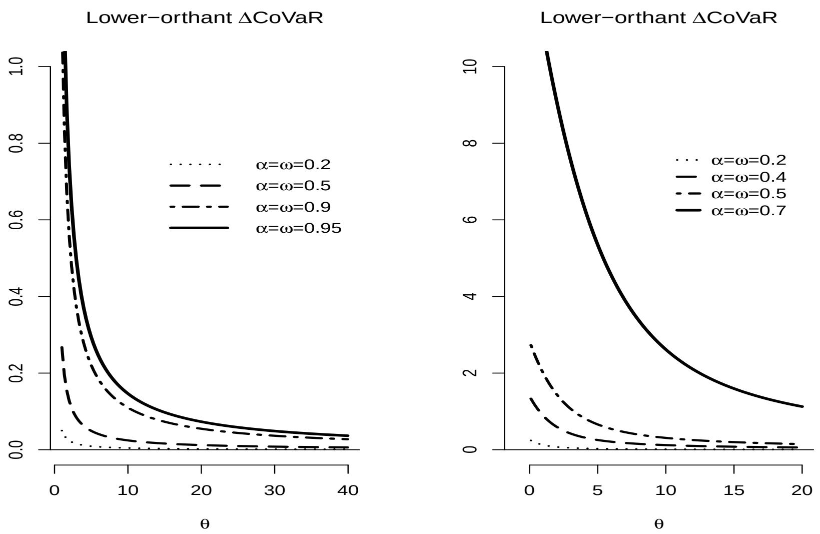

Figure 2.5: $\Delta\underline{\mathrm{CoVaR}}^1(X_1|\mathbf{X})$ for a bivariate Clayton copula, $\theta > 0$, with $X_1 \sim X_2 \sim Pareto(2,1)$ (left panel). $\Delta\underline{\mathrm{CoVaR}}^1(X_1|\mathbf{X})$ for a bivariate Gumbel copula, $\theta \geq 1$, with $X_1 \sim X_2 \sim Exp(2)$ (right panel).

In Mainik and Schaanning (2014) (see for instance their Figure 4), the systemic risk measure defined in (1.3) is studied in the bivariate elliptical distribution case. Here we analyse, the systemic risk measure $\Delta\underline{\mathrm{CoVaR}}$ defined in (2.16) in the bivariate Archimedean case (see Figure 2.5). We remark that the observed behaviours of both these measures are very similar. Indeed, in both cases, the measures are close to zero for the comonotonic dependence parameter (i.e., $\theta = +\infty$ for Archimedean copulas and $\rho = 1$ for elliptical copulas). Furthermore, the two measures are always larger than 0, when the dependence structure is positive. Finally, they are non-decreasing functions with respect to the risk level $\alpha = \omega$.

2.6 Semi-parametric estimation for Multivariate CoVaR

Semi-parametric estimators are given in this section by assuming Archimedean copula for the proposed multivariate CoVaR measures. Moreover, illustrations are provided for simulated data.

Firstly, let us assume that $\mathbf{X}$ has an Archimedean copula structure. The generator of an Archimedean copula depends on the dependence parameter θ of the copula (see, e.g., Table 4.1. in Nelsen (2006)). Consequently, a semi-parametric estimator of the generator is obtained by considering a maximum pseudo-likelihood estimator of the dependence parameter θ associated with this generator. Following these considerations and using Equation (2.9), we introduce a semi-parametric estimator for the multivariate lower-orthant CoVaR (see Definition 2.6.1) by using a semi-parametric estimation for θ and the empirical quantile estimation.

Definition 2.6.1. *Let $\boldsymbol{X}$ be a $d-$dimensional random vector with Archimedean copula with generator ϕ_θ and $\alpha \in (0,1)$. A semi-parametric estimator of the i-th component of the multivariate lower-orthant* CoVaR is defined as

$$\underline{\widehat{\mathrm{CoVaR}}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \widehat{\mathrm{VaR}}_{\omega_i}\left[\hat{F}^{-1}_{X_i}(\phi^{-1}_{\hat{\theta}_n}(S_i\phi_{\hat{\theta}_n}(\alpha)))\right], \quad \text{for } i = 1, \dots, d, \tag{2.17}$$

where $\boldsymbol{\omega} \in [0,1]^d$, S_i is a random variable with $Beta(1, d-1)$ distribution, $\widehat{\mathrm{VaR}}_\omega(X)$ is the empirical estimator of $\mathrm{VaR}_\omega(\mathrm{X})$, $\phi_{\hat{\theta}_n}$ is the semi-parametric estimator of ϕ_θ, and $\hat{F}^{-1}_{X_i}$ is the empirical estimator of $F^{-1}_{X_i}$ for $i = 1, \dots, d$.

Secondly, let us assume that $\mathbf{X}$ has an Archimedean survival copula structure. From Equation (2.11), we introduce a semi-parametric estimation of multivariate upper-orthant CoVaR (see Definition 2.6.2) using the semi-parametric estimation of the generator of the Archimedean survival copula and the empirical estimation of the quantile functions.

Definition 2.6.2. *Let $\boldsymbol{X}$ be a $d-$dimensional random vector with Archimedean survival copula with generator φ_θ and $\alpha \in (0,1)$. A semi-parametric estimator of the i-th component of the multivariate upper-orthant* CoVaR is defined as

$$\overline{\widehat{\mathrm{CoVaR}}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \widehat{\mathrm{VaR}}_{\omega_i}\left[\hat{\overline{F}}^{-1}_{X_i}(\varphi^{-1}_{\hat{\theta}_n}(S_i\varphi_{\hat{\theta}_n}(1-\alpha)))\right], \quad \text{for } i = 1, \dots, d, \tag{2.18}$$

where $\boldsymbol{\omega} \in [0,1]^d$, S_i is a random variable with $Beta(1, d-1)$ distribution, $\widehat{\mathrm{VaR}}_\omega(X)$

is the empirical estimator of $\mathrm{VaR}_\omega(\mathrm{X})$, $\varphi_{\hat{\theta}_n}$ is the semi-parametric estimator of φ_θ, and $\hat{\overline{F}}_{X_i}^{-1}$ is the empirical estimator of $\overline{F}_{X_i}^{-1}$ for $i = 1, \ldots, d$.

The estimator of the dependence parameter θ considered in Definitions 2.6.1 and 2.6.2 is obtained by a pseudo-likelihood estimation procedure. Genest et al. (1995) investigate the properties of the semi-parametric estimator for θ and study the efficiency, consistency, and asymptotic normality of $\hat{\theta}_n$. Proposition 2.1 in Genest et al. (1995) shows that, under regularity conditions, $\hat{\theta}_n$ is consistent and $n^{1/2}(\hat{\theta}_n - \theta)$ converges in distribution to a normal distribution with known variance. The regularity conditions of Proposition 2.1 in Genest et al. (1995) are satisfied, among others, by Archimedean copulas. Therefore, since ϕ_θ (φ_θ, respectively) is a continuous function, $\phi_{\hat{\theta}_n}$ ($\varphi_{\hat{\theta}_n}$, respectively) is consistent from Proposition 2.1 in Genest et al. (1995). On the other hand, empirical quantile estimator $\hat{F}_{X_i}^{-1}(p)$ is consistent if quantile $F_{X_i}^{-1}(p)$ is unique (see Serfling (1980), page 75). The empirical quantile estimator $\hat{F}_{X_i}^{-1}(p)$ is also asymptotically normal if F_{X_i} possesses a left- or right-hand derivative at the point $F_{X_i}^{-1}(p)$ (see Serfling (1980), page 77). However, due to Definitions 2.6.1 and 2.6.2, CoVaR estimators are the quantiles of non-independent observations as we explain in the following. We obtain the generator estimator and the margin quantile estimator by the same data-set and, then, we apply a plug-in procedure where we cannot know how to control the error. In this case, we need a Central Limit Theorem for dependent random variables. Consequently, consistency and asymptotic normal properties of these estimators need a supplementary study, by using the above results in Genest et al. (1995) and in Serfling (1980), which lies beyond the scope of this chapter.

2.7 Simulation study

The aim of this section is to evaluate the performance of the estimators introduced in Definitions 2.6.1 and 2.6.2. In particular, we focus on Definition 2.6.1 (the multivariate upper-orthant CoVaR estimator could similarly be studied). For this purpose, several simulated cases of the bivariate lower-orthant CoVaR estimator are studied. Although we restrict ourselves to the bivariate case, these illustrations could be adaptable in any dimension.

In the following, the ratio $\widehat{\underline{\mathrm{CoVaR}}}^1_{\alpha,\boldsymbol{\omega}}(X,Y)/\underline{\mathrm{CoVaR}}^1_{\alpha,\boldsymbol{\omega}}(X,Y)$ is considered for different values of α and ω and two different sizes of the sample: $n = 600$ (Figures 2.6 and 2.8) and $n = 1000$ (Figures 2.7 and 2.9). We generate our simulated data from the following two models: Ali-Mikhail-Haq copula with $\theta = 0.5$ and uniform marginals (Figures 2.6 and 2.7); and Gumbel copula with $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2 (Figures 2.8 and 2.9, Tables 2.3, 2.4 and 2.5).

We analyse misspecification model errors, in order to study the bias and the variance of the estimation when the parametric form of the copula is inappropriate to the data. To this end we use Clayton, Gumbel and Frank copulas in Figures 2.6 and 2.7; Joe, Clayton and Frank copulas in Figures 2.8 and 2.9. Obviously, the true model is included in the boxplot analysis, that is, Ali-Mikhail-Haq copula in Figures 2.6 and 2.7 and Gumbel copula in Figures 2.8 and 2.9.

In Figures 2.6 and 2.7, boxplots associated with Ali-Mikhail-Haq, Clayton and Frank copulas are similar in terms of bias and variance. Conversely, the Gumbel boxplot is obviously the worst boxplot. This is clearly related to the domain of attraction (in the upper tails) of these copula structures (asymptotically dependent structure for Gumbel copula, asymptotically independent structure for Ali-Mikhail-Haq, Clayton, and Frank copulas, see Remark 2.7.1).

In Figures 2.8 and 2.9, the Gumbel copula is the true (best) model. The Joe copula behaves asymptotically similar to the Gumbel copula. Conversely, the Frank and Clayton copulas are clearly different.

Remark 2.7.1. *Recall that a copula has upper tail dependence if the upper tail dependence parameter λ_U for this copula is in $(0,1]$. If $\lambda_U = 0$, the copula has no upper tail dependence, that is, it is independent in the tail. The Clayton, Frank and Ali-Mikhail-Haq copulas are independent in the tail (i.e., $\lambda_U^{Clayton} = \lambda_U^{Frank} = \lambda_U^{AMH} = 0$). The Gumbel and Joe copulas have upper tail dependence (i.e., $\lambda_U^{Gumbel} = \lambda_U^{Joe} = 2 - 2^{1/\theta}$). For more details see Section 1.4 and Nelsen (2006).*

Finally, for both the Ali-Mikhail-Haq copula with uniform marginals, and the Gumbel copula with Pareto marginals, the larger the sample size n, the better the estimation is.

In the following, we denote $\overline{\underline{\widehat{\mathrm{CoVaR}}}_{\alpha,\omega}}(X,Y) = \big(\overline{\underline{\widehat{\mathrm{CoVaR}}}^{1}_{\alpha,\omega}}(X,Y), \overline{\underline{\widehat{\mathrm{CoVaR}}}^{2}_{\alpha,\omega}}(X,Y)\big)$ as the mean (coordinate by coordinate) of $\underline{\widehat{\mathrm{CoVaR}}}_{\alpha,\omega}(X,Y)$ on M Monte Carlo simulations.

Henceforth, the empirical standard deviation (coordinate by coordinate) is defined as $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2)$ with

$$\hat{\sigma}_1 = \sqrt{\frac{1}{M-1}\sum_{j=1}^{M}\left(\underline{\widehat{\mathrm{CoVaR}}}^{1}_{\alpha,\omega}(X,Y)_j - \overline{\underline{\widehat{\mathrm{CoVaR}}}^{1}_{\alpha,\omega}}(X,Y)\right)^2}.$$

$RMSE = (RMSE_1, RMSE_2)$ corresponds to the relative mean square error (coordinate by coordinate) with

$$RMSE_1 = \sqrt{\frac{1}{M}\sum_{j=1}^{M}\left(\frac{\underline{\widehat{\mathrm{CoVaR}}}^{1}_{\alpha,\omega}(X,Y)_j - \underline{\mathrm{CoVaR}}^{1}_{\alpha,\omega}(X,Y)}{\underline{\mathrm{CoVaR}}^{1}_{\alpha,\omega}(X,Y)}\right)^2},$$

where M is the number of Monte Carlo simulations ($M = 500$ in this section). Similarly, $RMSE_2$ and $\hat{\sigma}_2$ are defined.

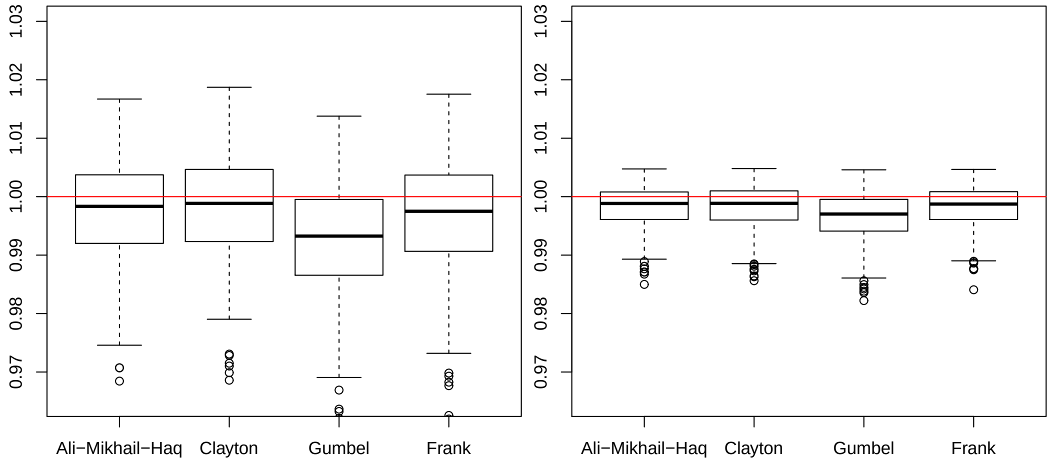

Figure 2.6: (X, Y) follows a bivariate Ali-Mikhail-Haq copula with parameter $\theta = 0.5$ and uniform marginals. Boxplot for the ratio $\widehat{\underline{\mathrm{CoVaR}}}^1_{\alpha,\omega}/\underline{\mathrm{CoVaR}}^1_{\alpha,\omega}$ for $n = 600$ with $\alpha = 0.75$ and $\omega = 0.9$ (left panel); $\alpha = 0.9$, and $\omega = 0.95$ (right panel). Theoretical values are $\underline{\mathrm{CoVaR}}^1_{0.75,0.9} = 0.9698$ and $\underline{\mathrm{CoVaR}}^1_{0.9,0.95} = 0.9946$. We take $M = 500$ Monte Carlo simulations.

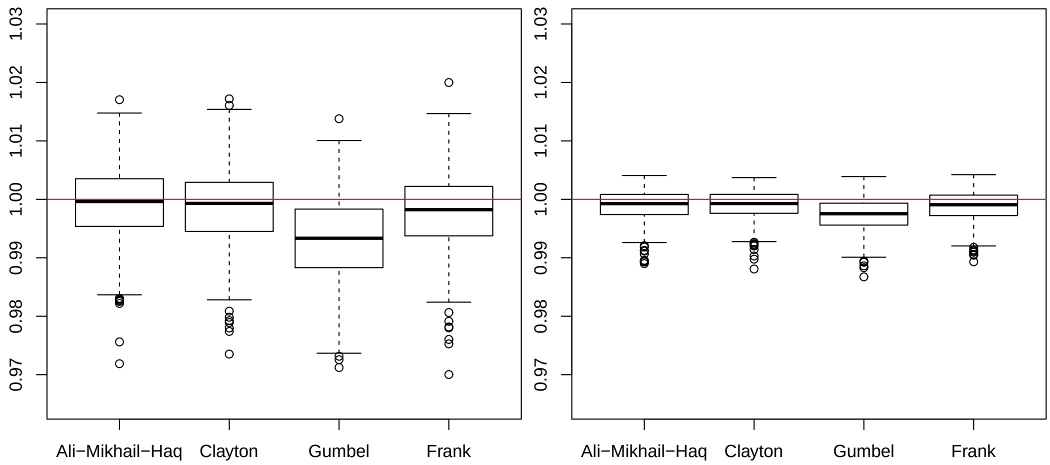

Figure 2.7: (X, Y) follows a bivariate Ali-Mikhail-Haq copula with parameter $\theta = 0.5$ and uniform marginals. Boxplot for the ratio $\widehat{\underline{\mathrm{CoVaR}}}^1_{\alpha,\omega}/\underline{\mathrm{CoVaR}}^1_{\alpha,\omega}$, for $n = 1000$ with $\alpha = 0.75$ and $\omega = 0.9$ (left panel); $\alpha = 0.9$ and $\omega = 0.95$ (right panel). Theoretical values are $\underline{\mathrm{CoVaR}}^1_{0.75,0.9} = 0.9698$ and $\underline{\mathrm{CoVaR}}^1_{0.9,0.95} = 0.9946$. We take $M = 500$ Monte Carlo simulations.

$RMES_1$ and $\hat{\sigma}_1$ are shown in Table 2.3 (Table 2.4, respectively) in terms of ω (α, respectively) with $\alpha = 0.7$ fixed ($\omega = 0.75$ fixed, respectively) for the Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape

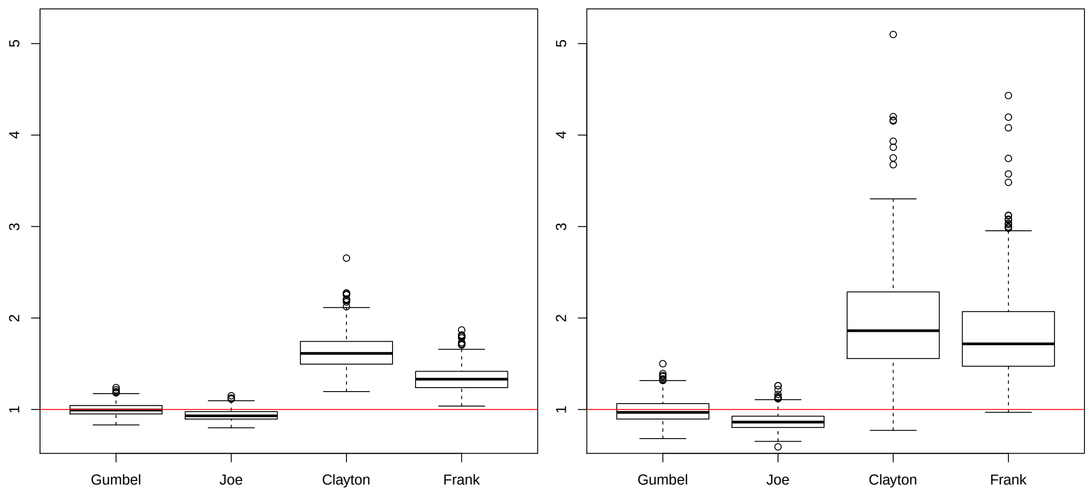

Figure 2.8: (X, Y) follows a bivariate Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2. Boxplot for the ratio $\underline{\widehat{\mathrm{CoVaR}}}^1_{\alpha,\omega}/\underline{\mathrm{CoVaR}}^1_{\alpha,\omega}$ for $n = 600$ with $\alpha = 0.75$ and $\omega = 0.9$ (left panel); $\alpha = 0.9$ and $\omega = 0.95$ (right panel). Theoretical values are $\underline{\mathrm{CoVaR}}^1_{0.75,0.9} = 3.3911$ and $\underline{\mathrm{CoVaR}}^1_{0.9,0.95} = 6.5535$. We take $M = 500$ Monte Carlo simulations.

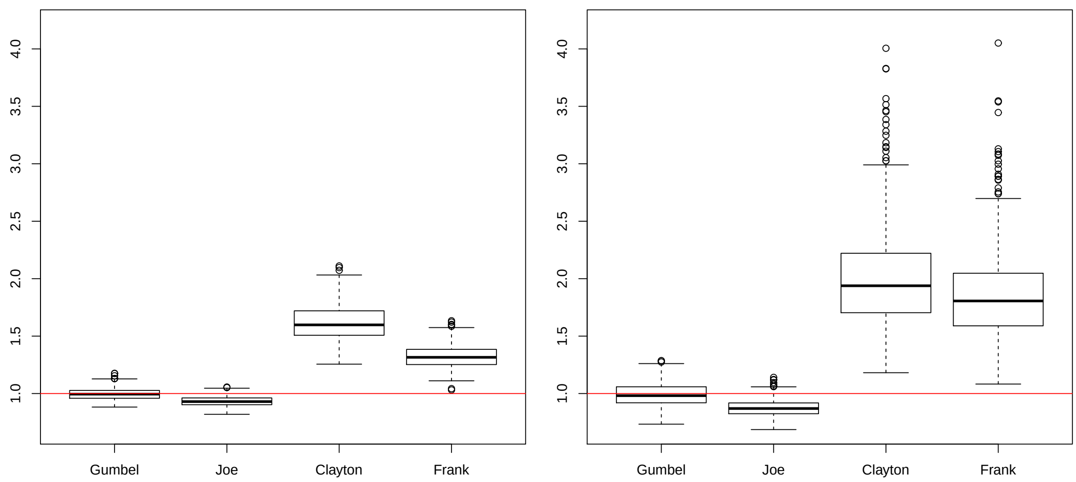

Figure 2.9: (X, Y) follows a bivariate Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2. Boxplot for the ratio $\underline{\widehat{\mathrm{CoVaR}}}^1_{\alpha,\omega}/\underline{\mathrm{CoVaR}}^1_{\alpha,\omega}$ for $n = 1000$ with $\alpha = 0.75$ and $\omega = 0.9$ (left panel); $\alpha = 0.9$ and $\omega = 0.95$ (right panel). Theoretical values are $\underline{\mathrm{CoVaR}}^1_{0.75,0.9} = 3.3911$ and $\underline{\mathrm{CoVaR}}^1_{0.9,0.95} = 6.5535$. We take $M = 500$ Monte Carlo simulations.

parameter 2. It can be observed that, the more α and ω increase, the more $RMES_1$ and $\hat{\sigma}_1$ increase. Similarly, for the Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2, $RMES_1$ and $\hat{\sigma}_1$ in terms of the sample size n for $\alpha = 0.9$ and $\omega = 0.98$ fixed are given in Table 2.5. As expected, $RMES_1$ and $\hat{\sigma}_1$ decrease when the sample size increases.

ω	Gumbel	Joe	Clayton	Frank
0.70	0.034 (0.081)	0.039 (0.078)	0.253 (0.145)	0.095 (0.098)
0.75	0.037 (0.091)	0.044 (0.084)	0.304 (0.175)	0.116 (0.122)
0.80	0.037 (0.096)	0.049 (0.088)	0.369 (0.210)	0.144 (0.134)
0.86	0.043 (0.122)	0.060 (0.111)	0.491 (0.333)	0.211 (0.207)
0.90	0.046 (0.140)	0.069 (0.122)	0.618 (0.437)	0.280 (0.264)
0.95	0.059 (0.212)	0.097 (0.173)	0.914 (0.864)	0.464 (0.503)
0.98	0.080 (0.360)	0.135 (0.281)	1.394 (2.040)	0.812 (1.248)
0.99	0.106 (0.559)	0.169 (0.415)	1.911 (3.900)	1.188 (2.831)

Table 2.3: (X, Y) follows a bivariate Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2. Evolution of $RMSE_1$ and $\hat{\sigma}_1$ (in parentheses) in terms of ω for $\alpha = 0.7$ fixed. We take 500 Monte Carlo simulations.

α	Gumbel	Joe	Clayton	Frank
0.75	0.038 (0.105)	0.049 (0.094)	0.324 (0.219)	0.153 (0.154)
0.80	0.045 (0.138)	0.059 (0.126)	0.338 (0.286)	0.190 (0.212)
0.85	0.051 (0.183)	0.065 (0.161)	0.372 (0.376)	0.251 (0.320)
0.90	0.068 (0.298)	0.079 (0.259)	0.403 (0.591)	0.315 (0.510)
0.95	0.086 (0.538)	0.099 (0.471)	0.428 (1.172)	0.380 (1.094)
0.98	0.149 (1.477)	0.152 (1.315)	0.458 (2.893)	0.442 (2.839)
0.99	0.211 (2.985)	0.197 (2.625)	0.575 (6.076)	0.582 (6.234)

Table 2.4: (X, Y) follows a bivariate Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2. Evolution of $RMSE_1$ and $\hat{\sigma}_1$ (in parentheses) in terms of α for $\omega = 0.75$ fixed. We take 500 Monte Carlo simulations.

n	Gumbel	Joe	Clayton	Frank
500	0.196 (1.590)	0.215 (1.097)	1.862 (9.903)	1.636 (9.030)
1000	0.137 (1.124)	0.187 (0.754)	1.732 (7.838)	1.447 (6.210)
1500	0.108 (0.885)	0.173 (0.576)	1.643 (5.754)	1.402 (4.827)
2000	0.103 (0.843)	0.171 (0.560)	1.651 (5.177)	1.388 (4.203)
2500	0.084 (0.684)	0.170 (0.472)	1.622 (4.610)	1.369 (3.918)
3000	0.074 (0.611)	0.163 (0.412)	1.564 (3.932)	1.363 (3.479)
5000	0.059 (0.482)	0.162 (0.360)	1.599 (3.392)	1.356 (2.741)

Table 2.5: (X, Y) follows a bivariate Gumbel copula with parameter $\theta = 2$ and Pareto marginals with location parameter 1 and shape parameter 2. Evolution of $RMSE_1$ and $\hat{\sigma}_1$ (in parenthesis) in terms of the size of the sample n for $\alpha = 0.9$, $\omega = 0.98$ fixed. We take 500 Monte Carlo simulations.

2.8 Application in Loss-ALAE data-set

The estimators of the multivariate CoVaR measures proposed in Definitions 2.2.1 and 2.2.2 are now calculated in a real insurance case: **Loss-ALAE data** (in the log scale). The considered data-set contains $n = 1500$ observations. Each claim is composed of an indemnity payment (the loss, X) and an allocated loss-adjustment expense (ALAE, Y). These ALAEs are insurance company expenses similar to the fees paid to lawyers and other experts to defend claims. This data-set is studied in-depth in Frees and Valdez (1998).

In Table 2.6 and 2.7, the $\widehat{\underline{\mathrm{CoVaR}}}_{\alpha,\omega}(X,Y)$ and $\widehat{\overline{\mathrm{CoVaR}}}_{\alpha,\omega}(X,Y)$ for Loss ALAE data are presented by considering different risk levels α, ω and different Archimedean copula models C. Frees and Valdez (1998), using the AIC criterion, proposed a Gumbel copula for Loss-ALAE data a Gumbel copula with parameter $\hat{\theta} = 1.453$. In Table 2.6, we provide the $\widehat{\underline{\mathrm{CoVaR}}}_{\alpha,\omega}(X,Y)$ estimators for Loss-ALAE data using the Gumbel model by Frees and Valdez (1998) (bold column). Furthermore, $\widehat{\underline{\mathrm{CoVaR}}}_{\alpha,\omega}(X,Y)$ from other Archimedean models are displayed in Table 2.6. Estimated parameters θ are obtained using the `R` function `fitCopula`. Analogously, for the survival structure of Loss-ALAE data, the Ali-Mikhail-Haq copula with parameter $\hat{\theta} = 0.96$ is chosen. Hence, the $\widehat{\overline{\mathrm{CoVaR}}}_{\alpha,\omega}(X,Y)$ using Ali-Mikhail-Haq copula (bold column) and some other Archimedean models are collected in Table 2.7.

