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Weighted Inequalities and Multiparameter Harmonic Analysis

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Sevilla, 2014

A mi madre querida, Dolo,
que siempre sabe encontrar la cesta de
castañuelas que calma mis tempestades

Obrigada? *sem dúvida* perguntareis, *Como está,* senhora estudante? Estou obrigada porque *nestas adversidades, tanto as provadas quanto as previsíveis,* é que conheci-vos. Minhas histórias matemáticas são assim, *disfarçam muito, vão se juntando umas com as outras, parece que não sabem aonde querem ir, e de repente, por causa de duas ou três, ou quatro* de vocês, de repente saem... *Simples em si mesmas*, uma proposição, um lema, um teorema *e aí temos a comoção a subir irresistível à superfície da pele e dos olhos, a estalar a compostura dos sentimentos.*[1]

A Carlos y Yannis, por vuestro apoyo incondicional y vuestra confianza ciega en mi. Both of you well know how much I owe you.

To Tuomas and Juha, for making Helsinki and Aalto my favorite places for mathematics.

To Andrei and Michael, for their careful reading on this thesis.

A Javi, por tener siempre una respuesta atenta a cada mail de matemáticas y de metamatemáticas.

A Liguang y Paul, por acortar con las cuentas los miles de kilómetros que nos separan.

A Ezequiel, Carmen y Mari Carmen, por discutir conmigo matemáticas con amistad, simpatía y mucha entusiasmo.

A los compañeros y amigos de los Departamentos de Análisis y Ecuaciones de la Universidad de Sevilla, por cuidarme y hacerme reír en los despachos y las comidas de años.

A mis amigos, *por ser mis amigos y, en la calle* y en casa, darme las mejores horas. Alberto, Ana Rana, Angélica, Anita, Bego, Carlitos, Débora, Diego, Guille, Helene, Lobo, Luis, María, María Arte, Mandi, Miguelito, Miren, Muni, Rocco, Sebas, Silvi, Silvia, Timo y Zaida.

A mi familia. A mi Jonás querido, Dolo, Yaiza y Gádor, por ser mi apoyo en cada paso, por calmarme, cuidarme y quererme todos los días de mi vida. A Bill, por traducir con dulzura cada frase de duda. A mis primas y tías, por los ánimos inagotables que siempre me hacen llegar. A mi familia de Utrera, por darme cariño del que sólo en Utrera se sabe dar.

A Pedro, por darme *de beber* amor *en cada esquina*, amor.

[1] Tesa leyendo *Ensaio sobre a Cegueira* de José Saramago.

El tema central de esta tesis es el estudio de desigualdades con pesos para algunos de los operadores *clásicos* del análisis armónico. De entre estos operadores, los que más nos interesan, son aquellos que son invariantes por dilataciones multiparamétricas. El principal representante de estos objetos es el operador maximal fuerte, elemento protagonista de esta tesis. Las dos cuestiones fundamentales que abordamos son las siguientes:

- Las propiedades de acotación de dichos operadores clásicos en espacios de Lebesgue con pesos. En particular, nos centramos en el estudio del problema de dos pesos para el operador maximal geométrico asociado a una cierta base *general*. Los resultados obtenidos se refieren fundamentalmente a las denominadas bases de Muckenhoupt, para las que se define una condición suficiente para el problema de dos pesos de tipo *bump*. Además, se estudia con detalle la desigualdad de Fefferman-Stein para el operador maximal fuerte. Finalmente, se caracteriza también el problema de un peso para estos operadores maximales *generales* en términos de condiciones *débiles de tipo restringido*.

- El cálculo preciso de la norma de estos operadores clásicos en función de la constante A_p del peso. Mostramos primero una estrategia para probar la optimalidad del exponente de la constante A_p del peso que evita el desarrollo de ejemplos específicos. Por último, aunque esta cuestión para el operador maximal fuerte continúa abierta, presentamos ciertos resultados parciales que se pueden entender como el primer paso hacia una teoría de pesos multiparamétrica cuantitativa.

Abstract

This PhD dissertation is focused on the study of weighted norm inequalities for classical operators in harmonic analysis; in particular, those commuting with multiparameter families of dilations. We address two main issues which are intimately connected to each other: the boundedness properties of these operators in weighted spaces and the sharp bounds on their operator norms in terms of the constant associated with the weight.

Concerning the first subject, we examine the boundedness properties of various maximal operators in weighted Lebesgue spaces. Let $\mathfrak{B}$ a collection of open sets in $\mathbb{R}^n$. Given a measure μ, $M_{\mathfrak{B}}^{\mu}$ denotes its associated geometric maximal operator. When μ is the Lebesgue measure, we simply write $M_{\mathfrak{B}}$. We first study those pairs of weights on $\mathbb{R}^n$, (w, v), for which $M_{\mathfrak{B}}$ is bounded from $L^p(v)$ to $L^p(w)$, for $1 < p < \infty$. For this two-weight problem, restricted to those $\mathfrak{B}$ that are Muckenhoupt bases, we define a sufficient condition in terms of power and logarithmic bumps. We also study the corresponding problem in the multilinear case.

Second, we consider the case where $v = M_{\mathfrak{R}}w$ and $\mathfrak{R}$ is the collection of n-dimensional rectangles with sides parallel to the coordinate axes. In this case, we obtain the endpoint Fefferman-Stein inequality, as $p \to 1$, for those weights w that are in the *strong*-A_∞ class; that is, the class of weights that satisfy the A_p condition with respect to rectangles for some $p \geqslant 1$.

Finally, we characterize the boundedness of $M_{\mathfrak{B}}^{\mu}$ on $L^p(v)$ provided μ satisfies an appropriate doubling condition with respect to $\mathfrak{B}$, p is large enough and v is any arbitrary locally finite measure. In this case, $\mathfrak{B}$ can be any homothecy invariant collection of convex sets in $\mathbb{R}^n$ and the characterization of the boundedness is in terms of a *very* weak restricted type condition. As a consequence of this result, we discuss applications to Muckenhoupt weights as well as results in differentiation theory.

In relation to the sharp bounds, we are interested in proving the optimality of weighted inequalities of the form:

$$\|Tf\|_{L^p(w)} \lesssim_{n,p,T} [w]_{A_p}^{\beta} \|f\|_{L^p(w)},$$

for a certain operator T and w an A_p weight. We show that whenever the above estimate is true, then necessarily β satisfies a lower bound which is a function of the asymptotic behaviour of the unweighted L^p-norm $\|T\|_{L^p(\mathbb{R}^n)}$ as p goes to 1 and $+\infty$. By combining these results with known weighted inequalities, we derive the sharpness of this exponent β, without building any specific example, for maximal, Calderón–Zygmund and fractional integral operators. We then study in detail the above estimate, as well as its weak version, in the case where $T = M_{\mathfrak{R}}$ and w is a weight in the *strong*-A_p class. Although for the latter operator no such optimal quantitative estimates are currently known, we describe some partial results we have obtained.

Contents

Introduction

The main goal of this dissertation is to study weighted inequalities associated with operators that commute with multiparameter families of dilations. Multiparameter harmonic analysis addresses the study of such classes of operators. Indeed, it is an outgrowth of harmonic analysis when attempting to understand the function theory associated to product domains. The first results can be found in the early 1930's with the works of Saks, Zygmund, Jessen and Marcinkiewicz, where the main properties of the strong maximal function were identified. A pretty well developed summary of the main achievements in multiparameter analysis until 1985 can be found in [Fef86]. Nowadays, the multiparameter theory is a quite active research area with many different topics under study and it is not our intention to give an overview of it. What we would like to remark is that many classical problems of one parameter harmonic analysis remain open in the multiparameter setting, or have only weaker analogues. As we will see soon, this is also the case for the multiparameter weighted inequalities.

Let us be more specific about the central operator of this thesis. For a locally integrable function f, we will define the strong maximal function of f on $\mathbb{R}^n$ as the maximal average of f with respect to rectangles with sides parallel to the coordinate axes; that is,

$$M_s f(x) \coloneqq \sup_{\substack{R \in \mathfrak{R} \\ R \ni x}} \frac{1}{|R|} \int_R |f(y)| dy, \quad x \in \mathbb{R}^n,$$

where $\mathfrak{R}$ denotes the family of n-dimensional rectangles with sides parallel to the coordinate axes. If we replace the rectangles by cubes in the above definition, we obtain the Hardy-Littlewood operator M. Since this object is the multiparameter version of the Hardy-Littlewood operator, it can be considered as the prototype for multiparameter analysis. While this object commutes with n-parameter dilations, the Hardy-Littlewood operator does not. Both, M and M_s, are bounded on $L^p(\mathbb{R}^n)$ with respect to the Lebesgue measure; however, their endpoint behaviour, as $p \to 1$ is completely different. The replacement of the Lebesgue measure by a weight, in the definition of M_s, opens up new challenges in the study of the boundedness properties of the strong maximal operator. Some of them will be faced in this thesis.

In order to describe our contributions in the study of the weighted inequalities for the strong maximal operator and other multiparameter objects, we give a brief overview of the main achievements in the one parameter weighted theory. We will be comparing the results achieved for the Hardy-Littlewood maximal operator with the analogous for the strong maximal operator.

One-weight norm inequalities. Given an operator that is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, it is a natural question to ask under which conditions on a measure μ, the same operator is bounded on $L^p(\mu)$; that is, if we let T denote some *classical* operator and we wish to characterize the measures μ on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} [Tf(x)]^p \, d\mu(x) \leqslant C \int_{\mathbb{R}^n} |f(x)|^p \, d\mu(x).$$

If we restrict ourselves to those measures that are non-negative and absolutely continuous with respect to the Lebesgue measure, the answer to this question is the essence of the theory of weighted norm inequalities. Although the roots of this theory can be found in the works of Helson, Szegö, Rosenblum and others, it was really in the 1970's when the modern theory begun. First, Muckenhoupt [Muc72] defined the class of weights w for which the Hardy-Littlewood operator maps $L^p(w)$ into itself. Shortly afterwards, Hunt, Muckenhoupt, and Wheeden [HMW73] established that the same class of weights also characterizes the boundedness of the Hilbert transform on $L^p(w)$. A bit later, Coifman and C. Fefferman [CF74] gave a more flexible argument which also applied to more general Calderón-Zygmund operators. The family of weights that was identified through all these initial investigations was the A_p-class; that is, non-negative locally integrable functions w that satisfy:

$$[w]_{A_p} \coloneqq \sup_Q \left(\frac{1}{|Q|} \int_Q w(y) dy \right) \left(\frac{1}{|Q|} \int_Q w(y)^{1-p'} dy \right)^{p-1} < +\infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

After these first papers, similar results were soon obtained for a variety of other operators and there exists now a vast literature on one-weight norm inequalities. As a very partial list, we refer the reader to [GCRdF85, Duo01, Gra09, CUMP11] and the references therein.

What about the operators and geometric objects that behave well with respect to more general dilation groups? The theory of one-weight inequalities for the strong maximal operator is pretty well developed. Indeed, Bagby and Kurtz [BK85] proved that the boundedness of the strong maximal operator was characterized in terms of the *strong* A_p-class, that is defined as the A_p-class replacing the cubes by rectangles:

$$[w]_{A_p^*} \coloneqq \sup_R \left(\frac{1}{|R|} \int_R w(y) dy \right) \left(\frac{1}{|R|} \int_R w(y)^{1-p'} dy \right)^{p-1} < +\infty,$$

for all rectangles $R \in \mathfrak{R}$. As it is described in [GCRdF85, Chapter 4.6], this class of weights characterizes in general the boundedness of the basic n-parameter operators.

Two-weight norm inequalities. It is a natural problem to try to generalize the above results to L^p spaces with different weights; more precisely, given an operator T, we want to determine sufficient and necessary conditions on a pair of weights (w, v) on $\mathbb{R}^n$ such that a two-weight norm inequality of the following form is valid:

$$\int_{\mathbb{R}^n} [Tf(x)]^p w dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^p v dx,$$

for $1 < p < \infty$. We can also formulate the weak version of this problem; that is, to characterize the couples of weights (w, v) such that

$$w(\{x \in \mathbb{R}^n : T(f)(x) > \lambda\}) \leqslant \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p v(x)dx, \quad \lambda > 0.$$

Both these problems are referred to as the two-weight norm inequalities and one of the chapters of this thesis is devoted to the first one. The understanding of the two-weight question has turned out to be considerably more difficult than the one-weight problem. Muckenhoupt and Wheeden [MW76] so discovered that the two-weight A_p condition is necessary but not sufficient for the strong L^p-boundedness of both the maximal operator and the Hilbert transform. Since then, progress has been made and small improvements in results has often required the development of very sophisticated and valuable techniques. There have been two main approaches to the resolution of the two-weight inequality question, which involve either *testing conditions* or *bump conditions*. Testing conditions were originally introduced by Sawyer [Saw82a] to characterize the two-weight strong (p, p) inequality for the Hardy-Littlewood maximal function. He proved that the Hardy-Littlewood maximal operator M satisfies the strong inequality

$$\int_{\mathbb{R}^n} [Mf(x)]^p w dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^p v dx,$$

$1 < p < \infty$, if and only if

$$\int_Q [M(v^{1-p'}\mathbf{1}_Q)(x)]^p w dx \leqslant C \int_Q v^{1-p'}(x)dx, \tag{1}$$

for every cube Q. In [Saw88], he extended this approach to linear operators with positive kernels, like the fractional integral operators and Poisson integrals. After the original work of Sawyer, no progress was made on the study of similar questions for other operators, until the innovative work of Nazarov, Treil and Volberg [NTV97, NTV03]. In these latter papers, they developed the theory of singular integrals on non-homogeneus spaces that allowed them to prove Sawyer's type conditions for several operators. In [NTV08], they defined Sawyer-type conditions that are necessary and sufficient for families of Haar multipliers. Partial information about the two-weight problem for singular integrals can be found in [PTV]. The study of testing conditions for some classical operators is still an open problem with current interest, as shows the very recent resolution by Lacey [Lac13] for the Hilbert transform, following previous results by Lacey, Sawyer, Shen and Uriarte-Tuero in [LSSUT]. Still more recently, Hytönen [Hyt] has completely solved this problem for the Hilbert transform.

Testing conditions in the multiparameter setting have only been studied for the strong maximal function. Indeed, the two-weight problem for this operator was first solved by Sawyer [Saw82b]. See also [Jaw86] for a generalization of this result. He proved the same characterization as in the case of the Hardy maximal operator, but some extra assumptions were required. First, condition (1) needs to be taken in a union of rectangles instead of in just rectangle. More precisely, the testing condition is the following:

$$\int_G |M_s(\sigma\mathbf{1}_G)|^p w \leqslant C \int_G v^{1-p'}(x)dx$$

for every set $G \subset \mathbb{R}^n$ that is a union of rectangles in $\mathfrak{R}$. Secondly, the *weighted* strong maximal operator associated to the measure $\nu^{1-p'}$ is required to be bounded on $L^p(\nu^{1-p'})$. Observe that this weighted operator defined with respect to cubes is always bounded independently of the measure. However, since the Besicovitch covering argument as well as the Calderón-Zygmund decomposition fails in the case of rectangles, we cannot guarantee the boundedness of this operator in general. These two extra requirements express the differences between the one parameter and the multiparameter case.

The fact that Sawyer-type conditions involve the operator under study itself, makes difficult to use them in applications. It is not easy to find pairs of weights satisfying these conditions. Moreover, unlike the A_p-weights, the testing conditions are just defined for individual operators: if the operator is changed, the work of finding or checking pairs of weights must be repeated. This drawback explains the development of a different line of research that looks for sufficient conditions, close in form to the A_p condition. That kind of condition should be easier to test in practice. This approach is known as *seeking* bump conditions because the norms involved in the two-weight condition are *bumped up* in the Lebesgue integrability scale. These conditions first appeared in connection with estimates for integral operators related to the spectral theory of Schrödinger operators: see C. Fefferman [Fef83] and Chang, Wilson and Wolff [CWW85]. Independently, Neugebauer [Neu83] introduced the power bump conditions for the Hardy-Littlewood maximal operator. Motivated by these previous works, Pérez generalized and improved them in [Pér95b] and in [Pér95c]. In fact, he proved that if (w,ν) satisfies

$$\left(\frac{1}{|Q|}\int_Q w\right)\|\nu^{-1}\|_{\Phi,Q}<\infty \tag{2}$$

for all cubes Q then

$$\int_{\mathbb{R}^n}[Mf(x)]^p w dx \leqslant C\int_{\mathbb{R}^n}|f(x)|^p \nu dx,$$

for all $1<p<\infty$. Here the second term in (2) is a *bumped up average* of ν^{-1} with respect to a certain young function Φ that is a *bit bigger* than $t^{p'-1}$. After this initial paper, in [Pér94a] Pérez extended this result for fractional integral operators. In contrast to testing conditions, it is, in general, easier to determine if a couple of weights satisfies (2). However, it may be also very difficult if Φ is a very *complicated* Young function.

Though (2) is just a sufficient condition for the two-weight problem, the growth condition on Φ is necessary in the following sense: if the maximal operator M maps $L^p(\nu)$ into $L^p(w)$ when (u,ν) satisfies (2), then necessarily Φ is bigger than $t^{p'-1}$. For this reason, over the last few years bump conditions have become very popular as *optimal* sufficient conditions for two-weight norm inequalities. For example, the result for the Hilbert transform was proved in [CUMP07] and by different methods in [CUMP12] for any Calderón-Zygmund operator with C^1-kernel. Very recently the solution was extended in [CURV13] to the Lipschitz case and proved in full generality in [Ler13b]. In general, the type of bump conditions that appears for standard operators T are

$$\sup_Q\|w\|_{\Psi,Q}\|\nu^{-1}\|_{\Phi,Q}<\infty,$$

where Ψ and Φ satisfy some kind of *growth condition*. This type of conditions for linear operators appeared for first time in [Pér94a].

The main contribution of this thesis to the two-weight problem for the strong maximal function is exactly the investigation of appropriate bump conditions for the boundedness of M_s from $L^p(v)$ to $L^p(w)$. This is the first time that a bump condition is introduced for the study of weighted inequalities for multiparameter operators.

Fefferman-Stein inequalities Among two-weight norm inequalities, those that involve pairs of weights of the form (w, Mw) are especially important. These kind of weighted norm inequalities have been deeply studied for different operators, but we are interested in here with those that involve maximal operators; namely, inequalities of the form:

$$\int_{\mathbb{R}^n} (Mf)^p w \leqslant C \int_{\mathbb{R}^n} |f|^p Mw, \quad 1 < p < \infty, \tag{3}$$

where M denotes some maximal operator. Estimates like (3) are referred as Fefferman-Stein inequalites. They are important since, among other things, they can be used to derive the boundedness of vector-valued maximal operators. This inequality was first proved for the Hardy-Littlewood maximal function by C. Fefferman and Stein, [FS71], for *every* non-negative, locally integrable weight w. Indeed, the main application in [FS71] was exactly the vector-valued extension of the classical Hardy-Littlewood maximal theorem. For the strong maximal function the same inequality is true provided that $w \in A_r^*$, for some $r > 1$. This result was proved by Lin [Lin84] and the extra condition on the weight is an instance of the differences between the two operators.

Quantitative weighted theory. Another very important issue in the theory of weighted norm inequalities is the study of optimal quantitative estimates for the norm $\|T\|_{L^p(w)}$ whenever $w \in A_p$; that is, the description of the precise bounds of T in terms of the A_p constant $[w]_{A_p}$ of the weight. The first author who studied that question for the Hardy-Littlewood operator was Buckley [Buc93]. He proved the following quantitative estimate

$$\|M\|_{L^p(w)} \leqslant C_{n,p}[w]_{A_p}^{\frac{1}{p-1}}, \qquad w \in A_p, \tag{4}$$

where the constant $C_{n,p}$ does not depend on the weight and $1/(p-1)$ cannot be replaced with any smaller exponent. After that, Petermichl and Volberg [PV02], settling the corresponding optimal bound for the Beurling-Ahlfors transform, resolved a conjecture by Astala, Iwaniec and Saksman [AIS01, Eq. 45]. This result has important applications to the regularity theory of Beltrami equations. This result brought a lot of attention to the subject.The next important paper in the area is due to Petermichl [Pet07], where it is described the optimal bounds for the Hilbert transform. Then she extended the result for the Riesz trasforms in [Pet08]. Since then, the corresponding question concerning the sharp dependence of the norm of a general Calderón-Zygmund operator on the A_p constant of the weight has led to an overwhelming amount of activity and development of relevant tools. It was in 2010 when Hytönen, [Hyt12], exhibited the optimal bound for general Calderón-Zygmund operators T; namely, he showed that

$$\|T\|_{L^p(w)} \leqslant C_{n,p}[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}, \qquad w \in A_p.$$

Subsequent important developments and simplifications of his result can be found in [HP], [HLP] and [Ler13a]. This subject and allied matters are still under intense investigation as for example in [LM13] and [Ler13c].

Multiparameter sharp weighted inequalities have not been developed, as there exists a serious obstruction in carrying over the already described achievements of classical weighted theory to the multiparameter setting. As we will see through this thesis this is somehow another manifestation of the failure of the Besicovitch covering argument.

Main contributions. This thesis presents a first effort to extend some important recent achievements in weighted norm inequalities to the multiparameter setting. Besides answering some of the questions that we will describe soon, we believe that we also establish a certain research direction towards a more quantitative multiparameter weighted theory. More precisely, the novelties are the following:

- The definition of the two-weight problem for the strong maximal operator with the bump approach.
- The resolution of the endpoint Fefferman-Stein inequality for the strong maximal operator in any dimension.
- A new characterization of the boundedness of the strong maximal function defined with respect to a doubling measure. This characterization is in terms of a very weak restricted type condition.
- A partial solution to the multiparameter analogue of Buckley's result (4).

The study of these problems in the multiparameter setting emphasizes the differences between the one parameter and the multiparameter analysis. The difficulties arise because the objects under study are quite different. Since we are interested in operators that behave well with respect to more general dilation groups, the tools developed for the study of weighted inequalities do not generally work. For example, covering arguments like the Besicovitch covering lemma or the Calderón-Zygmund decomposition, that are used repeatedly in the one parameter theory, fail in the case of rectangles. Indeed, while the (centered) Hardy-Littlewood maximal operator, defined with respect to a general measure, is always bounded independently of the measure, the analogous multiparameter operator is not. The tools and techniques used to solve the problems that we address here are somehow classical in the multiparameter theory. However, the more efficient use of these tools together with a deep understanding of the properties of rectangles in weighted spaces, have made the resolution of some of the aforementioned problems possible. Also we believe that the arguments that we exhibit here could give insights when considering other multiparameter questions.

We now give a brief account of the specific questions considered in this thesis together with their corresponding solutions.

Problem 0.1. Given p, $1 < p < \infty$, determine a sufficient bump condition on a pair of weights (w, v) so that the the strong maximal operator M_s is bounded from $L^p(v)$ to $L^p(w)$; that is, for which we have the following inequality:

$$\int_{\mathbb{R}^n} [M_s f(x)]^p w dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^p v dx. \tag{5}$$

This problem finds an answer in Theorem 2.5. In particular we prove that if (w, v) satisfy the bump condition

$$\sup_R \left(\frac{1}{|R|} \int_R w \right) \|v^{-1}\|_{\Phi,R} < \infty,$$

for some Φ such that $\Phi_n(\Phi(t))$ is bigger, in order of magnitude, than $t^{p'-1}$ and $w \in A_p^*$ for some p, then we have the two -weight norm inequality (5). Here $\Phi_n(t) := t(1 + (\log^+ t)^{n-1})$.