(α,ω)	Clayton (0.51)	Frank (3.07)	Ali-Mikhail-Haq (0.79)	Gumbel (1.453)	Joe (1.64)
(0.75, 0.90)	(12.42, 10.96)	(12.43, 10.95)	(12.48, 11.01)	(**11.92**, **10.61**)	(11.84, 10.53)
(0.90, 0.95)	(13.12, 11.94)	(13.13, 11.99)	(13.13, 11.98)	(**12.95**, **11.50**)	(12.82, 11.27)
(0.95, 0.98)	(13.81, 12.82)	(13.82, 12.78)	(13.81, 12.94)	(**13.56**, **12.17**)	(13.12, 12.07)

Table 2.6: Coordinates of risk measure $\widehat{\underline{\mathrm{CoVaR}}}_{\alpha,\omega}$ for Loss ALAE data, using different copula structures and risk levels (α,ω). The $\hat{\theta}$ for each copula is in parentheses.

(α,ω)	Clayton (0.78)	Frank (3.07)	Ali-Mikhail-Haq (0.96)	Gumbel (1.37)	Joe (1.39)
(0.75, 0.90)	(10.31, 9.37)	(10.31, 9.38)	(**10.31**, **9.36**)	(10.31, 9.34)	(10.31, 9.33)
(0.90, 0.95)	(11.48, 10.14)	(11.44, 10.13)	(**11.46**, **10.14**)	(11.43, 10.13)	(11.41, 10.13)
(0.95, 0.98)	(12.03, 10.72)	(11.99, 10.69)	(**12.03**, **10.72**)	(12.00, 10.69)	(12.00, 10.70)

Table 2.7: Coordinates of risk measure $\widehat{\overline{\mathrm{CoVaR}}}_{\alpha,\omega}$ for Loss ALAE data, using different survival copula structures and risk levels (α,ω). The $\hat{\theta}$ for each copula is in parentheses.

We remark that Loss-ALAE data-set contains repeated values. This fact might adversely affect the proper identification of the model of this data-set (see Pappadà et al. (2016a)). The deeply study of the model of this data-set would be a future work that lies beyond the scope of the present chapter.

2.9 Conclusions

In this chapter, two multivariate extensions of the classic CoVaR are provided for continuous random vectors. These two risk measures are constructed by using the level set approach used in Embrechts and Puccetti (2006), Cousin and Di Bernardino (2013) and Cousin and Di Bernardino (2014). Since the defined CoVaR are the minimizers of suitable expected losses (see (P7)), then they verify the *elicitability property*, which provides a natural methodology to perform backtesting. Moreover, since the two proposed measures are based on the corresponding quantile functions, they are more robust to extreme values than any other central tendency measures.

The positive homogeneity and translation invariance properties are shown for the two proposed multivariate CoVaRs. The relations between the univariate VaR and our CoVaR are also analysed as well as the relations between the multivariate VaR proposed by Cousin and Di Bernardino (2013) and our multivariate CoVaR. Interestingly enough, both multivariate CoVaRs coincide with the univariate VaR when a comonotonic random vector is considered, and they verify the additivity property under π-comonotonic conditions. The behaviour of the multivariate CoVaR with respect to the risk level, the usual stochastic order of marginal distributions, and the dependence structure are studied. Unsurprisingly, the effect in the multivariate lower-orthant CoVaR (upper-orthant CoVaR, respectively) with respect to a change in the risk level, a change in the dependence structure, or the usual stochastic order of marginal distributions, tends to be the same as for the multivariate lower-orthant VaR (upper-orthant VaR, respectively) proposed in Cousin and Di Bernardino (2013). Important results and analytical expressions for our multivariate risk measures are obtained for random vectors with Archimedean copulas. In particular, certain subadditivity inequality is presented in the Archimedean case under regular variation conditions. A systemic risk measure based on the multivariate ΔCoVaR is illustrated. Moreover, under Archimedean copula conditions, estimators of the two proposed multivariate CoVaRs are provided for simulated data and real insurance data. As we point out in Section 2.6, consistency and asymptotic normal properties of these estimators need a supplementary study that constitutes a future line to develop.

As a future perspective, the evaluation of the proposed measures in certain multidimensional portfolios and the comparison between the results for these measures and the results for multivariate existent measures could be studied (see Cousin and Di Bernardino (2013), Cousin and Di Bernardino (2014), and Cai and Li (2005)). Moreover, a deep study of the subadditivity property for the two multivariate CoVaRs and of our proposed ΔCoVaR measure could be done. It would also be interesting to develop an estimation procedure for ΔCoVaR.

Chapter 3

Non-parametric extreme estimation of Multivariate CoVaR

3.1 Introduction

In this chapter, we consider the multivariate return level based on the critical layers as proposed in Definition 2.2.1 in Chapter 1. This risk measure is defined by the $(1-p)$-quantile of the random variable $T_i := [X_i | \mathbf{X} \in \partial \underline{L}(\alpha)]$ in (2.6), that is,

$$\underline{\mathrm{CoVaR}}^i_{\alpha,1-\boldsymbol{\omega}}(\mathbf{X}) := U_{T_i}\left(\frac{1}{p}\right), \quad \text{for } p \in (0,1), \tag{3.1}$$

where $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{i-1}, p, \omega_{i+1}, \dots, \omega_d)$ is a marginal risk vector with $\omega_i, p \in [0,1]$, $U_{T_i}(t) := F^{-1}_{T_i}(1 - \frac{1}{t} \,|\, \alpha)$, for $t > 1$ (see (1.5)), and $F^{-1}_{T_i}(\cdot \,|\, \alpha)$ denotes the left-continuous inverse of $F_{T_i}(\cdot \,|\, \alpha)$ defined in Lemma 2.2.1. From now on, $\underline{\mathrm{CoVaR}}^i_{\alpha,1-\boldsymbol{\omega}}(\mathbf{X})$ in Equation (3.1) is denoted by x^i_p. The goal of this chapter is to estimate the multivariate extreme return levels $x^i_{p_n}$. To this end, two problematic points can be identified:

i) In Chapter 2, we analyse this measure and introduce a semi-parametric estimation procedure. However, the aforementioned semi-parametric estimation (see Definition 2.6.1) and empirical quantile estimators perform well only if the threshold is not too high. These methods cannot handle extreme events, that is, $p_n \ll 1/n$, which are specifically required for hydrological and environmental risk measures.

ii) The considered conditional random variable T_i relies on the latent Multivariate Probability Integral Transformation (MPIT) $Z := F(X_1, \dots, X_d)$ (see (2.6)), which is not observed. Therefore, in order to apply a quantile estimation procedure, Z has to be previously estimated. This type of plug-in procedure increases the variance of the final estimation and introduces statistical difficulties (see, e.g., Di Bernardino et al. (2011)).

In order to overcome the drawback outlined in item i), in the present chapter we provide an estimator of $x^i_{p_n}$, for a fixed α and when $p_n \to 0$, as $n \to +\infty$, by using Extreme Value Theory (EVT). For the dependence structure of the multivariate risk vector $\mathbf{X}$, we consider Archimedean copulas. The rationale for employing Archimedean copulas is motivated by the fact that, under this assumption, the distribution of T_i and its tail index can be easily obtained (see Proposition 3.2.1). Frequently, hydrological phenomena are characterized by upper tail dependence as described by Gumbel-Logistic models (e.g., see Fawcett and Walshaw (2012), Chebana and Ouarda (2011a), de Waal et al. (2007)). Furthermore, in this framework, one can avoid having to previously estimate the latent variable Z (see item ii)). Indeed, the proposed estimator procedure is only based on quantities that can be directly estimated by using the observed d-dimensional independent and identically distributed (iid) sample of $(\mathbf{X}_j)$, for $j = 1, \dots, n$ (see Equation (3.10)).

Following these considerations, under a regular variation condition for the generator ϕ and the von Mises condition for the marginal X_i, we develop an extreme extrapolation technique in order to estimate $x^i_{p_n}$ (see, e.g., Cai et al. (2015)).

This chapter is organized as follows. In Section 3.2, we derive the tail index of T_i. Under suitable assumptions, a non-parametric estimation procedure for $x^i_{p_n}$ is obtained when $p_n \to 0$ as $n \to +\infty$ for a fixed α (Section 3.3). The main result is the asymptotic convergence of our estimator with $p_n \to 0$, as $n \to +\infty$ (Section 3.4). In Section 3.5, the performance of the proposed extreme estimator is illustrated on simulated data. An adaptive version of the proposed extreme estimator is provided in Section 3.6. Finally, Section 3.7 concludes with an application to a 3−dimensional rainfall data-set in order to illustrate how the proposed estimation procedure can help in the evaluation of multivariate extreme return levels. The conclusions are provided in Section 3.8.

3.2 Study of T_i tail index

In this section, we aim to study the tail behaviour of T_i, for $i = 1, \dots, d$. We assume the existence of the limit in $[1, \infty]$ of

$$\rho = -\lim_{s \uparrow 1} \frac{(1-s)\phi'(s)}{\phi(s)}. \tag{3.2}$$

Equation (3.2) is equivalent to a regular variation of ϕ at 1 with index ρ, that is, $\phi \in RV_\rho(1)$ (see Charpentier and Segers (2009) for details). Furthermore, $\rho \geq 1$ due to the convexity of ϕ. When $\rho > 1$, the upper tail of the copula exhibits asymptotic dependence, while if $\rho = 1$, then the upper tail exhibits asymptotic independence. Under condition (3.2) for the generator ϕ, we now study the maximum domain of attraction (see Section 1.2) of T_i, for $i = 1, \dots, d$.

Proposition 3.2.1 (The von Mises condition for T_i). *Let $(X_1, \dots, X_d)$ be a random vector with Archimedean copula with twice differentiable generator ϕ. Assume that $\phi \in RV_\rho(1)$, with $\rho \in [1, +\infty]$. Let $i \in \{1, \dots, d\}$ and F_{X_i} be the twice differentiable distribution function of X_i. Assume that F_{X_i} verifies the von Mises condition with index $\gamma_i \in \mathbb{R}$. Denote $T_i := [X_i | \boldsymbol{X} \in \partial \underline{L}(\alpha)]$ with distribution function $F_{T_i}(\cdot|\alpha)$ given by Lemma 2.2.1.*

i) If $\rho \in [1, +\infty)$, then $F_{T_i}(\cdot|\alpha)$ verifies the von Mises condition with tail index $\gamma^{T_i} = \frac{\gamma_i}{\rho}$. Specifically, $T_i \in MDA\left(\gamma^{T_i}\right)$.

ii) If $\rho = +\infty$, then $F_{T_i}(\cdot|\alpha)$ verifies the von Mises condition with tail index $\gamma^{T_i} = 0$. In particular, $T_i \in MDA\,(0)$.

Proof. We first prove item i). Let $x_{F_{T_i}}(\alpha)$ be the right endpoint of F_{T_i}. Since, by assumptions, $\phi \in RV_\rho(1)$, $\phi' \in RV_{\rho-1}(1)$ and F_{X_i} verifies the von Mises condition with index γ_i (see Definition 1.2.2), therefore

$$
\begin{aligned}
\lim_{x \uparrow x_{F_{T_i}}(\alpha)} \frac{(1 - F_{T_i}(x|\alpha)) F''_{T_i}(x|\alpha)}{(F'_{T_i})^2(x|\alpha)} &= \lim_{z \uparrow 1} \frac{d-2}{d-1} \left[\left(1 - \frac{\phi(z)}{\phi(\alpha)} \right)^{-(d-1)} - 1 \right] \\
&+ \frac{\phi(\alpha) \left[(-\rho + 1) - (\gamma_i + 1) \right]}{(d-1) \phi'(z)(1-z)} \left[- \left(1 - \frac{\phi(z)}{\phi(\alpha)} \right)^{2-d} + 1 \right] \\
&- \left(-\frac{1}{\rho} \right) \frac{1}{d-1} \left[(-\rho + 1) - (\gamma_i + 1) \right].
\end{aligned}
$$

Since $\phi(1) = 0$, the first summand approaches 0 when z approaches 1. We denote $C = \left(-\frac{1}{\rho} \right) \frac{[(-\rho+1)-(\gamma_i+1)]}{d-1}$. For the second summand it is verified that:

$$
\lim_{z \uparrow 1} \frac{\phi(\alpha) C}{\phi(z)} \left[- \left(1 - \frac{\phi(z)}{\phi(\alpha)} \right)^{2-d} + 1 \right] = \frac{2\rho + 2\gamma_i - d\rho - d\gamma_i}{\rho(d-1)}.
$$

Hence,

$$
\lim_{x \uparrow x_{F_{T_i}}(\alpha)} \frac{(1 - F_{T_i}(x|\alpha)) F''_{T_i}(x|\alpha)}{(F'_{T_i})^2(x|\alpha)} = - \left(\frac{\gamma_i}{\rho} + 1 \right).
$$

The random variable T_i therefore verifies the von Mises condition with $\gamma^{T_i} = \frac{\gamma_i}{\rho}$. Similar to the proof of item i), the von Mises condition for T_i when $\rho = +\infty$ is satisfied with $\gamma^{T_i} = 0$. Therefore item ii) is also proved. From Theorem 1.2.2, other assertions of Proposition 3.2.1 are shown directly. □

Remark 3.2.1. *Note that γ^{T_i} does not depend on the risk level α nor on the dimension d. However, γ^{T_i} depends on the domain of attraction of the respective margin X_i and on the regularly varying index ρ of the generator of the Archimedean copula considered. It should be borne in mind that assumptions of Proposition 3.2.1 can be easily satisfied* *(see example 3.2.1). In Table 1 in Charpentier and Segers (2009), various copula models with associated ρ index can be found. Furthermore, the classic von Mises condition is verified for a large class of unidimensional marginal distributions F_{X_i}.*

Illustrations of Proposition 3.2.1 are given in Example 3.2.1.

Example 3.2.1. *Certain tail indexes for T_i are now derived. The ρ indexes for the classic bivariate Archimedean copulas are collected in Table 1 in Charpentier and Segers (2009). From this table and from Proposition 3.2.1, Table 3.1 below is constructed. Table 3.1 contains the tail index γ^{T_i} of T_i when X_i is in the Weibull domain (i.e., $\gamma_i < 0$), Gumbel domain (i.e., $\gamma_i = 0$), and Fréchet domain (i.e., $\gamma_i > 0$), for different values of ρ. In Table 3.2, certain specific models are considered.*

ρ	$\gamma_i < 0$	$\gamma_i = 0$	$\gamma_i > 0$
$(1, +\infty)$	γ_i/ρ	0	γ_i/ρ
1	γ_i	0	γ_i
$+\infty$	0	0	0

Table 3.1: The tail index γ^{T_i} when $(X_1, \dots, X_d)$ follows an Archimedean copula with $\phi \in RV_\rho(1)$ and F_{X_i} verifies the von Mises condition with index γ_i.

Copula	$U(0,1)$	$Exp(\lambda)$	$Par(\delta, 1)$
Gumbel	$-1/\theta$	0	$1/\delta\theta$
Ali-Mikhail-Haq	-1	0	$1/\delta$
Copula 18 in Table 4.1 (Nelsen (2006))	0	0	0

Table 3.2: The tail index γ^{T_i} for certain specific models.

Remark 3.2.2. *Let $\boldsymbol{X}$ be a d-dimensional random vector with survival distribution function $\overline{F}$ and survival Archimedean copula $\hat{C}$ with generator φ, $i = 1, \dots, d$. We can now consider the conditional distribution of*

$$T_i' := [X_i \,|\, \boldsymbol{X} \in \partial\overline{L}(\alpha)], \quad \text{for } \alpha \in (0,1). \tag{3.3}$$

Let $\boldsymbol{V} = (V_1, \dots, V_d)$ be the vector whose components V_i represent the survival margins of $\boldsymbol{X}$. By taking into account the above notation and the expression in Equation (1.8), we obtain

$$[\boldsymbol{V}|\hat{C}(\boldsymbol{V}) = 1-\alpha] \stackrel{d}{=} (\varphi^{-1}(S_1\varphi(1-\alpha)), \dots, \varphi^{-1}(S_d\varphi(1-\alpha))), \tag{3.4}$$

where $\boldsymbol{S} = (S_1, \dots, S_d)$ is uniformly distributed on the unit simplex. From Equation (3.4), we adapt the discussion in Remark 3.8 in Brechmann (2014) to this case and we *obtain that the survival distribution of T_i' is given by*

$$\overline{F}_{T_i'}(x|\alpha) = \begin{cases} \left(1 - \frac{\varphi(\overline{F}_{X_i}(x))}{\varphi(1-\alpha)}\right)^{d-1}, & \text{if } x \leq Q_i(\alpha); \\ 0, & \text{if } x > Q_i(\alpha), \end{cases}$$

where $\overline{F}_{X_i}$ is the survival marginal distribution of X_i and $Q_i(\alpha)$ is the associated quantile function at level $\alpha \in (0,1)$. The von Mises condition in Definition 1.2.2 is now studied for the survival distribution function of T_i'. Furthermore, since we are interested in the behaviour of the distribution tail when the distribution tends to the maximum probability 1, we study the left-tail of the survival distribution.

By assuming that φ is twice differentiable, $\varphi \in RV_\rho(1)$ with $\rho' \in [1, +\infty]$, and that $\overline{F}_{X_i}$ verifies the von Mises condition with index $\gamma_i' \in \mathbb{R}$, it holds that:

i) If $\rho' \in [1, +\infty)$, then $\overline{F}_{T_i'}(\cdot|\alpha)$ verifies the von Mises condition with tail index $\gamma^{T_i'} = \frac{\gamma_i'}{\rho'}$;

ii) If $\rho' = +\infty$, then $\overline{F}_{T_i'}(\cdot|\alpha)$ verifies the von Mises condition with tail index $\gamma^{T_i'} = 0$.

The $(1-p)$-quantile of T_i' represents the multivariate upper-orthant CoVaR *defined in Definition 2.2.2 in Chapter 1. By using similar arguments to that of this chapter and by considering this remark, we can also develop an extreme estimation procedure for the multivariate upper-orthant* CoVaR.

The relationship between the quantile functions U_{T_i} and U_{X_i} is established in the following result.

Proposition 3.2.2 (Relation between U_{T_i} and U_{X_i}). *Let $(X_1, \ldots, X_d)$ be a random vector with Archimedean copula with generator ϕ. Assume that $\phi \in RV_\rho(1)$, with $\rho \in [1, +\infty]$. Denote $T_i := [X_i | \boldsymbol{X} \in \partial \underline{L}(\alpha)]$ with distribution function $F_{T_i}(\cdot|\alpha)$ given by Lemma 2.2.1. Let $k = k(n) \to \infty$, $k/n \to 0$, as $n \to \infty$, and*

$$k_U(n) := n \left\{ 1 - \phi^{-1} \left[\left(1 - \left(1 - \frac{k(n)}{n} \right)^{1/(d-1)} \right) \phi(\alpha) \right] \right\}. \tag{3.5}$$

Therefore,
i) $k_U(n)$ is an intermediate sequence, that is, $k_U(n) \to \infty$, $k_U/n \to 0$ as $n \to \infty$.
ii) $U_{T_i}\left(\frac{n}{k}\right) = U_{X_i}\left(\frac{n}{k_U}\right)$, where U_{X_i} is the marginal quantile function of X_i (see (1.5)).

Proof. For item i), since $k(n)/n \to 0$, as $n \to \infty$, and $\phi^{-1}(0) = 1$, $k_U/n \to 0$ holds, as $n \to \infty$. Furthermore, we have the following asymptotic approximation

$$k_U(n) \sim n \left(\phi(\alpha)(d-1)\right)^{1/\rho} \left(\frac{k(n)}{n}\right)^{1/\rho}, \tag{3.6}$$

as $n \to +\infty$. From Equation (3.6), it holds that $k(n)/k_U(n) \to 0$, as $n \to \infty$, for $\rho \in (1, +\infty]$. Therefore, $k_U(n) \to +\infty$ as $n \to +\infty$.
Since $U_{X_i}(t) = F_{T_i}^{\leftarrow}(1 - 1/t)$ and using Lemma 2.2.1,

$$U_{T_i}\left(\frac{n}{k(n)}\right) = U_{X_i}\left(\frac{1}{1 - \phi^{-1}\left[\left(1 - (1 - k(n)/n)^{1/(d-1)}\right)\phi(\alpha)\right]}\right).$$

Therefore,

$$U_{T_i}\left(\frac{n}{k(n)}\right) = U_{X_i}\left(\frac{n}{k_U(n)}\right)$$

where $k_U(n) = n\left\{1 - \phi^{-1}\left[\left(1 - \left(1 - \frac{k(n)}{n}\right)^{1/(d-1)}\right)\phi(\alpha)\right]\right\}$. Consequently, item *ii)* of Proposition 3.2.2 is also proved. □

3.3 Proposed non-parametric estimator for multivariate CoVaR

Henceforth, we will focus on the case: $\gamma_i > 0$ and $\rho \in [1, +\infty)$ (in particular, this implies $\gamma^{T_i} > 0$). This choice is motivated by our applications in hydrology and especially in those of real rainfall data-sets. Indeed, in these real-life applications, we can easily observe heavy tailed distributions (see, for instance, Pavlopoulos et al. (2008) and Papalexiou et al. (2013)). Salvadori and De Michele (2001) also show some particular scaling features of extreme value distributions for rainfall data. For the marginal distribution X_i, we therefore assume that there exists $\gamma_i > 0$ such that, for all $x > 0$, $U_{X_i} \in RV_{\gamma_i}$.

In this case, Propositions 3.2.1 and 3.2.2 yield, as $n \to \infty$,

$$x^i_{p_n} = U_{T_i}\left(\frac{1}{p_n}\right) \sim U_{X_i}\left(\frac{n}{k_U}\right)\left(\frac{k}{n\,p_n}\right)^{\gamma_i/\rho} = U_{X_i}\left(\frac{n}{k_U}\right)\left(\frac{k}{n\,p_n}\right)^{\gamma^{T_i}}, \tag{3.7}$$

where k_U is the same as in Equation (3.5), and $k = k(n) \to \infty$, $k(n)/n \to 0$, as $n \to \infty$.

Let $(X_1, \ldots, X_d)$ be a d-dimensional random vector with continuous distribution function F and Archimedean copula with generator ϕ. The goal is to estimate $x^i_{p_n}$ in (3.7) based on d-dimensional *iid* observations, $(\mathbf{X}_j)$, for $j = 1, \ldots, n$, from F, where $p_n \to 0$, as $n \to +\infty$. Let $X^i_{n-\lfloor k_U \rfloor, n}$ be the $(n - \lfloor k_U \rfloor)$-th order statistic of $(X^i_1, \ldots, X^i_n)$. Therefore, the natural estimator of $U_{X_i}\left(\frac{n}{k_U}\right)$ is its empirical counterpart, that is, $X^i_{n-\lfloor k_U \rfloor, n}$ (e.g., see de Haan and Ferreira (2006)).

From Equation (3.7), in order to define the estimator of $x^i_{p_n}$, it thus remains to estimate γ_i and ρ. We estimate γ_i with the Hill estimator (see Hill (1975)):

$$\widehat{\gamma}_i = \frac{1}{k_1} \sum_{j=0}^{k_1-1} \log X^i_{n-j,n} - \log X^i_{n-k_1,n}, \tag{3.8}$$

where k_1 is an integer sequence such that $k_1(n) \to \infty$, $k_1/n \to 0$, $n \to \infty$, and such that $X^i_{n-k_1,n}$ is the intermediate order statistic at level $n - k_1$. We now deal with the estimation of the regularly varying index ρ of the Archimedean generator ϕ. From Charpentier and Segers (2008), if the distribution function of a random vector $\mathbf{X}$ is given by a $d-$dimensional Archimedean copula C with generator ϕ, then the distribution function of every bivariate subvector (X_i, X_j) is given by the bivariate Archimedean copula with the same generator. As a consequence, to estimate ρ, we focus on the bivariate subvector (X_i, X_j). Furthermore, under our assumption, one can prove that the upper tail dependence coefficient in Equation (1.10) is given by $\lambda_U = 2 - 2^{1/\rho}$ (e.g., see Corollary 2.1. in Di Bernardino and Rullière (2014)). Therefore, we use the following estimator of ρ:

$$\widehat{\rho} := \frac{\log(2)}{\log(2 - \widehat{\lambda}_U)}, \tag{3.9}$$

where $\widehat{\lambda}_U$ is the estimator of the *upper tail dependence coefficient* proposed by Schmidt and Stadtmüller (2006) (see Equation (1.11)).

Let $\widehat{\gamma}^{T_i} := \frac{\widehat{\gamma}_i}{\widehat{\rho}}$. We can therefore estimate $x^i_{p_n}$ in (3.7) by

$$\widehat{x}^i_{p_n} = X^i_{n-\lfloor k_U \rfloor, n} \left(\frac{k}{n\, p_n} \right)^{\widehat{\gamma}^{T_i}}. \tag{3.10}$$

Remark 3.3.1. *Notice that the proposed estimator in Equation* (3.10) *does not rely on the latent MPIT $Z := F(X_1, \ldots, X_d)$, which is not directly observed. Under assumptions of Proposition 3.2.1, the application of the proposed extrapolation technique precludes the necessity to previously estimate Z. Indeed, the estimator $\widehat{x}^i_{p_n}$ in Equation* (3.10) *is only based on quantities that can be directly estimated by using the observed d-dimensional* iid *sample $(\boldsymbol{X}_j)$, for $j = 1, \ldots, n$. In Section 3.5, we provide a comparison with an empirical quantile estimation of T_i constructed by using the empirical multivariate distribution function $F_n(\boldsymbol{X}_j)$ (see Equation* (3.20)*).*

3.4 Asymptotic convergence

3.4.1 Preliminary Results

In order to prove asymptotic normality of $\widehat{\gamma}^{T_i}$, we need to quantify the rates of convergence defined due to $U_{X_i} \in RV_{\gamma_i}$, $\gamma_i > 0$. We therefore assume the following second-order strengthening of the above regularity condition: $U_{X_i} \in 2RV_{\gamma_i, \tau_i}(A_i)$, $\gamma_i > 0$ and

$\tau_i < 0$ (see Appendix A for the definition of 2RV).

Corollary 3.4.1 and Theorem 3.4.1 below are crucial in the proof of our main result Theorem 3.4.3. Under a second-order condition for the bivariate *upper tail copula* $\Lambda_U(x, y)$ (see the condition in (3.11)), we can obtain an asymptotic normality result for the estimator $\widehat{\rho}$ (see Corollary 3.4.1 below). The proof of Corollary 3.4.1 follows from Corollary 2 in Schmidt and Stadtmüller (2006) and the Delta Method technique.

Corollary 3.4.1 (Asymptotic normality of $\widehat{\rho}$)**.** *Let F be a bivariate distribution function of (X_i, X_j) with continuous marginal distribution functions F_{X_i} and F_{X_j}. Let C be the Archimedean copula of (X_i, X_j) with generator $\phi \in RV_\rho(1)$, with $\rho \in (1, +\infty)$. Let $k_2 = k_2(n) \to \infty$ and $k_2/n \to 0$ as $n \to \infty$. Assume that the bivariate upper tail copula $\Lambda_U(x, y)$ exists and has continuous partial derivatives. Furthermore, let $A_\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be an auxiliary function such that $A_\rho(t) \to 0$ as $t \to \infty$ and*

$$\lim_{t\to\infty} \frac{\Lambda_U(x, y) - t\,\hat{C}(x/t,\, y/t)}{A_\rho(t)} = g(x,\, y) < \infty, \tag{3.11}$$

locally uniformly for $(x, y)^2 \in \overline{\mathbb{R}}^2_+ := [0, \infty]^2 \backslash \{(\infty, \infty)\}$ for some non-constant function g, where $\hat{C}$ represents the survival copula. Therefore, if $\sqrt{k_2} A_\rho(n/k_2) \to 0$ as $n \to \infty$, then

$$\sqrt{k}_2(\widehat{\rho} - \rho) \xrightarrow{d} N\left(0, \sigma^2\right),$$

where $N\left(0, \sigma^2\right)$ is a centred normal-distributed random variable with

$$\sigma^2 = \sigma_U^2 \left(\frac{\log(2)}{(2 - \lambda_U)\log^2(2 - \lambda_U)} \right)^2$$

and

$$\begin{aligned} \sigma_U^2 &= \lambda_U + \left(\frac{\partial}{\partial x}\Lambda_U(1,1)\right)^2 + \left(\frac{\partial}{\partial y}\Lambda_U(1,1)\right)^2 \\ &+ 2\lambda_U \left(\left(\frac{\partial}{\partial x}\Lambda_U(1,1) - 1\right)\left(\frac{\partial}{\partial y}\Lambda_U(1,1) - 1\right) - 1\right). \end{aligned} \tag{3.12}$$

Note that the asymptotic variance in Corollary 3.4.1, vanishes in the asymptotically independent case. Therefore, in the case $\Lambda_U = 0$, it is verified that $\widehat{\lambda}_U \xrightarrow{\mathbb{P}} 0$ (for more details see Theorem A.1. and Corollary A.1. in Di Bernardino et al. (2013)). Consequently, $\frac{\widehat{\rho}}{\rho} \xrightarrow{\mathbb{P}} 1$.

In Table 3.3, the second-order condition for the bivariate *upper tail copula* $\Lambda_U(x, y)$ in Equation (3.11) is illustrated for certain classic Archimedean copula models with $\Lambda_U(x, y) = x + y - (x^\theta + y^\theta)^{1/\theta}$. We consider the Gumbel copula, Joe copula, and Copulas (12), (14), (15) and (21) in Table 4.1 of Nelsen (2006). Observe that the property in Equation (3.11) is not verified for Copula (2) in Table 4.1 in Nelsen (2006).

In the following, we adapt the well-known Central Limit Theorem for the interme-

Copula	$\phi(t)$	$A_\rho(t)$
Gumbel	$(-\log(t))^\theta$	t^{-1}
Joe	$-\log(1-(1-t)^\theta)$	$t^{-\theta}$
(12)	$(1/t-1)^\theta$	t^{-1}
(14)	$(t^{-1/\theta}-1)^\theta$	t^{-1}
(15)	$(1-t^{1/\theta})^\theta$	t^{-1}
(21)	$1-(1-(1-t)^\theta)^{1/\theta}$	$t^{-\theta}$

Table 3.3: Bivariate Archimedean copula models with $\rho=\theta$ and $\lambda_U = 2-2^{1/\theta}$.

diate order statistics in our setting. This result follows easily from Theorems 2.4.1 and 2.4.2 in de Haan and Ferreira (2006). Further details are given in Theorem 2.1 in Drees (1998).

Theorem 3.4.1 (Theorem 2.1 in Drees (1998))**.** *Let $(X_1,\dots,X_d)$ be a random vector with Archimedean copula C with twice differentiable generator ϕ. Assume that $\phi \in RV_\rho(1)$, with $\rho \in [1,+\infty]$. Let $i \in \{1,\dots,d\}$. Assume that $U_{X_i} \in 2RV_{\gamma_i,\tau_i}(A_i)$, $\gamma_i>0$ and $\tau_i<0$. Let $k=k(n)\to\infty$, $k/n\to 0$, $n\to\infty$ such that $\lim_{n\to\infty}\sqrt{k_U}\,A_i(n/k_U)$ exists and is finite with the sequence k_U defined as in Equation* (3.5)*. Therefore, it holds that, as $n\to+\infty$,*

$$\sqrt{k_U(n)}\left(\frac{X^i_{n-\lfloor k_U\rfloor,n}}{U_{X_i}\left(\frac{n}{k_U(n)}\right)}-1\right)\xrightarrow{d}\gamma_i\,N(0,1).$$

Proof. From Proposition 3.2.2, it is verified that $U_{T_i}\left(\frac{n}{k}\right)=U_{X_i}\left(\frac{n}{k_U}\right)$, and $k_U(n)\to\infty$, $k_U/n\to 0$ as $n\to\infty$. Since, by assumptions, $U_{X_i}\in 2RV_{\gamma_i,\tau_i}(A_i)$, $\gamma_i>0$, $\tau_i<0$, and $\sqrt{k_U}A_i(n/k_U)\to\lambda'<\infty$, as $n\to\infty$, then, from Theorem 2.4.2 in de Haan and Ferreira (2006), the result is attained. □

3.4.2 Asymptotic convergence for $\frac{\widehat{\gamma_i}}{\widehat{\rho}}$

The asymptotic normality of $\widehat{\gamma}^{T_i}$ is obtained in Theorem 3.4.2.

Theorem 3.4.2 (Asymptotic normality of $\widehat{\gamma}^{T_i}$; upper tail dependence case; $\rho>1$)**.** *Let $(X_1,\dots,X_d)$ be a random vector with Archimedean copula C with twice differentiable generator ϕ. Assume that $\phi\in RV_\rho(1)$, with $\rho\in(1,+\infty)$. Let $i\in\{1,\dots,d\}$ and F_{X_i} be the twice differentiable distribution function of X_i. Assume that:*

1. *For (X_i,X_j), with $i\neq j$, the tail copula Λ_U exists, has continuous partial derivatives, and satisfies the second-order condition given in Equation* (3.11) *with auxiliary function $A_\rho(\cdot)$.*

2. *$U_{X_i}\in 2RV_{\gamma_i,\tau_i}(A_i)$, $\gamma_i>0$ and $\tau_i<0$.*

3. $k_1 = k_1(n) \to \infty$, $k_1/n \to 0$, $n \to \infty$ *such that* $\sqrt{k_1}A_i(n/k_1) \to \lambda$ *as* $n \to \infty$ *with* λ *finite.*

4. $k_2 = k_2(n) \to \infty$, $k_2/n \to 0$, $n \to \infty$ *such that* $\sqrt{k_2}A_\rho(n/k_2) \to 0$ *as* $n \to \infty$.

Let $r = \lim_{t\to+\infty} \frac{\sqrt{k_1(n)}}{\sqrt{k_2(n)}} \in [0,\infty]$ *and* $\gamma^{T_i} := \frac{\gamma_i}{\rho}$. *Therefore, as* $n \to \infty$,

$$\min(\sqrt{k_1}, \sqrt{k_2}) \left(\frac{\widehat{\gamma}^{T_i}}{\gamma^{T_i}} - 1 \right) \xrightarrow{d} \begin{cases} \Theta_1 + r\Theta_2, & \text{if } r \leq 1; \\ \frac{1}{r}\Theta_1 + \Theta_2, & \text{if } r > 1, \end{cases}$$

where $\Theta_1 \sim N(\mu/\gamma_i, 1)$ *with* $\mu = \lambda/(1-\tau_i)$, *and* $\Theta_2 \sim N(0, \sigma^2/\rho^2)$, *with* $\sigma^2 = \sigma_U^2 \left(\frac{\log(2)}{(2-\lambda_U)\log^2(2-\lambda_U)} \right)^2$, $\lambda_U := \Lambda(1,1)$ *the upper tail dependence coefficient, and* σ_U^2 *defined in* (3.12).

Proof. Since $U_{X_i} \in 2RV_{\gamma_i,\tau_i}(A_i)$ with $\gamma_i > 0$ and $\tau_i < 0$, from Theorem 3.2.5 in de Haan and Ferreira (2006) and Slutsky's Theorem (e.g., see Serfling (1980)), it is verified that

$$\sqrt{k_1} \left(\frac{\widehat{\gamma}_i}{\gamma_i} - 1 \right) \xrightarrow{d} N\left(\mu/\gamma_i, 1\right), \tag{3.13}$$

with $\mu = \lambda/(1-\tau_i)$. From Charpentier and Segers (2008), if the distribution function of a random vector $(X_1, \ldots, X_d)$ is given by a $d-$dimensional Archimedean copula C with generator ϕ, then the distribution function of every bivariate subvector (X_i, X_j), $i \neq j$, is obtained by the bivariate Archimedean copula with the same generator. As a consequence, to estimate ρ such that $\phi \in RV_\rho(1)$, we focus on the bivariate case. In addition, since the conditions of Corollary 3.4.1 are satisfied under the assumptions of Theorem 3.4.2, then by using the Delta Method, it is verified that

$$\sqrt{k_2} \left(\frac{\rho}{\widehat{\rho}} - 1 \right) \xrightarrow{d} N\left(0, \sigma^2/\rho^2\right) \tag{3.14}$$

with σ^2 provided in the statement of Theorem 3.4.2. We can now write

$$\frac{\widehat{\gamma}_i^T}{\gamma_i^T} = \frac{\widehat{\gamma}_i}{\gamma_i} \times \frac{\rho}{\widehat{\rho}} =: M_1 \times M_2$$

and we deal with the two factors separately:

- From (3.13), $M_1 = \frac{\Theta_1}{\sqrt{k_1}} + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_1}}\right) + 1$ with $\Theta_1 \sim N\left(\mu/\gamma_i, 1\right)$;

- From (3.14), $M_2 = \frac{\Theta_2}{\sqrt{k_2}} + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_2}}\right) + 1$ with $\Theta_2 \sim N\left(0, \sigma^2/\rho^2\right)$.

Hence,

$$\left(\frac{\widehat{\gamma}^{T_i}}{\gamma^{T_i}} - 1 \right) = M_1 \times M_2 - 1 = \frac{\Theta_1}{\sqrt{k_1}} + \frac{\Theta_2}{\sqrt{k_2}} + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_1}}\right) + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_2}}\right).$$

Since $k_1 \to \infty$ and $k_2 \to \infty$, as $n \to \infty$, the result is provided. □

Remark 3.4.1 (Asymptotic behaviour of $\widehat{\gamma}^{T_i}$; upper tail independence case; $\rho = 1$). *Notice that, if $\rho = 1$ (i.e. tail copula $\Lambda_U \equiv 0$) then the asymptotic variance in Theorem 3.4.2 vanishes. However, in the upper tail independence case, the consistency of the proposed estimator $\widehat{\gamma}^{T_i}$ can be obtained. To be precise, if $\phi \in RV_1(1)$ and the second and third conditions of Theorem 3.4.2 hold, then $\frac{\widehat{\gamma}^{T_i}}{\gamma^{T_i}} \xrightarrow{\mathbb{P}} 1$, for $n \to \infty$.*

3.4.3 Main result

In the following theorem, our main result is presented, i.e., the asymptotic normality for $\widehat{x}^i_{p_n}$ in (3.10).

Theorem 3.4.3 (Asymptotic normality of $\widehat{x}^i_{p_n}$; upper tail dependence case; $\rho > 1$). *Let $(X_1, \ldots, X_d)$ be a random vector with Archimedean copula with twice differentiable generator ϕ. Assume that $\phi \in RV_\rho(1)$, with $\rho \in (1, +\infty)$. Let $i \in \{1, \ldots, d\}$ and F_{X_i} be the twice differentiable distribution function of X_i. Assume that F_{X_i} verifies the von Mises condition with index $\gamma_i > 0$. Denote $T_i := [X_i | \boldsymbol{X} \in \partial \underline{L}(\alpha)]$ with distribution function $F_{T_i}(\cdot|\alpha)$ given by Lemma 2.2.1.*

Assume:

1. *For (X_i, X_j), with $i \neq j$, the upper tail copula Λ_U exists, has continuous partial derivatives, and satisfies the second-order condition given in Equation (3.11) with the auxiliary function $A_\rho(\cdot)$.*
2. *$U_{X_i} \in 2RV_{\gamma_i, \tau_i}(A_i)$, $\gamma_i > 0$ and $\tau_i < 0$.*
3. *$k = k(n) \to \infty$, $k/n \to 0$, $n \to \infty$, such that Theorem 3.4.1 is satisfied.*
4. *$k_1 = k_1(n) \to \infty$, $k_1/n \to 0$, $n \to \infty$, such that $\sqrt{k_1} A_i(n/k_1) \to \lambda$ as $n \to \infty$.*
5. *$k_2 = k_2(n) \to \infty$, $k_2/n \to 0$, $n \to \infty$, such that $\sqrt{k_2} A_\rho(n/k_2) \to 0$ as $n \to \infty$.*

Let $r = \lim_{n \to +\infty} \frac{\sqrt{k_1(n)}}{\sqrt{k_2(n)}}$, $r' = \lim_{n \to +\infty} \frac{\sqrt{k_U(n)} \log(d_n)}{\sqrt{k_1(n)}}$ and $r'' = \lim_{n \to +\infty} \frac{\sqrt{k_U(n)} \log(d_n)}{\sqrt{k_2(n)}}$ with r, r' and $r'' \in [0, \infty]$.

Hence, as $n \to \infty$, if $r \leq 1$ and $\lim_{n \to +\infty} \frac{\log(d_n)}{\sqrt{k_1(n)}} = 0$,

$$\min\left(\sqrt{k_U}, \frac{\sqrt{k_1}}{\log(d_n)}\right)\left(\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} - 1\right) \xrightarrow{d} \begin{cases} B + r'(\Theta_1 + r\Theta_2), & r' \leq 1; \\ \\ \frac{1}{r'} B + \Theta_1 + r\Theta_2, & r' > 1, \end{cases}$$

and, if $r > 1$ and $\lim_{n \to +\infty} \frac{\log(d_n)}{\sqrt{k_2(n)}} = 0$, then

$$\min\left(\sqrt{k_U}, \frac{\sqrt{k_2}}{\log(d_n)}\right)\left(\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} - 1\right) \xrightarrow{d} \begin{cases} B + r''(\frac{1}{r}\Theta_1 + \Theta_2), & r'' \leq 1; \\ \\ \frac{1}{r''} B + \frac{1}{r}\Theta_1 + \Theta_2, & r'' > 1, \end{cases}$$

where $d_n := k/(np_n)$, $B \sim N(0, \gamma_i^2)$, $\Theta_1 \sim N(\mu/\gamma_i, 1)$ *with* $\mu = \lambda/(1-\tau_i)$ *and* $\Theta_2 \sim N(0, \sigma^2/\rho^2)$, *with* $\sigma^2 = \sigma_U^2 \left(\frac{\log(2)}{(2-\lambda_U)\log^2(2-\lambda_U)}\right)^2$, $\lambda_U := \Lambda(1,1)$ *is the upper tail dependence coefficient and* σ_U^2 *is the same as in* (3.12).

Proof. We write

$$\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} = \frac{X^i_{n-\lfloor k_U \rfloor, n}}{U_{X_i}\left(\frac{n}{k_U}\right)} \times \left(\frac{k}{n\, p_n}\right)^{\widehat{\gamma}^{T_i} - \gamma^{T_i}} =: N_1 \times N_2.$$

From Theorem 3.4.1,

$$N_1 \xrightarrow{d} \frac{B}{\sqrt{k_U}} + 1 + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_U}}\right), \quad \text{where } \; B \sim N(0, \gamma_i^2). \tag{3.15}$$

Let $r = \lim_{t \to +\infty} \frac{\sqrt{k_1(n)}}{\sqrt{k_2(n)}} \in [0, \infty]$. From Theorem 3.4.2, as $n \to \infty$,

$$\min(\sqrt{k_1}, \sqrt{k_2}) \left(\frac{\widehat{\gamma}^{T_i}}{\gamma^{T_i}} - 1\right) \xrightarrow{d} \begin{cases} \Theta_1 + r\Theta_2, & \text{if } r \leq 1; \\ \frac{1}{r}\Theta_1 + \Theta_2, & \text{if } r > 1, \end{cases}$$

where $\Theta_1 \sim N(\mu/\gamma_i, 1)$ with $\mu = \lambda/(1-\tau_i)$, $\lim_{n \to \infty} \sqrt{k_1(n)} A_i(n/k_1(n)) = \lambda < +\infty$, and $\Theta_2 \sim N(0, \sigma^2/\rho^2)$ with σ^2 as provided in Corollary 3.4.1.

Therefore, it is verified that

$$\frac{\min(\sqrt{k_1}, \sqrt{k_2})}{\log(d_n)} \left(d_n^{\widehat{\gamma}^{T_i} - \gamma^{T_i}} - 1\right) \xrightarrow{d} \begin{cases} \Theta_1 + r\Theta_2, & \text{if } r \leq 1; \\ \frac{1}{r}\Theta_1 + \Theta_2, & \text{if } r > 1, \end{cases}$$

where $d_n = \frac{k}{np_n}$. The interested reader is also referred to the proof of Theorem 4.3.8 in de Haan and Ferreira (2006). Consequently,

$$N_2 \xrightarrow{d} \begin{cases} \frac{\log(d_n)}{\sqrt{k_1}}\left(\Theta_1 + r\Theta_2\right) + 1 + o_{\mathbb{P}}\left(\frac{\log(d_n)}{\sqrt{k_1}}\right), & \text{if } r \leq 1; \\ \frac{\log(d_n)}{\sqrt{k_2}}\left(\frac{1}{r}\Theta_1 + \Theta_2\right) + 1 + o_{\mathbb{P}}\left(\frac{\log(d_n)}{\sqrt{k_2}}\right), & \text{if } r > 1. \end{cases} \tag{3.16}$$

On combining the asymptotic relations (3.15) and (3.16), if $r \leq 1$, then

$$\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} - 1 = \frac{B}{\sqrt{k_U}} + \frac{\log(d_n)}{\sqrt{k_1}}\left(\Theta_1 + r\Theta_2\right) + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_U}}\right) + o_{\mathbb{P}}\left(\frac{\log(d_n)}{\sqrt{k_1}}\right). \tag{3.17}$$

Similarly, if $r > 1$, then

$$\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} - 1 = \frac{B}{\sqrt{k_U}} + \frac{\log(d_n)}{\sqrt{k_2}}\left(\frac{1}{r}\Theta_1 + \Theta_2\right) + o_{\mathbb{P}}\left(\frac{1}{\sqrt{k_U}}\right) + o_{\mathbb{P}}\left(\frac{\log(d_n)}{\sqrt{k_2}}\right). \tag{3.18}$$

Hence, Theorem 3.4.3 comes from Equations (3.17) and (3.18).

□

Remark 3.4.2 (Asymptotic consistency of $\widehat{x}^i_{p_n}$; upper tail independence case; $\rho = 1$). *Notice that, if $\rho = 1$ (i.e. tail copula $\Lambda_U \equiv 0$), then the asymptotic variance σ_U^2 in Theorem 3.4.3 vanishes. However, in the upper tail independence case, the consistency of the proposed estimator $\widehat{x}^i_{p_n}$ can be obtained. To be precise, if $\phi \in RV_1(1)$ and the second, third, and fourth conditions of Theorem 3.4.3 hold, then $\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} \xrightarrow{\mathbb{P}} 1$, for $n \to \infty$.*

3.5 Simulation study

The aim of this section is to evaluate the performance of $\widehat{x}^i_{p_n}$ in finite-size samples. Although we restrict ourselves to a 3-dimensional case in this study, these illustrations could be adaptable in any dimension d.

The performance of our extreme estimator $\widehat{x}^i_{p_n}$ is also compared with a pseudo-empirical estimator (denoted $\widehat{x}^{pseudo}_{p_n}$), an empirical estimator ($\widehat{x}^{emp}_{p_n}$), and the semi-parametric empirical estimator ($\widehat{\underline{\mathrm{CoVaR}}}^i_{\alpha,1-\boldsymbol{\omega}}(\mathbf{X})$) defined in (2.17). In order to attain $\widehat{x}^{pseudo}_{p_n}$, it is assumed that the distribution function of T_i is known (see Lemma 2.2.1). We can then sample from the random variable T_i by using the fact that

$$T_i \stackrel{d}{=} F^{-1}_{X_i}\left\{\phi^{-1}\left[\left(1 - U^{1/(d-1)}\right)\phi(\alpha)\right]\right\},$$

where U is a uniform random variable. Finally, the pseudo-empirical estimator $\widehat{x}^{pseudo}_{p_n}$ can be defined as the $(n - \lfloor n\, p_n \rfloor)$-th order statistic of the sample obtained from T_i,

$$\widehat{x}^{pseudo}_{p_n} = T^i_{n-\lfloor n\, p_n\rfloor, n}. \tag{3.19}$$

On the other hand, an empirical estimator ($\widehat{x}^{emp}_{p_n}$) can be proposed without the need for any information about T_i. To this end, we sample from the latent random variable $T_i = [X_i | F(\mathbf{X}) = \alpha]$ by using the empirical multivariate distribution function. Let $(\mathbf{X}_j)$, $j = 1, \ldots, n$, be a d-dimensional *iid* sample of $\mathbf{X}$. For all $t \in \mathbb{R}^d$, the d-dimensional empirical distribution function of $\mathbf{X}$ is defined as $F_n(t) := \frac{1}{n}\sum_{j=1}^n \mathbf{1}_{\{\mathbf{X}_j \le t\}}$. $\widetilde{T}_i$ is then obtained by collecting the points $(\mathbf{X}^i_j)$, for $j = 1, \ldots, n$, such that $F_n(\mathbf{X}_j) \in [\alpha - h, \alpha + h]$ for a sufficiently small positive value h. The quantity h is adjusted to each considered model and each sample size. The competitor estimator $\widehat{x}^{emp}_{p_n}$ is given by

$$\widehat{x}^{emp}_{p_n} = \widetilde{T}^i_{n-\lfloor n\, p_n\rfloor, n}. \tag{3.20}$$

Bear in mind that $\phi_{\widehat{\theta}_n}$ included in (2.17) is the semi-parametric estimator of the generator of the copula ϕ_θ obtained by considering the maximum pseudo-likelihood estimator of the parameter θ associated with ϕ_θ (e.g., see Genest et al. (1995)). We now consider the following 3-dimensional distributional models:

1. Joe copula and Fréchet margins: $F_i(t) = \exp\{-t^{-\beta}\}, i = 1, 2, 3$, and $C(u_1, u_2, u_3) =$

$1-[1-\exp\{\log(1-(1-u_1)^\theta)+\log(1-(1-u_2)^\theta)+\log(1-(1-u_3)^\theta)\}]^{1/\theta}$. In this section, we take the dependence copula parameter $\theta = 3$ and the marginal parameter $\beta = 3$. Bear in mind that the assumptions of Theorem 3.4.3 are satisfied. Indeed, $\phi \in RV_3(1)$ and $U_{X_i} \in 2RV_{1/3,-1}$, for $i = 1, 2, 3$. In addition, the associated tail-logistic model is given by $\Lambda_U(x, y) = x + y - (x^3 + y^3)^{1/3}$, which satisfies the second-order condition in Equation (3.11) , and $\lambda_U = 0.74$.

2. Independence copula with Fréchet margins:

$$F_i(t) = \exp\{-t^{-\beta}\}, \text{ i=1,2,3, and } C(u_1, u_2, u_3) = u_1\, u_2\, u_3.$$

In this case, $\phi \in RV_1(1)$ and the upper tail copula is given by $\Lambda_U = \lambda_U = 0$. However, using Remark 3.4.2, the consistency of $\widehat{\gamma}^{T_i}$ and $\widehat{x}^i_{p_n}$ is illustrated in this section.

3. Gumbel copula with Pareto margins: $F_i(t) = 1 - (\delta_1/(t + \delta_1))^{\delta_2}, i = 1, 2, 3$, and $C(u_1, u_2, u_3) = \exp\left\{-\left((-\log(u_1))^\theta + (-\log(u_2))^\theta + (-\log(u_3))^\theta\right)^{1/\theta}\right\}$. In the simulation study, we take the dependence copula parameter $\theta = 2$ and the marginal parameters $\delta_1 = 1$ and $\delta_2 = 2$. Bear in mind that the assumptions of Theorem 3.4.3 are satisfied. Indeed, $\phi \in RV_2(1)$ and U_{X_i} verifies the 2RV condition with $\gamma_i = 1/2$ and $\tau = -1/2$, for $i = 1, 2, 3$. Furthermore, $\Lambda_U(x, y) = x + y - (x^2 + y^2)^{1/2}$ verifies the second-order condition in Equation (3.11), and $\lambda_U = 0.59$.

Various sample sizes are taken and we consider $p_n = 1/n$ and $p_n = 1/2n$, the critical layer level $\alpha = 0.9$, and 500 Monte Carlo simulations.

Note that specific values for auxiliary sequences of our procedure (k_1, k_2, and k) are chosen for each sample size as indicated in the figures. In order to choose k_1, the estimator of γ_i is plotted against various values of k_1. By balancing the potential estimation bias and variance, a common practice is to choose k_1 from the first stable region of the plots (see e.g., Cai et al. (2015)). Finally, in order to gain stability in the estimates, the obtained values are averaged in this region. A similar procedure is developed for the auxiliary sequence k_2 (for the estimation of ρ). The sequence k is selected by observing the stability of the final ratio $\widehat{x}^i_{p_n}/x^i_{p_n}$.

Boxplots of ratio $\widehat{\gamma}^{T_i}/\gamma^{T_i}$ In Figure 3.1, we present the boxplots of ratio $\widehat{\gamma}^{T_i}/\gamma^{T_i}$ for the three distributional models considered, and for different sample sizes.

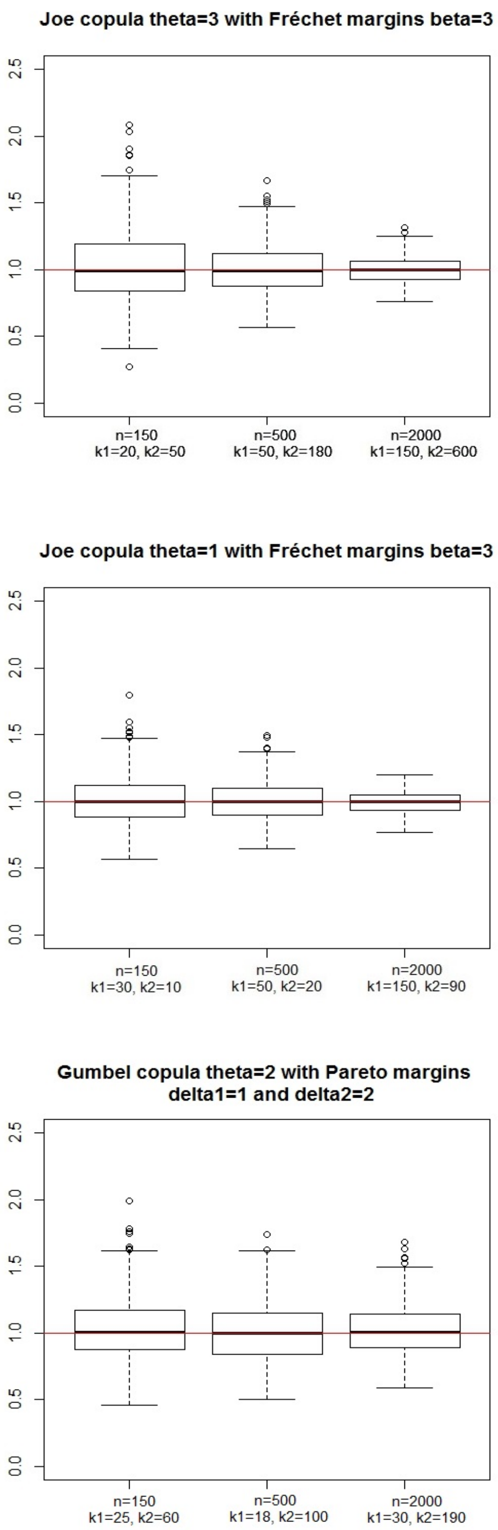

Figure 3.1: Boxplots of the ratio $\widehat{\gamma}^{T_i}/\gamma^{T_i}$: for Joe copula with $\theta = 3$ and Fréchet margins with $\beta = 3$ (first row); for Independence copula and Fréchet margins with $\beta = 3$ (second row); and for Gumbel copula with $\theta = 2$ and Pareto margins with $\delta_1 = 1$ and $\delta_2 = 2$ (third row). We consider $n = 150$, $n = 500$, and $n = 2000$, and 500 Monte Carlo simulations.