We can also pose similar problems replacing M_s by a multilinear maximal operator associated with more general basis of open sets. Our contribution in this case is described in Theorem 2.18.

Problem 0.2. Find sufficient conditions for those weights w such that

$$w(\{x \in \mathbb{R}^n : M_s f(x) > \lambda\}) \leqslant C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \Big(1 + \big(\log^+ \frac{|f(x)|}{\lambda}\big)^{n-1}\Big) M_s w(x) dx, \quad \lambda > 0.$$

Note that by interpolation, this estimate implies a particular version of (5), where $v \equiv M_s w$.

Mitsis showed in [Mit06] that the estimate was true in dimension $n = 2$ for any weight $w \in \cup_{p \geqslant 1} A_p^*$. Theorem 2.22 of this thesis extends Mitsis's result to all dimensions.

Problem 0.3. Let $\mathfrak{B}$ be a homothecy invariant collection of convex sets in $\mathbb{R}^n$. Given a *doubling* measure μ, we want to study the mapping properties of the geometric maximal operator $M_{\mathfrak{B}}^{\mu}$ acting on $L^p(\nu)$. Here ν denotes a non negative, locally finite measure and the operator $M_{\mathfrak{B}}^{\mu}$ is defined as

$$M_{\mathfrak{B}}^{\mu} f(x) := \sup_{\substack{B \in \mathfrak{B} \\ B \ni x \\ \mu(B) > 0}} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y),$$

if $x \in \cup_{B \in \mathfrak{B}} B$ and $M_{\mathfrak{B}}^{\mu} f(x) := 0$ if $x \notin \cup_{B \in \mathfrak{B}} B$.

The authors in [HS09] prove that if $d\mu \equiv d\nu \equiv dx$, where dx denotes the Lebesgue measure, then the operator $M_{\mathfrak{B}}$ satisfies

$$\Big|\{x \in \mathbb{R}^n : M_{\mathfrak{B}}(\mathbf{1}_E)(x) > \frac{1}{2}\}\Big| \leqslant c|E|$$

for every measurable set E if and only if $M_{\mathfrak{B}}$ is bounded on $L^p(\mathbb{R}^n)$ for some $p > 1$. Theorem 3.11 of this thesis extends this result for any $\mathfrak{B}$-*doubling* measure μ. It is shown that the operator $M_{\mathfrak{B}}^{\mu}$ satisfies

$$\mu\{x \in \mathbb{R}^n : M_{\mathfrak{B}}^{\mu}(\mathbf{1}_E)(x) > \frac{1}{2}\} \leqslant c\mu(E)$$

for every measurable set E if and only if $M_{\mathfrak{B}}^{\mu}$ is bounded on $L^p(\nu)$ for some $p > 1$. As a consequence of this result we provide an alternative characterization of the class of Muckenhoupt weights $A_{\infty,\mathfrak{B}}$ for homothecy invariant Muckenhoupt bases $\mathfrak{B}$ consisting of convex sets. See Theorem 3.12 for this characterization. In particular this theorem gives a new characterization of the boundedness of the weighted strong maximal operator M_s^w whenever w is a product-doubling weight. In addition we discuss applications in differentiation theory. See Corollary 3.13.

Problem 0.4. Let T be an operator and w an A_p-weight, $1 < p < \infty$. Prove the optimality (in terms of the exponent on the A_p constant) of weighted inequalities of the form:

$$\|Tf\|_{L^p(w)} \leqslant C_{n,p,T}[w]_{A_p}^{\beta}\|f\|_{L^p(w)}, \tag{6}$$

$$\|Tf\|_{L^{p,\infty}(w)} \leqslant C_{n,p,T}[w]_{A_p}^{\beta'}\|f\|_{L^p(w)}. \tag{7}$$

Theorem 4.2 establishes a necessary lower bound for the exponent β in (6). More precisely, $\beta > \Psi$, where Ψ is a function related to the asymptotic behaviour of the unweighted L^p norm $\|T\|_{L^p(\mathbb{R}^n)}$ as p goes to 1 and $+\infty$.

Problem 0.5. Let T be an operator that commutes with n-parameter dilations and $w \in A_p^*$, $1 < p < \infty$. Prove the optimality (in terms of the exponent on the A_p^* constant) of weighted inequalities of the form:

$$\|Tf\|_{L^p(w)} \leqslant C_{n,p,T}[w]_{A_p^*}^{\beta}\|f\|_{L^p(w)}, \tag{8}$$

$$\|Tf\|_{L^{p,\infty}(w)} \leqslant C_{n,p,T}[w]_{A_p^*}^{\beta'}\|f\|_{L^p(w)}. \tag{9}$$

Proposition 4.7 and Theorem 4.8 describe new optimal estimates (8) and (9) for T the strong maximal function and $w \in A_p^*$ a power weight and a product weight respectively. In particular, for $w(x) = w_1(x_1)\cdots w_n(x_n) \in A_p^*$ we obtain the following sharp weighted inequalities:

$$\|M_s f\|_{L^p(w)} \leqslant C_{n,p}[w]_{A_p^*}^{\frac{1}{p-1}}\|f\|_{L^p(w)}.$$

$$\|M_s f\|_{L^{p,\infty}(w)} \leqslant C_{n,p}[w]_{A_p^*}^{\frac{1}{p-1}(1-\frac{1}{np})}\|f\|_{L^p(w)}.$$

Outline of contents. There are four chapters in this thesis and the content of each of one is briefly as follows. In Chapter 1 we give a overview of the theory that contains some definitions, general results and techniques that are necessary for the rest of the manuscript. This chapter does not contain any new result. Chapter 2 is devoted to study the two-weight problem for the strong maximal function and its multilinear version. Namely, we present the solution of Problem 0.1 and 0.2. In Chapter 3 we characterize the boundedness of $M_{\mathfrak{B}}^{\mu}$ in terms of a Tauberian condition, provided μ satisfies an appropriate doubling condition. This chapter is largely devoted to solve Problem 0.3. Chapter 4 contains the solution of Problem 0.4 and some partial results concerning Problem 0.5. More precisely, we first describe a new approach to test sharpness of weighted estimates that can be applied to many classical operators in harmonic analysis. Although most of the optimal results obtained with that approach were already known, the method allows us to avoid the use of specific examples and deal with all the operators at once. We also describe new sharp results concerning the strong maximal operator. At the end of each chapter, we discuss some open questions and considerations related with the contents of each chapter.

Notations

Symbol	**Meaning**
$A \lesssim B$	$A \leqslant CB$ for some numerical constant $C > 0$
$A \lesssim_m B$	$A \leqslant CB$ for some constant $C > 0$ that depends on m
$A \simeq B$	$A \lesssim B$ and $B \lesssim A$
$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{R}^n$	the n-dimensional Euclidean space
dx	Lebesgue measure
μ	non-negative measure
w	weight
$\|x\|$	$\sqrt{\|x_1\|^2 + \cdots + \|x_n\|^2}$ when $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$
$\bar{x}_j$	$(x_1, \ldots, \widehat{x_j}, \ldots, x_n) \in \mathbb{R}^{n-1}$, where the variable $\widehat{x_j}$ is missing
$C^\infty(\mathbb{R}^n)$	the space of smooth functions from $\mathbb{R}^n$ to $\mathbb{C}$
$\partial_j^m f$	the m-th partial derivative of $f(x_1, \cdots, x_n)$ with respect to x_j
$\partial^\beta f$	$\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$
$\mathscr{S}(\mathbb{R}^n)$	$\{f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} \|x^\alpha \partial^\beta f(x)\| < \infty \quad \forall \alpha, \beta\}$
$L^p(X, \mu)$	the Lebesgue space over the measure space (X, μ)
$L^p(X)$	the Lebesgue space over the measure space $(X, \|\cdot\|)$
$L^{p,\infty}(X, \mu)$	the weak Lebesgue space over the measure space (X, μ)
$L^{p,\infty}(\mathbb{R}^n)$	the weak Lebesgue space over the measure space $(\mathbb{R}^n, \|\cdot\|)$
$L^\Phi(X, \mu)$	the Orlicz space associated to the Young function Φ over (X, μ)
$\\|f\\|_{\Phi,E}$	$\inf\left\{\lambda > 0 \,:\, \frac{1}{\|E\|}\int_E \Phi\left(\frac{\|f(x)\|}{\lambda}\right) dx \leqslant 1\right\}$
$\mathbf{1}_E$	the characteristic function of the set $E \in \mathbb{R}^n$
$\log^+(t)$	$\max(0, \log t)$ for $t > 0$
$\Phi_n(t)$	$t(1 + (\log^+(t))^{n-1})$

$\bar{f}_q$	the quantity $(\sum_{j=1}^{\infty}(f_j)^q)^{1/q}$ where $f := \{f_j\}_{j=1}^{\infty}$
$\bar{\Phi}$	the complementary of a Young function Φ
$\|E\|$	the Lebesgue measure of the set $E \in \mathbb{R}^n$
$\mathscr{D}_R$	the mesh of *dyadic* rectangles associated to the rectangle R
$\mathfrak{B}$	a basis; that is, a collection of open sets in $\mathbb{R}^n$
$\mathfrak{Q}$	the basis of n-dimensional axes parallel cubes
$\mathfrak{R}$	the basis of n-dimensional axes parallel rectangles
$\mathfrak{G}$	the basis of n-dimensional rectangles with arbitrary orientation
M^{μ}	the Hardy-Littlewood maximal operator with respect to μ
M	the Hardy-Littlewood maximal operator
M_c^{μ}	the centered Hardy-Littlewood maximal operator with respect to μ
M^{Φ}	the Orlicz maximal operator
$\mathscr{M}$	the multilinear operator associated to the basis $\mathfrak{Q}$
M_s^{μ}	the strong maximal operator with respect to μ
M_s	the strong maximal operator
M_s^{Φ}	the strong Orlicz maximal operator
$\mathscr{M}_s$	the multilinear operator associated to the basis $\mathfrak{Q}$
$M_{\mathfrak{B}}^{\mu}$	the maximal operator associated to the basis $\mathfrak{B}$ with respect to μ
$M_{\mathfrak{B}}$	the maximal operator associated to the basis $\mathfrak{B}$
M_{α}^{μ}	the fractional maximal operator
I_{α}	the fractional integral operator
H	the Hilbert transform
H_s	the multiple Hilbert transform
R_j	the j-th Riesz transform
S_d	the dyadic square function
$[b, T](\cdot)$	$bT(\cdot) - T(b)(\cdot)$

Chapter 1

Preliminaries

In this chapter we introduce definitions and concepts used in this study. We start with the basic framework where the problems we have already stated will be studied. We also define the main operators and their basic mapping properties in the Lebesgue spaces. Then we extend these properties to the setting of weights and we describe in detail the classes of weights w for which these operators are bounded on $L^p(w)$. Some of the definitions and results will be presented without references or proofs, although in the last section we point out the general bibliography used.

1.1 Framework

In this section we recall some notions related to the theory of L^p spaces and $Orlicz$ spaces; both spaces define the environment where this thesis is developed.

Our basic setting is a measure space (X, Σ, μ), that is a set X together with a σ-algebra Σ of sets in X and a non-negative measure μ on X. The measure μ is always assumed to be σ-finite.The most important space of functions in this thesis is the space $L^p(X, \mu)$, $1 \leqslant p < \infty$, defined as the collection of measurable functions from X to $\mathbb{C}$ whose p-th powers are integrable; the norm of $f \in L^p(X, \mu)$ is defined as

$$\|f\|_{L^p(X,\mu)} \coloneqq \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}.$$

For $p = \infty$, $L^\infty(X, \mu)$ denotes the Banach space of essentially bounded functions from X to $\mathbb{C}$; that is, the space of functions f such that

$$\|f\|_{L^\infty(X,\mu)} \coloneqq \operatorname*{ess\,sup}_{x\in X} |f(x)| < \infty,$$

where

$$\operatorname*{ess\,sup}_{x\in X} |f(x)| \coloneqq \inf\{\alpha > 0 : \mu(\{x \in X : |f(x)| > \alpha\}) = 0\}.$$

We also work on the weak L^p-spaces, denoted by $L^{p,\infty}(X, \mu)$. For $1 \leqslant p < \infty$, the space $L^{p,\infty}(X, \mu)$ is the collection of all measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_{L^{p,\infty}(X,\mu)} \coloneqq \sup_{\lambda>0} \lambda \mu(\{x \in X : |f(x)| > \lambda\})^{\frac{1}{p}} < \infty.$$

For $1 < p < \infty$, p' denotes the dual or conjugate exponent of p defined by the relation $1/p + 1/p' = 1$. In general X will be $\mathbb{R}^n$ or a subset of $\mathbb{R}^n$. In the cases where μ is the Lebesgue measure we often do not give the measure and we simply write $L^p(X)$ and $L^{p,\infty}(X)$ or L^p and $L^{p,\infty}$ when $X = \mathbb{R}^n$. When μ is absolutely continuous with respect to the Lebesgue measure and $d\mu = w dx$, then we write $L^p(w)$ and $L^{p,\infty}(w)$. Most of the time we will be working on these spaces and w will be called a *weight* (see Section 1.3).

We introduce some basic facts about the theory of Orlicz spaces. These spaces define a more flexible setting than that provided by L^p spaces. These will play a central role in our approach to two-weight norm inequalities and may be understood as natural generalizations of the L^p spaces. Indeed, we can say that $f \in L^p$ if and only if $\Phi(|f|) \in L^1$, where $\Phi(t) = t^p$. The theory of Orlicz spaces applies this idea to more general convex functions Φ, that are called Young functions. More precisely, a function $\Phi : [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, convex and strictly increasing, satisfying $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. Note that since Φ is convex, Φ' exists almost everywhere and is increasing; therefore,

$$\Phi(t) \leqslant \int_0^t \Phi'(s)ds \leqslant t\Phi'(t). \tag{1.1}$$

Given a Young function Φ, we define the Orlicz space $L^\Phi(X,\mu)$ to be the set of measurable functions f such that

$$\int_X \Phi\left(\frac{f}{\lambda}\right) d\mu < \infty$$

for some $\lambda > 0$. We define a norm for $L^\Phi(X,\mu)$ introducing the Luxembourg norm

$$\|f\|_{\Phi,X} := \inf\left\{\lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu \leqslant 1\right\}.$$

In the case that X is a subset E of $\mathbb{R}^n$ and $d\mu = dx$, we simply write L^Φ. In this particular case, the following definition will be frequently used:

Definition 1.1. We define the *localized* L^Φ-norm of a function f over a set E of $\mathbb{R}^n$ as

$$\|f\|_{\Phi,E} := \inf\left\{\lambda > 0 : \frac{1}{|E|}\int_E \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leqslant 1\right\}. \tag{1.2}$$

It will be helpful to observe the following feature of this so-called norm. If Φ_1 and Φ_2 are two Young functions then:

$$\Phi_1(t) \leqslant \Phi_2(t) \quad \text{for all } t \in (0,\infty) \Rightarrow \|f\|_{\Phi_1,E} \leqslant \|f\|_{\Phi_2,E} \tag{1.3}$$

for all positive functions f.

Associated with each Young function Φ, one can define a complementary function

$$\bar{\Phi}(s) := \sup_{t>0}\{st - \Phi(t)\} \tag{1.4}$$

for $s \geqslant 0$. Such $\bar{\Phi}$ is also a Young function and we have

$$\Phi^{-1}(t)\bar{\Phi}^{-1}(t) \sim t \quad \text{for all } t \in (0,\infty) \tag{1.5}$$

and

$$st \leqslant C\left[\Phi(t) + \bar{\Phi}(s)\right]$$

for all $s, t \geqslant 0$. Also the $\bar{\Phi}$-norms are related to the L^{Φ}-norms via the *the generalized Hölder inequality*, namely

$$\frac{1}{|E|}\int_E |f(x)\, g(x)|\, dx \leqslant 2\, \|f\|_{\Phi,E}\, \|g\|_{\bar{\Phi},E}. \tag{1.6}$$

There are certain classes of Young functions that have a particular relevance to the material in the next chapter. For this reason we present the next definitions:

Definition 1.2. Φ is said to be *doubling* if there exists a positive constant α such that

$$\Phi(2t) \leqslant \alpha\Phi(t)$$

for all $t \geqslant 0$.

Definition 1.3. We say that a Young function Φ is *submultiplicative* if for each $t, s > 0$

$$\Phi(ts) \leqslant \Phi(t)\Phi(s).$$

We finish this section with some illustrative examples of Young functions.

- If $\Phi(t) = t^p$, $1 \leqslant p < \infty$, then $L^{\Phi}(X, \mu) = L^p(X, \mu)$. In the case $d\mu(x) = dx$ we use the notation $\| \quad \|_{p,E}$ for the corresponding *localized* norm on a set E.
- Define $\Phi(t) = \exp(t^2) - 1$. If the measure space is finite, $L^{\Phi}(X, \mu)$ consists of the exponentially square-integrable functions.
- Let $1 \leqslant r < \infty$ and $s \in \mathbb{R}$. If $s \geqslant 0$ then $\Phi(t) = t^r(\log(e + x))^s$ is aYoung function. Observe that for any set $E \in \mathbb{R}^n$ we have the following relation:

  $$\|\cdot\|_{L^r(E)} \leqslant \|\cdot\|_{\Phi,E} \leqslant \|\cdot\|_{L^{r+\epsilon}(E)}$$

  for any $\epsilon > 0$. In this case, we will say that the *localized* L^{Φ}-norm is *stronger* than the L^r- norm but it is *weaker* than the $L^{r+\epsilon}$-norm.

1.2 The main operators

In this section we describe the main operators we will be working with in this thesis. We first recall some basic definitions and we introduce some notation. We say that an operator T is *linear* if

$$T(f+g) = T(f) + T(g) \qquad \text{and} \qquad T(\lambda f) = \lambda T(f)$$

for all functions f, g and all $\lambda \in \mathbb{C}$. The operator is *sublinear* if

$$|T(f+g)| \leqslant |T(f)| + |T(g)| \qquad \text{and} \qquad |T(\lambda f)| = |\lambda||T(f)|$$

for all functions f, g and all $\lambda \in \mathbb{C}$. Given two Lebesgue spaces $L^p(X, \mu)$ and $L^q(Y, \nu)$, we say that a linear or sublinear operator T is bounded from $L^p(X, \mu)$ to $L^q(Y, \nu)$ (and we write $T : L^p(X,\mu) \to L^q(Y,\nu))$ if for all functions $f \in L^p(X,\mu)$ we have

$$\|Tf\|_{L^q(Y,\nu)} \lesssim_{p,T,\nu,\mu} \|f\|_{L^p(X,\mu)}.$$

The operator norm of T, denoted by $\|T\|_{L^p(X,\mu)\to L^q(Y,\nu)}$, is defined as

$$\|T\|_{L^p(X,\mu)\to L^q(Y,\nu)} \coloneqq \sup_{f\neq 0} \frac{\|Tf\|_{L^q(Y,\nu)}}{\|f\|_{L^p(X,\mu)}}.$$

We use the shorthand notation $\|T\|_{L^p(X,\mu)}$ when T maps $L^p(X,\mu)$ into itself.

The operators T that we present here include maximal operators, Calderón-Zygmund operators, fractional integrals and square functions. Maximal operators will be described in more detail because these operators are central objects in this thesis. We will be mainly interested in the mapping properties of these operators in the Lebesgue spaces $L^p(\mathbb{R}^n,\mu)$; that is, in estimates of the type

$$\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p,T} \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p \leqslant +\infty, \tag{1.7}$$

together with appropriate endpoint bounds as $p \to 1^+$.

Maximal operators. We will often use $\mathfrak{B}$ to denote a collection of open sets in $\mathbb{R}^n$, that is, a *basis*. We mainly focus in our work on two special bases of open sets; namely the basis $\mathfrak{Q}$, consisting of all n-dimensional cubes with sides parallel to the coordinate axes, and the basis $\mathfrak{R}$ consisting of all rectangles with sides parallel to the coordinate axes.

Definition 1.4. Let μ be a non-negative measure on $\mathbb{R}^n$, finite on compact sets, and let $\mathfrak{B}$ be a basis. For $f \in L^1_{\mathrm{loc}}(\mu)$ we define the *maximal operator with respect to* μ by

$$M^\mu_{\mathfrak{B}} f(x) \coloneqq \sup_{\substack{B\in\mathfrak{B}\\ B\ni x\\ \mu(B)>0}} \frac{1}{\mu(B)} \int_B |f(y)|d\mu(y),$$

if $x \in \cup_{B\in\mathfrak{B}} B$ and $M^\mu_{\mathfrak{B}} f(x) \coloneqq 0$ if $x \notin \cup_{B\in\mathfrak{B}} B$.

When μ is the Lebesgue measure we drop the superscript μ and we just write $M_{\mathfrak{B}}$. For the basis $\mathfrak{Q}$ the corresponding maximal operator is the *Hardy-Littlewood maximal operator* denoted by M. The maximal operator associated to the basis $\mathfrak{R}$ is referred to as the *strong maximal operator*, and is denoted by M_s.

The boundedness properties of $M_{\mathfrak{B}}$ depend strongly on the geometry of the basis $\mathfrak{B}$. Note that $M_{\mathfrak{B}}$ is a bounded operator in L^∞ for any basis $\mathfrak{B}$. However, the existence of L^p bounds, $1 < p < \infty$, and of endpoint bounds as $p \to 1^+$ cannot be guaranteed in the generality of $\mathfrak{B}$. Indeed, if $\mathfrak{B}$ is the family of *all* rectangles in $\mathbb{R}^n$, allowing all rotations, dilations and translations, then $M_{\mathfrak{B}}$ is called the *universal maximal operator* which is known to be unbounded for any $p < +\infty$; see [dG81]. There are some bases $\mathfrak{B}$ for which the boundedness properties of $M_{\mathfrak{B}}$ are well understood. In order to illustrate this, we present the following definitions.

Definition 1.5. A basis $\mathfrak{B}$ is *homothecy invariant*, if it satisfies

(i) For every $B \in \mathfrak{B}$ and every $y \in \mathbb{R}^n$ we have that $\tau_y B \in \mathfrak{B}$, where $\tau_y B \coloneqq \{x + y : x \in B\}$.

(ii) For every $B \in \mathfrak{B}$ and $s > 0$ we have that $\operatorname{dil}_s B \in \mathfrak{B}$, where $\operatorname{dil}_s B := \{sx : x \in B\}$.

Definition 1.6. $M_{\mathfrak{B}}$ satisfies a *Tauberian condition* with respect to a fixed $\gamma \in (0,1)$ if there exists some constant $c_{\mathfrak{B},\gamma} > 0$ such that, for every measurable set $E \subset \mathbb{R}^n$, we have

$$\left|\{x \in \mathbb{R}^n : M_{\mathfrak{B}}(\mathbf{1}_E)(x) > \gamma\}\right| \leqslant c_{\mathfrak{B},\gamma}|E|. \tag{$A_{\mathfrak{B},\gamma}$}$$

It is essential to notice that the previous estimate is supposed to hold only for a fixed $\gamma \in (0,1)$. However, in practice, many times one has a Tauberian condition of the form ($A_{\mathfrak{B},\gamma}$) for every $\gamma \in (0,1)$ and typically $c_{\mathfrak{B},\gamma}$ blows up to infinity as $\gamma \to 0^+$.

For a homothecy invariant basis $\mathfrak{B}$ of convex sets, the L^p boundedness of $M_{\mathfrak{B}}$ is characterized in terms of Tauberian conditions.