Boxplots of ratio $\widehat{x}^i_{p_n}/x^i_{p_n}$ Using Remark 3.4.2, an illustration of the consistency of the proposed estimator is provided in the independent copula case. In Figure 3.2, the obtained boxplots for the ratio $\widehat{x}^i_{p_n}/x^i_{p_n}$ are presented for $p_n = 1/n$ and $p_n = 1/2n$ for $n = 150$ (left panel) and $n = 1000$ (right panel). For $p_n = 1/n$, we also provide the boxplots of the ratios $\underline{\widehat{\mathrm{CoVaR}}}^i_{\alpha,1-\boldsymbol{\omega}}(\mathbf{X})/x^i_{p_n}$, $\widehat{x}^{pseudo}_{p_n}/x^i_{p_n}$ and $\widehat{x}^{emp}_{p_n}/x^i_{p_n}$, with the empirical competitor estimators previously defined in Equations (2.17), (3.19) and (3.20), respectively.

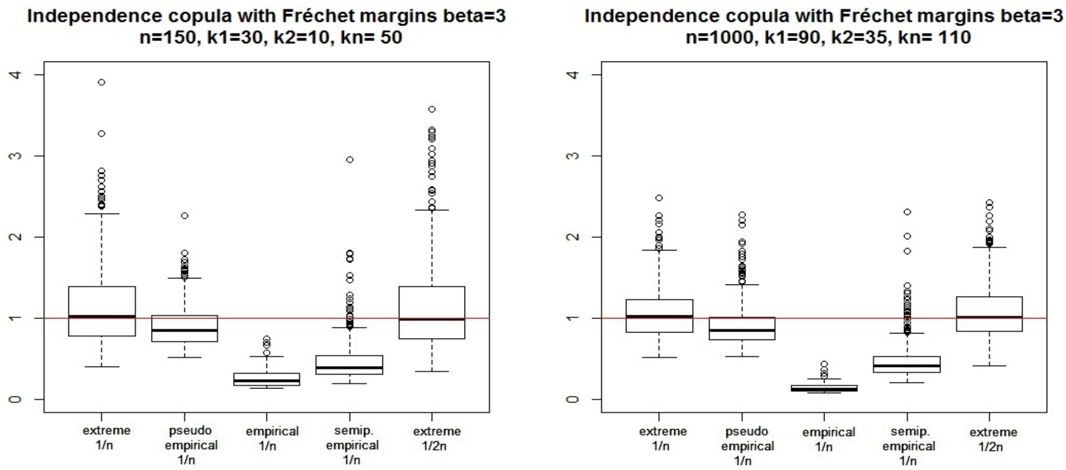

Figure 3.2: Independence copula and Fréchet margins with $\beta = 3$. Boxplots for the ratio $\widehat{x}^i_{p_n}/x^i_{p_n}$ with $p_n = 1/n$, $p_n = 1/2n$, and $n = 150$ (left panel), $n = 1000$ (right panel). Boxplots for the competitor empirical estimators with $p_n = 1/n$ are also displayed. We consider $\alpha = 0.9$ and 500 Monte Carlo simulations.

The obtained boxplots for the ratio $\widehat{x}^i_{p_n}/x^i_{p_n}$ are presented for $p_n = 1/n$ and $p_n = 1/2n$, in the Joe and Gumbel copula models in Figures 3.3 and 3.4, respectively. For $p_n = 1/n$, the comparison with the empirical competitor estimators is also provided.

In Figures 3.2, 3.3 and 3.4, it can be observed that the empirical ($\widehat{x}^{emp}_{p_n}$), pseudo-empirical ($\widehat{x}^{pseudo}_{p_n}$) and semi-parametric empirical ($\underline{\widehat{\mathrm{CoVaR}}}^i_{\alpha,1-\boldsymbol{\omega}}(\mathbf{X})$) competitor estimators underestimate the conditional quantile $x^i_{p_n}$ and are consistently outperformed by the proposed EVT estimator $\widehat{x}^i_{p_n}$. In addition, the empirical estimators are not applicable for $p < 1/n$. Furthermore, the behaviour of the EVT estimator $\widehat{x}^i_{p_n}$ remains stable when p_n changes from $1/n$ to $1/2n$.

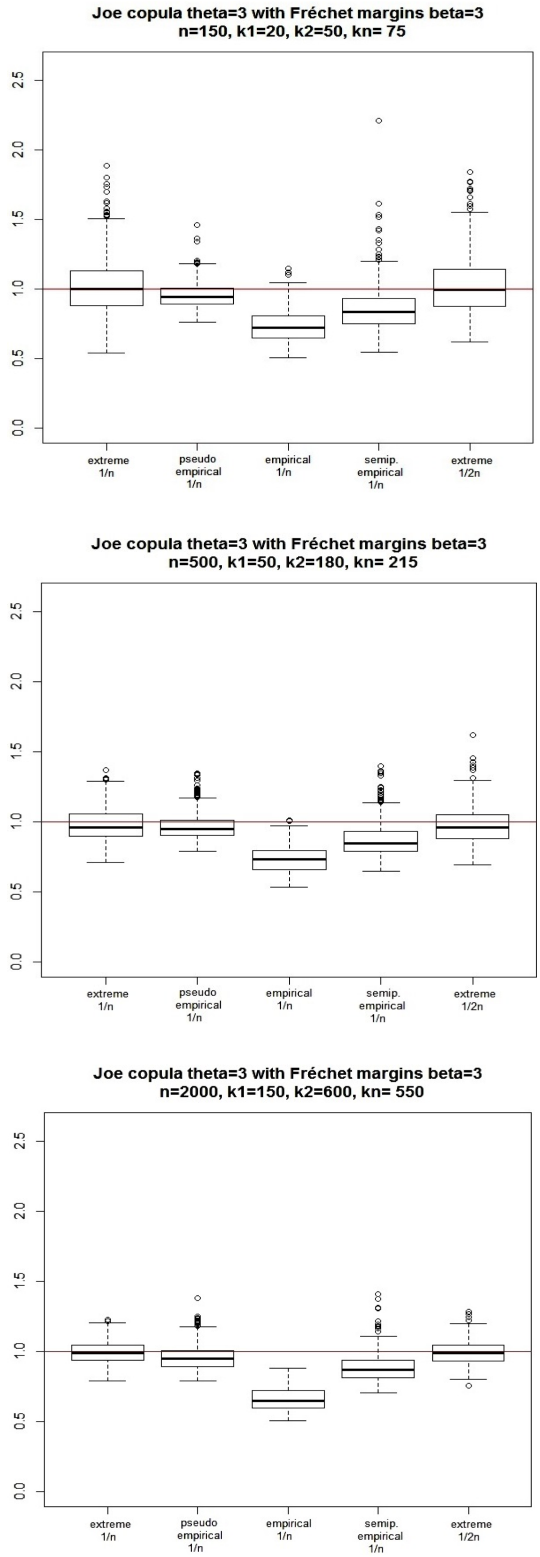

Figure 3.3: Trivariate Joe copula with $\theta = 3$ and Fréchet margins with $\beta = 3$. Boxplots for the ratio $\widehat{x}^i_{p_n}/x^i_{p_n}$ with $p_n = 1/n$, $1/2n$ for $n = 150$ (first row), $n = 500$ (second row) and $n = 2000$ (third row). Boxplots for the competitor empirical estimators with $p_n = 1/n$ are also displayed. We consider $\alpha = 0.9$ and 500 Monte Carlo simulations.

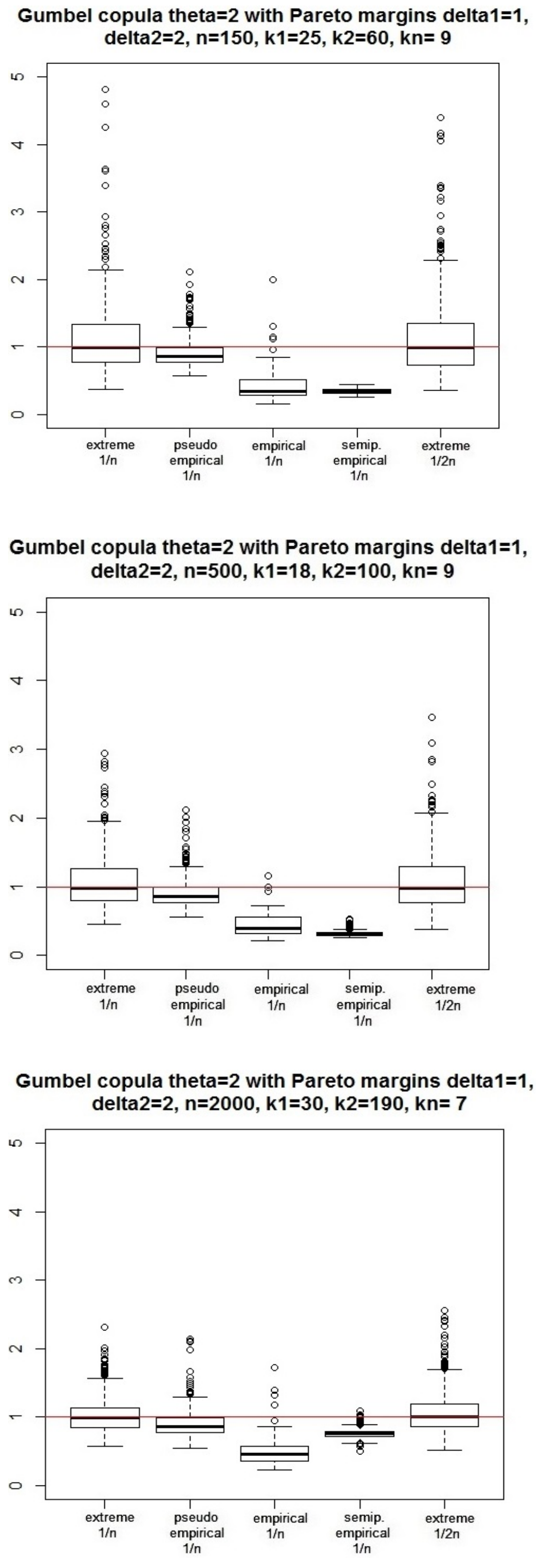

Figure 3.4: Trivariate Gumbel copula with $\theta = 2$ and Pareto margins with $\delta_1 = 1$ and $\delta_2 = 2$. Boxplots for the ratio $\widehat{x}^i_{p_n}/x^i_{p_n}$ with $p_n = 1/n$, $1/2n$ for $n = 150$ (first row), $n = 500$ (second row) and $n = 2000$ (third row). Boxplots for the competitor empirical estimators with $p_n = 1/n$ are also displayed. We consider $\alpha = 0.9$ and 500 Monte Carlo simulations.

Asymptotic normality Finally, the asymptotic normality in Theorem 3.4.3 is illustrated in Figure 3.5 for the Joe copula model. The Q-Q plots in Figure 3.5 represent the sample quantiles of $\min\left(\sqrt{k_U}, \frac{\sqrt{k_1}}{\log(d_n)}, \frac{\sqrt{k_2}}{\log(d_n)}\right)\left(\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} - 1\right)$ versus the theoretical normal quantiles for various sample sizes with $p_n = 1/n$. Since the scatterplots line up on the line in Figure 3.5, this indicates that the sample quantiles coincide largely with the theoretical quantiles from the asymptotic distribution. Hence, Theorem 3.4.3 provides an adequate approximation for finite sample sizes.

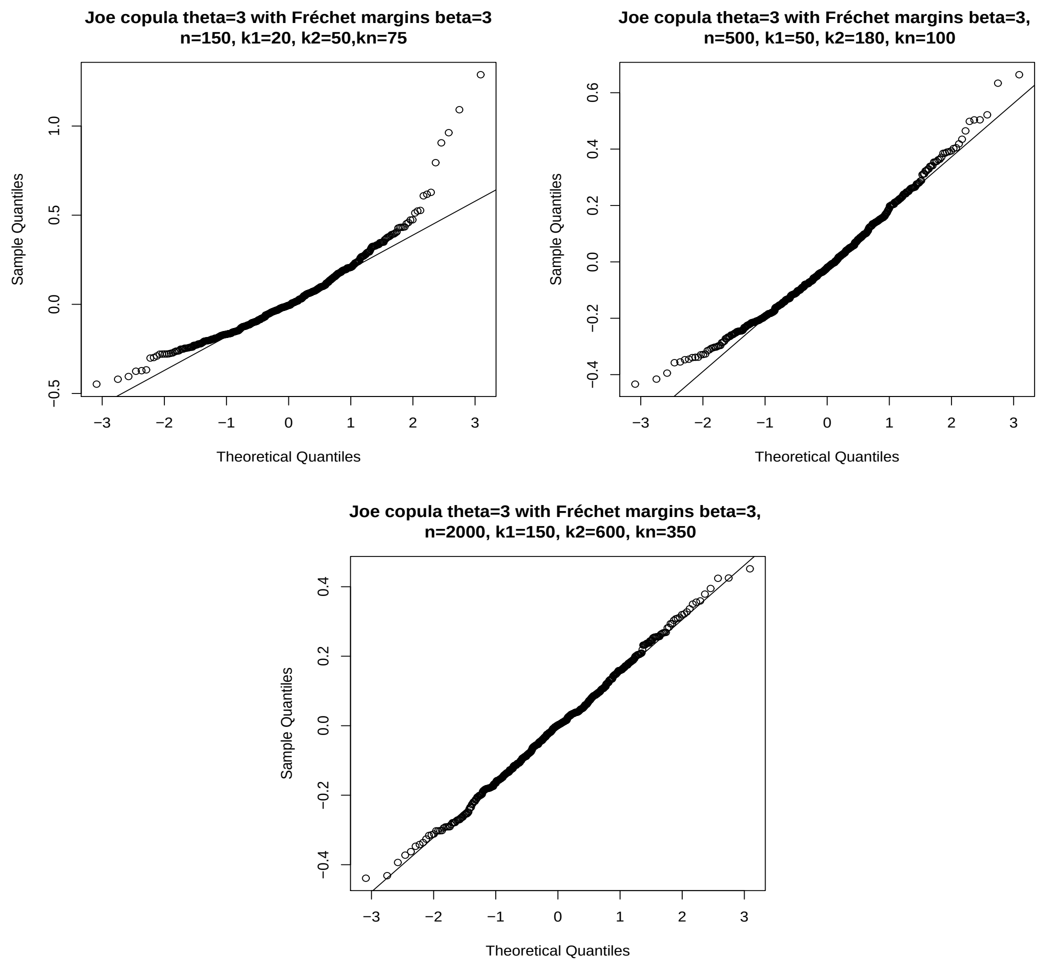

Figure 3.5: Joe copula with parameter $\theta = 3$ and Fréchet margins with $\beta = 3$. Q-Q plots for $\min\left(\sqrt{k_U}, \frac{\sqrt{k_1}}{\log(d_n)}, \frac{\sqrt{k_2}}{\log(d_n)}\right)\left(\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} - 1\right)$ for $p_n = 1/n$. We take $n = 150$ (left panel, first row), $n = 500$ (right panel, first row) and $n = 2000$ (second row). We consider $\alpha = 0.9$ and 500 Monte Carlo simulations.

Behaviour of ratio $\widehat{x}^i_{p_n}$ in terms of α In Figure 3.6, the boxplots of ratio $\widehat{x}^i_{p_n}/x^i_{p_n}$ are presented for a Joe copula with $\theta = 4$ and Fréchet margins with $\beta = 4$ by considering different values of the critical layer level α. Note that the convergence rate k_U in

Proposition 3.2.2 (see Equation (3.5)) satisfies $\frac{\partial k_U(\alpha)}{\partial \alpha} \leq 0$ for fixed values of sample size n and dimension d. Therefore, as can be observed in Figure 3.6, the performance of the proposed estimators decreases when α increases.

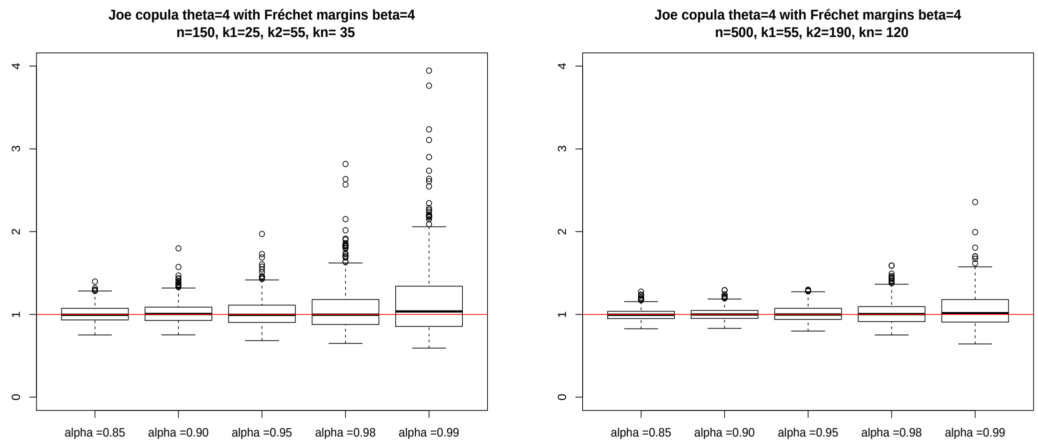

Figure 3.6: Trivariate Joe copula with $\theta = 4$ and Fréchet margins with $\beta = 4$. Boxplots for the ratio $\widehat{x}^i_{p_n}/x^i_{p_n}$ with $p_n = 1/n$, $n = 150$ (left panel) and $n = 500$ (right panel). Various values of α and 500 Monte Carlo simulations are considered.

3.6 Adaptive version

The intermediate sequence $k_U(n)$ in Proposition 3.2.2 is an unknown sequence that depends on the generator of the considered Archimedean copula. In this section, a plug-in procedure based on the estimation of k_U is presented. This can be seen as an adaptive version of the results in Section 3.4. For this purpose, the notion of the self-nested diagonal of a copula and the associated non-parametric estimator are used in the following (for further details see Di Bernardino and Rullière (2013)).

The definition of the self-nested diagonals of a copula δ_r is recalled in (1.9). Di Bernardino and Rullière (2013) introduce the following estimation of a self-nested diagonal δ_r given by using an interpolation procedure (see Lemma 3.6 in the aforementioned paper).

Definition 3.6.1 (Definition 4.2 in Di Bernardino and Rullière (2013)). *Let $\widehat{\delta}_1$ be an estimator of δ_1, and $\widehat{\delta}_{-1}$ be an estimator of the inverse function δ_{-1}. Estimators of δ_h and δ_{-h} can be obtained for any $h \in \mathbb{N}\backslash\{0\}$ by setting*

$$\begin{cases} \widehat{\delta}_h(u) & = & \widehat{\delta}_1 \circ \ldots \circ \widehat{\delta}_1(u), & (h \textit{ times}) \\ \widehat{\delta}_{-h}(u) & = & \widehat{\delta}_{-1} \circ \ldots \circ \widehat{\delta}_{-1}(u), & (h \textit{ times}) \\ \widehat{\delta}_0(u) & = & u. & \end{cases}$$

At any order $r \in \mathbb{R}$, an estimator of $\widehat{\delta}_r$ of δ_r is

$$\widehat{\delta}_r(u) = z\left(\left(z^{-1} \circ \widehat{\delta}_h(u)\right)^{1-\eta}\left(z^{-1} \circ \widehat{\delta}_{h+1}(u)\right)^{\eta}\right), \quad \textit{for } u \in [0,1], \tag{3.21}$$

with $\eta = r - \lfloor r \rfloor$ and $h = \lfloor r \rfloor$ where $\lfloor r \rfloor$ denotes the integer part of r, and where z is a strictly monotone function driving the interpolation (ideally the inverse of the generator of the copula).

Several different estimators for δ_1 can be found in the literature. In particular, one can propose $\widehat{\delta}_1(u) = F_{Y,n}(u)$, where $F_{Y,n}(u)$ is the empirical distribution function of $Y := \max(U_1, U_2, \ldots, U_d)$ with $\mathbf{U}$ the vector whose copula is C. Similarly, we consider $\widehat{\delta}_{-1}(u) = F^{-1}_{Y,n}(u)$, with $F^{-1}_{Y,n}(u)$ as the empirical quantile function of Y. Illustrations of the estimator $\widehat{\delta}_r$ in Definition (3.6.1) are presented in Example 3.6.1.

Using the self-nested diagonal family δ_r, we write the sequence $k_U(n)$ in Proposition 3.2.2 as:

$$k_U(n) \;=\; n\,(1 - \delta_{r(n)}(\alpha)),$$

where $r(n)$ is a negative real sequence $r(n) := \log\left(1 - \left(1 - \frac{k(n)}{n}\right)^{1/(d-1)}\right) / \log(d)$. Therefore, by using the non-parametric estimator $\widehat{\delta}_r$ in Definition 3.6.1, we introduce the estimator

$$\widehat{k}_U(n) \;=\; n\,(1 - \widehat{\delta}_{r(n)}(\alpha)), \quad \text{for} \;\; \alpha \in (0,1). \tag{3.22}$$

The following consistency result for $\widehat{k}_U(n)$ can now be proved.

Lemma 3.6.1. *Let $k_U(n)$ be the intermediate sequence defined as in Proposition 3.2.2. Let $\widehat{\delta}_1(u) = F_{Y,n}(u)$, with $F_{Y,n}(u)$ as the empirical distribution function of the random variable $Y := \max(U_1, U_2, \ldots, U_d)$, and $\widehat{\delta}_{-1}(u) = F^{-1}_{Y,n}(u)$, with $F^{-1}_{Y,n}(u)$ as the empirical quantile function of Y. Let $\widehat{k}_U(n)$ be the associated estimator proposed in Equation (3.22) for a fixed $\alpha \in (0,1)$, where z is a strictly monotone function driving the interpolation. Therefore,*

$$\frac{\widehat{k}_U(n)}{k_U(n)} \xrightarrow{\mathbb{P}} 1, \quad \textit{as} \;\; n \to \infty.$$

Proof. Firstly, we prove that $\frac{\widehat{\delta}_h(u)}{\delta_h(u)} \xrightarrow{\mathbb{P}} 1$, for $u \in (0,1)$ and for fixed $h \in \mathbb{Z}$, where δ_h is introduced in (1.9) and $\widehat{\delta}_h(u)$ is defined in Definition 3.6.1. Consider that $h \in \mathbb{Z}^+$. Since $\widehat{\delta}_1(u) := F_{Y,n}(u)$, where $F_{Y,n}(u)$ is the empirical distribution function of $Y := \max(U_1, U_2, \ldots, U_d)$, then from Glivenko Cantelli's Theorem, it is verified that

$$\sup_{u \in [0,1]} |\widehat{\delta}_1(u) - \delta_1(u)| = \sup_{u \in [0,1]} |F_{Y,n}(u) - F_Y(u)| \xrightarrow{\mathbb{P}} 0, \qquad \text{as } n \to \infty.$$

By induction, we assume that $\sup_{u\in[0,1]} |\widehat{\delta}_{m-1}(u) - \delta_{m-1}(u)| \xrightarrow{\mathbb{P}} 0$. Since C is a Lipschitz function (see Definition 6.2.6 in Nelsen (2006)), from Theorem 1 in Kasy (2015) and from the uniform convergence of $\widehat{\delta}_1(u)$, then $\sup_{u\in[0,1]} |\widehat{\delta}_m(u) - \delta_m(u)| \xrightarrow{\mathbb{P}} 0$, as $n \to \infty$. Let $h \in \mathbb{Z}^-$. We have $\widehat{\delta}_{-1}(u) := F^{-1}_{Y,n}(u)$, where $F^{-1}_{Y,n}(u)$ is the empirical quantile function of Y. From Theorem 3 in Mason (1982),

$$\sup_{u\in(0,1)} |\widehat{\delta}_{-1}(u) - \delta_{-1}(u)| = \sup_{u\in(0,1)} |F^{-1}_{Y,n}(u) - F^{-1}_{Y}(u)| \xrightarrow{\mathbb{P}} 0, \qquad \text{as } n \to \infty.$$

By induction, we suppose that $\sup_{u\in(0,1)} |\widehat{\delta}_m(u) - \delta_m(u)| \xrightarrow{\mathbb{P}} 0$. Since C^{-1} is a uniformly continuous function in $[0,1]$, then from Theorem 1 in Kasy (2015) and from the uniform convergence of $\widehat{\delta}_{-1}(u)$, we obtain $\sup_{u\in(0,1)} |\widehat{\delta}_{m-1}(u) - \delta_{m-1}(u)| \xrightarrow{\mathbb{P}} 0$, as $n \to \infty$. Therefore, $\frac{\widehat{\delta}_h(u)}{\delta_h(u)} \xrightarrow{\mathbb{P}} 1$, for $u \in (0,1)$ and for fixed $h \in \mathbb{Z}$. Furthermore, by using Slutsky's Theorem (e.g., see Serfling (1980)), one can prove that $\frac{\widehat{\delta}_r(u)}{\delta_r(u)} \xrightarrow{\mathbb{P}} 1$, $\forall\, u \in (0,1)$ and $\forall\, r \in \mathbb{R}$ fixed. Therefore, since δ_r is also a continuous and bounded function in r, from Polya's Theorem (e.g., see Section A.1.1 in Embrechts et al. (1997)), then, for $u \in (0,1)$,

$$\sup_{r\in\mathbb{R}} |\widehat{\delta}_r(u) - \delta_r(u)| \xrightarrow{\mathbb{P}} 0, \qquad \text{as } n \to \infty.$$

From the application of this uniform consistency, we obtain the assertion of Lemma 3.6.1. □

By using the result in Lemma 3.6.1 and stating that $\sqrt{k_U(n)}\left(\left(\frac{\widehat{k}_U(n)}{k_U(n)}\right)^{-\gamma_i} - 1\right) \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, it can be proved that $X^i_{n-\lfloor \widehat{k}_U \rfloor, n}$ is asymptotically as efficient as $X^i_{n-\lfloor k_U \rfloor, n}$. To be more precise, an adaptive plug-in version of Theorem 2.1 in Drees (1998) can be obtained, i.e.,

$$\sqrt{\widehat{k}_U(n)}\left(\frac{X^i_{n-\lfloor \widehat{k}_U \rfloor, n}}{U_{X_i}\left(\frac{n}{k_U(n)}\right)} - 1\right) \xrightarrow{d} \gamma_i N(0,1), \quad \text{as} \quad n \to \infty. \tag{3.23}$$

Further details are given in Hall and Welsh (1985), Drees and Kaufmann (1998), and Danielsson et al. (2001). An adaptive version of Theorem 3.4.3 for $\widehat{x}^i_{p_n}$ can also be provided. The proof is a slightly modified version of the proof of Theorem 3.4.3, through the application of the result in Equation (3.23) instead of Theorem 3.4.1. Illustrations of this plug-in estimation version of $\widehat{x}^i_{p_n}$, by using $\widehat{k}_U$ instead of k_U, can be found in Section 3.7 for the real hydrological data-set considered. In particular, to estimate the adaptive sequence $\widehat{k}_U(n)$ in Section 3.7, the equality $z(x) = \exp(-x)$ was chosen. This choice is recommended in Di Bernardino and Rullière (2013) when there is positive dependence, since it is the best choice for any Gumbel copula, whatever the parameter of the copula (see Corollary 3.7 in Di Bernardino and Rullière (2013)). Another natural choice could be any estimator of the inverse of the generator of the copula. Finally, it should be borne in mind that this function z does not change values of any δ_k, for $k \in \mathbb{Z}$. Therefore, the global shape of δ_r, as a function of $r \in \mathbb{R}$, is not heavily influenced by the choice of z. For a in-depth analysis of the weak impact of the interpolation function z in the evaluation of δ_r, the reader is referred to Section 4.3.1 in Di Bernardino and Rullière (2013).