Theorem 1.7 (Hagelstein & Stokolos, [HS09]). *Let $\mathfrak{B}$ be a homothecy invariant basis consisting of convex sets in $\mathbb{R}^n$. Then the following are equivalent:*

(i) The operator $M_{\mathfrak{B}}$ satisfies a Tauberian condition ($A_{\mathfrak{B},\gamma}$) with respect to some fixed *$\gamma \in (0,1)$.*

(ii) There exists some $1 < p_o = p_o(\mathfrak{B},\gamma,n) < +\infty$ such that $M_{\mathfrak{B}} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for all $p > p_o$.

In virtue of the previous theorem, the Tauberian condition ($A_{\mathfrak{B},\gamma}$) for a single $\gamma \in (0,1)$ is equivalent to Tauberian conditions for *every* $\gamma \in (0,1)$ whenever $\mathfrak{B}$ is a homothecy invariant basis consisting of convex sets in $\mathbb{R}^n$.

We now turn our attention to the main bases of this thesis; that is, $\mathfrak{Q}$ and $\mathfrak{R}$. The next two theorems describe L^p and endpoint bounds for M and M_s, respectively.

Theorem 1.8 (Hardy & Littlewood, [?] & Wiener, []). *Let $1 < p < \infty$, then $M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. Moreover, the following endpoint holds.*

$$\left|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}\right| \lesssim_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\,dx, \quad \lambda > 0,$$

Theorem 1.9 (Jessen, Marcinkiewicz & Zygmund, [JMZ35]). *Let $1 < p < \infty$, then $M_s : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. Moreover, the following endpoint holds.*

$$\left|\{x \in \mathbb{R}^n : M_s f(x) > \lambda\}\right| \lesssim_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\Big(1 + \big(\log^+ \frac{|f(x)|}{\lambda}\big)^{n-1}\Big)\,dx. \tag{1.8}$$

Theorem 1.8 is the classical maximal theorem of Hardy and Littlewood. The distributional inequality (1.8) is due to Jessen, Marcinkiewicz and Zygmund from [JMZ35]. See also [CF75] for a geometric approach to the same result. By interpolation, the previous endpoint bounds imply the L^p bounds, for both M and M_s.

The boundedness properties of $M_{\mathfrak{B}}^{\mu}$ are much harder than the corresponding ones for the maximal operator $M_{\mathfrak{B}}$, with definitive information only for special cases of bases $\mathfrak{B}$ and measures μ. The boundedness properties for $M_{\mathfrak{B}}^{\mu}$ where $\mathfrak{B}$ is a homothecy invariant basis is one of the problems of this thesis and it is presented in Chapter 3. If $\mathfrak{B} = \mathfrak{Q}, \mathfrak{R}$, we drop the subscript $\mathfrak{B}$ and write M^{μ} and M_s^{μ}. As in the case of $d\mu = dx$, the one-parameter operator M^{μ} is easier to analyze than the operator M_s^{μ}. However, even in the one-parameter case, there is no complete characterization of the measures μ for which M^{μ}

is bounded on $L^p(\mathbb{R}^n, \mu)$. Below we present some known results for the basis $\mathfrak{Q}$ that are relevant for this thesis.

- *A special one-dimensional result.* In dimension $n = 1$, let μ be any non-negative Borel measure. We have that $M^\mu : L^1(\mu) \to L^{1,\infty}(\mu)$ and by interpolation $M^\mu : L^p(\mu) \to L^p(\mu)$ for all $1 < p \leqslant +\infty$. This result is very special to one dimension since the proof depends on a covering lemma for intervals of the real line. Observe that there is essentially no restriction on the measure μ. See for example [Sjö83] for the details of this result.

- *The centered, one-parameter maximal function with respect to a measure μ.* A common variation of M^μ is the weighted *centered* Hardy-Littlewood maximal function, given as

$$M_c^\mu f(x) := \sup_{r>0} \frac{1}{\mu(Q(x,r))} \int_{Q(x,r)} |f(y)| d\mu(y),$$

  where $Q(x, r)$ denotes the cube with sides parallel to the coordinate axes and side-length $r > 0$, centered at $x \in \mathbb{R}^n$. Then for any non-negative Borel measure μ we have that $M_c^\mu : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ and thus, by interpolation, $M_c^\mu : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for all $1 < p \leqslant +\infty$.

  The proof of this result depends on the Besicovitch covering lemma and it remains valid whenever the Besicovitch argument goes through. Thus, the condition that the maximal function defined above is centered is essential. For example, it was shown in [Sjö83] that if γ is the Gaussian measure in $\mathbb{R}^2$ then the non-centered weighted maximal operator M^γ does not map L^1 to $L^{1,\infty}$.

  The second essential hypothesis, hidden in the definition of M_c^μ, is that it is a *one-parameter* maximal operator, that is, we average with respect to a one-parameter family of cubes. Here one could replace cubes by Euclidean balls or more general "balls", given by translations and *one-parameter* dilations of a convex set in $\mathbb{R}^n$ symmetric about the origin.

  On the other hand, emphasizing the need for the one-parameter hypothesis mentioned previously, the boundedness fails for M_s^μ, even if we consider a *centered version* of it. The reason is that the family $\mathfrak{R}$ is an n-parameter family of sets for which the Besicovitch covering is not valid. See for example [Fef81b] for an example of a locally finite measure μ for which $M_{s,c}^\mu$ is unbounded on $L^p(\mu)$ for all $p < \infty$. In the next section we will describe in detail this case where μ is a weight.

- *The non-centered, one-parameter maximal function with respect to a doubling measure.* Let μ be a non-negative Borel measure. The following definition is standard.

  Definition 1.10. The measure μ is called *doubling* if there is a constant $\Delta_\mu > 0$ such that, for every cube $Q = Q(x, r) \subseteq \mathbb{R}^n$ we have

$$\mu(2Q) \leqslant \Delta_\mu \mu(Q),$$

  where $2Q = Q(x, 2r)$.

  It is an easy observation that for μ doubling, the non-centered weighted maximal operator M^μ is pointwise equivalent to its centered version, that is, $M^\mu f(x) \simeq M_c^\mu f(x)$, where the implicit constants depend only on Δ_μ. It follows from the discussion above that the maximal operator M^μ with respect to a doubling measure μ maps $L^1(\mu)$ to $L^{1,\infty}(\mu)$ and $L^p(\mu)$ to $L^p(\mu)$ for all $1 < p \leqslant \infty$.

Calderón-Zygmund operators. Let K be a function defined on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,x) : x \in \mathbb{R}^n\}$. We say that K is a *standard kernel* if it satisfies the size estimate

$$K(x,y) \leqslant \frac{c}{|x-y|^n}$$

and, for some $\delta > 0$, the regularity condition

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leqslant c\frac{|x-z|^\delta}{|x-y|^{n+\delta}}$$

whenever $2|x-z| < |x-y|$. We are now ready to define Calderón-Zygmund operators.

Definition 1.11. A linear operator T is a Calderón-Zygmund operator if

- T is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.
- There exists a standard kernel K such that

  $$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

  for any $f \in L^2(\mathbb{R}^n)$ and $x \notin supp(f)$.

The main examples of Calderón-Zygmund operators are those given as a convolution with a standard kernel $K(x,y) = k(x-y)$, where k is a locally integral on $\mathbb{R}^n \backslash \{0\}$. The archetypal operator of this type is the Hilbert transform H. It is given by convolution against the kernel $K(x) = 1/(\pi x)$ for $x \in \mathbb{R}^n$. More precisely, let f be in the Schwartz class $\mathscr{S}(\mathbb{R})$, and define the Hilbert transform as

$$Hf(x) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \frac{f(y)}{x-y} dy. \tag{1.9}$$

The n-dimensional analogue of the Hilbert transform is the Riesz transform. Indeed, for $1 \leqslant j \leqslant n$, the j-th Riesz transform of f is given by

$$R_j f(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)dy, \tag{1.10}$$

for all $f \in \mathscr{S}(\mathbb{R})$. Here Γ denotes the gamma function.

Boundedness of Calderón-Zygmund operators from the Lebesgue spaces $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ is completely understood. The proof can be found in [Gra08, Chapter 5].

Fractional operators. We first consider a fractional variant of the maximal operator M^μ.

Definition 1.12. Let $0 < \alpha < n$, we define the *fractional maximal operator* as

$$M^{\mu}_{\alpha} f(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)^{1-\alpha/n}} \int_Q |f(y)| \, d\mu(y). \tag{1.11}$$

Note that the case $\alpha = 0$ corresponds to the operator M^{μ}. By Hölder's inequality

$$\frac{1}{\mu(Q)^{1-\alpha/n}} \int_Q |f(y)| \, d\mu y \leqslant \left(\int_Q |f(y)|^{n/\alpha} d\mu y \right)^{\alpha/n},$$

which implies $M^{\mu}_{\alpha} : L^{n/\alpha}(\mu) \to L^{\infty}(\mu)$. When μ is the Lebesgue measure we simply write M_{α}. If μ is doubling (see Definition 1.10), then a similar argument to the one indicated for M^{μ} shows that M^{μ}_{α} is weak $(1, (n/\alpha)')$. Thus, by interpolation, $M^{\mu}_{\alpha} : L^p(\mu) \to L^q(\mu)$ if the exponents p and q are related by the relation

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}. \tag{1.12}$$

The fractional maximal operator is intimately related to the following operator.

Definition 1.13. Let $f \in \mathscr{S}(\mathbb{R}^n)$. For $0 < \alpha < n$, the *fractional integral operator* or *Riesz potential* I_{α} is defined by

$$I_{\alpha} f(x) \coloneqq \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

We notice that the function $|\cdot|^{n-\alpha}$ is locally integrable for $0 < \alpha < n$, so I_{α} is well defined. This operator is pointwise bigger than M_{α}. Moreover,it has the same boundedness properties as M_{α}; that is: $I_{\alpha} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ and $I_{\alpha} : L^1(\mathbb{R}^n) \to L^{q,\infty}(\mathbb{R}^n)$, where $1 \leqslant p < q$ satisfies (1.12).

Square functions. Let Δ denote the collection of dyadic cubes in $\mathbb{R}^n$. Given $Q \in \Delta$, let $\hat{Q}$ be its dyadic parent, that is, the unique dyadic cube containing Q whose side-length is twice that of Q.

Definition 1.14. The *dyadic square function* is the defined as the operator

$$S_d f(x) \coloneqq \left(\sum_{Q \in \Delta} (f_Q - f_{\hat{Q}})^2 \mathbf{1}_Q(x) \right)^{1/2}$$

where $f_Q = \fint_Q f(x) \, dx$.

This operator is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

1.2.1 Multiparameter operators

Most of the operators considered so far in this section, commute with one-parameter dilations of $\mathbb{R}^n$; namely, the conditions satisfied by them are not affected by the change of scale $x \to xt$, $t > 0$. Now, we gather some practical information concerning the operators which are invariant under n-parameter dilations in $\mathbb{R}^n$.

The most basic example of the multiparameter theory is the strong maximal operator M_s, that we have already presented. In fact, if we define the n-parameter dilation

$$x \to \delta(x) = (\delta_1 x_1, \delta_2 x_2, \cdots \delta_n x_n)$$

with $\delta_i > 0$ for every i, we have that for any arbitrary positive function f

$$M_s(f_\delta) = (M_s f)_\delta$$

with $f_\delta(x) = f(\delta x)$. Thus the operator M_s commutes with n-parameter dilations.

Another interesting example of a multiparameter operator is the *multiple Hilbert transform*, defined by

$$H_s f(x) := \frac{1}{\pi^n} \lim_{\epsilon_1, \epsilon_2, \cdots, \epsilon_n \to 0^+} \int_{|x_1 - y_1| > \epsilon_1} \cdots \int_{|x_n - y_n| > \epsilon_n} \frac{f(y)}{(x_1 - y_1) \cdots (x_n - y_n)} dy.$$

Its boundedness properties on $L^p(\mathbb{R}^n)$ follows directly by the studying of the n-composition of the corresponding one-dimensional operators. More precisely, given an operator T acting on functions $f : \mathbb{R}^n \to \mathbb{R}$, for every $j = 1, \cdots, n$, T_j denotes the operator defined by

$$T^j f(x) := T(f(x_1, \cdots, x_{j-1}, \cdot, x_{j+1}, \cdots, x_n))(x_j).$$

That is, the operator T acting on the j-th variable while keeping the remaining variables fixed. In particular, we have

$$H_s f(x) = H^1 \circ H^2 \circ \cdots \circ H^n f(x) \tag{1.13}$$

where H is exactly the *usual* Hilbert transform. Also, it is not difficult to obtain the following estimate:

$$M_s f(x) \leqslant M^1 \circ M^2 \circ \cdots \circ M^n f(x). \tag{1.14}$$

where M^j is the one dimensional (Hardy-Littlewood) maximal operator in the j-th coordinate. Since it is already known that the operators H^j and M^j are bounded on $L^p(\mathbb{R}^n)$, from identity (1.13) and inequality (1.14) we deduce the L^p boundedness of H_s and M_s, respectively.

1.3 Weights

A *weight* is a non-negative locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere. Given a weight w and a measurable set E, we use the notation

$$w(E) = \int_E w(x) dx$$

to denote the w-measure of the set E. In this section we are interested in estimates of the form

$$\|Tf\|_{L^p(w)} \lesssim_{n,p,w,T} \|f\|_{L^p(w)}, \quad 1 < p \leqslant +\infty, \tag{1.15}$$

where T is any of the operators presented in the last section. We also present the corresponding endpoint bounds as $p \to 1^+$ that are are typically harder (and stronger) than their L^p analogues (1.15). To study this estimate (1.15) we need to introduce the classes of Muckenhoupt weights A_p.

1.3.1 A_p-weights

Definition 1.15. We say that a weight w belongs to the class A_p, $1 < p < +\infty$, if

$$[w]_{A_p} \coloneqq \sup_{Q \in \mathfrak{Q}} \left(\frac{1}{|Q|} \int_Q w(y)dy \right) \left(\frac{1}{|Q|} \int_Q w(y)^{1-p'}dy \right)^{p-1} < +\infty. \tag{1.16}$$

For the limiting case $p = 1$ the class A_1 is defined to be the set of weights w such that

$$[w]_{A_1} \coloneqq \sup_{Q \in \mathfrak{Q}} \left(\frac{1}{|Q|} \int_Q w(y)dy \right) \operatorname*{ess\,sup}_Q (w^{-1}) < +\infty.$$

This is equivalent to w having the property

$$Mw(x) \leqslant [w]_{A_1} \cdot w(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

It follows from Hölder's inequality and the definitions above that for all $1 \leqslant p < q < +\infty$ we have that $A_p \subset A_q$, that is, the classes A_p are increasing in $p \geqslant 1$. It is thus natural to define the limiting class A_∞ as

$$A_\infty \coloneqq \bigcup_{p>1} A_p = \bigcup_{p \geqslant 1} A_p.$$

We notice here that there are several definitions of the A_∞ class that appeared in the literature and all of them are equivalent. Another classical definition is the one introduced by Hruščev [Hru84]; namely, that A_∞ consists of those w such that:

$$[w]_{A_\infty} \coloneqq \sup_{Q \in \mathfrak{Q}} \left(\frac{1}{|Q|} \int_Q w(y)dy \right) \exp \left(\frac{1}{|Q|} \int_Q \log w(y)^{-1}dy \right) < +\infty. \tag{1.17}$$

Since the above condition coincides with the formal limit conditions (1.16) as p tends to ∞, it explains perfectly the name A_∞. For more details in the equivalent definitions of the A_∞ class, we refer the interested reader to [DMRO13b].

For a weight w, belonging to the A_p-class, we have the next result:

Theorem 1.16 (Muckenhoupt, [Muc72]). *The following statements are true.*

(i) Let $1 < p < \infty$, $M : L^p(w) \to L^p(w)$ if and only if $M : L^p(w) \to L^{p,\infty}(w)$, if and only if $w \in A_p$.

(ii) $M : L^1(w) \to L^{1,\infty}(w)$ if and only if $w \in A_1$.

Therefore the A_p-class is equivalent to estimate (1.15) in the case $T = M$. In the case of $T = H$, (1.15) was obtained by R. Hunt, B. Muckenhoupt and R. Wheeden for A_p-weights. Then, R. Coifman and C. Fefferman extended this result for any Calderón-Zygmund operator T. In this case the A_p-class turns out to be sufficient for boundedness, although not always necessary.

Theorem 1.17 (Coifman & C. Fefferman, [CF74]). *If T is a Calderón-Zygmund operator, then the following statements are true:*

- *For any* $w \in A_p$, $1 < p < \infty$, $T : L^p(w) \to L^p(w)$.
- *For any* $w \in A_1$, $T : L^1(w) \to L^{1,\infty}(w)$.

For the case of T being the square dyadic function, (1.15) is also characterized in terms of the A_p condition. The case of the fractional operators T requires a *variant* of the A_p class of weights. More precisely,

Theorem 1.18 (Muckenhoupt & Wheeden, [MW74]). I_α *and* M_α *are bounded from* $L^p(w^p)$ *to* $L^q(w^q)$ *if and only if the exponents* p *and* q *are related by the equation* (1.12) *and* w *satisfies the so called* $A_{p,q}$ *condition; that is,* $w \in A_{p,q}$ *if*

$$[w]_{A_{p,q}} := \sup_Q \left(\frac{1}{|Q|}\int_Q w^q \, dx\right)\left(\frac{1}{|Q|}\int_Q w^{-p'} \, dx\right)^{q/p'} < \infty. \tag{1.18}$$

1.3.2 Weights associated to bases

We say that w is a *weight* associated to the basis $\mathfrak{B}$ if w is a non-negative locally integrable function on $\mathbb{R}^n$ and $w(B) := \int_B w(x)dx < +\infty$ for every $B \in \mathfrak{B}$. For the bases $\mathfrak{Q}$ we have already seen in Theorem 1.16 that the validity of (1.15) for any $1 < p < +\infty$ is characterized by the membership of w to the class A_p. Thus, we extend Definition 1.15 to general bases.

Definition 1.19. We say that a weight w belongs to the class $A_{p,\mathfrak{B}}$, $1 < p < +\infty$, if

$$[w]_{A_{p,\mathfrak{B}}} := \sup_{B\in\mathfrak{B}} \left(\frac{1}{|B|}\int_B w(y)dy\right)\left(\frac{1}{|B|}\int_B w(y)^{1-p'}dy\right)^{p-1} < +\infty.$$

For the limiting case $p = 1$ the class $A_{1,\mathfrak{B}}$ is defined to be the set of weights w such that

$$[w]_{A_{1,\mathfrak{B}}} := \sup_{B\in\mathfrak{B}} \left(\frac{1}{|B|}\int_B w(y)dy\right)\operatorname*{ess\,sup}_B(w^{-1}) < +\infty.$$

This is equivalent to w having the property

$$M_{\mathfrak{B}}w(x) \leqslant [w]_{A_{1,\mathfrak{B}}} \cdot w(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

It is again natural to define the limiting class $A_{\infty,\mathfrak{B}}$ as

$$A_{\infty,\mathfrak{B}} := \bigcup_{p>1} A_{p,\mathfrak{B}} = \bigcup_{p\geqslant 1} A_{p,\mathfrak{B}}.$$

For the basis $\mathfrak{Q}$, this definition is exactly Definition 1.15.

For general bases $\mathfrak{B}$ and associated weights w, very little is known concerning the boundedness of $M_{\mathfrak{B}}$ and $M_{\mathfrak{B}}^w$ on $L^p(w)$. In fact, we do not have a clear characterization like Theorem 1.16. However, the following *abstract* theorem from [Pér91] gives a necessary and sufficient condition for the boundedness of the $M_{\mathfrak{B}}^w$, in terms of the maximal operator $M_{\mathfrak{B}}$ in the special case that $w \in A_{\infty,\mathfrak{B}}$.

Theorem 1.20 (Pérez, [Pér91])**.** *Let $\mathfrak{B}$ be a basis. The following are equivalent:*

(i) For every $1 < p < +\infty$ and every $w \in A_{p,\mathfrak{B}}$ we have that

$$M_{\mathfrak{B}} : L^p(w) \to L^p(w).$$

(ii) For every $1 < p < +\infty$ and every $w \in A_{\infty,\mathfrak{B}}$ we have that

$$M_{\mathfrak{B}}^{w} : L^p(w) \to L^p(w).$$

In Chapter 3 we characterize (1.15) for $M_{\mathfrak{B}}$ whenever $\mathfrak{B}$ is a homothecy invariant basis of convex sets. As it was described in the unweighted case (see Theorem 1.7), the characterization is in terms of *Tauberian conditions*. Though we do not present the result here, we state the weighted version of Definition 1.6 that will be used in the following chapters.

Definition 1.21. We will say that the maximal operator $M_{\mathfrak{B}}$ satisfies a *weighted Tauberian condition* with respect to some $\gamma \in (0,1)$ and a weight w if there exists a constant $c_{\mathfrak{B},\gamma,w} > 0$ such that, for all measurable sets $E \subset \mathbb{R}^n$ we have

$$w(\{x \in \mathbb{R}^n : M_{\mathfrak{B}}(\mathbf{1}_E)(x) > \gamma\}) \leqslant c_{\mathfrak{B},\gamma,w}\, w(E). \qquad (A_{\mathfrak{B},\gamma,w})$$

In this case, we will say that the weight w *satisfies* ($A_{\mathfrak{B},\gamma,w}$) for *some fixed level* $\gamma \in (0,1)$.

Next, we analyze two particular bases $\mathfrak{B}$ for which estimate (1.15) has been already studied in more detail.

The basis $\mathfrak{R}$ and the strong A_p-weights.

Definition 1.19 for the basis $\mathfrak{R}$ gives what is referred to as the *strong* A_p-class and we use the shorthand notation $A_p^* \coloneqq A_{p,\mathfrak{R}}$. Since the A_p^*-weights will play an essential role in this dissertation, we describe this class in detail. An important feature of strong A_p-weights, $1 \leqslant p \leqslant \infty$, is that if we fix any $t \in \mathbb{R}$ then the weight

$$w^t(x') \coloneqq w(x',t), \quad x' \in \mathbb{R}^{n-1},$$

is an A_p^*-weight on $\mathbb{R}^{n-1}$, *uniformly* in $t \in \mathbb{R}$. In practice, uniformly means that all the constants connected with the properties of the A_p^*-weight w^t can be taken to be independent of t. We can express this property in a *complementary* way. Given a weight $w \in A_p^*$, we write $\bar{x}_j \coloneqq (x_1, \dots, \widehat{x_j}, \dots, x_n) \in \mathbb{R}^{n-1}$, where the variable $\widehat{x_j}$ is missing, and we set $w_{\bar{x}_j}(t) \coloneqq w(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n)$, $t \in \mathbb{R}$. The point here is that we "freeze" the variable $\bar{x}_j \in \mathbb{R}^{n-1}$ and consider the one-dimensional weight $w_{\bar{x}_j}$. We then have the following:

Lemma 1.22. *Let $1 \leqslant p < \infty$. Then $w \in A_p^*$ if and only if, for every $1 \leqslant j \leqslant n$ we have that the one-dimensional weight $w_{\bar{x}_j} \in A_p$, uniformly in $\bar{x}_j \in \mathbb{R}^{n-1}$. Furthermore, for each $1 \leqslant j \leqslant n$ we have*

$$\sup_{\bar{x}_j} [w_{\bar{x}_j}]_{A_p} \leqslant [w]_{A_p^*}. \qquad (1.19)$$

Remark 1.23. We would like to stress that this lemma highlights the one-dimensional nature of the elements of the A_p^*-class. It was already proved in [Kur80] for the case $1 < p < \infty$ and extended to $p = 1$ in [BK85]. Here, we have restated the lemma showing the relation between the constants $[w]_{A_p^*}$ and $[w^j]_{A_p}$, that follows directly if we keep track of the constants. The first implication of this lemma is a consequence of the Lebesgue differentiation theorem. The second goes through the boundedness of the strong maximal operator. See [GCRdF85, Theorem 6.2] for a complete account of it.