Example 3.6.1. *In Figure 3.7, illustrations of $\widehat{\delta}_r$ with $r \in \mathbb{R}$ are provided for two different interpolation functions. As in Di Bernardino and Rullière (2013), a 2−dimensional Gumbel copula is generated with $\theta = 3$ and sample size $n = 2000$ and $n = 7000$. We consider $z(x) = \exp(-x)$ (first row of Figure 3.7) and $z(x) = \exp(-x^{1/\theta})$ (i.e., the inverse of the Gumbel generator copula, see second row of Figure 3.7) with $r = -3.5$, -2.4, -1.2, 0.6, 1.2, 2.4, 3.5. As pointed out before, it can be observed that the modification of the interpolation function z does not produce significant differences in the estimation of δ_r.*

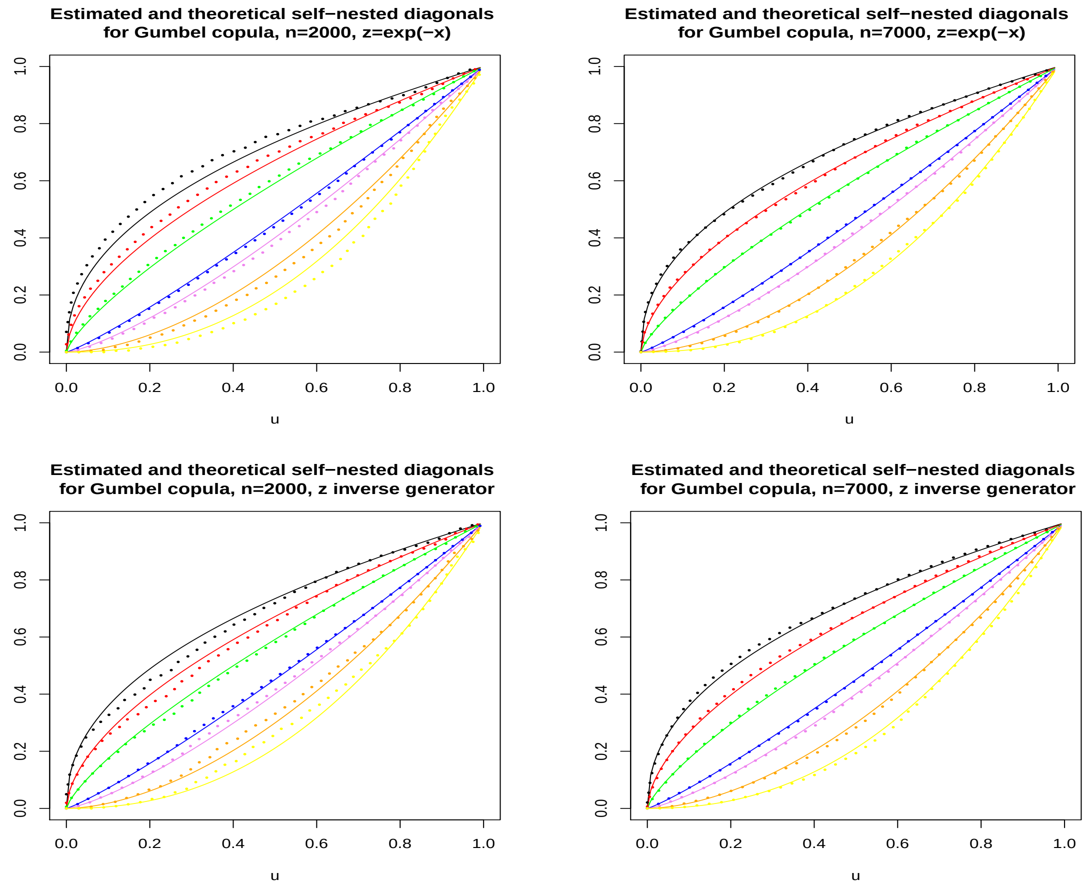

Figure 3.7: Gumbel copula with dependence parameter $\theta = 3$. Estimation of $\delta_r(x)$ by considering $z(x) = \exp(-x)$ (first row) and $z(x) = \exp(-x^{1/\theta})$ (second row) with $r = -3.5, -2.4, -1.2, 0.6, 1.2, 2.4, 3.5$, for $n = 2000$ (left panel) and $n = 7000$ (right panel).

Finally, an illustration of Lemma 3.6.1 is provided in Example 3.6.2.

Example 3.6.2. *In Figure 3.8, the boxplots of the ratio $\widehat{k_U}(n)/k_U(n)$ are gathered for a Joe copula $\theta = 3$ with Fréchet margins $\beta = 3$ by considering various sample sizes,* *with $k(n) = \sqrt{n}$ (left panel) and $k(n) = n^{0.9}$ (right panel). In this case, z is chosen as the inverse of the generator of the considered Joe copula.*

Boxplots of $\widehat{k}_U(n)/k_U(n)$

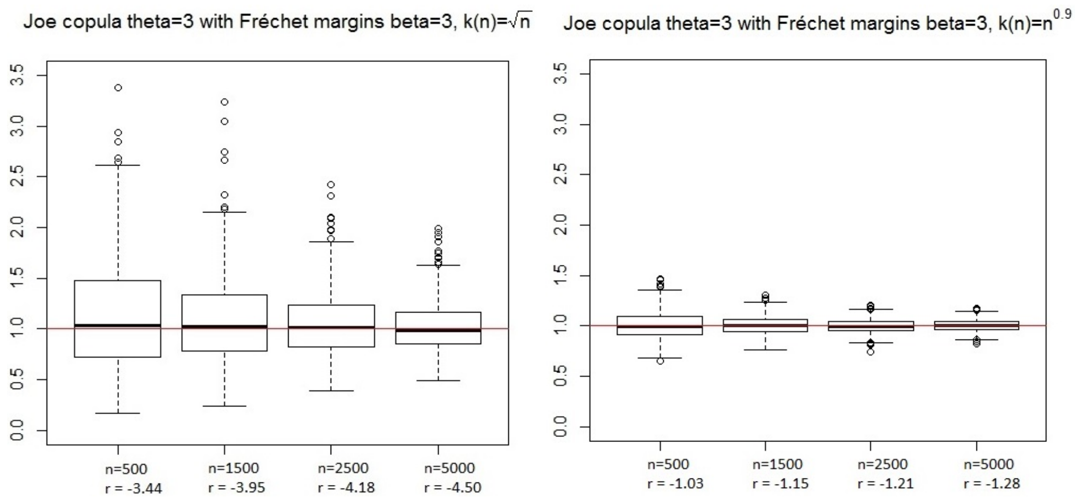

Figure 3.8: Joe copula with dependence parameter $\theta = 3$ and Fréchet margins with $\beta = 3$. Boxplots for the ratio $\widehat{k}_U(n)/k_U(n)$ for various values of n, $\alpha = 0.9$, $k(n) = \sqrt{n}$ (left panel) and $k(n) = n^{0.9}$ (right panel). 500 Monte Carlo simulations are taken.

3.7 Application in Bièvre region data-set

In this section, we focus on the estimation of the risk of a flood in the *Bièvre* region in the south of Paris (France) by using both the proposed multivariate extreme return level (see Equation (3.1)) and the classic univariate return level (see Section 1.1).

Presentation of the hydrological data-set The data-set contains the monthly mean of the rainfall measurements recorded in 3 different meteorological stations of the *Bièvre* region, from 2003 to 2013. The unit of measurement is *mm*. The size of the data-set is $n = 125$. The localization of the 3 stations is presented in Figure 3.9 and the data-set is represented in Figure 3.10. Let X_i denote the temporal series of the monthly mean of the rainfall measurements for station i, for $i = 1, 2, 3$. Station 1 is called *Geneste* (denoted X_1), station 2 *Loup Pendu* (X_2), and station 3 *Trou salé* (X_3). The data-set considered was provided by the *Syndicat Intercommunal pour l'Assainissement de la Valle de la Bièvre* (SIAVB, see http://www.siavb.fr/).

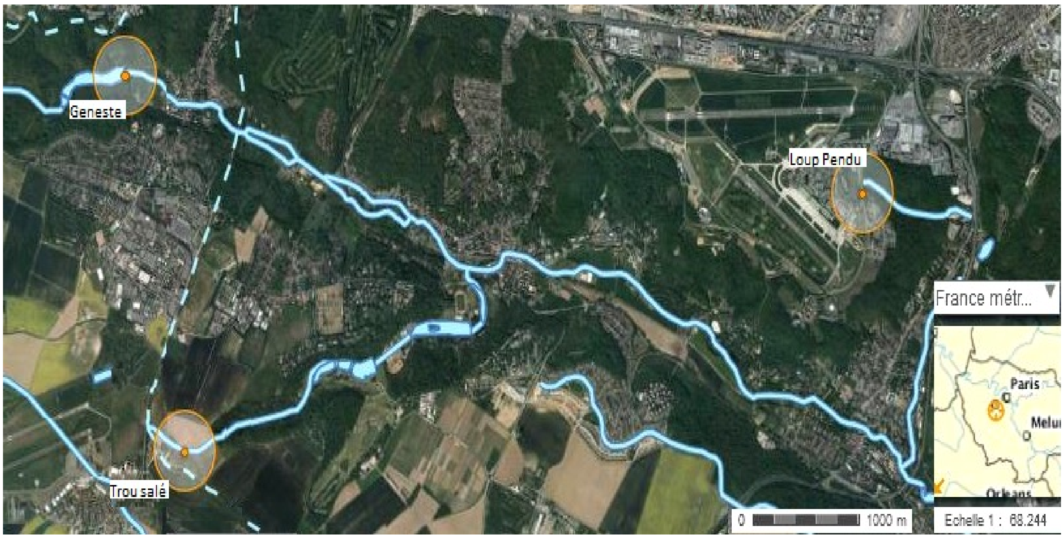

Figure 3.9: Localization of the three meteorological stations in the *Bièvre* region (in the south of Paris, France).

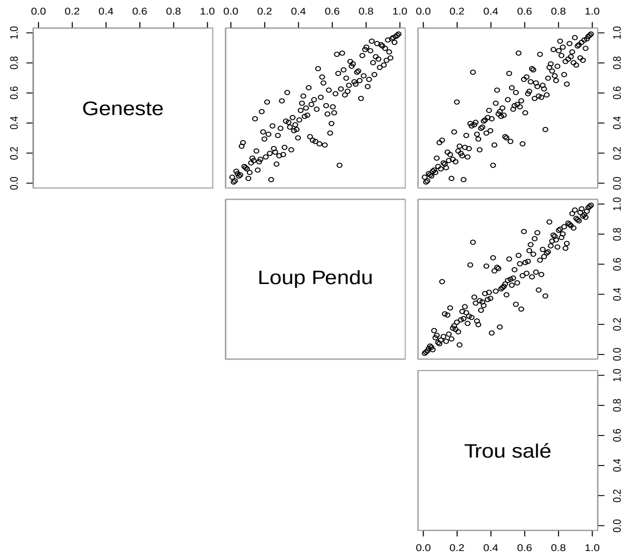

Figure 3.10: Rank-scatterplot for pairs of margins for the *Bièvre* region data-set.

The notion of autocorrelation can be defined as the correlation between observations of a time series separated by h time units. Autocorrelation plots are used as a tool for testing randomness in a data-set. When the autocorrelations in the plot are near zero for any and all time-lag separations, it could be possible to assume that there is no periodicity in the data-set. In Figure 2 in Di Bernardino and Prieur (2014), the autocorrelation function for each station of *Bièvre* region data-set is gathered. If we observe the aforementioned figure, apparently, certain seasonality should be considered for each station. However, the autocorrelation function of each station given in Figure 2 in Di Bernardino and Prieur (2014) is bounded between -0.2 and 0.2. Therefore, in our case, the periodicity of the data-set is regarded as statistically insignificant and we suppose the plausibility of the temporal independence assumption for these 3-dimensional monthly rainfall measurements. Remark that, as an improvement of this work, a deep study of the model of this data-set would be done by taking into account the certain seasonality observed in the data-set. However, this issue lies beyond the scope of the present chapter and may be addressed in future research.

For the sake of completeness, a test of exchangeability is performed (e.g., see Genest et al. (2012)) for copula of the three pairs (X_1, X_2), (X_1, X_3), and (X_2, X_3): we obtain $p-$values of 0.511, 0.206, and 0.181, respectively. The test is performed with the function `exchTest` of the `R` package `copula` and suggests exchangeability for all pairs. That is, we can reorder the variables U_i and U_j, $i, j \in \{1, 2, 3\}, i \neq j$, inside the pair (U_i, U_j), without changing their copula function (see Figure 3.10). Furthermore, by using a goodness-of-fit test for various parametrical multivariate distributions, Di Bernardino and Prieur (2014) proposed a 3-dimensional Gumbel copula with dependence parameter $\theta = 3.93$ for this data-set. The critical layers $\partial \underline{L}(\alpha)$ of this data-set for different values of α are displayed in Figure 3.11.

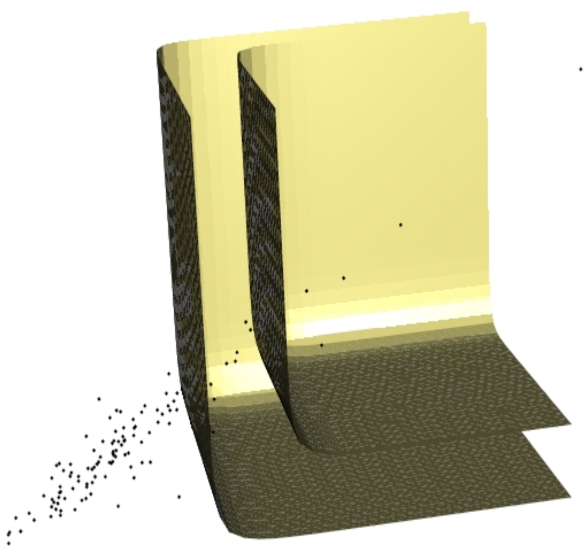

Figure 3.11: Associated critical layers $\partial \underline{L}(\alpha)$, for $\alpha = 0.75$ and 0.9.

Univariate versus multivariate return levels Given the temporal series X_i of the monthly mean of the rainfall measurements for station i, one can define the classic univariate return level with associated probability p as the quantity:

$$x_p^{i,univ} = U_{X_i}\left(\frac{1}{p}\right), \quad \text{for } i = 1, 2, 3,$$

where $p = 1/12T$ and T is called the *return period*, measured in years (see Section 1.1). As proposed by Salvadori et al. (2011), the return level associated with the three stations can be obtained by considering the vector $\overrightarrow{x}_p^{univ} := \left(x_p^{1,univ}, x_p^{2,univ}, x_p^{3,univ}\right)$, that is, the aggregation of univariate quantiles. Remark that not all hydrologists agree with this approach, simply it is a fast way to spot a multivariate return level.

However, $\overrightarrow{x}_p^{univ}$ does not take into account the dependence structure between the three temporal series. As discussed in Section 1.1, while the return level in the univariate setting is usually identified without ambiguity (see, for instance, Corbella and Stretch (2012), and Salvadori et al. (2011)), in the multivariate setting it is a troublesome task (Salvadori et al. (2011), Vandenberghe et al. (2012), and Gräler et al. (2013)).

In this present chapter, a possible procedure is proposed for the identification of the contribution of the margins to the global (regional) multivariate risk (Salvadori et al. (2011)). As mentioned at the beginning of this section, the information concerning the dependence structure of the three rainfall measurements considered is integrated in order to calculate the associated multivariate return levels. We consider $x_p^i := U_{T_i}\left(\frac{1}{p}\right)$, for $i = 1, 2, 3$, where $T_i := [X_i \,|\, (X_1, X_2, X_3) \in \partial \underline{L}(\alpha)]$, with $\alpha \in (0, 1)$ (see Equation (3.1)), and we define our multivariate return level as $\overrightarrow{x}_p := \left(x_p^1, x_p^2, x_p^3\right)$. In this case, x_p^i represents the return level associated with the i-th station conditioned to the fact that the 3-dimensional rainfall data-set belongs to the iso-surface $\partial \underline{L}(\alpha)$.

Estimation procedure and obtained results In the following, the return levels $x_p^{i,univ}$ and x_p^i on the considered rainfall data are estimated, for $i = 1, 2, 3$. We consider here $\alpha = 0.9$. In order to estimate x_p^i, we first deal with the estimation of $\widehat{\gamma}^{T_i}$ for each station. In Figure 3.12 (left panel, first row), the Hill estimator $\widehat{\gamma}_i$ is presented versus k_1 for each station; $k_1 \in [7, 27]$ is chosen since this window is the first stable region of this plot. Similarly, the stable region chosen for the considered $\widehat{\rho}$ corresponds to $k_2 \in [25, 50]$ (see Figure 3.12, right panel, first row). Furthermore, the adaptive sequence $\widehat{k}_U(n)$ is estimated as described in Section 3.6. In addition, the stable region chosen for $\widehat{x}_{p_n}^i$ is $k \in [30, 80]$ (see Figure 3.12, second row). Finally, to gain stability, the estimations $\widehat{\gamma}_i$, $\widehat{\rho}$ and $\widehat{x}_{p_n}^i$ are averaged in the chosen stable regions (see also Cai et al. (2015)).

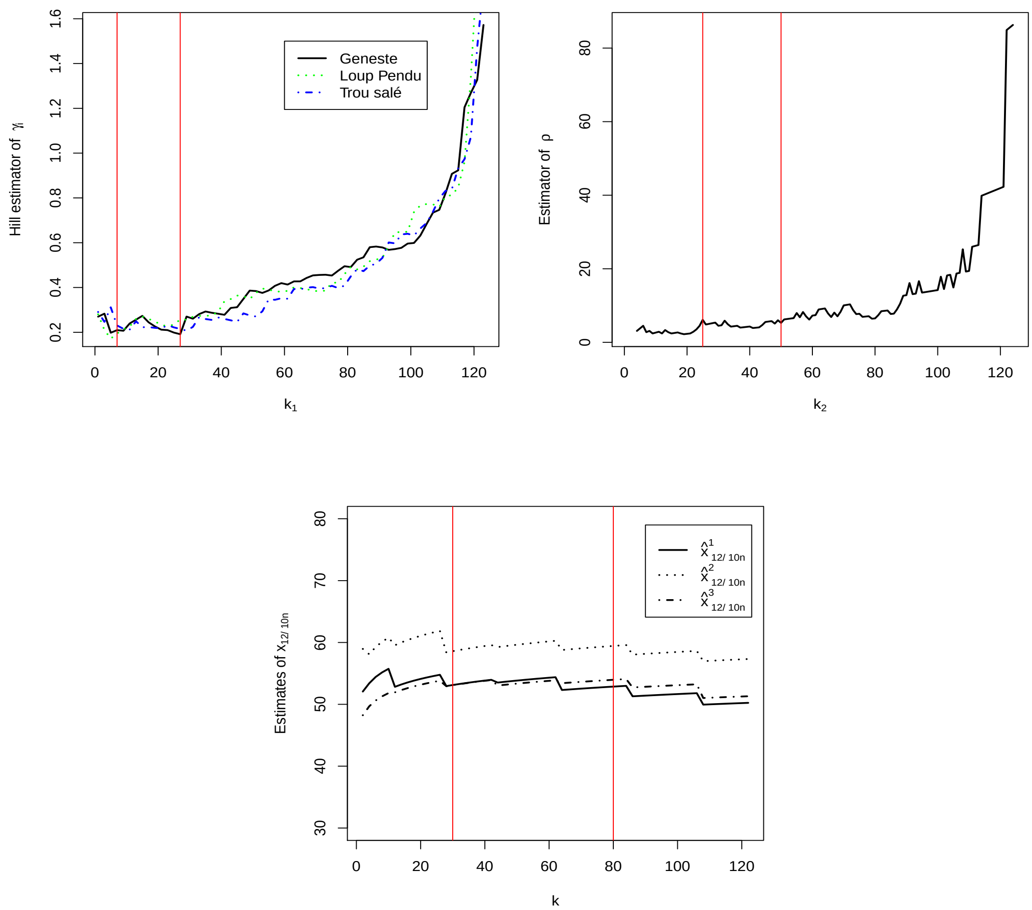

Figure 3.12: Hill estimators $\widehat{\gamma}_i$ versus k_1, for $i = 1, 2, 3$ (left panel, first row). Estimator $\widehat{\rho}$ based on the subvector $(X_1,\ X_3)$ versus k_2 (right panel, first row). Estimates $\widehat{x}^i_{p_n}$ against various values of the intermediate sequence k, for $i = 1, 2, 3$, and $p_n = 12/10n$ (second row). The chosen window for each intermediate sequence is shown with red lines.

The obtained extreme estimation for $\overrightarrow{x}_p$ is presented in Table 3.4 for different probability levels p. In Table 3.4, the empirical estimator $\widehat{x}^{i,emp}_{12/10n}$ given in Equation (3.20) is also included. Unsurprisingly, $\widehat{x}^{i,emp}_{12/10n}$ provide smaller quantifications for risk than the others estimators (see the simulation study in Section 3.5). Furthermore, using the extreme quantile estimator proposed in Theorem 4.3.8 in de Haan and Ferreira (2006), the univariate return level $x^{i,univ}_p$ is estimated for different probability levels p (see Table 3.5).

Note that in Table 3.4, $\widehat{\gamma}_i > 0$ (i.e., the monthly mean of the rainfall measurements for each station i belongs to the Fréchet MDA) and $\widehat{\rho} > 1$ (i.e., upper tail dependency). Values shown in Table 3.4 (in Table 3.5, respectively) represent the estimated multivariate return levels (univariate return levels, respectively) in *mm* with an associated return period of around 8 years (for $p = 12/10n$), 17 years (for $p = 12/20n$), 56 years (for $p = 12/65n$) and 86 years (for $p = 12/100n$).

i	Station	$\widehat{\gamma}_i$	$\widehat{\rho}$	$\widehat{x}^{i,emp}_{12/10n}$	$\widehat{x}^{i}_{12/10n}$	$\widehat{x}^{i}_{12/20n}$	$\widehat{x}^{i}_{12/65n}$	$\widehat{x}^{i}_{12/100n}$
1	Geneste	0.227	4.852	52.33912	53.36817	55.12814	58.25515	59.44177
2	Loup Pendu	0.239	4.852	58.55755	59.36239	61.43124	65.11608	66.51731
3	Trou salé	0.222	4.852	41.49933	53.55529	55.28661	58.36007	59.52553

Table 3.4: Estimated extreme multivariate return level $\widehat{x}^{i}_{p_n}$, for different values of p_n. The Hill estimator $\widehat{\gamma}_i$ is calculated by taking the average for $k_1 \in [7, 27]$; $\widehat{\rho}$ is obtained by taking the average for $k_2 \in [25, 50]$; $\widehat{x}^{i}_{p_n}$ are calculated by taking the average for $k \in [30, 80]$. The empirical estimator $\widehat{x}^{i,emp}_{12/10n}$ in Equation (3.20) is also displayed.

i	Station	$\widehat{x}^{i,univ}_{12/10n}$	$\widehat{x}^{i,univ}_{12/20n}$	$\widehat{x}^{i,univ}_{12/65n}$	$\widehat{x}^{i,univ}_{12/100n}$
1	Geneste	68.72638	80.44542	105.14140	115.94974
2	Loup Pendu	77.65664	91.70115	121.65781	134.89910
3	Trou Salé	69.53792	81.14697	105.50862	116.13393

Table 3.5: Estimated extreme univariate return level $\widehat{x}^{i,univ}_{p}$, for different values of p_n.

In Tables 3.4 and 3.5, a major contribution of the second station (i.e., X_2) can be observed. One can interpret that the manager of the *Bièvre* region needs to pay more attention to this station since it contributes towards producing a flood in the region to a greater degree, both in the univariate and multivariate return level cases.

3.8 Conclusions

The proposed approach to the multivariate return level in the present chapter has the advantage of using a mathematically consistent way of defining the multivariate probability of dangerous events by relying on the iso-curves $\partial \underline{L}(\alpha)$. However, there is no universal choice of an appropriate approach to all real-world problems. It is necessary to address the problem from a probabilistic point of view and to be aware of the practical implications of the approach chosen.

It is also evident in our hydrological study, but not necessarily the case, that the more variables/information included, the smaller the design quantiles become (see multivariate and univariate return levels in Tables 3.4 and 3.5). Indeed, marginal components of the multivariate levels (i.e., $\widehat{x}^{i}_{p}$) are considerably lower than the corresponding univariate return levels (i.e., $\widehat{x}^{i,univ}_{p}$) (see Tables 3.4 and 3.5). This fact can be intuitively interpreted: the probability of an extreme event which simultaneously exceeds a return level in all margins is liable to be much lower than the probability of an event which exceeds the same level in any one of the margins considered alone. Therefore, the univariate levels $x_p^{i,univ}$ should be set much larger in order to obtain the same small exceedance probability p. Salvadori and De Michele (2013) discuss this *dimensionality paradox* and provide a theoretical explanation. The interested reader is also referred to Salvadori et al. (2011) and Gräler et al. (2013) for analogous considerations.

In particular, in our study, the large discrepancy between the estimated $\widehat{x}_p^i$ and $\widehat{x}_p^{i,univ}$ depends on the parameter setting considered ($\alpha = 0.9$, $p_n = \frac{12}{10n}, \frac{12}{20n}, \frac{12}{65n}, \frac{12}{100n}$, with $n = 125$) and on the theoretical properties of the considered multivariate return level x_p^i. For further details about the properties of this risk measure, the interested reader is referred to Propositions 2.3.3 - 2.3.5 and Corollary 2.5.4 in Chapter 2.

Furthermore, one should also be aware of the fact that our T_i-quantile approach (see Equation (3.1)) is applied only to variables that are positively associated and that have a focus on extremes in terms of large values. In all other cases, adaptations should be made in order to operate in the proper region of the copula domain (such as the *directional* multivariate return levels proposed by Torres et al. (2015)).

From a practical perspective, it is impossible to provide a general suggestion for an appropriate approach to estimate multivariate design events that is applicable to a broad set of design exercises. Hitherto, many applications have been based on the concept of univariate return level, since the concept of multivariate return level has a different meaning and is potentially less conservative (as can be observed by comparing Tables 3.4 and 3.5).

On the other hand, when the analyst estimates the extension of flood inundation, a joint return period approach could prove appealing. Indeed, an ensemble of equally rare scenarios (i.e. those with the same return probability p) could help to assess the variability of the flood maps.