In the case of $\mathfrak{R}$, as it happened for $\mathfrak{Q}$, there are different definitions of A_∞^* and they are all equivalent. One of them, that we will use throughout this thesis is the following: $w \in A_\infty^*$ if there exist constants $\delta, c > 0$ such that, given any rectangle $R \in \mathfrak{R}_n$ and a measurable subset $S \subset R$, then

$$\frac{w(S)}{w(R)} \leqslant c\left(\frac{|S|}{|R|}\right)^{\delta}. \tag{1.20}$$

Remark 1.24. Let $w \in A_\infty^*$. By the previous definition we see that there exists some $\epsilon = \epsilon(w) > 0$ such that, for every rectangle $R \in \mathfrak{R}_n$ and all measurable sets $F \subset \mathbb{R}^n$, we have

$$|R \cap F| \leqslant \epsilon |R| \Rightarrow w(R \cap F) \leqslant \frac{1}{2} w(R) \Rightarrow w(R \setminus F) \geqslant \frac{1}{2} w(R).$$

In fact, it suffices to choose $\epsilon > 0$ so that $c\epsilon^\delta \leqslant \frac{1}{2}$, where c, δ are the constants associated to $w \in A_\infty^*$ from (1.20). Since for any $t \in \mathbb{R}$ the weight $w^t := w(\cdot, t)$ is an A_∞^*-weight on $\mathbb{R}^{n-1}$, uniformly in t, the parameter $\epsilon > 0$ can be chosen sufficiently small so that we have the previous property also for w^t, rectangles $R' \in \mathfrak{R}_{n-1}$ and sets $F' \subset \mathbb{R}^{n-1}$, uniformly in t.

The following theorem shows estimate (1.15) for the case of $T = M_s$.

Theorem 1.25 (Bagby & Kurtz, [BK85])**.** *The following statements are true.*

(i) *Let* $1 < p < +\infty$. *Then* $M_s : L^p(w) \to L^p(w)$ *if and only if* $M_s : L^p(w) \to L^{p,\infty}(w)$, *if and only if* $w \in A_p^*$.

(ii) *If* $w \in A_1^*$ *then*

$$w(\{x \in \mathbb{R}^n : M_s f(x) > \lambda\}) \lesssim_{n,w} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\Big(1 + \big(\log^+ \frac{|f(x)|}{\lambda}\big)^{n-1}\Big) w(x)\,dx, \quad \lambda > 0. \tag{1.21}$$

Thus, the boundedness properties of M_s on $L^p(w)$, $1 < p < +\infty$, are completely characterized for the basis $\mathfrak{R}$. However, the endpoint estimate (1.21) for the strong maximal operator is not so transparent. Indeed, the presence of the logarithmic terms on the right hand side of the (1.21) results in the condition A_1^* being sufficient, but not necessary, for the validity of (1.21). A necessary condition for (1.21), which is weaker than $w \in A_1^*$, appears in [BK85] but the authors show that it is not sufficient. On the other hand, the weighted endpoint estimate (1.21) has been characterized in [Gog92] in terms of a certain covering property for rectangles. See also [KK91, Theorem 4.3.1] for a detailed proof of this fact. Similar characterizations of the boundedness properties of maximal operators on general $L^p(\mu)$-spaces in terms of covering properties are contained for example in [Jaw86, Theorem 2.2] and [GCRdF85, Lemma IV.6.11], while the approach goes back to [C76] and [CF75]. There is however no characterization in the spirit of the Muckenhoupt A_p^*-classes, of the weights w such that (1.21) holds.

The study of estimate (1.15) for the weighted strong maximal operator M_s^w is more complicated. For the weighted strong maximal operator M_s^w, R. Fefferman showed in [Fef81b] that it maps $L^p(w)$ to $L^p(w)$ whenever $w \in A_\infty^*$:

$$\|M_s^w f\|_{L^p(w)} \lesssim_{w,n} c_{p,n}\|f\|_{L^p(w)} \quad 1 < p \leqslant \infty. \tag{1.22}$$

The corresponding endpoint inequality

$$w(\{x \in \mathbb{R}^n : M_s^w f(x) > \lambda\}) \lesssim_{n,w} \int_{\mathbb{R}^n} \Big(1 + \big(\log^+ \frac{|f(x)|}{\lambda}\big)^{n-1}\Big) w(x)dx$$

is also true whenever $w \in A_\infty^*$. This was proved by Jawerth and Torchinsky in [JT84] and independently by Long and Shen [LS88]. A weaker sufficient condition for the boundedness of M_s^w appears in [JT84] but it is quite technical and we will not describe it here; it shows however that $w \in A_\infty^*$ is *not* a necessary condition for the boundedness of M_s^w on $L^p(w)$, nor for the corresponding endpoint distributional estimate above.

The membership of a weight w in the class A_p^* also characterizes the boundedness of the multiple Hilbert transform H_s on $L^p(w)$, $1 < p < \infty$. Indeed, these weights are the *natural* ones in the multiparameter setting. See [GCRdF85, Chapter IV.6] for more details on this subject.

Muckehoupt bases

Theorem 1.20 as well as Theorem 1.16 and Theorem 1.25 motivate the following definition, which is from [Pér91].

Definition 1.26. A basis $\mathfrak{B}$ is a *Muckenhoupt basis* if for every $1 < p < +\infty$ and every $w \in A_{p,\mathfrak{B}}$, we have that

$$M_{\mathfrak{B}} : L^p(w) \to L^p(w).$$

With this definition, Theorem 1.20 states that $\mathfrak{B}$ is a Muckenhoupt basis if and only if the weighted maximal operator satisfies $M_{\mathfrak{B}}^w : L^p(w) \to L^p(w)$ for every $1 < p < +\infty$ and every $w \in A_{\infty,\mathfrak{B}}$. On the other hand, Theorem 1.16 and Theorem 1.25 show, respectively, that $\mathfrak{Q}$ and $\mathfrak{R}$ are Muckenhoupt bases. Another interesting example of a Muckenhoupt basis is given by the Córdoba-Zygmund basis in $\mathbb{R}^3$, consisting of rectangles with sides parallel to the coordinate axes and sidelengths of the form (s, t, st), $s, t > 0$.

Remark 1.27. We point out here that if $\mathfrak{B}$ is a Muckenhoupt basis, then the following are equivalent:

- $w \in A_{\infty,\mathfrak{B}}$.
- w satisfies ($A_{\mathfrak{B},\gamma,w}$) for *some fixed level* $\gamma \in (0, 1)$.

Although this is an original result contained in the thesis, it is announced here because we will use it several times in the next chapter. It will be detailed in Chapter 3.

1.4 References

In this chapter we have just provided a rough description of the concepts and tools we will need in this dissertation. We refer the interested reader to the following monographs for a more detailed information on these issues. The main features of the Lebesgue spaces and the boundedness properties of the operators presented in Section 1.2 can be found in [Gra08]. A classical reference for Orlicz spaces is the book by Bennett and Sharpley [BS88]. For a very clear explanation on the properties of a Young function and the construction of its complementary function, we refer to [Wil08, Chapter 10]. We also refer to the recent book [CUMP12, Chapter 5.1] for a description of the *localized* L^{Φ}-norm. A general analysis on the boundedness properties of the maximal operators $\mathrm{M}_{\mathfrak{B}}^{\mu}$ is presented in [Jou83]. A general introduction to multiparameter harmonic analysis is contained in [Fef86]. For a definition of the multiparameter version of the Hilbert transform as well as of more general multiparameter singular integral operators see for example [Fef81a] and [FS82]. Finally, a complete account of the results related to weighted norm inequalities (Section 1.3) can be found in [Duo01, Chapter 7], [GCRdF85, Chapter IV] and [Gra09, Chapter 9]. In particular, the definition of Muckenhoupt weights for general bases is in [GCRdF85, Chapter IV.4].

Chapter 2

Two-Weight norm inequalities for maximal operators

The main purpose of this chapter is to show our contribution to Problem 0.1 and Problem 0.2. Though we are mainly focus on the strong maximal operator and the basis of rectangles $\mathfrak{R}$, results related to Problem 0.1 are also extended to the multilinear setting and to other more general bases of open sets.

2.1 Strong two-weight problem and bump conditions

We remember that in Problem 0.1 we are interested in finding sufficient conditions on (w, v) so that M_s is bounded from $L^p(v)$ to $L^p(w)$. We recall first a few facts about the two-weight norm inequalities for the Hardy-Littlewood maximal function. Though this operator is rather different than the strong maximal one, its understanding motivates our approach.

Sawyer [Saw82a] characterized those pairs of weights (w, v) for which the Hardy-Littlewood maximal operator M is bounded from $L^p(v)$ to $L^p(w)$ for $1 < p < \infty$. He showed that:

Theorem 2.1 (Sawyer, [Saw82a]). $M : L^p(v) \to L^p(w)$ *if and only if for every cube* $Q \in \mathfrak{Q}$, (w, v) *satisfies the testing condition*

$$\int_Q (M(\mathbf{1}_Q \sigma))^p w dx \lesssim \sigma(Q), \tag{2.1}$$

where σ *denotes the dual weight of* v*; that is,* $\sigma := v^{1-p'}$.

On the other hand, it is also known that the two weight Muckenhoupt condition A_p

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u_1 dx\right)\left(\frac{1}{|Q|}\int_Q u_2^{1-p'} dx\right)^{p-1} < \infty$$

is necessary but not sufficient for the maximal operator to be of strong type (p, p) with respect to the pair of measures (u_1, u_2). As it was pointed out in the introduction, the fact that Sawyer's condition involves the maximal operator itself makes it often difficult to test in practice. Therefore, though this condition characterizes completely the two weight problem, it would be very useful to look for sufficient conditions close in form to the A_p condition. The first step in this direction was done by Neugebauer [Neu83]. He proved that if the pair of weights (w, v) is such that for $r > 1$

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w^{pr}dx\right)^{1/pr}\left(\frac{1}{|Q|}\int_Q v^{-p'r}dx\right)^{1/p'r} < \infty,$$

then

$$\int_{\mathbb{R}^n} (wMf)^p \, dx \lesssim_{n,p,r,w,v} \int_{\mathbb{R}^n} (vf)^p \, dx \tag{2.2}$$

for all nonnegative functions f. We henceforth study the problem in the form (2.2), using the weights (w^p, v^p) in order to simplify our calculations. Using Definition 1.1, Neugebauer's condition can be restated in terms of the localized L^p norm as follows:

$$\sup_Q \|w\|_{L^{pr}(Q)}\|v^{-1}\|_{L^{p'r}(Q)} < \infty. \tag{2.3}$$

Notice that the A_p condition can also be rewritten with the new formulation as follows

$$\sup_Q \|w\|_{L^{p}(Q)}\|v^{-1}\|_{L^{p'}(Q)} < \infty. \tag{2.4}$$

This tells us that if we replace the $L^p, L^{p'}$-average norms in (2.4) by some stronger ones, as in (2.3), then we get a condition that is sufficient for (2.2) to hold. At the same time, this new condition preserves the geometric structure of the classical A_p. Conditions like (2.3) are known as power bump conditions. These conditions can be generalized replacing the localized $L^p, L^{p'}$-norms in (2.4) by stronger localized norms, which are not necessarily as big as the $L^{pr}, L^{p'r}$-norms. Indeed, Pérez ([Pér95b], [Pér91] proved that it was enough to only substitute the norm associated with the weight v^{-1} by a stronger one defined in terms of certain Banach function spaces X with an appropriate boundedness property. To be more precise, we define the B_p condition, first introduced in [Pér95b], as follows

Definition 2.2. Let $1 < p < \infty$. We say that a Young function Φ satisfies the B_p condition and we write $\Phi \in B_p$, if there is a positive constant c such that

$$\int_c^\infty \frac{\Phi(t)}{t^p}\frac{dt}{t} < \infty. \tag{2.5}$$

As the following theorem shows, this growth condition turns out to be optimal sufficient for the boundedness of the Hardy-Littlewood function from $L^p(v^p)$ to $L^p(w^p)$.

Theorem 2.3 (Pérez, [Pér95b]). *Let $1 < p < \infty$, and let Φ be a doubling Young function such that the complementary Young function $\bar{\Phi}$ satisfies the condition (2.5).*

(i) *Let (w, v) be a couple of weights such that*

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w^p\right)^{1/p}\|v^{-1}\|_{\Phi,Q} < \infty. \tag{$A_{p,\Phi}$}$$

Then

$$\int_{\mathbb{R}^n} (wMf)^p \, dx \lesssim_{n,p,\Phi,w,v} \int_{\mathbb{R}^n} (vf)^p \, dx,$$

for all non-negative functions f.

(ii) *Condition (2.5) is also necessary in the following sense. Suppose that* Φ *is a Young function such that* M *maps* $L^p(v^p)$ *into* $L^p(w^p)$ *whenever the couple of weights* (w, v) *satisfies condition* ($A_{p,\Phi}$)*. Then,* $\bar{\Phi} \in B_p$*.*

Let us clarify a bit more the role of the B_p condition in ($A_{p,\Phi}$). Note that if a Young function Φ verifies the B_p condition, then $\Phi(t) \lesssim t^p$, say for $t \geqslant 1$. Indeed, since $\Phi \in B_p$ there exists $c > 0$ such that

$$\int_c^\infty \frac{\Phi(s)}{s^p} \frac{ds}{s} < \infty.$$

Actually, we have that

$$\int_1^\infty \frac{\Phi(s)}{s^p} \frac{dt}{t} < \infty.$$

Indeed observe that if $c > 1$, then

$$\int_1^\infty \frac{\Phi(s)}{s^p} \frac{ds}{s} \leqslant \int_1^c \frac{\Phi(s)}{s^p} \frac{ds}{s} + \int_c^\infty \frac{\Phi(s)}{s^p} \frac{ds}{s} < \infty.$$

So we get that

$$\infty > \int_1^\infty \frac{\Phi(s)}{s^p} \frac{ds}{s} > \int_t^{2t} \frac{\Phi(s)}{s^p} \frac{ds}{s} \geqslant \frac{\Phi(t)}{2^p t^p} \int_t^{2t} \frac{ds}{s} = \frac{\Phi(t)}{t^p} \frac{\ln 2}{2^p}$$

for $t \geqslant 1$.

Therefore, the hypothesis of Theorem 2.3 implies that $\bar{\Phi}(t) \gtrsim t^p$ and then $\Phi(t) \gtrsim t^{p'}$, by the equality (1.4) of the complementary Young function. Therefore (1.3) assures that

$$\|v^{-1}\|_{L^{p'}(Q)} \leqslant \|v^{-1}\|_{\Phi,Q}$$

for all Q. We conclude that condition ($A_{p,\Phi}$) is in general stronger than the A_p condition. However, upon choosing an appropriate Young function Φ we can make it so that ($A_{p,\Phi}$) is arbitrarily close to the A_p condition, in the logarithmic scale. In fact, note that there exists a big range of Young functions in B_p for which the corresponding norm is *a bit bigger* than the localized $L^{p'}$-norm and do produce sufficient conditions for the two-weight problem. In this sense, is very easy to describe Young functions $\Phi \in B_p$ such that

$$\|v^{-1}\|_{L^{p'}(Q)} \leqslant \|v^{-1}\|_{\Phi,Q} \leqslant \|v^{-1}\|_{L^q(Q)}, \quad q > p'.$$

The Orlicz spaces have more refined scales of integrability than the L^p spaces. Indeed, this flexibility plays an important role in the definition of optimal sufficient conditions (see (ii) in the last theorem).

The first purpose of this chapter is to study the strong two-weight norm inequalities for M_s in terms of bump conditions; that is, to describe a sufficient condition for the boundedness of the strong maximal function using an appropriate B_p condition.

As we explained in the introduction, the two-weight problem for the strong maximal operator was already characterized by Sawyer [Saw82b] in terms of testing conditions. The problem was also studied in [Pér93] with a more similar approach to the one that we present here. It was proved that if (u, v) is a couple of weights satisfying for some $r > 1$

$$\sup_R \left(\frac{1}{|R|}\int_R w^p \, dx\right)^{1/p} \left(\frac{1}{|R|}\int_R v^{-p'r} \, dx\right)^{1/p'} < \infty \tag{2.6}$$

and $w^p \in A_\infty^*$, then $M_s : L^p(v^p) \to L^p(w^p)$. In this case, the strong weighted estimate is obtained from weak type ones using interpolation and the fact that there exists a reverse Hölder's inequality for the weights that verify (2.6). However, this good property disappears if we substitute the $L^{p'r}$-norm associated with the weight v^{-1} by a weaker one. Therefore, we will need a different strategy in order to solve the two-weight problem with general bump conditions. In order to state the result we need to define the appropriate class of Young functions that enables to obtain bump conditions in the case of rectangles.

Definition 2.4. Let $1 < p < \infty$. A Young function Φ is said to satisfy the strong B_p^* condition, if there exists a positive constant c such that

$$\int_c^\infty \frac{\Phi_n(\Phi(t))}{t^p} \frac{dt}{t} < \infty, \tag{2.7}$$

where $\Phi_n(t) := t[1 + (\log^+ t)^{n-1}]$ for all $t > 0$. In this case, we say that $\Phi \in B_p^*$.

We are now ready to state our two-weight theorem.

Theorem 2.5. *Let $1 < p < \infty$, and let Φ be a Young function such that the complementary Young function $\bar{\Phi}$ satisfies condition (2.7).*

(i) *Let (w, v) be a couple of weights such that $w^p \in A_\infty^*$ and*

$$\sup_R \left(\frac{1}{|R|}\int_R w^p\right)^{1/p} \|v^{-1}\|_{\Phi,R} < \infty. \tag{$A_{p,\Phi}^*$}$$

Then

$$\int_{\mathbb{R}^n} (wM_s f)^p \, dx \lesssim_{n,p,\Phi,w,v} \int_{\mathbb{R}^n} (vf)^p \, dx$$

for all non-negative functions f.

(ii) *Condition (2.7) is also necessary. Namely, suppose that Φ is a Young function such that $M_s : L^p(v^p) \to L^p(w^p)$ whenever the couple of weights (w, v) satisfies property ($A_{p,\Phi}^*$). Then $\bar{\Phi} \in B_p^*$.*

If this result is compared with Theorem 2.3, then there are some differences between them. First, Theorem 2.5 does not require the Young function Φ to be doubling. As we will see in Section 2.3, the doubling condition is not necessary. Moreover, Theorem 2.5 not only needs a more restrictive class of young functions (the class B_p^*), but also we ask for an extra condition on the weight w. Observe that in (ii) of the above theorem, the hypothesis $w^p \in A_\infty^*$ does not play any role. This leads one to think that may be this condition is not necessary. The last section in this chapter discusses this matter.

The starting point of the proof of this theorem is the study of the growth condition (2.7) and its connection with the boundedness of the strong maximal function. Then we need to understand the geometry of the rectangles in order to deal with their covering properties throughout the proof. This last point is also necessary for the second problem we have posed. In the next two sections we address these issues in order to present the proof of Theorem 2.5 in Section 2.4.2.

2.2 On covering properties

It is very well-known that the boundedness properties of a maximal function $M_{\mathfrak{B}}$ are closely connected to the covering properties of the corresponding basis $\mathfrak{B}$. It is for this reason that we present in this section some results related to covering arguments. For the bases $\mathfrak{B}$ we are interested in here, the covering lemmas of Vitali type do not work. Thus it is necessary to deal with some alternative tools such as:

- Selection procedures that allow to remove the unnecessary elements of the basis and preserve the elements with *good* sparseness properties.
- Control of the overlap of the selected elements of the basis. This control can be for example in norm.

Since we mainly work in weighted spaces, our proofs will focus on the weighted versions of the covering lemmas. The solution of Problem 0.2 requires a very precise understanding of the geometry of the basis, while for Problem 0.1 it is enough a more loose approach. Considering that, we first analyze in detail the basis $\mathfrak{R}$ and we then extend some of the results to a more general context.

2.2.1 The basis of rectangles $\mathfrak{R}$

In this section we recall some sparseness properties of n-dimensional rectangles, introduced in [CF75]. Here we adopt the slightly different approach from [LS88]. We begin with some definitions and notation that we need in order to present these properties.

Definition 2.6. Let $\mathscr{R} = \{R_j\}_{1\leqslant j\leqslant N}$ be a finite sequence of rectangles from $\mathfrak{R}$. We will say that $\mathscr{R}$ satisfies the sparseness property (P_1) or it is ϵ-*scattered*, $0<\epsilon<1$, if

$$|R_j \cap \bigcup_{i<j} R_i| \leqslant \epsilon |R_j|, \quad j = 1, 2, \dots, N. \tag{P_1}$$

For some of our purposes, we will need to consider a refinement of the sparseness property. For $t \in \mathbb{R}$ and $E \subset \mathbb{R}^n$ we introduce the slice operator

$$P_t(E) \coloneqq \{x' \in \mathbb{R}^{n-1} : (x', t) \in E\}.$$

Thus $P_t(E)$ is the 'slice' of E by a hyperplane perpendicular to the n-th coordinate axis at level $t \in \mathbb{R}$. The $(n-1)$-dimensional projection is

$$P_{||}(E) \coloneqq \{x' \in \mathbb{R}^{n-1} : (x', t) \in E \quad \text{for some} \quad t \in \mathbb{R}\}.$$

We will also use the one-dimensional projection $P^\perp$ defined for $E \subset \mathbb{R}^n$ as

$$P^\perp(E) := \{t \in \mathbb{R} : (x', t) \in E \quad \text{for some} \quad x' \in \mathbb{R}^{n-1}\}.$$

If $R \in \mathfrak{R}$ observe that we have

$$R = P_{||}(R) \times P^\perp(R) = P_t(R) \times P^\perp(R), \quad \text{for all} \quad t \in P^\perp(R).$$

For any interval $I \subset \mathbb{R}$, let I^* be the interval with the same center and three times the length of I, $|I^*| = 3|I|$. For $R \in \mathfrak{R}$ we then use the notation

$$R^* := P_{||}(R) \times (P^\perp(R))^*.$$

Thus R^* is the rectangle with the same center as R and whose sides parallel to the first $n-1$ coordinate axes have the same lengths as the corresponding sides of R; the side of R which is parallel to the n-th coordinate axis has length equal to three times the length of the corresponding side of R.

Now we consider the second sparseness property.

Definition 2.7. Let $\mathscr{R} = \{R_j\}_{1 \leqslant j \leqslant N}$ be a finite sequence of rectangles from $\mathfrak{R}$. We will say that $\mathscr{R}$ satisfies the sparseness property (P_2) if

$$\begin{cases} P^\perp(R_1) \geqslant P^\perp(R_2) \geqslant \cdots \geqslant P^\perp(R_N), \\ |R_j \cap \bigcup_{i<j} R_i^*| \leqslant \epsilon |R_j|, \quad j = 1, 2, \ldots, N, \end{cases} \tag{P_2}$$

where $0 < \epsilon < 1$.

For $t \in \mathbb{R}$ we now consider the collection $\mathscr{T}(t) = \mathscr{T} = \{P_t(R_j)\}_{1 \leqslant j \leqslant N} \subset \mathfrak{R}_{n-1}$ which is produced by slicing all the n-dimensional rectangles of $\mathscr{R}$ by a hyperplane perpendicular to the n-th coordinate axis, at the level t. The collection $\mathscr{T}$ depends on t but we will many times suppress this fact in what follows.

The main point about the collections $\mathscr{R}$ and $\mathscr{T}$ and the sparseness property (P_2) is contained in the following standard fact:

Lemma 2.8. *Suppose that the sequence $\mathscr{R} = \{R_j\}_{1 \leqslant j \leqslant N}$ has the sparseness property (P_2). Then, for all $t \in \mathbb{R}$, the $(n-1)$-dimensional collection of rectangles $\mathscr{T}(t) = \{P_t(R_j)\}_{1 \leqslant j \leqslant N}$ has the sparseness property (P_1), uniformly in t; that is:*

$$|P_t(R_j) \cap \bigcup_{i<j} P_t(R_i)| \leqslant \epsilon |P_t(R_j)|, \quad j = 1, 2, \ldots, N.$$

Proof. We fix some $1 \leqslant j \leqslant N$ and $t \in P^\perp(R_j)$. Denoting $I := \{i < j : P_t(R_j) \cap P_t(R_i) \neq \emptyset\}$ we have by the second condition in (P_2) that

$$\epsilon |R_j| \geqslant |R_j \cap \bigcup_{i<j} R_i^*| = |\bigcup_{i<j} (R_j \cap R_i^*)| \geqslant |\bigcup_{i \in I} (R_j \cap R_i^*)| \tag{2.8}$$

Observe that for $i \in I$ we have that $\emptyset \neq P^\perp(R_j) \cap P^\perp(R_i) \ni t$ and by the first condition in (P_2) we have $|P^\perp(R_i)| \geqslant |P^\perp(R_j)|$. A moment's reflection shows that if I_1, I_2 are intervals in $\mathbb{R}$, $|I_2| \geqslant |I_1|$ and $I_1 \cap I_2 \neq \emptyset$ then $I_1 \subseteq I_2^*$. We conclude that $P^\perp(R_j) \subseteq P^\perp(R_i^*)$. Thus the n-dimensional rectangle $R_j \cap R_i^*$ is of the form $P^\perp(R_j) \times P_{||}(R_j \cap R_i^*)$. However, $i \in I$ implies that $P_t(R_j \cap R_i) = P_t(R_j) \cap P_t(R_i) \neq \emptyset$, so we conclude that $P_{\|}(R_j \cap R_i^*) = P_t(R_j \cap R_i)$ and thus

$$R_j \cap R_i^* = P^{\perp}(R_j) \times P_t(R_j \cap R_i). \tag{2.9}$$

Now estimate (2.8) and identity (2.9) give

$$\begin{aligned}
\epsilon|P^{\perp}(R_j)| \times |P_t(R_j)| = \epsilon|R_j| \geqslant \Big|\bigcup_{i\in I} P^{\perp}(R_j) \times P_t(R_j \cap R_i)\Big| \\
= |P^{\perp}(R_j)| \times \Big|\bigcup_{i\in I} P_t(R_j \cap R_i)\Big| \\
= |P^{\perp}(R_j)| \times \Big|P_t(R_j) \cap \bigcup_{i\in I} P_t(R_i)\Big| \\
= |P^{\perp}(R_j)| \times |P_t(R_j) \cap \bigcup_{i<j} P_t(R_i)|.
\end{aligned}$$

This proves the lemma for $t \in P^{\perp}(R_j)$. For $t \notin P^{\perp}(R_j)$ the conclusion follows trivially. □

The importance of this Lemma is contained in the following covering argument.

Lemma 2.9 (Córdoba and R. Fefferman, [CF75]). *Let $\Sigma = \{R_k\}_{1\leqslant j\leqslant M}$ be any finite collection of rectangles contained in $\mathfrak{R}$. Then there exists a subcollection of rectangles $\mathscr{R} = \{R_j^s\}_{1\leqslant j\leqslant N} \subset \Sigma$ satisfying property (P_2) with parameter $\epsilon = 1/2$, such that:*

(i) $\big|\bigcup_{j=1}^{M} R_j^s\big| \lesssim_n |\bigcup_{j=1}^{N} R_j^s|$.

(ii) $\| \exp(\theta \sum_{j=1}^{N} \mathbf{1}_{R_k^s})^{\frac{1}{n-1}} \|_{L^1(\bigcup_{j=1}^{N} R_j)} \leqslant 2|\bigcup_{j=1}^{N} R_j^s|, \quad \theta > 0.$

(iii) $\| \sum_{j=1}^{N} \mathbf{1}_{R_j^s} \|_{L^p(\bigcup_{j=1}^{N} R_j^s)} \lesssim_{n,p} |\bigcup_{j=1}^{N} R_j^s|^{\frac{1}{p}}$.

The proof of this Lemma is contained in [CF75] and it is based upon Lemma 2.8. Note that we consider a specific ϵ because it is enough for our purposes, but the same theorem can be stated for any $0 < \epsilon < 1$.

We now present a weighted version of inequalities (i) and (iii). The first lemma assures that the w-measure of any collection of rectangles is comparable to the w-size of a subcollection that verifies property (P_2). The second one gives a precise quantitative bound on the overlap of the rectangles in $\mathscr{R}$ under the sparseness property (P_2).

Lemma 2.10. *Let $\{R_j\}_{j\in J}$ be a family of rectangles in $\mathfrak{R}$. Suppose further that $w \in A_\infty^*$. Then there exists a finite subcollection of rectangles $\mathscr{R} = \{R_j^s\}_{1\leqslant j\leqslant N}$ such that:*

(i) *The collection $\mathscr{R}$ has the property (P_2) with parameter ϵ.*

(ii) *We have the estimate*

$$w(\cup_j R_j) \lesssim_{\epsilon,w,n} w(\cup_j R_j^s).$$

Note that same lemma can also be stated with property (P_1) instead of (P_2).

Proof. First we reorder the rectangles R_j so that $P^{\perp}(R_1) \geqslant P^{\perp}(R_2) \geqslant \cdots \geqslant P^{\perp}(R_M)$. We choose $R_1^s := R_1$ and assume that the rectangles $R_1^s, R_2^s, \ldots, R_\tau^s$, have been selected. Also let $1 \leqslant j_o < M$ so that $R_\tau^s = R_{j_o}$. We then choose $R_{\tau+1}^s$ to be the rectangle with the smallest index among the rectangles $S \in \{R_{j_o+1}, \ldots, R_M\}$ that satisfy

$$|S \cap \bigcup_{j \leqslant \tau} (R_j^s)^*| \leqslant \epsilon |S|.$$

Since the collection $\{R_j\}_{1 \leqslant j \leqslant M}$ is finite, the selection process will end after a finite number of N steps, and the collection $\{R_k^s\}_{1 \leqslant k \leqslant N}$ will automatically satisfy (i). Now assume that some $S \in \{R_1, \ldots, R_M\}$ was not selected. We can then find some positive integer $N_o \in \{1, 2, \ldots, N\}$ such that

$$|S \cap \bigcup_{j \leqslant N_o} (R_j^s)^*| > \epsilon |S|.$$

Thus we get for all $x \in S$

$$M_s(\mathbf{1}_{\cup_{j \leqslant N}(R_j^s)^*})(x) \geqslant M_s(\mathbf{1}_{\cup_{j \leqslant N_o}(R_j^s)^*})(x) > \epsilon,$$

which means that

$$\bigcup_{\substack{1 \leqslant j \leqslant N \\ R_j \text{ not selected}}} R_j \subseteq \{x : M_s(\mathbf{1}_{\cup_{j \leqslant N}(R_j^s)^*})(x) > \epsilon\}.$$

However, since $w \in A_\infty^*$ we know that $M_s : L^{p_o}(w) \to L^{p_o,\infty}(w)$ for some $p_o > 1$. We conclude that

$$w\Big(\bigcup_{\substack{1 \leqslant j \leqslant N \\ R_j \text{ not selected}}} R_j \Big) \lesssim_{\epsilon, w, n} w(\cup_{j \leqslant N}(R_j^s)^*) \lesssim w(\cup_{j \leqslant N} R_j^s).$$

Thus

$$w(\cup_j R_j) \lesssim_{\epsilon, w, n} w(\cup_j R_j^s).$$

□

Lemma 2.11. *Let $w \in A_\infty^*$ and suppose that the finite sequence $\mathscr{R} = \{R_j\}_{1 \leqslant j \leqslant N} \subset \mathfrak{R}$ satisfies property (P_2) with ϵ sufficiently small, depending on the weight w. We set $\Omega := \cup_{j=1}^N R_j$. For $1 < p < \infty$ we have*

$$\Big(\int_\Omega \Big| \sum_{j=1}^N \mathbf{1}_{R_j} \Big|^p w(x) dx \Big)^{\frac{1}{p}} \lesssim_{w,n} c_{p,n} w(\Omega)^{\frac{1}{p}}$$

with $c_{p,n} = O_n(p^{n-1})$ as $p \to +\infty$.

Proof. For a sequence $\{R_j\}_{1 \leqslant j \leqslant N}$ as before, consider the sequence $\mathscr{T}(t)$ of $(n-1)$-dimensional rectangles, by slicing the collection $\mathscr{R}$ with a hyperplane perpendicular to the n-th coordinate axis, at level $t \in \mathbb{R}$. Let $\Omega_t := P_t(\Omega)$ denote the corresponding slice of Ω at level t and set $T_j := P_t(R_j)$ in order to simplify the notation. By Lemma 2.8 the collection $\mathscr{T}(t) = \{T_j\}_{1 \leqslant j \leqslant N}$ has the property (P_1). We set $E_j := T_j \setminus \cup_{i<j} T_i$. For fixed $t \in \mathbb{R}$, the function $w^t(x') = w(x', t)$, $x' \in \mathbb{R}^{n-1}$, is an A_∞^*-weight in $\mathbb{R}^{n-1}$, uniformly in

$t \in \mathbb{R}$; see Subsection 1.3.2 in Chapter 1. By the property (P_1) and the fact that $w^t \in A^*_\infty$ uniformly in t, we will have that $w^t(T_j) \geqslant w^t(E_j) \geqslant \frac{1}{2}w^t(T_j)$ if $\epsilon > 0$ was selected sufficiently small in property (P_2), and thus also in (P_1), according to Remark 1.24. Define the linear operator

$$L_{w^t} f(x') := \sum_{j=1}^N \frac{1}{w^t(T_j)} \left(\int_{T_j} f(y') w^t(y') dy' \right) \mathbf{1}_{E_j}(x'), \quad x' \in \mathbb{R}^{n-1}.$$

For any locally integrable function f on $\mathbb{R}^{n-1}$ we have that $L_{w^t} f(x') \leqslant M_s^{w^t} f(x')$, $x' \in \mathbb{R}^{n-1}$. Also observe that for f, g locally integrable we have

$$\begin{aligned}\int_{\Omega_t} L_{w^t} f(x') g(x') w^t(x') dx' &= \int_{\Omega_t} \sum_{j=1}^N \frac{1}{w^t(T_j)} \left(\int_{E_j} g(y') w^t(y') dy' \right) \mathbf{1}_{T_j}(x') f(x') w^t(x') dx' \\ &= \int_{\Omega_t} L^*_{w^t} g(x') f(x') w^t(x') dx'.\end{aligned}$$

Furthermore $L^*_{w^t}(\mathbf{1}_{\Omega_t}) = \sum_{k=1}^N \frac{w^t(E_k)}{w^t(T_k)} \mathbf{1}_{T_k} \geqslant \frac{1}{2} \sum_{k=1}^N \mathbf{1}_{T_k}$. For any locally integrable function g on $\mathbb{R}^{n-1}$ we thus have

$$\begin{aligned}\int_{\Omega_t} g(x') \sum_{j=1}^N \mathbf{1}_{T_j}(x') w^t(x') dx' &\lesssim \int_{\Omega_t} g(x') L^*_{w^t}(\mathbf{1}_{\Omega_t})(x') w^t(x') dx' \\ &= \int_{\Omega_t} L_{w^t} g(x') w^t(x') dx' \leqslant \|M_s^{w^t} g\|_{L^{p'}(w^t, \mathbb{R}^{n-1})} w^t(\Omega_t)^{\frac{1}{p}} \\ &\lesssim_{w,n} (p'-1)^{-(n-1)} \|g\|_{L^{p'}(w^t, \mathbb{R}^{n-1})} w^t(\Omega_t)^{\frac{1}{p}},\end{aligned}$$

where in the last step we use the precise asymptotic estimate for $c_{p,n}$ in (1.22):

$$c_{p,n} = O_n((p-1)^{-n}) \quad \text{as} \quad p \to 1^+. \tag{2.10}$$

Taking now the supremum over $g \in L^{p'}(\mathbb{R}^{n-1})$ with $\|g\|_{L^{p'}(w^t, \mathbb{R}^{n-1})} \leqslant 1$ gives the estimate

$$\int_{\Omega_t} \Big| \sum_{j=1}^N \mathbf{1}_{T_j}(x') \Big|^p w^t(x') dx \lesssim_{w,n} p^{(n-1)p} w^t(\Omega_t), \tag{2.11}$$

as $p \to \infty$. It is essential to note here that this estimate is uniform in $t \in \mathbb{R}$. Thus integrating over $t \in P^\perp(\Omega)$, gives the claim. $\square$

Observe that since we use a slicing argument and work in one dimension less, we get the dependance on p with exponent $n-1$, which will be essential for the proof of Theorem 2.22.

2.2.2 Covering properties for a general basis $\mathfrak{B}$

We begin with an extension of the Definition 2.6.

Definition 2.12. Let $\mathfrak{B}$ be a basis and $0<\gamma<1$. A finite sequence $\{A_j\}_{i=1}^M \subset \mathfrak{B}$ of sets of finite Lebesgue measure is called γ-scattered with respect to the Lebesgue measure if for all $1<j\leqslant M$,

$$\left|A_j \bigcap \bigcup_{i<j} A_i\right| \leqslant \gamma|A_j|.$$

The main point behind this definition is the following property: it is possible to select a sub-collection of the given sets so that the (Lebesgue) size of the collected ones is still comparable to the original collection. The connection between both concepts, Tauberian and γ-scattered is provided by the following lemma. In particular, we see that Definition 2.12 also assures the same sparseness property in terms of the w-size if w satisfies a certain weighted Tauberian condition (see Definition 1.21).

Lemma 2.13 (Jawerth, [Jaw86]). *Let $\mathfrak{B}$ be a basis and let w be a weight associated to this basis. Suppose further that w satisfies the Tauberian condition ($A_{\mathfrak{B},\gamma,w}$) for some $0<\gamma<1$. Then given any finite sequence $\{A_j\}_{j=1}^N$ of sets $A_j \in \mathfrak{B}$,*

(a) there exists a subsequence $\{A_i^s\}_{i\in I}$ of $\{A_j\}_{j=1}^M$ which is γ-scattered with respect to the Lebesgue measure;

(b) $A_i^s = A_i$, $i \in I$;

(c) for any $1\leqslant i<k\leqslant N+1$,

$$w\Big(\bigcup_{i<k} A_i\Big) \lesssim_\gamma \Big[w\Big(\bigcup_{i<j} A_i\Big) + w\Big(\bigcup_{j\leqslant i<k} A_i^s\Big)\Big].$$

In this lemma we are assuming that $A_i^s=\emptyset$ whenever $i\notin I$. For a detailed proof of this lemma we refer the interested reader to [GLPT11, Lemma 5.1]. Note that this result is an extension to a context of general basis of Lemma 2.10 and the result is pretty strong. Actually, it guarantees not only that the family A_i^s is γ-scattered with respect to the Lebesgue measure, but also that is λ-scattered ($\lambda=\lambda(\gamma)$) with respect to both measures, w and σ. Remember also that according to Remark 1.27 the Tauberian condition ($A_{\mathfrak{B},\gamma,w}$) is equivalent to $A_{\infty,\mathfrak{B}}$ whenever $\mathfrak{B}$ is a Muckenhoupt basis (see Definition 1.26). Thus we recover the result in Lemma 2.10 for the case of rectangles.

2.3 Characterization of the B_p^* condition

In this section we describe the strong B_p^* class, that will provide the appropriate class of Young functions Φ to define the two-weight bump condition ($A_{p,\Phi}^*$). Let us point out first some considerations about the classical B_p condition.

2.3.1 The classical B_p condition

Given a certain Young function Φ we can define the corresponding Orlicz maximal operator

$$M^\Phi f(x) := \sup_{\substack{Q\in\mathfrak{Q}\\ Q\ni x}} \|f\|_{\Phi,Q}, \tag{2.12}$$

The L^p boundedness of this operator is intimately connected with the B_p condition and the proof of Theorem 2.3 relies upon this fact. More precisely, Pérez proved the following key observation: when $1 < p < \infty$ and Φ is a doubling (see Definition 1.2) Young function, then

$$M^\Phi : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \quad \text{if and only if} \quad \Phi \in B_p. \tag{2.13}$$

We first remark that the hypothesis of Φ being doubling was only used to prove the necessity of the B_p condition, but we show now that it can be removed. Indeed, we have the following original result:

Proposition 2.14. *If M^Φ is bounded on $L^p(\mathbb{R}^n)$, then Φ satisfies B_p.*

Proof. if we assume that for any non-negative function f the operator M^Φ is bounded on $L^p(\mathbb{R}^n)$ and we take $f = \mathbf{1}_{[0,1]^n}$, we have

$$\int_{\mathbb{R}^n} M^\Phi(\mathbf{1}_{[0,1]^n})(y)^p \, dy < \infty. \tag{2.14}$$

Now, it is easy to see that there exist positive dimensional constants a, b such that, whenever $|y| > a$ we have that

$$M^\Phi(\mathbf{1}_{[0,1]^n})(y) \gtrsim \frac{1}{\Phi^{-1}(\frac{|y|^n}{b})}.$$

This inequality can be proved using the same argument as in the case of $\Phi(t) = t$ described in [WZ77, p. 104]. Hence

$$\begin{aligned}
\int_{\mathbb{R}^n} M^\Phi(\mathbf{1}_{[0,1]^n})(y)^p \, dy \quad & \gtrsim_p \int_0^\infty t^p \left| \left\{ y \in \mathbb{R}^n : |y| > a, \frac{1}{\Phi^{-1}(\frac{|y|^n}{b})} > t \right\} \right| \frac{dt}{t} \\
& \simeq_p \int_0^\infty t^p \left| \left\{ y \in \mathbb{R}^n : a < |y| < \Phi\left(\frac{1}{t}\right)^{1/n} b^{1/n} \right\} \right| \frac{dt}{t} \\
& \simeq_{n,p} \int_0^\infty t^p \left(b\Phi\left(\frac{1}{t}\right) - a^{1/n} \right) \frac{dt}{t}.
\end{aligned}$$

Since Φ is increasing and $\Phi(t) \to \infty$ as $t \to \infty$, we can choose some $t_0 > 0$ such that for every $t \leqslant t_0$,

$$b\Phi\left(\frac{1}{t}\right) - a^{1/n} \geqslant \frac{b}{2}\Phi\left(\frac{1}{t}\right).$$

Then

$$\begin{aligned}
\infty \quad & > \int_0^\infty t^{p-1} \left(b\Phi\left(\frac{1}{t}\right) - 1 \right) dt \\
& \gtrsim_{n,p} \int_0^{t_0} t^{p-1} \Phi\left(\frac{1}{t}\right) dt \simeq_{n,p} \int_{1/t_0}^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t}.
\end{aligned}$$

$\square$

It would be interesting to ask whether a similar connection can be established between the B_p condition and the boundedness of the Orlicz maximal operator M_s^Φ associated with rectangles rather than cubes. For each locally integrable function f and a Young function Φ we define the *strong Orlicz maximal operator* as

$$M_s^\Phi f(x) := \sup_{\substack{R\in\mathfrak{R}\\ R\ni x}} \|f\|_{\Phi,R}.$$

In particular, when $\Phi(t) = t$ the maximal operator M_s^Φ is exactly the classical strong maximal operator. The necessity of the B_p condition for the L^p boundedness of M_s^Φ follows trivially from the equivalence (2.13). However, the B_p condition (2.5) is not sufficient as the following example shows

Example 2.15. Let $f = \mathbf{1}_{[0,1]^n}$, and

$$\Phi(t) = \frac{t^p}{(\log(1+t))^{1+\delta}}, \quad \text{for all } t \in (0,\infty)$$

with $0 < \delta < 1$. It is easy to verify that such a function Φ satisfies (2.5) but not (2.7). For simplicity in the notation, we consider only the case when $n = 2$. If $|x_i|$ is big enough for $i = 1, 2$, say $|x_i| > 4$, then

$$M_s^\Phi f(x_1,x_2) \gtrsim \frac{1}{\Phi^{-1}(|x_1||x_2|)} \simeq \frac{1}{(|x_1||x_2|)^{1/p}(\log(1+|x_1||x_2|))^{(1+\delta)/p}},$$

since $\Phi^{-1}(t) \sim t^{1/p}(\log(1+t))^{(1+\delta)/p}$ for all $t \in (0,\infty)$. Then Fubini's theorem gives

$$\begin{aligned}
\int_{\mathbb{R}^2} \left(M_s^\Phi f(x)\right)^p \, dx &\gtrsim \int_4^\infty \int_4^\infty \left(\frac{1}{\Phi^{-1}(|x_1||x_2|)}\right)^p dx_2 dx_1 \\
&\gtrsim \int_4^\infty \int_4^\infty \frac{1}{x_1x_2\left(\log\left(1+x_1x_2\right)\right)^{1+\delta}} dx_2 dx_1 \\
&\gtrsim \int_4^\infty \int_4^\infty \frac{1}{\left(1+x_1x_2\right)\left(\log\left(1+x_1x_2\right)\right)^{1+\delta}} dx_2 dx_1 \\
&\sim \frac{1}{\delta}\int_4^\infty \frac{1}{x_1(\log(1+4x_1))^\delta} dx_1 \\
&\geqslant \frac{1}{\delta}\int_{16}^\infty \frac{1}{(1+x_1)(\log(1+x_1))^\delta} dx_1 = \infty.
\end{aligned}$$

However,

$$\|f\|_{L^p(\mathbb{R}^2)} = \|\mathbf{1}_{[0,1]^2}\|_{L^p(\mathbb{R}^2)} = 1.$$

Hence M_s^Φ is not bounded on $L^p(\mathbb{R}^2)$.The general case for $n > 2$ is similar and we omit the details.

Though B_p condition (2.5) is not sufficient for the L^p boundedness of M_Φ^s, it becomes so if we restrict to those Young functions that are submultiplicative (see Definition 1.3). We will show this result at the end of the next section.

2.3.2 The case of rectangles

The condition B_p^*, as defined in (2.4), is a stronger condition than B_p. In fact, we do not only need a control on the growth of the function at infinity, but also on the growth of the logarithmic of the function. As the next theorem shows, this smaller class does characterize the boundedness of M_s^Φ in terms of the B_p^* class.

Theorem 2.16. *Let $1 < p < \infty$. Suppose that Φ is a Young function. Then the following statements are equivalent:*

(i) $\Phi \in B_p^*$.