Chapter 4

The Component-wise Excess design realization for Archimedean copulas

4.1 Introduction

Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a random vector with distribution function F and survival distribution function $\overline{F}$. By taking into account the notion of critical layers in Definition 1.1.4, Salvadori et al. (2011) provide the definition of design realization given in Definition 1.1.5. As we remark in Section 1.1, Definition 1.1.5 is based on the idea of introducing a suitable function that "weights" the realizations lying on the critical layer of interest. Salvadori et al. (2011) point out that a realization lying on the critical layer may be of major interest when all of its marginal components are exceeded with the largest probability. Therefore, Salvadori et al. (2011) consider as the weight function the multivariate survival distribution function of the considered risk vector (see Definition 4.1.1).

Definition 4.1.1 (Constrained optimization problem for Component-wise Excess design realization). *The Component-wise Excess design realization δ_{CE} of level α is defined as*

$$\delta_{CE}(\alpha) = \underset{\boldsymbol{x} \in \partial \underline{L}(\alpha)}{\arg\max} \; \mathbb{P}(\boldsymbol{X} \in [\boldsymbol{x}, \infty)). \tag{4.1}$$

Definition 4.1.1 suggests searching the point(s) $\boldsymbol{x} = (x_1, \ldots, x_d) \in \partial \underline{L}(\alpha)$ in order to maximize the probability that a dangerous realization $\boldsymbol{y} = (y_1, \ldots, y_d)$ verifies all the following component-wise inequalities: $y_1 \geq x_1, \ldots, y_d \geq x_d$.

Furthermore, Salvadori et al. (2011) estimate the Component-wise Excess design realization to study the behaviour of a dam by using a parametric Gumbel setting. As will be proved below, by considering an Archimedean copula, the constrained optimization problem in Definition 4.1.1 becomes tractable (see Section 4.2). The objectives in this chapter are:

- to provide the explicit expression of the Component-wise Excess design realization given in Salvadori et al. (2011) by assuming a general Archimedean copula dependency between the risks considered;
- to construct an extreme estimation procedure for the obtained Component-wise Excess design realization by using Extreme Value Theory (EVT);
- to study and to compare the performance of our estimator on simulated data and with the estimator of Salvadori et al. (2011) on the same real dam data-set as studied in the aforementioned paper.

To this end, we will assume the following assumption setting:

Assumption 4.1.1 (Assumption setting)**.**

a) The $d-$dimensional random risk vector $\mathbf{X}$ follows an Archimedean copula with a twice differentiable generator $\phi_{\boldsymbol{\theta}}$ and a vector of dependence parameters $\boldsymbol{\theta}$. In addition, the $d-$dimensional random risk vector has a partially strictly-increasing joint distribution function F.

b) The marginal X_i considered has an absolutely continuous distribution F_{X_i}. We denote as $F_{X_i}^{-1}$ the left-continuous inverse of F_{X_i}, for $i \in \{1, \ldots, d\}$.

c) The marginal X_i considered has a heavy tailed distribution, for $i \in \{1, \ldots, d\}$ (see details in Section 4.4).

d) $x \to \phi_{\boldsymbol{\theta}}(1 - 1/x) \in 2RV_{-\rho,\beta}(A_Y)$, with $\rho \in [1, +\infty)$ and $\beta < 0$.

Condition *a*) in Assumption 4.1.1 is useful to obtain a more tractable constrained optimization problem for Component-wise Excess design realization in Definition 4.1.1 (see Proposition 4.2.1). Furthermore, condition *c*) is used in Section 4.4 to construct the extreme quantile estimator by using classic extrapolation techniques for heavy tailed distributions. Condition *c*) is also motivated by our applications in hydrology. Indeed, in these real-life applications, we can easily observe heavy tailed distributions (Pavlopoulos et al. (2008) and Papalexiou et al. (2013)). Moreover, peak and volume variables in the considered data-set in Section 4.7 turn out to be statistically compatible with heavy-tailed distributions in Salvadori et al. (2016). However, it is crucial to notice that our procedure can also be adapted in the case where X_i follows a light tailed distribution (de Haan and Ferreira (2006)). The notions of regularly varying (RV) and second-order regularly varying (2RV) functions are recalled in Appendix A.

The structure of the chapter is as follows: In Section 4.2, we provide the analytical expression of the Component-wise Excess design realization in an Archimedean copula framework. The tail behaviour of the random variable that constitutes our extreme estimator for the Component-wise Excess design realization is studied in Section 4.3. The proposed extreme non-parametric estimator for the Component-wise Excess design realization is presented in Section 4.4. In addition, a Central Limit Theorem for this estimator is provided under certain regularly varying conditions in Section 4.5. The performance of our estimator is analysed in Section 4.6. Our estimator is compared with that of Salvadori et al. (2011) on a real dam data-set in Section 4.7. Finally, we include conclusions of this chapter in Section 4.8.

4.2 Explicit expression of Component-wise Excess design realization

In this section, we solve the constrained maximization problem in Equation (4.1) in order to obtain the closed-form expression of the Component-wise excess design realization in our Archimedean copula framework.

Proposition 4.2.1 (Component-wise Excess design realization in the Archimedean copula setting)**.** *Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be the considered random risk vector. Assume that $\boldsymbol{X}$ satisfies conditions* a) *and* b) *in Assumption 4.1.1. The solution $\delta_{CE}(\alpha)$ of the constrained maximization problem in Equation* (4.1) *is given by*

$$\delta_{CE}(\alpha) = \left\{ F_{X_1}^{-1}\left(\phi_{\boldsymbol{\theta}}^{-1}\left(\frac{\phi_{\boldsymbol{\theta}}(\alpha)}{d}\right)\right), \ldots, F_{X_d}^{-1}\left(\phi_{\boldsymbol{\theta}}^{-1}\left(\frac{\phi_{\boldsymbol{\theta}}(\alpha)}{d}\right)\right)\right\}. \tag{4.2}$$

Proof. Let $\alpha \in (0,1)$. Let C and $\overline{C}$ be, respectively, the copula and the joint survival function associated with the random vector $\mathbf{V} = (V_1, \ldots, V_d)$ with $V_i \overset{d}{=} F_{X_i}(X_i)$, for $i \in \{1, \ldots, d\}$. Note that the copula version of the constrained optimization problem in Equation (4.1) can be written as

$$\delta_{CE}(\alpha) = \underset{\mathbf{v} \in \partial L_C(\alpha)}{\arg\max}\ \overline{C}(v_1, \ldots, v_d), \tag{4.3}$$

with $\partial L_C(\alpha) = \{\mathbf{v} \in [0,1]^d : C(\mathbf{v}) = \alpha\}$.

Equivalently, one can write the constrained optimization problem (4.3) as

$$\begin{aligned} &\underset{\mathbf{s}}{\arg\max} && \overline{C}(\phi_{\boldsymbol{\theta}}^{-1}(s_1), \ldots, \phi_{\boldsymbol{\theta}}^{-1}(s_d)) \\ &\text{s.t.:} && \sum_{i=1}^{d} s_i = \phi_{\boldsymbol{\theta}}(\alpha), \\ &&& s_i \geq 0, \quad \text{for} \quad i = 1, \ldots, d. \end{aligned} \tag{4.4}$$

Our aim is to find the solution of the constrained optimization problem in (4.4). From Theorem 2.21 in Boche and Jorswieck (2007), if $\overline{C}(\phi_{\boldsymbol{\theta}}^{-1}(s_1), \ldots, \phi_{\boldsymbol{\theta}}^{-1}(s_d))$ is a Schur-concave function (see Definition 1.4.1), the global maximum for the problem in (4.4) is achieved by $\boldsymbol{s}^* = \left(\frac{\phi_{\boldsymbol{\theta}}(\alpha)}{d}, \ldots, \frac{\phi_{\boldsymbol{\theta}}(\alpha)}{d}\right)$. Therefore, our aim is to study the Schur-concavity of the following function: $\overline{C}(\phi_{\boldsymbol{\theta}}^{-1}(s_1), \ldots, \phi_{\boldsymbol{\theta}}^{-1}(s_d))$.

We now prove that $\overline{C}$ associated with an Archimedean copula is a Schur-concave function. To this end, it is helpful to realize that by using the symmetry property, one can take $d = 2$ without loss of generality. The interested reader is referred to Marshall et al. (2011) (Section A.5) for further details. That is, it is sufficient to prove that bivariate joint survival function $\overline{C}$ is a Schur-concave function. In addition, every Archimedean copula is Schur-concave (see Proposition 1.4.3). Furthermore, from Proposition 1.4.2, a bivariate copula C is Schur-concave if and only if $\hat{C}(u, v)$ (see $\hat{C}$ definition in Section 1.4) associated with C is also a Schur-concave function.

Since, in our case, C is an Archimedean copula (see condition a) in Assumption 4.1.1), then $\hat{C}(u, v)$ is also a Schur-concave function. We remark that $\overline{C}(u, v) = \hat{C}(1 - u, 1 - v)$, for $(u, v) \in [0, 1]^2$ (see Section 1.4). Therefore, in order to obtain the desired result, we have to prove that $\hat{C}(f(u), f(v)))$, with $f(u) = 1 - u$, $u \in [0, 1]$, is a Schur-concave function. This last statement holds true because $\hat{C}(f(u), f(v))$ is a composition of an increasing Schur-concave function ($\hat{C}$) and a concave function (f). The interested reader is referred to Marshall et al. (2011) (Section B.2) for further details. Using similar arguments, since $\overline{C}$ is a decreasing Schur-concave function and $\phi_{\boldsymbol{\theta}}^{-1}$ is a convex function, therefore $\overline{C}(\phi_{\boldsymbol{\theta}}^{-1}(s_1), \ldots, \phi_{\boldsymbol{\theta}}^{-1}(s_d))$ is a Schur-concave function.

Finally, by taking $s_i = \phi(v_i)$, for $i = 1, \ldots, d$, from Theorem 2.21 in Boche and Jorswieck (2007), the global maximum in the constrained optimization problem (4.4) is achieved through $\boldsymbol{v}^* = \left(\phi_{\boldsymbol{\theta}}^{-1}\left(\frac{\phi_{\boldsymbol{\theta}}(\alpha)}{d}\right), \ldots, \phi_{\boldsymbol{\theta}}^{-1}\left(\frac{\phi_{\boldsymbol{\theta}}(\alpha)}{d}\right)\right)$. By using the Probability Integral Transform Theorem (see Section 1.5.8.3 in Denuit et al. (2005)) for each margin, we obtain the result. More precisely, the global optimum point for the constrained optimization problem in Equation (4.1) is given by $\left(F_{X_1}^{-1}(v_1^*), \ldots, F_{X_d}^{-1}(v_d^*)\right)$. □

As a limit case, we remark that, for $d = 1$, the solution of the maximization problem in Equation (4.1) goes to the univariate marginal quantile.

We now present an illustration of Proposition 4.2.1. We consider the following bivariate models:

- Ali-Mikhail-Haq copula with dependence parameter $\theta = 0.5$ and Uniform margins;
- Clayton copula with dependence parameter $\theta = 3$ and Exponential margins with $\lambda = 1$;
- Gumbel copula with dependence parameter $\theta = 4$ and Pareto margins with $\delta_1 = 1$ and $\delta_2 = 2$.

In Figure 4.1, we show critical layers $\partial \underline{L}(\alpha)$ (red lines), the survival functions $\overline{F}(x_1, x_2)$ (black surfaces), $\delta_{CE}(\alpha)$ (green dots) for different values of α and the associated maximum points (blue dots) for the three models considered.

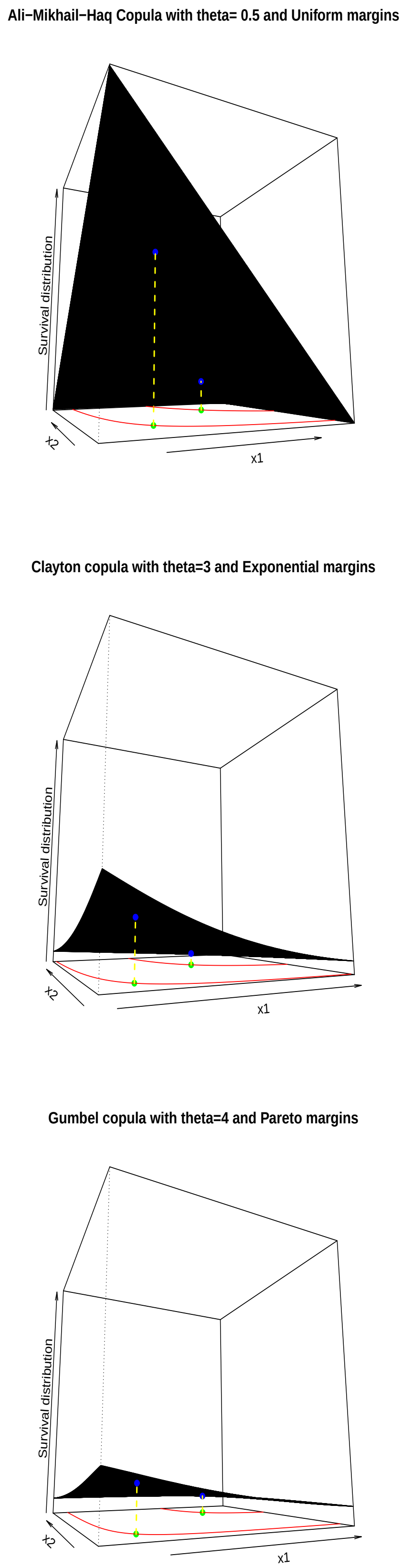

Figure 4.1: Critical layers $\partial \underline{L}(\alpha)$ (red lines), survival functions (black surfaces), $\delta_{CE}(\alpha)$ (green dots) and the associated maximum points (blue dots) for the considered models. We take $\alpha = 0.1$ and 0.5 (first row); $\alpha = 0.6$ and 0.8 (second row); $\alpha = 0.8$ and 0.9 (third row).

4.3 First and second order tail index for $Y \stackrel{d}{=} \max\{V_1, \ldots, V_d\}$

Let $Y \stackrel{d}{=} \max\{V_1, \ldots, V_d\}$ be the random variable with distribution function F_Y. The (first-order) tail behaviour of Y is studied in the following proposition. The interested reader is also referred to Theorem 1.2.2.

Proposition 4.3.1 (The von Mises condition for Y). *Assume that $\boldsymbol{X} = (X_1, \ldots, X_d)$ satisfies conditions* a) *and* b) *in Assumption 4.1.1. Assume that $\phi_{\boldsymbol{\theta}} \in RV_\rho(1)$, with $\rho \in [1, +\infty)$. Let $Y \stackrel{d}{=} \max\{V_1, \ldots, V_d\}$ with $V_i \stackrel{d}{=} F_{X_i}(X_i)$, for $i \in \{1, \ldots, d\}$. Therefore, F_Y verifies the von Mises condition with tail index $\gamma_Y = -1$. In particular, $Y \in MDA(-1)$.*

Proof. The vector $\mathbf{V} := (V_1, \ldots, V_d)$ has an Archimedean copula as its distribution function, therefore

$$P[\max\{V_1, \ldots, V_d\} \leq v] = \phi_{\boldsymbol{\theta}}^{-1}\left(d\, \phi_{\boldsymbol{\theta}}(v)\right), \ v \in (0,1). \tag{4.5}$$

Hence, we obtain

$$\begin{gathered} F'_Y(t) = (\phi_{\boldsymbol{\theta}}^{-1})'[d\,\phi_{\boldsymbol{\theta}}(t)]\, d\,\phi'_{\boldsymbol{\theta}}(t) \quad \text{and} \\ F''_Y(t) = (\phi_{\boldsymbol{\theta}}^{-1})''[d\,\phi_{\boldsymbol{\theta}}(t)]\, d^2\, (\phi'_{\boldsymbol{\theta}}(t))^2 + (\phi_{\boldsymbol{\theta}}^{-1})'[d\,\phi_{\boldsymbol{\theta}}(t)]\, d\,\phi''_{\boldsymbol{\theta}}(t). \end{gathered}$$

The limit in Definition 1.2.2 can now be calculated.

$$\begin{aligned} &\lim_{t\uparrow 1} \tfrac{(1-F_Y(t))F''_Y(t)}{(F'_Y(t))^2} \\ &= \lim_{t\uparrow 1} \frac{(1-\phi_{\boldsymbol{\theta}}^{-1}(d\,\phi_{\boldsymbol{\theta}}(t)))\left((\phi_{\boldsymbol{\theta}}^{-1})''[d\,\phi_{\boldsymbol{\theta}}(t)]\, d^2\,(\phi'_{\boldsymbol{\theta}}(t))^2 + (\phi_{\boldsymbol{\theta}}^{-1})'[d\,\phi_{\boldsymbol{\theta}}(t)]\, d\,\phi''_{\boldsymbol{\theta}}(t)\right)}{\left((\phi_{\boldsymbol{\theta}}^{-1})'[d\,\phi_{\boldsymbol{\theta}}(t)]\, d\,\phi'_{\boldsymbol{\theta}}(t)\right)^2} \\ &= \lim_{t\uparrow 1}(1-\phi_{\boldsymbol{\theta}}^{-1}(d\,\phi_{\boldsymbol{\theta}}(t)))\,(\phi_{\boldsymbol{\theta}}^{-1})''[d\,\phi_{\boldsymbol{\theta}}(t)]\,(\phi'_{\boldsymbol{\theta}}(\phi_{\boldsymbol{\theta}}^{-1}(d\,\phi_{\boldsymbol{\theta}}(t))))^2 \\ &+ \lim_{t\uparrow 1}(1-\phi_{\boldsymbol{\theta}}^{-1}(d\,\phi_{\boldsymbol{\theta}}(t)))\,\phi'_{\boldsymbol{\theta}}(\phi_{\boldsymbol{\theta}}^{-1}(d\,\phi_{\boldsymbol{\theta}}(t)))\,\frac{\phi''_{\boldsymbol{\theta}}(t)}{d\,(\phi'_{\boldsymbol{\theta}}(t))^2}. \end{aligned}$$

Given the assumption $\phi_{\boldsymbol{\theta}} \in RV_\rho(1)$, then $\phi'_{\boldsymbol{\theta}} \in RV_{\rho-1}(1)$. Therefore,

$$\lim_{t\uparrow 1}(1-\phi_{\boldsymbol{\theta}}^{-1}(d\,\phi_{\boldsymbol{\theta}}(t)))\,(\phi_{\boldsymbol{\theta}}^{-1})''[d\,\phi_{\boldsymbol{\theta}}(t)]\,(\phi'_{\boldsymbol{\theta}}(\phi_{\boldsymbol{\theta}}^{-1}(d\,\phi_{\boldsymbol{\theta}}(t))))^2 = \rho - 1$$

and

$$\lim_{t\uparrow 1}(1-\phi_{\boldsymbol{\theta}}^{-1}(d\,\phi_{\boldsymbol{\theta}}(t)))\,\phi'_{\boldsymbol{\theta}}(\phi_{\boldsymbol{\theta}}^{-1}(d\,\phi_{\boldsymbol{\theta}}(t)))\,\frac{\phi''_{\boldsymbol{\theta}}(t)}{d\,(\phi'_{\boldsymbol{\theta}}(t))^2} = -\rho + 1.$$

Finally, we conclude that $\lim_{t\uparrow 1} \frac{(1-F_Y(t))F''_Y(t)}{(F'_Y(t))^2} = 0 \Rightarrow \gamma_Y = -1$. Hence the result follows. $\square$

The (second-order) tail behaviour of Y is studied in the following proposition.

Proposition 4.3.2. *Assume that* $\boldsymbol{X} = (X_1, \ldots, X_d)$ *satisfies conditions* a), b) *and* d) *in Assumption 4.1.1. Let* $Y \overset{d}{=} \max\{V_1, \ldots, V_d\}$ *with* $V_i \overset{d}{=} F_{X_i}(X_i)$, *for* $i \in \{1, \ldots, d\}$. *Therefore,* $1 - U_Y \in 2RV_{-1,\beta}(A_Y)$.

Proof. From Equation (4.5), it is known that

$$\overline{F}_Y(1-1/t) = 1 - \phi_{\boldsymbol{\theta}}^{-1}\left(d\,\phi_{\boldsymbol{\theta}}(1-1/t)\right).$$

Since $x \to \phi_{\boldsymbol{\theta}}(1-1/x) \in 2RV_{-\rho,\beta}(A_Y)$, then from Proposition 2.4 in Mao and Hu (2012), we can write

$$\overline{F}_Y(1-1/t) = 1 - \phi_{\boldsymbol{\theta}}^{-1}\left[d\,c\,t^{-\rho}\left(1 + \tfrac{1}{\beta}A_Y(t) + o(A_Y(t))\right)\right].$$

In addition, $x \to \phi_{\boldsymbol{\theta}}(1-1/x) \in 2RV_{-\rho,\beta}(A_Y)$ implies $\phi_{\boldsymbol{\theta}} \in RV_\rho(1)$. From Remark C in Di Bernardino and Rullière (2014), it is verified that $1 - \phi_{\boldsymbol{\theta}}^{-1}(1/x) \in RV_{-1/\rho}$. We now obtain

$$\overline{F}_Y(1-1/t) = (dc)^{1/\rho}\,t^{-1}\left(1 + \tfrac{1}{\beta}A_Y(t) + o(A_Y(t))\right)^{1/\rho}.$$

By using the Taylor expansion,

$$\overline{F}_Y(1-1/t) = (dc)^{1/\rho}\,t^{-1}\left(1 + \tfrac{1}{\beta\,\rho}A_Y(t) + o(A_Y(t))\right).$$

It can be observed that $|\tilde{A}_Y| := |\frac{1}{\rho}A_Y| \in RV_\beta$ from the assumptions. From Proposition 2.4 in Mao and Hu (2012), the result is given. $\square$

4.4 Non-parametric extreme estimator for Component-wise Excess design realization

Assume that the considered risk vector $\mathbf{X}$ satisfies conditions in Assumption 4.1.1. Using Proposition 4.2.1, we now propose an extreme non-parametric estimation procedure for each component of the Component-wise Excess design realization $\delta_{CE}(\alpha)$ in Equation (4.2) for extreme value $\alpha := \alpha_n \to 1$, for $n \to \infty$, where n is the sample size considered and $d \geq 2$.

From conditions *a*) and *b*) in Assumption 4.1.1, the i-th component of $\delta_{CE}(\alpha)$ in Equation (4.2) can be written as the $(1-p)$-quantile of the random variable X_i with $p = 1 - \phi_{\boldsymbol{\theta}}^{-1}\left(\frac{\phi_{\boldsymbol{\theta}}(\alpha)}{d}\right)$, for $i \in \{1, \ldots, d\}$. Let $\alpha := \alpha_n \to 1$, therefore $p := p_n \to 0$, as $n \to +\infty$.

The goal of this section is to estimate

$$x^i_{p_n} := U_{X_i}\left(\frac{1}{p_n}\right), \qquad \text{for } i \in \{1, \ldots, d\}, \tag{4.6}$$

for $p_n := 1 - \phi_{\boldsymbol{\theta}}^{-1}\left(\frac{\phi_{\boldsymbol{\theta}}(\alpha_n)}{d}\right) \to 0$, as $n \to +\infty$, and $U_{X_i}(t) := F_{X_i}^{-1}\left(1 - \frac{1}{t}\right)$, for $t > 1$. Therefore, the final estimator is based on the following "plug-in procedure": we first deal with the estimation of the random level p_n, and secondly with the estimation of $x_{p_n}^i$ in (4.6). These two steps are detailed below.

First step: estimation of the random risk level p_n Let $V_i = F_{X_i}(X_i)$, for $i \in \{1, \dots, d\}$. The vector $\mathbf{V} := (V_1, \dots, V_d)$ has the considered Archimedean copula as its distribution function, therefore

$$P[\max\{V_1, \dots, V_d\} \leq v] = \phi_{\boldsymbol{\theta}}^{-1}\left(d\, \phi_{\boldsymbol{\theta}}(v)\right), \ v \in (0,1).$$

Let $Y :\overset{d}{=} \max\{V_1, \dots, V_d\}$, therefore $p_n = 1 - F_Y^{-1}(\alpha_n)$. The random risk level p_n can be written as a function of the α_n-quantile of Y, that is,

$$p_n = 1 - U_Y\left(\frac{1}{1-\alpha_n}\right) = 1 - x_{1-\alpha_n}^Y.$$

Under conditions $a)$, $b)$ and $d)$ in Assumption 4.1.1, from Proposition 4.3.1, $Y \in MDA(-1)$. Therefore, by using Theorem 1.1.13 in de Haan and Ferreira (2006), it is verified that $U_Y \in ERV_{-1}$ (see Appendix A for the definition of ERV). Following the same approximation technique as in Equation (3.1.6) in de Haan and Ferreira (2006), we obtain

$$x_{1-\alpha_n}^Y = U_Y\left(\frac{1}{1-\alpha_n}\right) \sim\ U_Y\left(\frac{n}{k_Y}\right) + a_Y\left(\frac{n}{k_Y}\right) \frac{\left(\frac{k_Y}{n\,(1-\alpha_n)}\right)^{\gamma_Y} - 1}{\gamma_Y}, \tag{4.7}$$

where k_Y is an intermediate sequence such that $k_Y = k_Y(n) \to \infty$, $k_Y(n)/n \to 0$, as $n \to \infty$. Let $Y_{n-k_Y,n}$ be the $(n-k_Y)$-th order statistic of the sample $(Y_1, \dots, Y_n)$. The natural estimator of $U_Y\left(\frac{n}{k_Y}\right)$ is its empirical counterpart $Y_{n-k_Y,n}$. In order to estimate $a_Y(n/k_Y)$ and γ_Y, we consider the probability-weighted moment estimators defined in Equations (3.6.9) and (3.6.10) in de Haan and Ferreira (2006). For the sake of clarity, these estimators are laid out below. We take

$$\widehat{\gamma}_Y = 1 - \left(\frac{P_n}{2\,Q_n} - 1\right)^{-1} \tag{4.8}$$

and

$$\widehat{a}_Y\left(\frac{n}{k_Y}\right) := \widehat{\sigma}_{PWM} = P_n\left(\frac{P_n}{2\,Q_n} - 1\right)^{-1}, \tag{4.9}$$

with $P_n := \frac{1}{k_Y}\sum_{i=0}^{k_Y-1} Y_{n-i,n} - Y_{n-k_Y,n}$ and $Q_n := \frac{1}{k_Y}\sum_{i=0}^{k_Y-1} \frac{i}{k_Y}\left(Y_{n-i,n} - Y_{n-k_Y,n}\right)$. The consistency results for the estimators $Y_{n-k_Y,n}$, $\widehat{\gamma}_Y$ and $\widehat{a}_Y(n/k_Y)$ can be found in Theorems 2.4.1 and 3.6.1 in de Haan and Ferreira (2006).

Hence, from Equation (4.7), we obtain

$$\widehat{p}_n = 1 - \widehat{x}^Y_{1-\alpha_n} = 1 - \left(Y_{n-k_Y,n} + \widehat{a}_Y \left(\frac{n}{k_Y}\right) \frac{\left(\frac{k_Y}{n\,(1-\alpha_n)}\right)^{\widehat{\gamma}_Y} - 1}{\widehat{\gamma}_Y} \right). \tag{4.10}$$

Second step: estimation of the extreme quantile $x^i_{p_n}$ From condition $c)$ in Assumption 4.1.1, there exists $\gamma_i > 0$ such that, for all $x > 0$, $U_{X_i} \in RV_{\gamma_i}$ (Appendix A).