(ii) *the operator M_s^Φ is bounded on $L^p(\mathbb{R}^n)$.*

(iii) *For all non-negative functions f and u the next inequality holds*

$$\int_{\mathbb{R}^n} [M_s(f)(x)]^p \frac{1}{[M_s^{\bar\Phi}(u^{1/p})(x)]^p}\, dx \lesssim_{n,p} \int_{\mathbb{R}^n} f(x)^p \frac{1}{u(x)}\, dx. \tag{2.15}$$

(iv) *Given a weight $w \in A_\infty^*$, the Orlicz maximal operator M_s^Φ satisfies the strong (p,p) inequality*

$$\int_{\mathbb{R}^n} [M_s^\Phi(f)(x)]^p w(x)\, dx \lesssim_{n,p,w} \int_{\mathbb{R}^n} f(x)^p M_s w(x)\, dx \tag{2.16}$$

for all non-negative functions f.

Particular examples of Young functions $\Phi \in B_p$ are described in detail in [Pér94b, section 4.3]. Example 2.15 together with theorem in hand, show that $B_p^* \subsetneq B_p$. A typical Young function that belongs to the class B_p^* is $B(t) = t^s$ with $1 \leqslant s < p$. Some more sophisticated examples are the following:

$$\Phi(t) \sim t^\alpha \log^{-\beta}(e+t) \qquad 1<\alpha<p,\ \beta \in \mathbb{R};$$

$$\Phi(t) \sim t^p \log^{-\beta}(e+t) \qquad \beta > n;$$

$$\Phi(t) \sim t^p \log^{-n}(e+t)\, [\log(\log(e+t))]^{-\gamma}, \quad \gamma > 1.$$

Proof of Theorem 2.16. We first assume that (i) holds and show (ii). To this end, for each $t > 0$, we split the function f into $f = f_t + f^t$ with $f_t := f\mathbf{1}_{|f|>t/2}$ and $f^t := f\mathbf{1}_{|f|\leqslant t/2}$. Then,

$$M_s^\Phi f \leqslant M_s^\Phi(f_t) + M_s^\Phi(f^t) \leqslant M_s^\Phi(f_t) + t/2$$

and

$$\{x \in \mathbb{R}^n : M_s^\Phi f(x) > t\} \subset \{x \in \mathbb{R}^n : M_s^\Phi(f_t)(x) > t/2\}.$$

Set

$$\Omega_t := \{x \in \mathbb{R}^n : M_s^\Phi(f_t)(x) > t/2\}.$$

For every $x \in \Omega_t$ let $R_x \in \mathfrak{R}$ be a rectangle such that

$$\|f_t\|_{\Phi,R_x} > t.$$

Without loss of generality we may assume that $\{R_x\}_{x\in\Omega_t}$ is a finite sequence $\{R_j\}_{1\leqslant j\leqslant M}$. By [GLPT11, Lemma 6.1], the condition $\|f_t\|_{\Phi,R_j} > t$ implies that

$$1 < \left\|\frac{f_t}{t}\right\|_{\Phi,R_j} \leqslant \frac{1}{|R_j|}\int_{R_j}\Phi\left(\frac{|f_t(x)|}{t}\right)\,dx. \tag{2.17}$$

Given the collection $\{R_j\}_{j=1}^M$, by the covering Lemma 2.9 we may extract a subcollection $\{R_j^s\}_{j=1}^N$. Then Property (i) of Lemma 2.9 together with inequality (2.17) gives:

$$\begin{aligned}|\Omega_t| \quad &\lesssim_n \left|\bigcup_{j=1}^N R_j^s\right| \lesssim_n \sum_{j=1}^N |R_j^s| \lesssim_n \sum_{j=1}^N \int_{R_j^s}\Phi\left(\frac{|f_t(x)|}{t}\right)\,dx\\ &\simeq_n \int_{\bigcup_{j=1}^N R_j^s}\sum_{j=1}^N \mathbf{1}_{R_j^s}(x)\Phi\left(\frac{|f_t(x)|}{t}\right)\,dx.\end{aligned}$$

We will use the following elementary estimate: For each $\theta > 0$ there exists a constant $c_\theta > 0$ such that for all $s, t \geqslant 0$ we have

$$st \leqslant c_\theta s[1 + (\log^+ s)^{n-1}] + \exp(\theta t^{\frac{1}{n-1}}) - 1, \quad n \geqslant 2. \tag{2.18}$$

The interested reader can find a detailed proof of this classical inequality in [Bag83]. Applying (2.18) together with Property (ii) of Lemma 2.9, we get for every $\epsilon > 0$:

$$\begin{aligned}\left|\bigcup_{j=1}^N R_j^s\right| \quad &\lesssim_n \epsilon\int_{\bigcup_{j=1}^N R_j^s}\left[C_\theta\Phi_n\left(\frac{1}{\epsilon}\Phi\left(\frac{|f_t(x)|}{t}\right)\right) + \exp\left(\theta\sum_{j=1}^N \mathbf{1}_{R_j^s}(x)\right) - 1\right]\,dx\\ &\lesssim_n \left\{\epsilon C_\theta\Phi_n(1/\epsilon)\int_{\bigcup_{j=1}^N R_j^s}\Phi_n\left(\Phi\left(\frac{|f_t(x)|}{t}\right)\right) + \epsilon\left|\bigcup_{j=1}^N R_j^s\right|\,dx\right\}.\end{aligned}$$

If ϵ is small enough, the second summand in the right hand side of the inequality above can be absorbed into the left-hand side. Then we obtain

$$\left|\bigcup_{j=1}^N R_j^s\right| \lesssim_n \int_{\bigcup_{j=1}^N R_j^s}\Phi_n\left(\Phi\left(\frac{|f_t(x)|}{t}\right)\right)\,dx.$$

Since $\Phi_n(\Phi(0)) = 0$, we have

$$\begin{aligned}|\Omega_t| \quad &\lesssim_n \int_{\bigcup_{j=1}^N R_j^s}\Phi_n\left(\Phi\left(\frac{|f_t(x)|}{t}\right)\right)\,dx\\ &\lesssim_n \int_{\{y\in\mathbb{R}^n:|f(x)|>t/2\}}\Phi_n\left(\Phi\left(\frac{|f(x)|}{t}\right)\right)\,dx.\end{aligned}$$

This inequality and the fact $\{x \in \mathbb{R}^n : M_s^\Phi f(x) > t\} \subset \Omega_t$, together with the change of variable $s = |f(y)|/t$, yields

$$\begin{aligned}
\|M_s^\Phi f\|_{L^p(\mathbb{R}^n)}^p &= p\int_0^\infty t^p |\{x \in \mathbb{R}^n : M_s^\Phi f(x) > t\}| \frac{dt}{t} \leqslant p\int_0^\infty t^p |\Omega_t| \frac{dt}{t} \\
&\lesssim_{n,p} \int_0^\infty \int_{\{x\in\mathbb{R}^n : |f(x)|>t/2\}} t^p \Phi_n\left(\Phi\left(\frac{|f(x)|}{t}\right)\right) dx\, \frac{dt}{t} \\
&\simeq_{n,p} \int_{\mathbb{R}^n} \int_0^{2|f(x)|} t^p \Phi_n\left(\Phi\left(\frac{|f(x)|}{t}\right)\right) \frac{dt}{t}\, dx \\
&\lesssim_{n,p} \int_{\mathbb{R}^n} \int_{1/2}^\infty |f(x)|^p \frac{\Phi_n(\Phi(s))}{s^p} \frac{ds}{s}\, dx \\
&\lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}^p,
\end{aligned}$$

where in the last approximate inequality we have used the hypothesis $\Phi \in B_p^*$. This proves (ii).

Let us assume that (ii) holds. Using the generalized Hölder inequality (1.6) we obtain

$$M_s(hg)(x) \leqslant 2M_s^\Phi h(x) M_s^{\bar\Phi} g(x).$$

The above inequality together with the boundedness of M_s^Φ on $L^p(\mathbb{R}^n)$ implies

$$\begin{aligned}
\int_{\mathbb{R}^n} [M_s(hg)(x)]^p \frac{1}{[M_s^{\bar\Phi} g(x)]^p}\, dx &\leqslant 2\int_{\mathbb{R}^n} [M_s^\Phi h(x)]^p\, dx \\
&\leqslant 2\|M_s^\Phi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}^p \int_{\mathbb{R}^n} h(x)^p\, dx.
\end{aligned}$$

Taking $h = fu^{-1/p}$ and $g = u^{1/p}$, we obtain the claim (iii).

To prove that (iii) implies (i), for any $N \in \mathbb{N}$, we let $f := \mathbf{1}_{[0,1/2]^n}$ and $u_N := \mathbf{1}_{[0,1/2]^n} + \frac{\mathbf{1}_{\mathbb{R}^n\setminus[0,1/2]^n}}{N}$ in (2.15). Note that u is asymptotically equal to f as $N \to \infty$. However, we need to extend the support of u_N throughout $\mathbb{R}^n$ in order to turn it into a weight. Hence we get

$$\int_{\mathbb{R}^n} \left(\frac{M_s(\mathbf{1}_{[0,1/2]^n})(x)}{M_s^{\bar\Phi}(\mathbf{1}_{[0,1/2]^n} + \frac{\mathbf{1}_{\mathbb{R}^n\setminus[0,1/2]^n}}{N})(x)}\right)^p dx < \infty.$$

Note that $M_s^{\bar\Phi}(f+g) \leqslant M_s^{\bar\Phi} f + M_s^{\bar\Phi} g$. This observation together with monotone convergence gives

$$\int_{\mathbb{R}^n} \left(\frac{M_s(\mathbf{1}_{[0,1/2]^n})(x)}{M_s^{\bar\Phi}(\mathbf{1}_{[0,1/2]^n})(x)}\right)^p dx < \infty. \tag{2.19}$$

As pointed out before, it is not difficult to see that for any point $(x_1, \cdots, x_n) \in \mathbb{R}^n$ such that $|x_j| > 1$ for all $j \in \{1, \cdots, n\}$, we have

$$M_s(\mathbf{1}_{[0,1/2]^n})(x) = \sup_{\substack{R\in\mathfrak{R} \\ R\ni x}} \frac{|R \cap [0,1/2]^n|}{|R|} \gtrsim_n \frac{1}{|x_1||x_2|\cdots|x_n|},$$

and

$$\begin{aligned}
M_s^{\bar{\Phi}}(\mathbf{1}_{[0,1/2]^n})(x) &= \sup_{\substack{R\in\mathfrak{R}\\ R\ni x}} \inf\left\{\lambda>0:\ \bar{\Phi}\left(\lambda^{-1}\right)\leqslant \frac{|R|}{|R\cap[0,1/2]^n|}\right\}\\
&= \sup_{\substack{R\in\mathfrak{R}\\ R\ni x}} \frac{1}{\bar{\Phi}^{-1}\left(\frac{|R|}{|R\cap[0,1/2]^n|}\right)}\\
&\gtrsim_{n,\Phi} \frac{1}{\bar{\Phi}^{-1}\left(|x_1||x_2|\cdots|x_n|\right)}.
\end{aligned}$$

Inserting these two estimates into (2.19) and using property (1.4), we deduce that

$$\begin{aligned}
\infty &> \int_1^\infty\cdots\int_1^\infty\left(\frac{\bar{\Phi}^{-1}\left(x_1x_2\cdots y_n\right)}{x_1x_2\cdots x_n}\right)^p dx_n\cdots dx_1\\
&\simeq_{n,\Phi} \int_1^\infty\cdots\int_1^\infty\left(\frac{1}{\Phi^{-1}\left(x_1x_2\cdots x_n\right)}\right)^p dx_n\cdots dx_1.
\end{aligned}$$

To evaluate the integral with respect to the variable x_n, we just use the property (1.1) that relates a Young function and its derivative. Then

$$\begin{aligned}
\int_1^\infty\left(\frac{1}{\Phi^{-1}\left(x_1x_2\cdots x_n\right)}\right)^p dx_n &= \frac{1}{x_1\cdots x_{n-1}}\int_{\Phi^{-1}(x_1\cdots x_{n-1})}^\infty \frac{\Phi'(z)}{z^p}\,dz\\
&\geqslant \frac{1}{x_1\cdots x_{n-1}}\int_{\Phi^{-1}(x_1\cdots x_{n-1})}^\infty \frac{\Phi(z)}{z^p}\frac{dz}{z}.
\end{aligned}$$

Now we integrate with respect to the variable x_{n-1} in order to obtain

$$\begin{aligned}
&\int_1^\infty\int_1^\infty\left(\frac{1}{\Phi^{-1}\left(x_1x_2\cdots x_n\right)}\right)^p dx_n\,dx_{n-1}\\
&\gtrsim_{n,\Phi} \int_1^\infty\frac{1}{x_1\cdots x_{n-1}}\int_{\Phi^{-1}(x_1\cdots x_{n-1})}^\infty \frac{\Phi(z)}{z^p}\frac{dz}{z}\,dx_{n-1}\\
&\simeq_{n,\Phi} \int_{\Phi^{-1}(x_1\cdots x_{n-2})}^\infty\int_1^{\frac{\Phi(z)}{x_1\cdots x_{n-2}}}\frac{1}{x_1\cdots x_{n-1}}\,dx_{n-1}\frac{\Phi(z)}{z^{p+1}}\,dz\\
&\simeq_{n,\Phi} \frac{1}{x_1\cdots x_{n-2}}\int_{\Phi^{-1}(x_1\cdots x_{n-2})}^\infty \ln\left(\frac{\Phi(z)}{x_1\cdots x_{n-2}}\right)\frac{\Phi(z)}{z^p}\frac{dz}{z}.
\end{aligned}$$

Integrating the right hand side with respect to the variable x_{n-2}

$$\begin{aligned}
&\int_1^\infty\frac{1}{x_1\cdots x_{n-2}}\int_{\Phi^{-1}(x_1\cdots x_{n-2})}^\infty \ln\left(\frac{\Phi(z)}{x_1\cdots x_{n-2}}\right)\frac{\Phi(z)}{z^p}\frac{dz}{z}\,dx_{n-2}\\
&= \frac{1}{x_1\cdots x_{n-3}}\int_{\Phi^{-1}(x_1\cdots x_{n-3})}^\infty\int_1^{\frac{\Phi(z)}{x_1\cdots x_{n-3}}}\frac{1}{x_{n-2}}\ln\left(\frac{\Phi(z)}{x_1\cdots x_{n-2}}\right)dx_{n-2}\frac{\Phi(z)}{z^p}\frac{dz}{z}\\
&\simeq \frac{1}{x_1\cdots x_{n-3}}\int_{\Phi^{-1}(x_1\cdots x_{n-3})}^\infty\left(\ln\left(\frac{\Phi(z)}{x_1\cdots x_{n-3}}\right)\right)^2\frac{\Phi(z)}{z^p}\frac{dz}{z}.
\end{aligned}$$

We continue this process by integrating in the next variables $x_{n-3},\cdots,x_1$ in turn and we obtain

$$\begin{aligned}\infty &> \int_1^\infty \cdots \int_1^\infty \left(\frac{1}{\Phi^{-1}(x_1 x_2 \cdots x_n)}\right)^p dx_n \cdots dx_1 \\ &\gtrsim_{n,\Phi} \int_{\Phi^{-1}(1)}^\infty (\ln(\Phi(z)))^{n-1} \frac{\Phi(z)}{z^p}\frac{dz}{z}\end{aligned}$$

which implies (i) inmediately.

In order to see that that (iv) implies (ii), we take $w = 1$ in the right side of (2.16). We postpone the proof of the implication (ii)$\to$ (iv) until Subsection 2.4.2, since the argument is very similar to the one presented in the proof of Theorem 2.18. This concludes the proof. □

As it was already announced there exists a subcollection of Young functions in B_p for which the operator M_s^Φ is bounded in $L^p(\mathbb{R}^n)$, for $p > 1$.

Proposition 2.17. *Let* $1 < p < \infty$*. Assume that* Φ *is a submultiplicative Young function such that* $\Phi \in B_p$*. Then the operator* M_s^Φ *is bounded on* $L^p(\mathbb{R}^n)$*.*

Proof. The proof is an immediate consequence of the fact that for a submultiplicative Young function such that $\Phi \in B_p$, there exits $\epsilon > 0$ for which $\Phi \in B_{p-\epsilon}$; see [Pér95b, Lemma 4.3]. Indeed, using the previous theorem we only need to prove that $\Phi \in B_p^*$. Note that

$$\int_c^\infty \frac{\Phi_n(\Phi(t))}{t^p}\frac{dt}{t} = \int_c^\infty \frac{\Phi(t)}{t^p}\frac{dt}{t} + \int_c^\infty \frac{\Phi(t)}{t^{p-\epsilon}}\frac{(\log^+\Phi(t))^{n-1}}{t^\epsilon}\frac{dt}{t}. \tag{2.20}$$

The first term in the right hand of(2.20) is clearly bounded. On the other hand,

$$\frac{(\log^+\Phi(t))^{n-1}}{t^\epsilon} \leqslant \frac{\Phi(t)^{\delta(n-1)}}{t^{\epsilon\delta(n-1)}},$$

for every $\delta > 0$. Since Φ is in the class B_p, it follows that $\Phi(t) \lesssim t^p$ for $t \geqslant 1$ and hence for $\delta = \frac{\epsilon}{p(n-1)}$ the above term is bounded. This further implies that the second term of (2.20) is bounded by a constant multiple of

$$\int_c^\infty \frac{\Phi(t)}{t^{p-\epsilon}}\frac{dt}{t},$$

which together with the aforementioned fact that $\Phi \in B_{p-\epsilon}$ gives the boundedness of the second term of (2.20). □

We observe that a typical example of a submultiplicative Young function that belongs to the class B_p is $\Phi(t) = t^r$ with $1 \leqslant r < p$. A more interesting example is given by the function $\Phi(t) = t^r(1 + \log_+ t)^\alpha$ with $1 \leqslant r < p$ and $\alpha > 0$. It is not difficult to see that such functions are submultiplicative and they are in the B_p class. The important difference to be noted here is that though the B_p^* class characterizes the boundedness of M_s^Φ, that Φ is submultiplicative and belongs to the class B_p, does not.

2.4 Solutions to Problem 0.1

In this section we present the proof of the strong two-weight Theorem 2.5 as a inmediate consequence of a more general result stated in terms of multilinear operators and very general bases.

2.4.1 Some multilinear results for general bases

In this section, we address similar questions to Problem 0.1, but involving the multilinear version of the strong maximal function and some other more general maximal functions. We start by introducing some notation that we will use through this section. By $\mathfrak{B}$ we denote, as usual, a collection of bounded open sets in $\mathbb{R}^n$. If $\{\Phi_j\}_{j=1}^m$ is a sequence of Young functions, we define the multi(sub)linear Orlicz maximal function by

$$\mathscr{M}_{\mathfrak{B}}^{\vec{\Phi}}(\vec{f})(x) := \sup_{B\in\mathfrak{B}, B\ni x} \prod_{j=1}^{m} \|f_j\|_{\Phi_j,B}.$$

where $\vec{f} = (f_1, \ldots, f_m)$ and $\vec{\Phi} = (\Phi_1, \ldots, \Phi_m)$. In particular, when $\Phi_j(t) = t$ for all $t \in (0, \infty)$ and all $j \in \{1, \cdots, m\}$, we simply write $\mathscr{M}_{\mathfrak{B}}^{\vec{\Phi}}$ as $\mathscr{M}_{\mathfrak{B}}$; that is,

$$\mathscr{M}_{\mathfrak{B}}(\vec{f})(x) = \sup_{B\in\mathscr{B}, B\ni x} \prod_{j=1}^{m} \frac{1}{|B|}\int_B |f_j(y)|\,dy.$$

When $m = 1$, we use $M_{\Phi}^{\mathfrak{B}}$ and $M_{\mathfrak{B}}$ to respectively denote $\mathscr{M}_{\mathfrak{B}}^{\vec{\Phi}}$ and $\mathscr{M}_{\mathfrak{B}}$. When $\mathfrak{B} \equiv \mathfrak{R}$ we will simply use the notation $\mathscr{M}_s^{\vec{\Phi}}$ and $\mathscr{M}_s$ to denote the strong Orlicz multi(sub)linear operator and the strong multi(sub)linear maximal function respectively.

The multilinear maximal operator was first defined by Lerner et al. in [LOPT09], with respect to the basis of cubes $\mathfrak{Q}$. The strong version and the general one were introduced in [GLPT11]. In particular, it was proved that a certain power bump variant of the multilinear version of the A_p condition is sufficient for the weak boundedness of $\mathscr{M}_s$. More precisely, for $1 < p_1, \cdots, p_m < \infty$ and $0 < p < \infty$ such that $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$, the multilinear strong maximal function maps

$$L^{p_1}(v_1) \times \cdots \times L^{p_m}(v_m) \to L^{p,\infty}(w),$$

provided that $(w, \vec{v}) = (w, v_1, \ldots, v_m)$ are weights that satisfy the power bump condition

$$\sup_{R\in\mathfrak{R}} \left(\frac{1}{|R|}\int_R w(x)\,dx\right) \prod_{j=1}^{m} \left(\frac{1}{|R|}\int_R v_j^{(1-p_j')r}\,dx\right)^{\frac{p}{p_j' r}} < \infty \tag{2.21}$$

for some $r > 1$. In the case that $w = \prod_{j=1}^m v_j^{p/p_j}$, the strong boundedness of $\mathscr{M}_s$ is also characterized; see [GLPT11, Corollary 2.4 and Theorem 2.5]. The weight theory for the multilinear operator $\mathscr{M}$ has been also fully developed by Lerner et al. [LOPT09] and generalized by Moen [Moe09].

Inspired by these previous works, our first goal in this section is to introduce the multilinear version of $(A_{p,\Phi})$ and $(A_{p,\Phi}^*)$ for weights $(w, \vec{v})$ associated with a general basis. Then, the $L^{p_1}(v_1)\times\cdots\times L^{p_m}(v_m) \to L^p(w)$ boundedness of $\mathscr{M}_{\mathfrak{B}}$ will be proved whenever

w is any arbitrary weight such that w^p satisfies a certain Tauberian condition. This result is the content of the following theorem.

Theorem 2.18. *Let $1 < p_1, \cdots, p_m < \infty$ and $0 < p < \infty$ such that $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$. Assume that $\mathfrak{B}$ is a basis and that $\{\Phi_j\}_{j=1}^m$ is a sequence of Young functions such that $\mathscr{M}_{\mathfrak{B}}^{\vec{\Phi}}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Let $(w, \vec{v}) = (w, v_1, \cdots, v_m)$ be weights such that w^p satisfies ($A_{\mathfrak{B},\gamma,w}$), and that*

$$\sup_{B\in\mathfrak{B}} \left(\frac{1}{|B|}\int_B w(x)^p\,dx\right)^{1/p} \prod_{j=1}^m \|v_j^{-1}\|_{\Phi_j,B} < \infty. \tag{2.22}$$

Then $\mathscr{M}_{\mathfrak{B}}$ is bounded from $L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$ to $L^p(w^p)$.

Remark 2.19. We observe that for all $x \in \mathbb{R}^n$ and for all non-negative functions $\vec{f} = (f_1, \ldots, f_m)$,

$$\mathscr{M}_{\mathfrak{B}}^{\vec{\Phi}}(\vec{f})(x) \leqslant \prod_{j=1}^m M_{\mathfrak{B}}^{\Phi_j}(f_j)(x).$$

Thus, if we assume that each $M_{\mathfrak{B}}^{\Phi_j}$ is bounded on $L^{p_j}(\mathbb{R}^n)$, then $\mathscr{M}_{\mathfrak{B}}^{\vec{\Phi}}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Therefore, the conclusion of Theorem 2.18 is that the operator $\mathscr{M}_{\mathfrak{B}}$ is bounded from $L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$ to $L^p(w^p)$, whenever $(w, \vec{v})$ satisfies (2.22).

Proof of Theorem 2.18. Let $N > 0$ be a large positive integer. We will prove the required estimate for the quantity

$$\int_{2^{-N} < \mathscr{M}_{\mathfrak{B}}(\vec{f}) \leqslant 2^{N+1}} \mathscr{M}_{\mathfrak{B}}(\vec{f})(x)^p\, w^p(x)\,dx$$

with a bound independent of N. We restrict for simplicity to the case that every f_j, $j \in \{1, \cdots, m\}$, is a bounded function with compact support. The general case will follows by density. In this situation, we claim that for each integer k with $|k| \leqslant N$, there exists a compact set K_k and a finite sequence $b_k = \{B_\alpha^k\}_{\alpha\geqslant 1}$ of sets $B_\alpha^k \in \mathfrak{B}$ such that

$$w^p(K_k) \leqslant w^p(\{\mathscr{M}_{\mathfrak{B}}(\vec{f}) > 2^k\}) \leqslant 2\, w^p(K_k)$$

The sequence of sets $\{\cup_{B\in b_k} B\}_{k=-N}^N$ is decreasing. Moreover,

$$\bigcup_{B\in b_k} B \subset K_k \subset \{\mathscr{M}_{\mathfrak{B}}(\vec{f}) > 2^k\},$$

and

$$\prod_{j=1}^m \frac{1}{|B_\alpha^k|}\int_{B_\alpha^k} |f_j(y)|\,dy > 2^k, \qquad \alpha \geqslant 1, \tag{2.23}$$

To see the claim, for each k we choose a compact set $\widetilde{K}_k \subset \{\mathscr{M}_{\mathfrak{B}}(\vec{f}) > 2^k\}$ such that

$$w^p(\widetilde{K}_k) \leqslant w^p(\{\mathscr{M}_{\mathfrak{B}}(\vec{f}) > 2^k\}) \leqslant 2\, w^p(\widetilde{K}_k).$$

For this $\widetilde{K}_k$, there exists a finite sequence $b_k = \{B^k_\alpha\}_{\alpha\geqslant 1}$ of sets $B^k_\alpha \in \mathfrak{B}$ such that every B^k_α satisfies (2.23) and such that

$$\widetilde{K}_k \subset \bigcup_{B\in b_k} B \subset \{\mathscr{M}_{\mathfrak{B}}(\vec{f}) > 2^k\}.$$

Now, we take a compact set K_k such that $\cup_{B\in b_k} B \subset K_k \subset \{\mathscr{M}_{\mathfrak{B}}(\vec{f}) > 2^k\}$. Finally, to ensure that $\{\cup_{B\in b_k} B\}_{k=-N}^N$ is decreasing, we begin the above selection from $k = N$ and once a selection is done for k we do the selection for $k-1$ with the next additional requirement

$$\widetilde{K}_{k-1} \supset K_k.$$

This proves the claim. Since $\{\cup_{B\in b_k} B\}_{k=-N}^N$ is a sequence of decreasing sets, we set

$$\Omega_k = \begin{cases} \bigcup_\alpha B^k_\alpha = \bigcup_{B\in b_k} B & \text{when} \quad |k| \leqslant N, \\ \emptyset & \text{when} \quad |k| > N. \end{cases}$$

Observe that these sets are decreasing in k, i.e., $\Omega_{k+1} \subset \Omega_k$ when $-N < k \leqslant N$.

We now distribute the sets in $\bigcup_k b_k$ over μ sequences $\{A_i(l)\}_{i\geqslant 1}$, $0 \leqslant l \leqslant \mu-1$, where μ will be chosen momentarily to be an appropriately large natural number. Set $i_0(0) = 1$. In the first $i_1(0) - i_0(0)$ entries of $\{A_i(0)\}_{i\geqslant 1}$, i.e., for

$$i_0(0) \leqslant i < i_1(0),$$

we place the elements of the sequence $b_N = \{B^N_\alpha\}_{\alpha\geqslant 1}$ in the order indicated by the index α. For the next $i_2(0) - i_1(0)$ entries of $\{A_i(0)\}_{i\geqslant 1}$, i.e., for

$$i_1(0) \leqslant i < i_2(0),$$

we place the elements of the sequence $b_{N-\mu}$. We continue in this way until we reach the first integer m_0 such that $N - m_0\mu \geqslant -N$, when we stop. For indices i satisfying

$$i_{m_0}(0) \leqslant i < i_{m_0+1}(0),$$

we place in the sequence $\{A_i(0)\}_{i\geqslant 1}$ the elements of $b_{N-m_0\mu}$. The sequences $\{A_i(l)\}_{i\geqslant 1}$, $1 \leqslant l \leqslant \mu-1$, are defined similarly, starting from b_{N-l} and using the families $b_{N-l-s\mu}$, $s = 0, 1, \cdots, m_l$, where m_l is chosen to be the biggest integer such that $N-l-m_l\mu \geqslant -N$.

Since w^p is a weight associated to $\mathfrak{B}$ and it satisfies the Tauberian condition ($A_{\mathfrak{B},\gamma,w}$), we can apply the coverging Lemma 2.13 to each $\{A_i(l)\}_{i\geqslant 1}$ for some fixed $0 < \gamma < 1$. Then we obtain sequences

$$\{A^s_i(l)\}_{i\geqslant 1} \subset \{A_i(l)\}_{i\geqslant 1}, \qquad 0 \leqslant l \leqslant \mu-1,$$

which are γ-scattered (see Definition 2.12) with respect to the Lebesgue measure. In view of the definition of the set Ω_k and the construction of the families we can use assertion (c) of Lemma 2.13 to obtain that for any $k = N - l - s\mu$ with $0 \leqslant l \leqslant \mu - 1$ and $1 \leqslant s \leqslant m_l$,

$$\begin{aligned} w^p(\Omega_k) = w^p(\Omega_{N-l-s\mu}) &\leqslant c(\gamma)\left[w^p(\Omega_{k+\mu}) + w^p\left(\bigcup_{i_s(l)\leqslant i < i_{s+1}(l)} A_i^s(l)\right)\right] \\ &\leqslant c(\gamma)\, w^p(\Omega_{k+\mu}) + c(\gamma)\sum_{i=i_s(l)}^{i_{s+1}(l)-1} w^p(A_i^s(l)). \end{aligned}$$

For the case $s = 0$, we have $k = N - l$ and

$$w^p(\Omega_k) = w^p(\Omega_{N-l}) \qquad \leqslant c(\gamma)\sum_{i=i_0(l)}^{i_1(l)-1} w^p(A_i^s(l)).$$

Now, all these sets $\{A_i^s(l)\}_{i=i_s(l)}^{i_{s+1}(l)-1}$ belong to b_k with $k = N - l - s\mu$ and therefore

$$\prod_{\alpha=1}^{m} \frac{1}{|A_i^s(l)|}\int_{A_i^s(l)} |f_j(x)|\, dx > 2^k. \tag{2.24}$$

It now readily follows that

$$\int_{2^{-N} < \mathscr{M}_{\mathfrak{B}}(\vec{f}) \leqslant 2^{N+1}} \mathscr{M}_{\mathfrak{B}}(\vec{f})(x)^p\, w^p(x)\, dx \leqslant 2^p \sum_{k=-N}^{N-1} 2^{kp} w^p(\Omega_k)$$

and then

$$\begin{aligned} \sum_{k=-N}^{N-1} 2^{kp} w^p(\Omega_k) &= \sum_{\ell=0}^{\mu-1}\sum_{0\leqslant s\leqslant m_\ell} 2^{p(N-l-s\mu)} w^p(\Omega_{N-l-s\mu}) \\ &= c(\gamma)\sum_{\ell=0}^{\mu-1}\sum_{1\leqslant s\leqslant m_\ell} 2^{p(N-l-s\mu)} w^p(\Omega_{N-l-s\mu+\mu}) \\ &+ c(\gamma)\sum_{\ell=0}^{\mu-1}\sum_{0\leqslant s\leqslant m_\ell} 2^{p(N-l-s\mu)} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} w^p(A_i^s(l)). \end{aligned} \tag{2.25}$$

Observe that the first term in the last equality of (2.25) is equal to

$$c(\gamma)2^{-p\mu}\sum_{\ell=0}^{\mu-1}\sum_{0\leqslant s\leqslant m_\ell - 1} 2^{p(N-l-s\mu)} w^p(\Omega_{N-l-s\mu}) \leqslant c(\gamma)2^{-p\mu}\sum_{k=-N}^{N-1} 2^{kp} w^p(\Omega_k).$$

If we choose μ so large that $c\,2^{-\mu p} \leqslant \frac{1}{2}$, the first term on the right hand side of (2.25) can be subtracted from the left hand side of (2.25). This yields

$$\int_{2^{-N} < \mathscr{M}_{\mathfrak{B}}(\vec{f}) \leqslant 2^{N+1}} \mathscr{M}_{\mathfrak{B}}(\vec{f})^p\, w^p\, dx \lesssim_{n,p} \sum_{\ell=0}^{\mu-1}\sum_{0\leqslant s\leqslant m_\ell}\sum_{i=i_s(l)}^{i_{s+1}(l)-1} 2^{p(N-l-s\mu)} w^p(A_i^s(l)).$$

By (2.24) and the generalized Hölder's inequality (1.6) we obtain

$$\begin{aligned}
&\sum_{\ell=0}^{\mu-1}\sum_{0\leqslant s\leqslant m_\ell}\sum_{i=i_s(l)}^{i_{s+1}(l)-1} 2^{p(N-l-s\mu)} w^p(A_i^s(l)) \qquad (2.26)\\
&\leqslant c(\gamma)\sum_{\ell=0}^{\mu-1}\sum_{0\leqslant s\leqslant m_\ell}\sum_{i=i_s(l)}^{i_{s+1}(l)-1} w^p(A_i^s(l))\left[\prod_{j=1}^m \frac{1}{|A_i^s(l)|}\int_{A_i^s(l)}|f_j|dx\right]^p\\
&\leqslant c(\gamma)\sum_{\ell=0}^{\mu-1}\sum_{0\leqslant s\leqslant m_\ell}\sum_{i=i_s(l)}^{i_{s+1}(l)-1} w^p(A_i^s(l))\left[\prod_{j=1}^m \|f_j v_j\|_{\bar{\Phi}_j,A_i^s(l)}\|v_j^{-1}\|_{\Phi_j,A_i^s(l)}\right]^p\\
&\leqslant c(\gamma)\sum_{\ell=0}^{\mu-1}\sum_{0\leqslant s\leqslant m_\ell}\sum_{i=i_s(l)}^{i_{s+1}(l)-1}\left[\prod_{j=1}^m \|f_j v_j\|_{\bar{\Phi}_j,A_i^s(l)}\right]^p |A_i^s(l))|,
\end{aligned}$$

where in the last step we use the assumption (2.22). For each l we let $I(l)$ be the index set of $\{A_i^s(l)\}_{0\leqslant s\leqslant m_\ell,\, i_s(l)\leqslant i<i_{s+1}(l)}$, and

$$E_1(l)=A_1^s(l) \quad \& \quad E_i(l)=A_i^s(l)\setminus\bigcup_{i<j}A_i^s(l) \qquad \forall i\in I(l).$$

Since the sequences $\{A_i^s(l)\}_{i\in I(l)}$ are γ-scattered with respect to the Lebesgue measure, for each i we have that $|A_i^s(l)|\leqslant\frac{1}{1-\gamma}|E_i(l)|$. Then we have the following estimate for the last term in (2.26)

$$\frac{1}{1-\gamma}\sum_{l=0}^{\mu-1}\sum_{i\in I(l)}\left[\prod_{j=1}^m \|f_j v_j\|_{\bar{\Phi}_j,A_i^s(l)}\right]^p |E_i(l)|. \qquad (2.27)$$

Since the collection $\{E_i(l)\}_{i\in I(l)}$ is a disjoint family and $\mathscr{M}_{\mathfrak{B}}^{\vec{\bar{\Phi}}}$ is bounded from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, we can estimate this last equation (2.27). Then

$$\begin{aligned}
&\sum_{l=0}^{\mu-1}\sum_{i\in I(l)}\int_{E_i(l)}\left[\mathscr{M}_{\mathfrak{B}}^{\vec{\bar{\Phi}}}((f_1v_1,\cdots f_mv_m))(x)\right]^p dx\\
&\lesssim_{n,p,\gamma}\int_{\mathbb{R}^n}\left[\mathscr{M}_{\mathfrak{B}}^{\vec{\bar{\Phi}}}((f_1v_1,\cdots f_mv_m))(x)\right]^p dx\\
&\lesssim_{n,p,\gamma}\prod_{j=1}^m\|f_jv_j\|_{L^{p_j}(\mathbb{R}^n)}^p.
\end{aligned}$$

Letting $N\to\infty$ yields the desired assertion of the theorem. □

We can reformulate this result for the particular case of Muckenhoupt bases; see Definition 1.26. For these bases the generalization of the power bump condition (2.21) assures the boundedness of $\mathscr{M}_{\mathfrak{B}}^{\vec{\bar{\Phi}}}$. Therefore we can deduce the following result.

Corollary 2.20. *Let $\mathfrak{B}$ be a Muckenhoupt basis. Let $\frac{1}{m}<p<\infty$ and $1<p_1,\dots,p_m<\infty$ such that $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$. If the weights $(w,\vec{v})=(w,v_1,\cdots,v_m)$ satisfy the power* *bump condition*

$$\sup_{B\in\mathfrak{B}}\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\prod_{j=1}^m\left(\frac{1}{|B|}\int_B v_j^{(1-p_j')r}\,dx\right)^{\frac{p}{p_j'r}}<\infty \tag{2.28}$$

for some $r>1$ *and* $w\in A_{\infty,\mathfrak{B}}$, *then* $\mathscr{M}_{\mathfrak{B}}$ *is bounded from* $L^{p_1}(v_1)\times\cdots\times L^{p_m}(v_m)$ *to* $L^p(w)$.

Proof. For each $j\in\{1,\cdots,m\}$, we set $\widetilde{v}_j:=v_j^{1/p_j}$ and $\Phi_j(t):=t^{p_j'r}$ for all $t\in(0,\infty)$. Set $\widetilde{w}:=w^{1/p}$. Then the power bump condition (2.28) can be rewritten as

$$\sup_{B\in\mathfrak{B}}\left\{\frac{1}{|B|}\int_B\widetilde{w}^p\,dx\right\}^{1/p}\prod_{j=1}^m\|\widetilde{v}_j^{-1}\|_{\Phi_j,B}<\infty.$$

In this case, for all $x\in\mathbb{R}^n$,

$$M_{\mathfrak{B}}^{\bar{\Phi}_j}f(x)=\sup_{B\in\mathfrak{B},B\ni x}\|f\|_{\bar{\Phi}_j,B}=\sup_{B\in\mathfrak{B},B\ni x}\left\{\frac{1}{|B|}\int_B|f(y)|^{(p_j'r)'}\,dy\right\}^{1/(p_j'r)'}.$$

Note that for each $M_{\bar{\Phi}_j}^{\mathfrak{B}}$ is bounded on $L^{p_j}(\mathbb{R}^n)$ if and only if $p_j/(p_j'r)'>1$. Since $r>1$ every $M_{\bar{\Phi}_j}^{\mathfrak{B}}$ is actually bounded. Moreover, Remark 2.19 this implies that $\mathscr{M}_{\mathfrak{B}}^{\vec{\bar{\Phi}}}$ is bounded from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Thus, by Theorem 2.18

$$\mathscr{M}_{\mathfrak{B}}:L^{p_1}(\widetilde{v}_1^{p_1})\times\cdots\times L^{p_m}(\widetilde{v}_m^{p_m})\to L^p(\widetilde{w}^p),$$

which completes the proof. □

Specializing to the multilinear strong maximal function Corollary 2.20 can be strengthened in the case that the pair $(w,\vec{v})$ satisfies a logarithmic bump condition.

Corollary 2.21. *Let* $1<p_1,\cdots,p_m<\infty$ *and* $\frac{1}{m}<p<\infty$ *such that* $\frac{1}{p}=\sum_{j=1}^m\frac{1}{p_j}$. *Let* $(w,\vec{v})=(w,v_1,\cdots,v_m)$ *such that* w *and* v_j, *for every* j, *are weights, and* w^p *satisfies the* A_∞^* *condition. Let* Φ_j *be a Young function such that* $\bar{\Phi}_j\in B_{p_j}^*$.

$$\sup_{R\in\mathfrak{R}}\left(\frac{1}{|R|}\int_R w(x)^p\,dx\right)^{1/p}\prod_{j=1}^m\|v_j^{-1}\|_{\Phi_j,R}<\infty.$$

Then $\mathscr{M}_s$ *is bounded from* $L^{p_1}(v_1^{p_1})\times\cdots\times L^{p_m}(v_m^{p_m})$ *to* $L^p(w^p)$.

Proof. From Theorem 2.16 and the assumption that each $\bar{\Phi}_j$ is a Young function satisfying the condition (2.7), it follows that every $M_s^{\bar{\Phi}_j}$ is bounded on $L^{p_j}(\mathbb{R}^n)$. Applying Remark 2.19 and Theorem 2.18 with $\mathfrak{B}=\mathfrak{R}$ we obtain the desired conclusion. □

2.4.2 The case of the strong maximal function

We conclude the section by applying the general results to the particular basis $\mathfrak{R}$, consisting on axes parallel rectangles. There were two pending tasks concerning this basis, the proof of Theorem 2.5 and one of the equivalences of the characterization of the B_p^* condition.

Both are a straightforward consequence of the results in Subsection 2.4.1. We first present the proof of Theorem 2.5 that follows directly from Corollary 2.21.

Proof of Theorem 2.5. We first notice that (i) is the linear case ($m = 1$) of Corollary 2.21. For the proof of (ii) we proceed as in [Pér95b, Proposition 3.2]. That is, consider any non-negative function g and define the couple of weights $(u, v) = (M_s^{\Phi}(g^{1/p})^{-1}, g^{-1/p})$. Obviously, (u, v) satisfies condition ($A^*_{p,\Phi}$). By hypothesis the following inequality is satisfied

$$\int_{\mathbb{R}^n} [M_s(f)(x)]^p \frac{1}{[M_s^{\Phi}(g^{1/p})(x)]^p}\, dx \lesssim \int_{\mathbb{R}^n} f(x)^p \frac{1}{g(x)}\, dx.$$

Therefore, by Theorem 2.16, this inequality implies that $\bar{\Phi} \in B_p^*$. □

Finally we give the proof of the pending equivalence of the characterization of the B_p^* condition. In particular, the fact that Theorem 2.16 (ii) implies Theorem 2.16 (iv) can be deduced by proceeding as in the proof of Theorem 2.18. Indeed, we will prove the estimate

$$\int_{2^{-N} < M_s^{\Phi}(f) \leqslant 2^{N+1}} M_s^{\Phi}(f)(x)^p\, w(x)\, dx \lesssim_{n,p,w} \int_{\mathbb{R}^n} f(x)^p M_s w(x)\, dx,$$

where N is a large integer. We use the same covering argument of Theorem 2.18 replacing (2.23) by

$$\frac{1}{|R_\alpha^k|} \int_{R_\alpha^k} |f(y)|\, dy > 2^k.$$

Repeating equations (2.24) and (2.25), we will get

$$\begin{aligned}
&\sum_{\ell=0}^{\mu-1} \sum_{0 \leqslant s \leqslant m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} 2^{p(N-l-s\mu)} w(A_i^s(l)) \\
&\lesssim_{n,p,w} \sum_{\ell=0}^{\mu-1} \sum_{0 \leqslant s \leqslant m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} w(A_i^s(l)) \|f\|^p_{\Phi, A_i^s(l)} \\
&\lesssim_{n,p,w} \sum_{\ell=0}^{\mu-1} \sum_{0 \leqslant s \leqslant m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} \left\| f \left(\frac{w(A_i^s(l))}{|A_i^s(l)|} \right)^{1/p} \right\|^p_{\Phi, A_i^s(l)} |A_i^s(l)| \\
&\lesssim_{n,p,w} \sum_{\ell=0}^{\mu-1} \sum_{0 \leqslant s \leqslant m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} \left\| f (M_s w)^{1/p} \right\|^p_{\Phi, A_i^s(l)} |A_i^s(l)|,
\end{aligned}$$

where in the penultimate step we used the generalized Hölder inequality (1.6). Finally, we will obtain the claimed conclusion using the hypothesis (ii), that is that the operator M_s^{Φ} is bounded on $L^p(\mathbb{R}^n)$. We omit the details.

2.5 Fefferman-Stein inequalities

As we described already in the introduction, by this we mean in general a two-weight norm inequality of the form

$$\int_{\mathbb{R}^n} (Mf)^p w \lesssim_{n,p,w} \int_{\mathbb{R}^n} |f|^p Mw dx, \quad 1 < p < \infty, \tag{2.29}$$

where M denotes some maximal operator. This inequality was first proved for the Hardy-Littlewood maximal function by C. Fefferman and Stein, [FS71], for *every* non-negative, locally integrable weight w. For the strong maximal function the same inequality is true provided that $w \in A_\infty^*$; see [Lin84] for a direct proof of this result and also [Pér93], where the Fefferman-Stein inequality is obtained as a corollary of a more general two weight-norm inequality. Note also that (2.29) follows from Theorem 2.5. Indeed, the pair of weights $(w, M_s w)$ satisfies the A_1^* condition and, trivially, the bump condition for any Young function Φ.

Observe that, as in the case of the boundedness of the weighted strong maximal function (see estimate (1.22)), we need an extra assumption on the weight in order to prove the Fefferman-Stein inequality for the strong maximal function. This should be contrasted to the corresponding result for the weighted centered Hardy-Littlewood maximal function, as well as (2.29), where no assumption on the weight is needed.

The form of the endpoint Fefferman-Stein inequality depends on the corresponding unweighted endpoint properties of the maximal operator under study. For the usual Hardy-Littlewood maximal function M the right statement is

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \lesssim_n \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)| Mw(x) dx.$$

This result follows easily by the usual covering arguments for cubes. It is also a direct consequence of the boundedness of M_c^w. Observe that both approaches are not longer applicable in the case of rectangles. In fact, the natural endpoint Fefferman-Stein inequality for the strong maximal function was proved by Mitsis, [Mit06], in dimension $n = 2$. In particular Mitsis showed that

$$w(\{x \in \mathbb{R}^2 : M_s f(x) > \lambda\}) \lesssim_w \int_{\mathbb{R}^2} \frac{|f(x)|}{\lambda} \Big(1 + \log^+ \frac{|f(x)|}{\lambda}\Big) M_s w(x) dx.$$

Our goal in this section is to derive the extension of this result to higher dimensions, answering Problem 0.2. This is the content of the following theorem.

Theorem 2.22. *Let $w \in A_\infty^*$. For all dimensions $n \geqslant 1$ we have*

$$w(\{x \in \mathbb{R}^n : M_s f(x) > \lambda\}) \lesssim_{n,w} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \Big(1 + \big(\log^+ \frac{|f(x)|}{\lambda}\big)^{n-1}\Big) M_s w(x) dx. \tag{2.30}$$

By interpolation, the Fefferman-Stein inequality of our main theorem above implies the strong L^p version of the Fefferman-Stein inequality from [Lin84], [Pér93]. Furthermore, since every A_1^*-weight is an A_∞^*-weight, we recover the endpoint inequality (1.21) for A_1^*-weights.