In this case, as $n \to \infty$,

$$x^i_{p_n} = U_{X_i}\left(\frac{1}{p_n}\right) \sim U_{X_i}\left(\frac{n}{k_i}\right)\left(\frac{k_i}{n\,p_n}\right)^{\gamma_i}, \tag{4.11}$$

where k_i is an intermediate sequence such that $k_i = k_i(n) \to \infty$, $k_i(n)/n \to 0$, as $n \to \infty$. We denote the $(n-k_i)$-th order statistic of the sample $(X^i_1, \dots, X^i_n)$ as $X^i_{n-k_i,n}$. We estimate the tail index γ_i in Equation (4.11) by using the Hill estimator (Hill (1975)) recalled in (3.8). We can therefore estimate $x^i_{p_n}$ in (4.11) by

$$\widehat{x}^i_{p_n} = X^i_{n-k_i,n}\left(\frac{k_i}{n\,p_n}\right)^{\widehat{\gamma}_i}. \tag{4.12}$$

Finally, by using a plug-in technique with Equations (4.10) and (4.12), we propose the following extreme estimator for the i-th component of $\delta_{CE}(\alpha)$ in Equation (4.2):

$$\widehat{x}^i_{\widehat{p}_n} = X^i_{n-k_i,n}\left(\frac{k_i}{n\,\widehat{p}_n}\right)^{\widehat{\gamma}_i}. \tag{4.13}$$

Remark 4.4.1. *Notice that Y is an unobservable random variable in real-life applications (see, e.g., Section 4.7). In this case, one has previously to construct a pseudo-observed version of Y. More precisely, consider a real data-set of observations $\{\mathbf{X}^{(k)} = (X^{(k)}_1, \dots, X^{(k)}_d)\}_{k\in\{1,\dots,n\}}$ of the random risk vector $\mathbf{X}$. Firstly, we define pseudo-observations $\{\mathbf{V}^{(k)} = (V^{(k)}_1, \dots, V^{(k)}_d)\}_{k\in\{1,\dots,n\}}$ by setting every component i for observation number k to*

$$V^{(k)}_i = \frac{1}{n+1}\sum_{j=1}^{n} 1_{\left\{X^{(j)}_i \le X^{(k)}_i\right\}},$$

with $i = 1, \dots, d$, $k \in \{1, \dots, n\}$. One can check that, for any $i = 1, \dots, d$, $k \in \{1, \dots, n\}$, $V^{(k)}_i \in (0,1)$. Therefore, it is obtained the desired univariate sample: $\{\widehat{Y}^k := \max\{V^{(k)}_1, \dots, V^{(k)}_d\}\}_{k\in\{1,\dots,n\}}$. Consistency of this method in a non-parametric setting has been established in Einmahl et al. (2001), Einmahl and Segers (2009). We remark that, since Y cannot be observed, in our procedure we neglect the uncertainty induced by the margins. More precisely, since the uncertainty induced by the estimation *of $\widehat{Y}^k$ is not taken into account in our procedure, then the main result in the following Theorem 4.5.1 is only valid under full knowledge of the margins. Therefore, the confidence intervals provided in Theorem 4.5.1 can be understood only as optimistic approximations. However, this issue lies beyond the scope of the present chapter and may be addressed in future work.*

4.5 Asymptotic convergence

In this section, we obtain the Central Limit Theorem for the proposed extreme estimator in (4.13). Theorem 4.5.1 provides two results: the consistency for the estimated random risk level p_n (see Equation (4.14)); the consistency of the final extreme quantile estimator $\widehat{x}^i_{\widehat{p}_n}$ (see Equation (4.16)).

Theorem 4.5.1. *Assume that $\boldsymbol{X} = (X_1, \ldots, X_d)$ satisfies conditions in Assumption 4.1.1. Let $Y \stackrel{d}{=} \max\{V_1, \ldots, V_d\}$ with $V_i \stackrel{d}{=} F_{X_i}(X_i)$, for $i \in \{1, \ldots, d\}$. Let $\alpha_n \to 1$, $p_n = 1 - x^Y_{1-\alpha_n}$ and $\widehat{p}_n$ remain as in Equation (4.10).*

Suppose:

1. *$k_Y = k_Y(n) \to \infty$, $k_Y/n \to 0$, and $\sqrt{k_Y} A_Y(n/k_Y) \to \lambda \in \mathbb{R}$, for $n \to \infty$ [1];*
2. *$n(1-\alpha_n) = o(k_Y)$ and $\log(n(1-\alpha_n)) = o(\sqrt{k_Y})$, for $n \to \infty$.*

Let $d^Y_n := k_Y/(n(1-\alpha_n))$ and $q_\gamma(t) := \int_1^t s^{\gamma-1}\log(s)ds$. Let $v_n = \frac{\sqrt{k_Y}}{a_Y\left(\frac{n}{k_Y}\right) q_{\gamma_Y}(d^Y_n)}$ and $\widehat{v}_n = \frac{\sqrt{k_Y}}{\widehat{a}_Y\left(\frac{n}{k_Y}\right) q_{\widehat{\gamma}_Y}(d^Y_n)}$. Therefore, for $n \to \infty$,

$$\widehat{v}_n(\widehat{p}_n - p_n) \xrightarrow{d} \Theta_1, \tag{4.14}$$

where $\Theta_1 = \Gamma + B + \Lambda + \frac{\lambda}{\beta - 1}$ and where B is a standard normal distribution and, Γ and Λ are normal distributions defined as in Theorems 3.6.1 and 4.3.1 in de Haan and Ferreira (2006).

Moreover, assume that:

3. *$U_{X_i} \in 2RV_{\gamma_i,\tau_i}(A_i)$, $\gamma_i > 0$ and $\tau_i < 0$;*
4. *$k_i = k_i(n) \to \infty$, $k_i/n \to 0$, and $\sqrt{k_i} A_i(n/k_i) \to \lambda_i \in \mathbb{R}$, for $n \to \infty$;*
5. *$np_n = o(k_i)$ and $\log(n p_n) = o(\sqrt{k_i})$, for $n \to \infty$.*

Let $d^i_n := k_i/(n p_n)$. Define $x^i_{p_n}$ and $\widehat{x}^i_{\widehat{p}_n}$ as in Equations (4.11) and (4.13), respectively. If, for $n \to \infty$,

$$\frac{\sqrt{k_i}}{\log(d^i_n)\, v_n} \to 0, \tag{4.15}$$

then it is verified that

$$\frac{\sqrt{k_i}}{\log(d^i_n)} \left(\frac{\widehat{x}^i_{\widehat{p}_n}}{x^i_{p_n}} - 1 \right) \xrightarrow{d} \Theta_2, \tag{4.16}$$

where Θ_2 is a normal random variable with mean $\lambda_i/(1-\tau_i)$ and variance γ_i^2.

[1] A_Y is the auxiliary function of U_Y, since $1 - U_Y \in 2ERV_{-1,\beta}(A_Y)$ (see proof of Theorem 4.5.1).

Proof. Firstly, note that

$$\widehat{v}_n(\widehat{x}^Y_{1-\alpha_n} - x^Y_{1-\alpha_n}) \xrightarrow{d} \Theta_1, \tag{4.17}$$

where $\Theta_1 = \Gamma + B + \Lambda + \frac{\lambda}{\beta - 1}$ and where B is a standard normal and, Γ and Λ are normal distributions as defined in Theorems 3.6.1 and 4.3.1 in de Haan and Ferreira (2006).

Indeed, from Proposition 4.3.2, condition $d)$ in Assumption 4.1.1 implies that $1 - U_Y \in 2ERV_{-1,\beta}(A_Y)$ and $a(t) = (1 - U_Y(t))$. Therefore, the asymptotic result in Equation (4.17) comes from Theorems 3.6.1 and 4.3.1, and from Corollary 4.3.2 in de Haan and Ferreira (2006).

Consequently, we obtain

$$\widehat{v}_n(\widehat{p}_n - p_n) \xrightarrow{d} -\Theta_1,$$

(Theorem on page 24 in Serfling (1980)).

Under conditions 3, 4 and 5 in Theorem 4.5.1, and by applying Theorem 4.3.8 in de Haan and Ferreira (2006), we determine that

$$\frac{\sqrt{k_i}}{\log(d^i_n)} \left(\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} - 1 \right) \xrightarrow{d} \Theta_2,$$

where Θ_2 is a normal random variable with mean $\lambda_i/(1 - \tau_i)$ and variance γ_i^2 (see Theorem 3.2.5 in de Haan and Ferreira (2006)).

We now write $\widehat{x}^i_{\widehat{p}_n}$ as a function of $\widehat{x}^i_{p_n}$. That is, we can write

$$\widehat{x}^i_{\widehat{p}_n} = X^i_{n-k_i,n} \left(\frac{k_i}{n \frac{\widehat{p}_n}{p_n} p_n} \right)^{\widehat{\gamma}_i} = \widehat{x}^i_{p_n} \left(\frac{\widehat{p}_n}{p_n} \right)^{-\widehat{\gamma}_i}.$$

Therefore, we obtain

$$\begin{aligned}
&\frac{\sqrt{k_i}}{\log(d^i_n)} \left(\frac{\widehat{x}^i_{\widehat{p}_n}}{x^i_{p_n}} - 1 \right) \\
&= \frac{\sqrt{k_i}}{\log(d^i_n)} \left(\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} \left(\frac{\widehat{p}_n}{p_n} \right)^{-\widehat{\gamma}_i} - 1 \right) \\
&= \frac{\sqrt{k_i}}{\log(d^i_n)} \left(\frac{\widehat{p}_n}{p_n} \right)^{-\widehat{\gamma}_i} \left(\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} - \left(\frac{\widehat{p}_n}{p_n} \right)^{\widehat{\gamma}_i} \right) \\
&= \frac{\sqrt{k_i}}{\log(d^i_n)} \left(\frac{\widehat{p}_n}{p_n} \right)^{-\widehat{\gamma}_i} \left(\frac{\widehat{x}^i_{p_n}}{x^i_{p_n}} - 1 \right) + \frac{\sqrt{k_i}}{\log(d^i_n)} \left[\left(\frac{\widehat{p}_n}{p_n} \right)^{-\widehat{\gamma}_i} - 1 \right].
\end{aligned} \tag{4.18}$$

On the other hand,

$$\frac{\widehat{p}_n}{p_n} = \frac{-\Theta_1}{v_n} + 1 + o_{\mathbb{P}}\left(\frac{1}{v_n} \right).$$

Hence, we obtain

$$\frac{\sqrt{k_i}}{\log\left(d_n^i\right)}\left[\left(\frac{\widehat{p}_n}{p_n}\right)^{-\widehat{\gamma}_i}-1\right]=\frac{\sqrt{k_i}}{\log d_n^i}\left[\left(\frac{-\Theta_1}{v_n}+1+o_{\mathbb{P}}\left(\frac{1}{v_n}\right)\right)^{-\widehat{\gamma}_i}-1\right].$$

By using Taylor's expansion and condition (4.15), it is verified that

$$\frac{\sqrt{k_i}}{\log\left(d_n^i\right)}\left[\left(\frac{\widehat{p}_n}{p_n}\right)^{-\widehat{\gamma}_i}-1\right]\xrightarrow{\mathbb{P}}0. \tag{4.19}$$

In addition, Equation (4.19) implies that

$$\left(\frac{\widehat{p}_n}{p_n}\right)^{-\widehat{\gamma}_i}\xrightarrow{\mathbb{P}}1. \tag{4.20}$$

Finally, by using (4.19) and (4.20) in Equation (4.18), from Slutsky's Theorem, we attain the result. □

It should be borne in mind that the quantity d_n^i in the convergence rate of Equation (4.16) remains unknown. However, an adaptive version of this consistency result is provided in Corollary 4.5.1.

Corollary 4.5.1. *Let $\widehat{d}_n^i := k_i/(n\,\widehat{p}_n)$. Conditions of Theorem 4.5.1 imply $\frac{\log(\widehat{d}_n^i)}{\log(d_n^i)}\xrightarrow{\mathbb{P}}1$, for $n\to\infty$. Hence an equivalent statement of Equation* (4.16) *is*

$$\frac{\sqrt{k_i}}{\log\left(\widehat{d}_n^i\right)}\left(\frac{\widehat{x}_{\widehat{p}_n}^i}{x_{p_n}^i}-1\right)\xrightarrow{d}\Theta_2,$$

where Θ_2 is defined as in Theorem 4.5.1.

The proof of Corollary 4.5.1 comes down trivially by using the convergency $\frac{\widehat{p}_n}{p_n}\xrightarrow{\mathbb{P}}1$ for $n\to\infty$ (see also proof of Theorem 4.5.1). We remark that the latter form proposed in Corollary 4.5.1 is more useful for constructing confidence intervals for $\widehat{x}_{\widehat{p}_n}^i$ (see Section 4.7).

4.6 Simulation study

A simulation and comparison study is implemented to investigate the finite sample performance of our estimator in this section. The estimation procedure presented in this section involves the notation progressively introduced in Section 4.4. To improve clarity, a comprehensive scheme of our extreme estimation procedure is presented in Algorithm 1. The nature of the considered parameters is specified; in particular, we distinguish between tuning parameters and estimated/calculated quantities. Firstly, we simulate under the chosen Archimedean copula model.

Simulations
Choose the sample size n.
Choose the dimension $d \geq 2$.
Simulate a sample $(u_1^k, \ldots, u_d^k)$ from a d-Archimedean copula, with $k \in \{1, \ldots, n\}$.
Build the sample $Y_k := \max\{u_1^k, \ldots, u_d^k\}$, for $k \in \{1, \ldots, n\}$.
Obtain marginal distribution samples $x_i^k = F_{X_i}^{-1}(u_i^k)$ for $i = 1, \ldots, d$ and $k = 1, \ldots, n$.

Algorithm 1 Comprehensive scheme for the estimator $\widehat{x}^i_{\widehat{p}_n}$

Input parameters
Choose the extreme risk level $\alpha = \alpha_n$.
Select the margin X_i, for $i \in \{1, \ldots, d\}$.

Estimations
Choose $k_Y(n)$ and estimate
- the tail index $\widehat{\gamma}_Y$ by using the estimator in Equation (4.8);
- the scale sequence $\widehat{a}_Y\left(\frac{n}{k_Y}\right)$ by using the estimator in Equation (4.9);
- the intermediate order statistic $Y_{n-k_Y,n}$.

A specific value for $k_Y(n)$ is chosen by following the stability strategy proposed in Remark 4.6.1.

Obtain $\widehat{p}_n$ as in Equation (4.10) by using quantities $\widehat{\gamma}_Y$, $\widehat{a}_Y\left(\frac{n}{k_Y}\right)$ and $Y_{n-k_Y,n}$.
Choose $k_i(n)$ and estimate
- the tail index $\hat{\gamma}_i$ by using the Hill estimator in Equation (3.8);
- the intermediate order statistic $X^i_{n-k_i,n}$.

A specific value for $k_i(n)$ is chosen by following the stability strategy proposed in Remark 4.6.1.

Obtain $\widehat{x}^i_{\widehat{p}_n}$ as in Equation (4.13) by using quantities $\widehat{p}_n$, $\hat{\gamma}_i$ and $X^i_{n-k_i,n}$.

Secondly, we follow Algorithm 1 to obtain our estimator.

We now consider the following models in dimensions $d = 2$ and $d = 5$:

i) The Gumbel copula with dependence parameter $\theta = 3$ and Fréchet margins with $\beta = 3$ (i.e., $F_i(t) = \exp\{-t^{-\beta}\}$).

ii) The Joe copula with dependence parameter $\theta = 2$ and Pareto margins with $\delta_1 = 1$ and $\delta_2 = 2$ (i.e., $F_i(t) = 1 - (\delta_1/(t + \delta_1))^{\delta_2}$).

It should be borne in mind that assumptions of Theorem 4.5.1 are verified in the considered cases *i*) and *ii*).

In Figure 4.2 we focus on the simulated model *i*). We present the boxplots of the ratio $\widehat{x}^i_{\widehat{p}_n}/x^i_{p_n}$ (Figure 4.2, first and third columns, respectively, for $d = 2$ and $d = 5$). Furthermore, the Q-Q plots present the normalized sample quantiles of $\frac{\sqrt{k_i}}{\log(d_n^i)}\left(\frac{\widehat{x}^i_{\widehat{p}_n}}{x^i_{p_n}} - 1\right)$, based on 500 Monte Carlo simulations, versus the theoretical standard normal quantiles (Figure 4.2, second and fourth columns, respectively, for $d = 2$ and $d = 5$). Analogously, the results for the simulated model *ii*) are gathered in Figure 4.3. We take $n = 500$, 100, 50 and $\alpha_n = 1 - 10/n$.

Remark 4.6.1. In Algorithm 1, the intermediate sequences (k_i and k_Y) are chosen for each sample size by using the following stability strategy. We first plot the estimator against various values of the associated intermediate sequence. By balancing the potential bias and variance, the usual practice is to choose the sequence from the first stable region of the plots (for further details, see Cai et al. (2015)). Furthermore, to gain stability in the estimates, we take the average of the estimates corresponding to those values of the sequence and regard this average as the final estimate value. In Figures 4.2 and 4.3, the chosen values for the auxiliary sequences are displayed in the main title of each figure.

The boxplots of the two proposed models, *i*) and *ii*), show the good performance of our estimator in terms of bias and variance (see first and third columns in Figures 4.2 and 4.3). In the Q-Q plots, we observe that the scatters line up on the line $y = x$ in each plot, which indicates that the sample quantiles coincide largely with the theoretical quantiles from the asymptotic distribution. Consequently, we conclude that the limit Theorem 4.5.1 provides an adequate approximation for finite sample sizes. The performance of our estimators remains stable when the dimension d increases. Finally, we propose a comparison with the performance of the empirical estimator. A "nested" empirical quantile is considered in order to estimate $\widehat{x}^i_{\widehat{p}_n}$. Firstly, we estimate p_n with the empirical quantile of Y at level $1 - \alpha_n$. Secondly, the empirical quantile of the X_i marginal distribution is estimated at the obtained random risk level $\widehat{p}_n$. Figure 4.4 shows the boxplots obtained of the ratio between the empirical estimators and the theoretical values of the $i-$th component of the Component-wise Excess design realization $\delta_{CE}(\alpha)$ for the *i*) and *ii*) models considered. We take $n = 500$, 100, 50, and $\alpha_n = 1 - 10/n$ (as in Figures 4.2 and 4.3). It can be observed in Figures 4.2-4.4 that the empirical competitor estimator always underestimates $\widehat{x}^i_{\widehat{p}_n}$ and is consistently outperformed by the proposed EVT estimator. Moreover, we observe that there are small differences when the sample size increases.

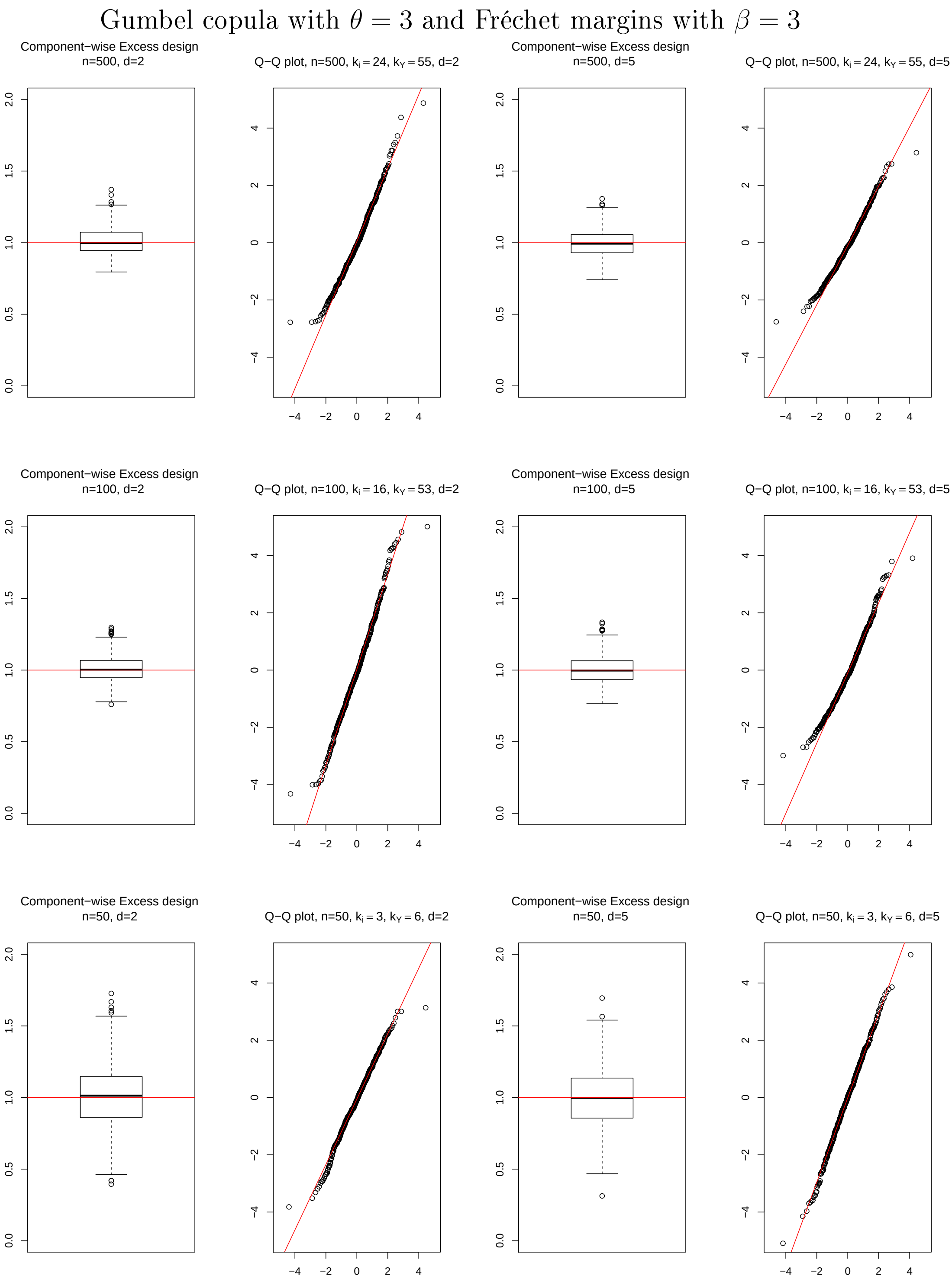

Figure 4.2: Model i). Boxplots of the ratio $\widehat{x}^i_{\widehat{p}_n}/x^i_{p_n}$ for $d = 2$ (first column) and $d = 5$ (third column). Q-Q plots for the normalized sample quantiles of $\frac{\sqrt{k_i}}{\log(d^i_n)}\left(\frac{\widehat{x}^i_{\widehat{p}_n}}{x^i_{p_n}} - 1\right)$ versus the theoretical standard normal quantiles for $d = 2$ (second column) and $d = 5$ (fourth column). We consider $n = 500$ (first row), $n = 100$ (second row), $n = 50$ (third row), and $\alpha_n = 1 - 10/n$. We take 500 Monte Carlo simulations.

Joe copula with $\theta = 2$ and Pareto margins with $\delta_1 = 1$ and $\delta_2 = 2$

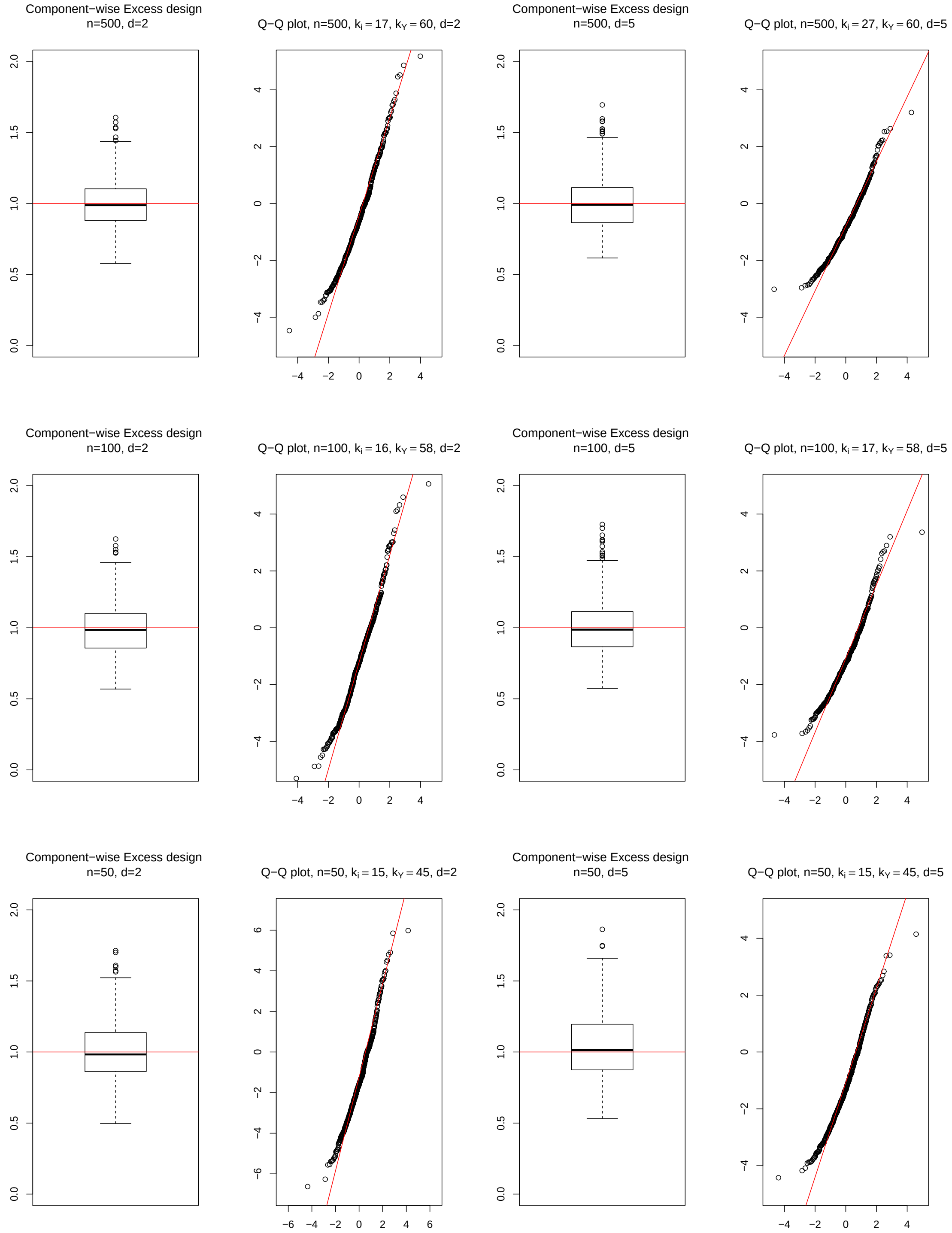

Figure 4.3: Model *ii*). Boxplots of the ratio $\widehat{x}^i_{\widehat{p}_n}/x^i_{p_n}$ for $d = 2$ (first column) and $d = 5$ (third column). Q-Q plots for the normalized sample quantiles of $\frac{\sqrt{k_i}}{\log(d^i_n)}\left(\frac{\widehat{x}^i_{\widehat{p}_n}}{x^i_{p_n}} - 1\right)$ versus the theoretical standard normal quantiles for $d = 2$ (second column) and $d = 5$ (fourth column). We consider $n = 500$ (first row), $n = 100$ (second row), $n = 50$ (third row), and $\alpha_n = 1 - 10/n$. We take 500 Monte Carlo simulations.

Empirical estimation for the Component-wise Excess design realization

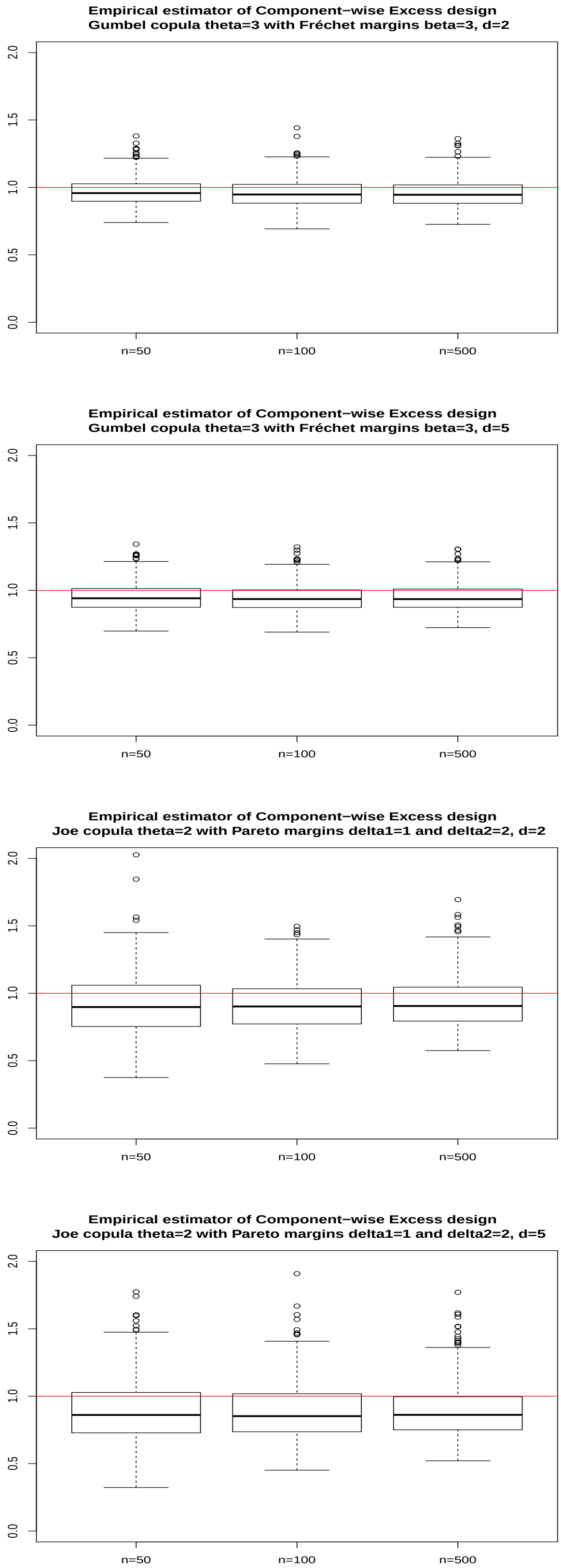

Figure 4.4: Boxplots of the ratio between the empirical estimator and the theoretical value of the $i-$th component of the Component-wise Excess design realization $\delta_{CE}(\alpha)$ for the $i)$ and $ii)$ models considered. We take n = 500, 100, 50; $\alpha_n = 1 - 10/n$; 500 Monte Carlo simulations; $d = 2$ (first and third rows) and $d = 5$ (second and fourth rows).

4.7 Application in Ceppo Morelli data-set

We now focus on estimating the Component-wise Excess design realization $\delta_{CE}(\alpha)$ in Equation (4.2) for an extreme value of α, for the flood peak, volume and initial water level of the Ceppo Morelli dam data-set. This dam is located in the Anza catchment valley, a sub-basin of the Toce river (Italy), and was built in order to produce hydro-electric energy. The data-set contains data on the maximum annual flood peak (Q) and volume (V), and the initial water level in the reservoir before the flood (L) covering a period of 49 years, from 1937 to 1994. Q is measured in $m^3 s^{-1}$, V in $10^6 \times m^3$, and L in m above sea-level. The joint distribution function and the joint survival distribution of (L, Q, V) are denoted by F_{LQV} and $\overline{F}_{LQV}$, respectively.

The selection of adequate flood design is a frequent problem for dam engineers. More than 40% of dam failures in the world are caused by overtopping. In order to prevent dam failures due to overtopping, the adequacy of the dam spillway must be assessed (De Michele et al. (2005)). For this purpose, the hydrological variables L, Q and V are of great interest. For further details of this data-set, the reader is referred to Salvadori et al. (2011) and references therein. Furthermore, Durante and Okhrin (2015) propose an inference procedure for the Ceppo Morelli dam data-set through the use of exchangeable Marshall copulas.

We represent the data in a trivariate rank-plot (Figure 4.5) and the rank-scatterplot for pairs of margins (Figure 4.6).

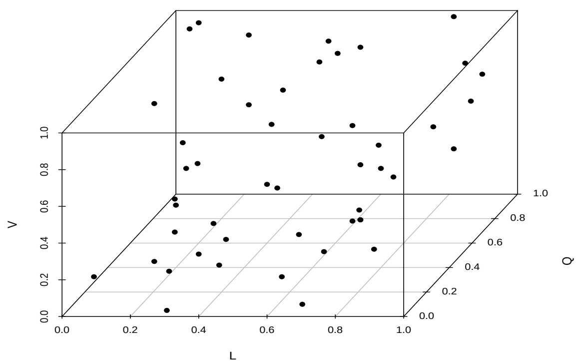

Figure 4.5: Trivariate rank-plot for (L, Q, V).

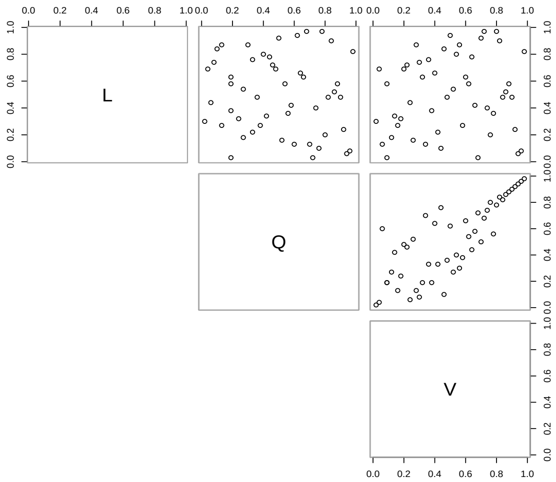

Figure 4.6: Rank-scatterplot for pairs of margins for the Ceppo Morelli dam data-set.

Salvadori et al. (2011) illustrated physical reasons for the assumption of independence between L and (Q, V). This fact is supported by Figures 4.5 and 4.6. Therefore, an Archimedean copula model could not be the most appropriate model for (L, Q, V). Indeed, Salvadori et al. (2011) propose a nested Archimedean copula for (L, Q, V) with a Gumbel copula between Q and V. However, we provide several goodness-of-fit tests on the data-set and the p-value for the 3-dimensional Gumbel model for (L, Q, V) is not statistically rejected. Hence, we can apply our estimation procedure to this data-set. We point out that a procedure to select the best model for this 3-dimensional data-set constitutes a highly interesting future line of study that lies beyond the scope of the present chapter. Furthermore, Generalized Extreme Value (GEV) distributions are supposed for Q and V (associated parameters are given in Table 1 in Salvadori et al. (2011)). The marginal distribution of L is obtained via a non-parametric Normal Kernel estimation in Salvadori et al. (2011). The Component-wise Excess design realizations for each margin, for a millinery return period with $\alpha \approx 0.946537$ obtained by Salvadori et al. (2011) are shown in the first row of Table 4.1 (denoted by δ^{S}_{CE}). In the second row of Table 4.1, we present the δ^{E}_{CE}, that is, the empirical estimated Component-wise Excess design realizations for each margin. Our extreme estimators $\widehat{x}_{\widehat{p}_n}$ are listed in the third row of Table 4.1. Furthermore, by using Theorem 4.5.1 and Corollary 4.5.1, confidence intervals at the 95% level for $\widehat{x}_{\widehat{p}_n}$ are also displayed.

In contrast with the estimator of Salvadori et al. (2011), which is based on a Gumbel model, in this work we propose a non-parametric estimation procedure for the risk measure $\delta_{CE}(\alpha)$. In our setting, only a general Archimedean copula framework is assumed and the heavy tailed behaviour of the margins in order to apply the proposed estimator (see Assumption 4.1.1).

Strategy	Q m^3s^{-1}	V 10^6m^3	L m above sea-level
δ^S_{CE}	352.76	25.21	781.25
δ^E_{CE}	337.82	19.41	781.13
$\widehat{x}_{\widehat{p}_n}$	359.68	26.01	781.21
$CI(\widehat{x}_{\widehat{p}_n}; 95\%)$	[327.93; 387.33]	[23.22; 28.60]	[781.19; 781.23]

Table 4.1: Estimates of the Component-wise Excess design realizations for a millinery return period, obtained by following different strategies for the Ceppo Morelli dam data-set. δ^S_{CE} denotes the estimator of Salvadori et al. (2011), δ^E_{CE} denotes the empirical estimator and $\widehat{x}_{\widehat{p}_n}$ denotes our extreme estimator proposed in Equation (4.13). Confidence intervals at the 95% level for our estimator are also displayed.

Unsurprisingly, the empirical estimator underestimates the Component-wise Excess design realization $\delta_{CE}(\alpha)$ (see second row in Table 4.1) and is consistently outperformed by the two other estimators (see first and third rows). Moreover, there is no significant statistical difference between our extreme estimator $\widehat{x}_{\widehat{p}_n}$ and that of Salvadori et al. (2011) δ^S_{CE}. We note that, by using theoretical results in Section 4.5, we are able to construct confidence intervals for $\widehat{x}_{\widehat{p}_n}$. The estimator of Salvadori et al. (2011) δ^S_{CE} and $\widehat{x}_{\widehat{p}_n}$ in Equation (4.13) are shown in Figure 4.7 with the critical iso-surface $\partial\underline{L}(\alpha)$ for $\alpha \approx 0.946537$.

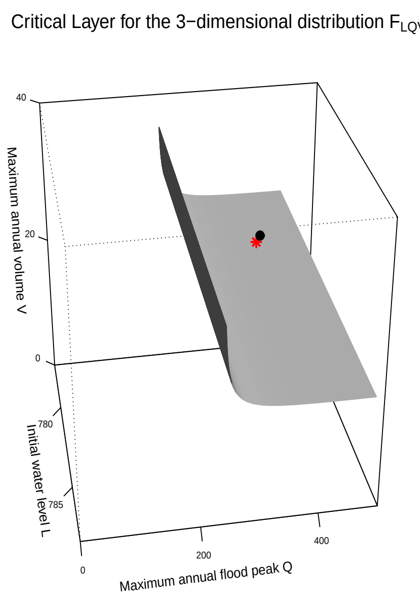

Figure 4.7: Critical iso-surface $\partial\underline{L}(\alpha)$ for $\alpha \approx 0.946537$. The *star* and the *dot* markers indicate, respectively, the estimator of Salvadori et al. (2011) δ^S_{CE} and $\widehat{x}_{\widehat{p}_n}$ in Equation (4.13) for the Ceppo Morelli dam data-set.

It should be noted that the Component-wise Excess design realizations shown in Table 4.1 represent points that have the greatest probability of being component-wise exceeded by an extreme realization with a return period longer than 1000 years. Therefore, these points could be interpreted as a "safety lower-bound". That is, the structure under design should, at least, withstand to realizations that have multivariate size of the Component-wise Excess design realization $\delta_{CE}(\alpha)$ for the millinery return period $\alpha \approx 0.946537$. Hence, the underestimation of these quantities can represent a major risk for dam managers and for environmental practitioners.

Furthermore, the target of the dam manager is to maintain a high water level, in order to achieve the maximum benefit through the production of electrical energy. From the pair (Q, V) of the calculated Component-wise Excess design realization, it is possible to obtain the associated flood hydrograph with peak Q and volume V. By using the flood hydrograph, one can calculate the maximum level of the dam associated with (L, Q, V), and one can check whether or not the crest level of the dam is exceeded by the reservoir level (see Salvadori et al. (2011)). The maximum water level obtained in Table 2 in Salvadori et al. (2011) for the associated values of (Q, V) in δ_{CE}^{S} (see the first row in Table 4.1) is 782.08 m above sea-level. Since the values obtained by using $\widehat{x}_{\widehat{p}_n}$ are very similar to those of Salvadori et al. (2011), we can compare our L realization with the maximum water level of 782.08 m above sea-level.

In this work, we focus on the estimation of the multivariate quantile of the hydrological load acting on the structure, that is, on the spillway of the dam. However, in order to evaluate the safety of the dam, one has to consider the interaction between the hydrological load and the structure. Volpi and Fiori (2014) point out the importance of considering the structure in hydraulic design and/or risk assessment problems in a multivariate environment and advise against the uncritical use of design event-based approaches. Indeed, the relationship between the structure and the hydrological loads acting on it is neglected in the study of the present chapter. Requena et al. (2016) and references therein, highlight the importance of considering the specific structure when designing or assessing flood risks in a multivariate context. Salvadori et al. (2016) illustrate the structural approach in a real sea-storm data-set.

4.8 Conclusions

In this chapter, we provide the explicit expression of the multivariate risk measure known as Component-wise Excess design realization given by Salvadori et al. (2011) in the Archimedean copula setting. Furthermore, this measure is estimated by using Extreme Value Theory techniques and the asymptotic normality of the proposed estimator is studied. In contrast with the estimator of Salvadori et al. (2011) based on a Gumbel model, we propose a non-parametric estimation procedure for the Component-wise Excess design realization. The performance of our estimator is evaluated on simulated data. Finally, we compare the performance of our estimator with the estimator of Salvadori et al. (2011) on the same real dam data-set as studied in the aforementioned paper. We conclude that there are no significant statistical differences between the values obtained from our extreme estimator and those of Salvadori et al. (2011).

Finally, it should be borne in mind that the improvement of Theorem 4.5.1 by using uncertainty induced by the margins could be developed in a future study. Furthermore, a procedure to select the best model for this 3-dimensional data-set of Ceppo Morelli dam constitutes a highly interesting future line of study.

Appendix A

Definitions of regularly varying functions

In the following, the notions of regularly varying functions are introduced. These definitions are useful in this thesis in order to provide the conditions to construct the extreme estimators.

Definition A.1 (RV function)**.** *A measurable function, $h : \mathbb{R}_+ \to \mathbb{R}$ that is eventually positive, is said to be of regular variation at infinity with index $\gamma \in \mathbb{R}\backslash\{0\}$, denoted by $h \in RV_\gamma$, if, for any $x > 0$,*

$$\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^\gamma. \tag{A.1}$$

If (A.1) holds with $\gamma = 0$ for any $x > 0$, then h is said to be slowly varying at infinity and is written as $h \in RV_0$.

There are a variety of concepts that extend RV, among which ERV, $2RV$ and $2ERV$ are the most significant concepts.

Definition A.2 (ERV function)**.** *A measurable function, $h : \mathbb{R}_+ \to \mathbb{R}$ is said to be of extended regular variation with index $\gamma \in \mathbb{R}$, denoted by $h \in ERV_\gamma$, if there exists a function $a : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $x > 0$,*

$$\lim_{t\to\infty} \frac{h(tx) - h(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma}, \tag{A.2}$$

where, for $\gamma = 0$, the right-hand-side in (A.2) *is interpreted as* $\log(x)$.

The function a is referred to as an auxiliary function for h.

Definition A.3 (2RV function)**.** *A measurable function, $h : \mathbb{R}_+ \to \mathbb{R}$ that is eventually positive, is said to be of second-order regular variation with the first-order parameter $\gamma \in \mathbb{R}$ and the second-order parameter $\tau \le 0$, denoted by $h \in 2RV_{\gamma,\tau}(A)$, if there exist* *some ultimately positive or negative function $A(t)$, with $A(t) \to 0$ as $t \to \infty$ such that*

$$\lim_{t\to\infty} \frac{\frac{h(tx)}{h(t)} - x^{\gamma}}{A(t)} = H(x), \tag{A.3}$$

with

$$H(x) = x^{\gamma} \int_1^x s^{\tau-1} ds, \ \forall x > 0.$$

Here, A is referred to as an auxiliary function of h.

If the functions a and A satisfy (A.2) and (A.3), respectively, then $a \in RV_{\gamma}$ and $|A| \in RV_{\tau}$.

Definition A.4 (2ERV function). *A measurable function, $h : \mathbb{R}_+ \to \mathbb{R}$ is said to be of second-order extended regular variation with the first-order parameter $\gamma \in \mathbb{R}$ and the second-order parameter $\tau \leq 0$, denoted by $h \in 2ERV_{\gamma,\tau}(A)$, if there exists some positive function $a(t)$ and some ultimately positive or negative function $A(t)$ with $A(t) \to 0$ as $t \to \infty$ such that*

$$\lim_{t\to\infty} \frac{\frac{h(tx)-h(t)}{a(t)} - \frac{x^{\gamma}-1}{\gamma}}{A(t)} = T_{\gamma,\tau}(x), \tag{A.4}$$

with

$$T_{\gamma,\tau}(x) = \int_1^x s^{\gamma-1} \int_1^s u^{\tau-1} du ds, \ \forall x > 0.$$

In Definition A.4, a and A are referred as the first-order and second-order auxiliary functions of h, respectively. It is easy to see that

$$T_{\gamma,\tau}(x) = \begin{cases} \frac{1}{\tau}\left(\frac{x^{\gamma+\tau}-1}{\gamma+\tau} - \frac{x^{\gamma}-1}{\gamma}\right), & \tau < 0, \\ \frac{1}{\gamma}\left(x^{\gamma}\log(x) - \frac{x^{\gamma}-1}{\gamma}\right), & \tau = 0 \neq \gamma \\ \frac{1}{2}(\log(x))^2, & \tau = 0 = \gamma. \end{cases}$$

From Theorem B.3.1 in de Haan and Ferreira (2006), if the functions a and A satisfy (A.4), then $|A| \in RV_{\tau}$ and $a \in 2RV_{\gamma,\tau}$ with auxiliary function A.

The notion of regular variation for a vector is presented in Definition A.5.

Definition A.5 (Multivariate Regularly Varying (MRV) Vector). *A random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ with joint distribution function F is said to be multivariate regularly varying ($\boldsymbol{X} \in MRV$) if there exists a Radon measure ν on $[\boldsymbol{0}, \infty]\backslash\{\boldsymbol{0}\}$, such that*

$$\lim_{t\to\infty} \frac{1 - F(t\boldsymbol{x})}{1 - F(t\boldsymbol{1})} = \nu([\boldsymbol{0}, \boldsymbol{x}]^c),$$

for all points $\boldsymbol{x} \in [\boldsymbol{0}, \infty)\backslash\{\boldsymbol{0}\}$, which are continuity points of the function $\nu([\boldsymbol{0}, \cdot]^c)$.

Observe also that for any non-negative MRV random vector $\mathbf{X}$, its non-degenerate univariate margins X_i have regularly varying right-hand-side tails, that is,

$$\overline{F}_{X_i}(t) := t^{-\beta} L(t), \ t \geq 0,$$

where $\beta > 0$ is the marginal heavy tail index and $L(t)$ is a slowly varying function, that is, $L(x\,t)/L(t) \to 1$ as $t \to \infty$ for any $x > 0$.

For more details about regular variation, the reader is referred to Sections B.1 and B.3 in de Haan and Ferreira (2006) and Mao and Hu (2012). Further details about multivariate regularly varying can be found in Resnick (2007), Resnick (2008) and Embrechts et al. (1997).

Appendix B

Alternative calculation for C.-E. design realization

In this Appendix, we provide a complementary development in order to verify that $\boldsymbol{v}^* = \left(\phi_{\boldsymbol{\theta}}^{-1}\left(\frac{\phi_{\boldsymbol{\theta}}(\alpha)}{d}\right), \ldots, \phi_{\boldsymbol{\theta}}^{-1}\left(\frac{\phi_{\boldsymbol{\theta}}(\alpha)}{d}\right)\right)$ is a critical point for the constrained optimization problem given in Equation (4.3).

From Equation (4.1), since $\boldsymbol{x} = (x_1, \ldots, x_d) \in \partial \underline{L}(\alpha)$, we can write, for $j \in \{1, \ldots, d\}$,

$$x_j = F_{X_j}^{-1}\left\{\phi_{\boldsymbol{\theta}}^{-1}\left(\phi_{\boldsymbol{\theta}}(\alpha) - \sum_{i=1, i\neq j}^{d} \phi_{\boldsymbol{\theta}}(F_{X_i}(x_i))\right)\right\} = g(\alpha, \boldsymbol{\theta}, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d),$$

where $F_{X_j}^{-1}$ denotes the left-continuous inverse of the margin distribution F_{X_j}, for $j \in \{1, \ldots, d\}$. As a consequence, from Equation (4.1), we can take $x_d = g(\alpha, \boldsymbol{\theta}, x_1, \ldots, x_{d-1})$ and the Component-wise Excess design realization in Definition 4.1.1 can be written as

$$\delta_{CE}(\alpha) = \underset{\substack{(x_1, \ldots, x_{d-1}): \\ F(x_1, \ldots, x_{d-1}, +\infty) > \alpha}}{\arg\max} \mathbb{P}[X_1 \geq x_1, \ldots, X_d \geq g(\alpha, \boldsymbol{\theta}, x_1, \ldots, x_{d-1})]. \tag{B.1}$$

Finally, from Sklar's Theorem (see Section 1.4) in our Archimedean framework, we obtain that condition $F(x_1, \ldots, x_{d-1}, +\infty) > \alpha$ holds true if and only if $C(u_1, \ldots, u_{d-1}, 1) > \alpha \Leftrightarrow \sum_{i=1}^{d-1} \phi_{\boldsymbol{\theta}}(u_i) \leq \phi_{\boldsymbol{\theta}}(\alpha)$. It is well-known that the iso-surface $\partial \underline{L}(\alpha)$ is lower-bounded by the univariate marginal quantiles, and therefore the restriction in the maximization problem (B.1) implies that $x_i \in [F_{X_i}^{-1}(\alpha), +\infty)$, for all $i \in \{1, \ldots, d-1\}$.

Note that the copula version of the optimization problem in Equation (B.1) can be written as the constrained optimization problem given in Equation (4.3). Let $v_d := \phi_{\boldsymbol{\theta}}^{-1}\left(\phi_{\boldsymbol{\theta}}(\alpha) - \sum_{i=1}^{d-1} \phi_{\boldsymbol{\theta}}(v_i)\right) = g(\alpha, \boldsymbol{\theta}, v_1, \ldots, v_{d-1})$. We substitute the $d-$th component in the joint survival function $\overline{C}$ (see the expression in Section 1.4), and obtain

$$
\begin{aligned}
&\overline{C}(v_1,\ldots,v_{d-1},g(\alpha,\boldsymbol{\theta},v_1,\ldots,v_{d-1}))\\
&\quad= \mathbb{P}[V_1 \geq v_1,\ldots,v_d \geq g(\alpha,\boldsymbol{\theta},v_1,\ldots,v_{d-1})]\\
&\quad= 1+\sum_{\forall i<d}\left[-v_i+(-1)^{d-1}\phi_{\boldsymbol{\theta}}^{-1}\left(\phi_{\boldsymbol{\theta}}(\alpha)-\phi_{\boldsymbol{\theta}}(v_i)\right)\right]\\
&\quad+ \sum_{l=2}^{d-1}\left\{\sum_{\substack{\forall(i_1,\ldots,i_l)\\ i_1<\cdots<i_l<d}}(-1)^l\phi_{\boldsymbol{\theta}}^{-1}\left(\sum_{\forall j\in(i_1,\ldots,i_l)}\phi_{\boldsymbol{\theta}}(v_j)\right)\right.\\
&\quad+ \left.(-1)^{d-l}\phi_{\boldsymbol{\theta}}^{-1}\left(\phi_{\boldsymbol{\theta}}(\alpha)-\sum_{\forall j\in(i_1,\ldots,i_l)}\phi_{\boldsymbol{\theta}}(v_j)\right)\right\}+(-1)^d\alpha. \qquad \text{(B.2)}
\end{aligned}
$$

The derivative with respect to each margin of the function in Equation (B.2) can now be calculated. We obtain

$$
\begin{aligned}
\frac{\partial\overline{C}(v_1,\ldots,v_{d-1},g(\alpha,\boldsymbol{\theta},v_1,\ldots,v_{d-1}))}{\partial v_k} &= -1-(-1)^{d-1}\frac{\phi'_{\boldsymbol{\theta}}(v_k)}{\phi'_{\boldsymbol{\theta}}\left(\phi_{\boldsymbol{\theta}}^{-1}\left(\phi_{\boldsymbol{\theta}}(\alpha)-\phi_{\boldsymbol{\theta}}(v_k)\right)\right)}+\\
&+ \sum_{\forall i<d,i\neq k}\frac{\phi'_{\boldsymbol{\theta}}(v_k)}{\phi'\left(\phi_{\boldsymbol{\theta}}^{-1}\left(\phi_{\boldsymbol{\theta}}(v_k)+\phi_{\boldsymbol{\theta}}(v_i)\right)\right)}\\
&- (-1)^{d-2}\frac{\phi'_{\boldsymbol{\theta}}(v_k)}{\phi'_{\boldsymbol{\theta}}\left(\phi_{\boldsymbol{\theta}}^{-1}\left(\phi_{\boldsymbol{\theta}}(\alpha)-\phi_{\boldsymbol{\theta}}(v_k)-\phi_{\boldsymbol{\theta}}(v_i)\right)\right)}+\\
&+ \sum_{l=2}^{d-2}\left\{\sum_{\substack{\forall(i_1,\ldots,i_l),i_j\neq k\\ i_1<\ldots<i_l<d}}(-1)^{l+1}\frac{\phi'_{\boldsymbol{\theta}}(v_k)}{\phi'_{\boldsymbol{\theta}}\left(\phi_{\boldsymbol{\theta}}^{-1}\left(\phi_{\boldsymbol{\theta}}(v_k)+\sum_{\forall j\in(i_1,\ldots,i_l)}\phi_{\boldsymbol{\theta}}(v_j)\right)\right)}\right.\\
&- \left.(-1)^{d-(l+1)}\frac{\phi'_{\boldsymbol{\theta}}(v_k)}{\phi'_{\boldsymbol{\theta}}\left(\phi_{\boldsymbol{\theta}}^{-1}\left(\phi_{\boldsymbol{\theta}}(\alpha)-\phi_{\boldsymbol{\theta}}(v_k)-\sum_{\forall j\in(i_1,\ldots,i_l)}\phi_{\boldsymbol{\theta}}(v_j)\right)\right)}\right\}.
\end{aligned}
$$

Notice that $\frac{\partial\overline{C}(v_1,\ldots,v_{d-1},g(\alpha,\boldsymbol{\theta},v_1,\ldots,v_{d-1}))}{\partial v_k}|_{v_1=v_1^*,\ldots,v_{d-1}=v_{d-1}^*}=0$, for all $k<d$. Therefore, for all $k<d$, $v_k^*=\phi_{\boldsymbol{\theta}}^{-1}\left(\frac{\phi_{\boldsymbol{\theta}}(\alpha)}{d}\right)$ is a stationary point for the system of first-order derivative equations provided above.

Hence, the $d-$dimensional point $\boldsymbol{v}^*=\left(\phi_{\boldsymbol{\theta}}^{-1}\left(\frac{\phi_{\boldsymbol{\theta}}(\alpha)}{d}\right),\ldots,\phi_{\boldsymbol{\theta}}^{-1}\left(\frac{\phi_{\boldsymbol{\theta}}(\alpha)}{d}\right)\right)$ is a stationary-point solution for the optimization problem in Equation (4.3) in the Archimedean copula framework.

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