2.5.1 A solution to Problem 0.2

We are now ready to present the proof of Theorem 2.22.

Proof of Theorem 2.22. We assume that $n \geqslant 2$ since in one dimension M_s is the usual Hardy-Littlewood maximal function and there is nothing (new) to prove. It suffices to prove the theorem for $\lambda = 1$. Let $F := \{x \in \mathbb{R}^n : M_s(f)(x) > 1\}$. For every $x \in F$ let $R_x \in \mathfrak{R}$ be a rectangle such that

$$\frac{1}{|R_x|}\int_{R_x} |f(y)|dy > 1. \tag{2.31}$$

Without loss of generality we may assume that $\{R_x\}_{x\in F}$ is a finite sequence $\{R_j\}_{1\leqslant j\leqslant M}$. Then applying Lemma 2.10 we obtain the subcollection $\mathscr{R} := \{R_j^s\}_{j=1}^N \subset \mathfrak{R}$. By (i) of that Lemma the collection $\mathscr{R}$ has the sparseness property (P_2). We assume that $\epsilon > 0$ was chosen small enough in Lemma 2.10, and thus in (P_2), so that Lemma 2.11 is valid. Observe that (P_2) also implies that

$$\Big|R_j^s \cap \bigcup_{i<j} R_i^s\Big| \leqslant \epsilon |R_j^s|.$$

By choosing $\epsilon > 0$ small enough we can also assume that $w(R_j^s \cap \bigcup_{i<j} R_i^s) \leqslant \frac{1}{2} w(R_j^s)$, according to Remark 1.24. Setting $E_j := R_j^s \setminus \bigcup_{i<j} R_i^s$ we will thus have

$$w(R_j^s) \geqslant w(E_j) \geqslant \frac{1}{2} w(R_j^s), \quad j = 1, 2, \dots, N, \tag{2.32}$$

and the choice of $\epsilon > 0$ depends only on the weight $w \in A_\infty^*$. Denoting $\Omega := \bigcup_{j=1}^N R_j^s$, we use (2.31) and (iii) of Lemma 2.10 to estimate

$$\begin{aligned} w(F) &\lesssim_{\epsilon,w,n} w(\Omega) \leqslant \sum_{j=1}^N w(R_j^s) \leqslant \sum_{j=1}^N \frac{w(R_j^s)}{|R_j^s|}\int_{R_j^s} |f(y)|dy \\ &= \int_\Omega f(x) \sum_{j=1}^N \frac{w(R_j^s)}{|R_j^s|} \mathbf{1}_{R_j^s}(x)dx. \end{aligned}$$

Define the linear operators

$$Tf(x) = \sum_{j=1}^N \frac{1}{|R_j^s|}\int_{R_j^s} f(y)dy \mathbf{1}_{E_j}(x), \quad T^*f(x) = \sum_{j=1}^N \frac{1}{|R_j^s|}\int_{E_j} f(y)dy \mathbf{1}_{R_j^s}(x), \quad x \in \mathbb{R}^n.$$

For locally integrable f, g we have

$$\int_\Omega Tf(x)g(x)dx = \int_\Omega T^*g(x)f(x)dx, \quad Tf(x) \leqslant M_s f(x), \quad x \in \mathbb{R}^n.$$

By (2.32) we have

$$T^*w(x) = \sum_{j=1}^{N} \frac{w(E_j^s)}{|R_j^s|}\mathbf{1}_{R_j^s}(x) \simeq \sum_{j=1}^{N} \frac{w(R_j^s)}{|R_j^s|}\mathbf{1}_{R_j^s}(x)$$

thus we can estimate for any $\delta > 0$

$$\begin{aligned} w(\Omega) &\lesssim \int_\Omega fT^*w \leqslant \int_{\{\Omega:T^*w\leqslant\delta M_s w\}} f(x)T^*w(x)dx + \int_{\{\Omega:T^*w>\delta M_s w\}} f(x)\frac{T^*w(x)}{M_s w(x)}M_s w(x)dx \\ &\leqslant \delta \int_{\mathbb{R}^n} |f(x)|M_s w(x)dx + \int_{\{\Omega:T^*w>\delta M_s w\}} f(x)\frac{T^*w(x)}{M_s w(x)}M_s w(x)dx. \end{aligned}$$

Applying the pointwise estimate (2.18) presented in Section 2.3.2 we get for every $\theta > 0$:

$$\begin{aligned} w(\Omega) \lesssim (\delta + c_\theta) &\int_{\mathbb{R}^n} |f(x)|\Big(1 + \big(\log^+|f(x)|\big)^{n-1}\Big)M_s w(x)dx \\ &+ \int_{\{\Omega:\ T^*w>\delta M_s w\}} \left(\exp\left[\theta\left(\frac{T^*w(x)}{M_s w(x)}\right)^{\frac{1}{n-1}}\right] - 1\right)M_s w(x)dx. \end{aligned}$$

We now estimate the last term,

$$\begin{aligned} Q &:= \int_{\{\Omega:\ T^*w>\delta M_s w\}} \left(\exp\left[\theta\left(\frac{T^*w(x)}{M_s w(x)}\right)^{\frac{1}{n-1}}\right] - 1\right)M_s w(x)dx \\ &= \sum_{j=1}^{\infty} \frac{\theta^j}{j!} \int_{\{\Omega:\ T^*w>\delta M_s w\}} \left(\frac{T^*w(x)}{M_s w(x)}\right)^{\frac{j}{n-1}} M_s w(x)dx \leqslant \sum_{1\leqslant j\leqslant n-1} + \sum_{j>n-1} := \mathrm{I} + \mathrm{II}. \end{aligned}$$

For I we just observe that since $j/(n-1) \leqslant 1$ and $T^*w/(\delta M_s w) > 1$ we have the elementary estimate

$$\begin{aligned} \left(T^*w/M_s w\right)^{\frac{k}{n-1}} &= \left(T^*w/(\delta M_s w)\right)^{\frac{j}{n-1}}\delta^{\frac{j}{n-1}} \\ &= \delta^{\frac{j}{n-1}-1}\left(T^*w/(\delta M_s w)\right)^{\frac{j}{n-1}-1}\frac{T^*w}{M_s w} \\ &\leqslant \delta^{\frac{j}{n-1}-1}\frac{T^*w}{M_s w}. \end{aligned}$$

So we have

$$\mathrm{I} \leqslant \sum_{1\leqslant j\leqslant n-1} \frac{\theta^j\delta^{\frac{j}{n-1}-1}}{j!}\int_\Omega T^*w(x)dx \leqslant \frac{\theta}{\delta}e^{\delta^{\frac{1}{n-1}}}\int_\Omega T1(x)w(x) \lesssim_{\delta,n} \theta w(\Omega).$$

Here we abuse notation by denoting $T1, T^*1(x)$ the action of T, T^*, respectively, on the constant function 1. For II we use the fact that $T^*w \simeq \sum_{j=1}^{N}\frac{w(R_j^s)}{|R_j^s|}\mathbf{1}_{R_j^s} \leqslant M_s w\sum_{j=1}^{N}\mathbf{1}_{R_j^s} \simeq$

$M_s w T^*1$. We have

$$
\begin{aligned}
\mathrm{II} &\leqslant \sum_{j>n-1} \int_\Omega \frac{\theta^j}{j!}\left(\frac{T^*w(x)}{M_s w(x)}\right)^{\frac{k}{n-1}-1} \frac{T^*w(x)}{M_s w(x)} M_s w(x)\,dx \\
&\lesssim_{n,w} \sum_{j>n-1} \frac{\theta^j}{j!}\int_\Omega (T^*1(x))^{\frac{j}{n-1}-1} T^*w(x)\,dx \\
&\lesssim_{n,w} \sum_{j>n-1} \frac{\theta^j}{j!}\int_\Omega (T^*1(x))^{\frac{j}{n-1}} T^*w(x)\,dx \quad \Big(\text{because } T^*1 \gtrsim 1 \text{ on } \Omega\Big) \\
&\lesssim_{n,w} \sum_{j>n-1} \frac{\theta^j}{j!}\int_\Omega T(T^*(1)^{\frac{j}{n-1}})(x) w(x)\,dx := \sum_{j>n-1} \frac{\theta^j}{j!} Q_j.
\end{aligned}
$$

Since $w \in A^*_{p_o}$ for some $1 < p_o < \infty$ and $Tf \leqslant M_s f$ we have $\|T(f)\|_{L^{p_o}(w)} \lesssim_{w,n} \|f\|_{L^{p_o}(w)}$. This together with Lemma 2.11 yields

$$
Q_j \lesssim_{n,w} w(\Omega)^{\frac{1}{p_o'}}\left(\int_\Omega |T^*1(x)|^{\frac{jp_o}{n-1}} w(x)\,dx\right)^{\frac{1}{p_o}} \lesssim_{n,w} [jp_o/(n-1)]^j w(\Omega).
$$

Overall we get

$$
\begin{aligned}
\mathrm{II} &\lesssim_{n,w} \sum_{j>n-1} \frac{\theta^j}{j!}\frac{(jp_o)^j}{(n-1)^j} w(\Omega) \lesssim_{n,w} \sum_{j>n-1} \frac{(\theta e p_o/(n-1))^j}{\sqrt{j}} w(\Omega) \\
&\lesssim_{n,w} \frac{(\theta e p_o/(n-1))^n}{\sqrt{n}} w(\Omega),
\end{aligned}
$$

if θ is small enough. Thus $Q \lesssim_{n,w} \theta w(\Omega)$. We have proved that for $\theta > 0$ small and fixing $\delta = 1$ in the previous estimates we have

$$
w(\Omega) \lesssim_{n,w} \theta w(\Omega) + (1+c_\theta)\int_{\mathbb{R}^n} |f(x)|\Big(1 + \big(\log^+ |f(x)|\big)^{n-1}\Big) M_s w(x)\,dx.
$$

Choosing $\theta > 0$ sufficiently small we thus have

$$
w(F) \lesssim w(\Omega) \lesssim_{n,w} \int_{\mathbb{R}^n} |f(x)|\Big(1 + \big(\log^+ |f(x)|\big)^{n-1}\Big) M_s w(x)\,dx,
$$

which is the desired estimate. □

We have actually proved the following weighted analogue of the Córdoba-Fefferman covering Lemma 2.9 presented in Section 2.2.1.

Lemma 2.23. *Let $w \in A^*_\infty$. Suppose that $\{R_j\}_{j\in J}$ is a finite sequence of rectangles from $\mathfrak{R}$. Then there exists a subcollection $\{R^s_j\}_{1\leqslant j\leqslant N} \subset \cup_{j\in J} R_j$ such that*

(i) $w(\cup_{j\in J} R_j) \lesssim_{n,w} w(\cup_{j=1}^N R^s_j)$.

(ii) *For every $\delta > 0$ there exists $\theta_o = \theta_o(n, w, \delta) > 0$ such that, for every $\theta < \theta_o$ we* *have*

$$\int_{\{\Omega:\ T^*w(x)>\delta M_s w(x)\}} \left(\exp\left[\theta\left(\frac{T^*w(x)}{M_s w(x)}\right)^{\frac{1}{n-1}}\right] - 1\right) M_s w(x)dx \lesssim_{n,w,\theta,\delta} w(\cup_{j=1}^N R_j^s).$$

Here $T^*w = \sum_{j=1}^N \frac{w(R_j^s)}{|R_j^s|}\mathbf{1}_{R_j^s}$.

2.6 Notes and references

References

The results described here have first appeared in [LL] and [LP14]. The two-weight problem for the strong maximal function in terms of bump conditions was proposed by Carlos Pérez and his paper [Pér95b]. The principal tools to deal with rectangles in the two-weight context have been inspired from [Jaw86]. Theorem 2.18 is a natural extension of Theorem 2.5 in [GLPT11]. Theorem 2.22 extends the result of Mitsis in [Mit06] to higher dimensions. The proof of Mitsis uses the combinatorics of two-dimensional rectangles, which allow one to get favorable estimates for the measures

$$|\{x \in R_j : \sum_{j=1}^N \mathbf{1}_{R_j}(x) = \ell\}|;$$

here $\{R_j\}_{1\leqslant j\leqslant N}$ is a sequence of rectangles which satisfy a certain sparseness property and ℓ is any integer in $\{1,2,\dots,N\}$. These combinatorics do not seem to be readily available in higher dimensions and so we have adopted a different approach, which relies on the boundedness of the weighted strong maximal function M_s^w and the precise estimate for its norm, (2.10). In particular, our approach is inspired by the arguments in [LS88], a paper which seems to have been overlooked by most of the works on weighted inequalities for the strong maximal function.

Extra condition on the weight w and covering arguments

There is a key difference between Theorem 2.5 and the analogous result for the Hardy-Littlewood operator (Theorem 2.3). Indeed, for the former not only do we need a more restrictive class of Young functions (the class B_p^*), but also we ask for an extra condition on the first weight of the pair (w,v). Is it possible to remove this extra condition on w or consider a weaker one?

The difference between the two-weight problems for M and M_s also appears if we study the problem in terms of testing conditions. Indeed, the analogous version of Theorem 2.1 for the strong maximal function was first described by Sawyer in [Saw82b] and requires some *extra* assumptions. The same result was also proved in the more general context of basis of open sets by Jawerth in [Jaw86]. More precisely we have:

Theorem 2.24 (Sawyer, [Saw82b]). *Let (w,v) be a couple of weights in $\mathbb{R}^n$ and $1<p<\infty$. Suppose that $\sigma(x)=v(x)^{1-p'}$ and assume that M_s^σ is bounded in $L^p(\sigma)$. Then the following two conditions are equivalent:*

(i) *The operator M_s is bounded from $L^p(v)$ to $L^p(w)$.*

(ii) *For every set* $G \subset \mathbb{R}^n$ *that is a union of rectangles in* $\mathfrak{R}$ *we have*

$$\int_G |M_s(\sigma \mathbf{1}_G)|^p w \lesssim \sigma(G). \tag{2.33}$$

In the case of the Hardy-Littlewood maximal function, if we restrict to its dyadic or center version, we have that M^σ always maps $L^p(\sigma)$ into itself; see Section 1.2 in Chapter 1. This is not longer true in the case of the strong maximal function, thus we require M_s^σ to be bounded on $L^p(\sigma)$.

This extra assumption should be compared to the extra condition in Theorem 2.5, where the condition was required to deal with the w-overlap of the rectangles. Indeed, it is known in general that the understanding of the boundedness properties of a maximal operator $M_{\mathfrak{B}}^\mu$ is equivalent to the study of the covering properties (with respect to the measure μ) of the elements in the basis $\mathfrak{B}$:

Theorem 2.25 (Córdoba, [C76]). *Given a basis* $\mathfrak{B}$ *in* $\mathbb{R}^n$*, and a measure* μ *such that* $0 < \mu(B) < \infty$ *for all* $B \in \mathfrak{B}$*. The following statements are equivalent:*

(i) $M_{\mathfrak{B}}^\mu$ *is weak type* (p', p') *with respect to* μ *and it is bounded on* $L^{p'}(\mu)$.

(ii) *From any finite sequence* $\{B_j\} \in \mathfrak{B}$*, we can select a subcollection* $\{B_j^s\}$ *such that*

$$\Big\| \sum_j \mathbf{1}_{B_j^s} \Big\|_{L^p(\mu)} \lesssim \mu\Big(\bigcup_j B_j\Big)^{\frac{1}{p}} \lesssim \mu\Big(\bigcup_j B_j^s\Big)^{\frac{1}{p}}.$$

As we have already discussed in the first chapter of this thesis, the geometry of rectangles in $\mathbb{R}^n$ is much more intricate than that of cubes in $\mathbb{R}^n$ and M_s^μ is not in general a bounded operator. The A_∞^* condition is sufficient in order to deal with rectangles and with their covering properties (see Lemma 2.10) in weighted spaces. A weaker sufficient condition for the boundedness of MR,w appears in [27] but it is quite technical and difficult to handle. Since it is known that A_∞^* is not a necessary condition for the boundedness of M_s^w on $L^p(w)$, an admissible candidate for a weaker condition may be the mix Tauberian condition

$$\mu(\{x \in \mathbb{R}^n : M_s^w(\mathbf{1}_E)(x) > \gamma\}) \lesssim_{n,w} w(E). \tag{$A^w_{\mathfrak{R},\gamma,w}$}$$

This condition is described in the next chapter and Corollary 3.2 shows that characterizes completely the $L^p(w)$-boundedness of M_s^w whenever w is doubling. This kind of condition is, for sure, weaker than the A_∞^*. However we were not even able to prove Theorem 2.5 assuming that M_s^w maps $L^p(w)$ into $L^p(w)$, what seems more *natural* if we compare our two-weight result with Theorem 2.24

We note that this discussion also relates to the Fefferman-Stein problem for M_s. In Theorem 2.22 we have also restricted to weights $w \in A_\infty^*$ to deal with the overlap of the rectangles, while in the case of M no condition was required on the weight.

The weak problem

Another interesting problem is the characterization of those pairs of weights (w, v) such that the strong maximal function is bounded from $L^p(v)$ to $L^{p,\infty}(w)$, or satisfies the weak

(p, p) inequality

$$w(\{x \in \mathbb{R}^n : M_s(f)(x) > \lambda\}) \lesssim_{n,p,[w,v]_{A_p^*}} \frac{1}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p v(x) dx, \quad \lambda > 0. \tag{2.34}$$

Complete answers for these questions are known for the Hardy-Littlewood maximal function. Indeed, the following very well-known theorem characterizes completely the weak problem:

Theorem 2.26 (Muckenhoupt, [Muc72]). *Given* p, $1 < p < \infty$ *the weak-type inequality*

$$w(\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}) \lesssim_{n,p,[w,v]_{A_p}} \frac{1}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p v(x) dx, \quad \lambda > 0.$$

holds if and only if $(w, v) \in A_p$.

For the strong maximal function, however, the A_p condition is necessary but it is not sufficient; see the manuscript [LP], still in progress. In this case, we need to deal with the covering properties and as it was pointed out in the last subsection, we can control the w-overlap of rectangles with an extra-condition. The following theorem shows the only known result for the weak problem

Theorem 2.27 (Pérez, [Pér93]). *Let* $1 < p < \infty$. *Suppose that* (w, v) *is a couple of weights such that* $w \in A_\infty^*$ *and for some* $1 < r < \infty$, $(w, v) \in A_{p,r}$*; that is:*

$$[w, v]_{A^*_{p,r}} = \sup_{R \in \mathfrak{R}} \left(\frac{1}{|R|} \int_R w\right) \left(\frac{1}{|R|} \int_R v^{r(1-p')}\right)^{\frac{p-1}{r}} < +\infty. \tag{2.35}$$

Then

$$M_s : L^p(v) \longrightarrow L^{p,\infty}(w).$$

Indeed, since this A_p^r condition verifies an open property (see [Pér93, Theorem 1.3]), the strong (p, p) boundedness can be also deduced. Moreover note that as a corollary of Theorem 2.5 we also have this result under the hypothesis that Φ satisfies a more general bump condition. However, it should be possible to have a weaker condition than in the strong problem. In that sense, it could seem reasonable to try to prove that if the pair of weights $(w, v) \in A_p^*$ and $w \in A_\infty^*$, then we have the M_s is weak (p, p), $p > 1$. However,at least for now, we have not been able to improve Theorem 2.27.

Moreover, it was not either known a weak version of Theorem 2.24. In [LP] we prove that the same characterization in terms of testing conditions, with a *weak* version of condition 2.33, answers to this problem.

The endpoint case

A problem that looks even more difficult is the endpoint version of the two-weight problem for the strong maximal function. More precisely, determine all the couple of weights (w, v) such that

$$w(\{x \in \mathbb{R}^n : M_s f(x) > \lambda\}) \lesssim_{n,w,v} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \Big(1 + \big(\log^+ \frac{|f(x)|}{\lambda}\big)^{n-1}\Big) v(x) dx. \tag{2.36}$$

The analogous estimate for the Hardy-Littlewood operator is the following weak $(1,1)$ inequality:

$$w(\{x \in \mathbb{R}^n : M_s f(x) > \lambda\}) \lesssim_{n,w,v} \int_{\mathbb{R}^n} |f(x)| v(x) dx.$$

This estimate is characterized in terms of the A_1 condition. But in the case of the strong maximal operator, even in the one weight problem, $v = w$, it turns out that A_1^* is not necessary for estimate (2.36). See Theorem 1.25 for a detailed explanation of this fact. For the case $v \equiv M_s w$ estimate (2.36) is the Fefferman-Stein inequality (2.30), that has been proved whenever $w \in A_\infty^*$. Again it is not known if this estimate remains true if w is any weight or if it verifies a weaker condition than the A_∞^*.

Chapter 3

Tauberian conditions and weights

Let $\mathfrak{B}$ be a homothecy invariant collection of convex sets in $\mathbb{R}^n$. Given an appropriately *doubling* measure μ, the purpose of this chapter is to characterize the boundedness properties of the geometric maximal operator $M^{\mu}_{\mathfrak{B}}$ in terms of Tauberian conditions. We first introduce a few facts related to the notion of doubling for the bases $\mathfrak{B}$. Then we prove the main result concerning a basis $\mathfrak{G}$ consisting of rectangles with arbitrary orientation whose maximal operator satisfies a Tauberian condition. The extension of this result to general bases of convex sets answers to Problem 0.3. We finally discuss some applications of this result in differentiation theory and in the study of weighted inequalities for M^{w}_{s} and $M_{\mathfrak{B}}$. In particular, we will focus in the case that $\mathfrak{B}$ is a Muckenhoupt basis.

3.1 Maximal operators $M^{\mu}_{\mathfrak{B}}$

Throughout this thesis we have studied the mapping properties of $M^{\mu}_{\mathfrak{B}}$ for certain bases $\mathfrak{B}$ and measures μ. For example, in Section 1.2 of Chapter 1, we presented some known results for the basis $\mathfrak{Q}$. Also in Subsection 1.3.2 of the same chapter, we described the particular case of the operator M^{w}_{s} where the measure was restricted to be a weight. Our goal in this chapter is to study the $L^p(\mu)$-boundedness of $M^{\mu}_{\mathfrak{B}}$ whenever $\mathfrak{B}$ is a homothecy invariant collection of convex sets in $\mathbb{R}^n$. The case where μ is the Lebesgue measure was already presented in Chapter 1, Theorem 1.7. In particular, it was shown that $M_{\mathfrak{B}} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ if and only if $M_{\mathfrak{B}}$ satisfies a Tauberian condition

$$\left|\{x \in \mathbb{R}^n : M_{\mathfrak{B}}(\mathbf{1}_E)(x) > \gamma\}\right| \leqslant c_{\mathfrak{B},\gamma}|E|.$$

The intention here is to extend this result to a more general context.

As a first step, we turn our attention to the maximal function $M^{\mu}_{\mathfrak{G}}$ defined with respect to a non-negative measure μ which is doubling with respect to the basis and a homothecy invariant basis $\mathfrak{G}$ in $\mathbb{R}^n$ consisting of rectangles. Observe that the rectangles here can have arbitrary orientation. Our first objective is to find a characterization of the measures ν such that $M^{\mu}_{\mathfrak{G}} : L^p(\nu) \to L^p(\nu)$ for some $p > 1$ in terms of a mixed μ, ν-Tauberian condition. Here ν denotes any arbitrary measure and we just assume is non-negative and locally finite. A Tauberian condition, in this more general context, takes the following form

$$\nu(\{x \in \mathbb{R}^n : M^{\mu}_{\mathfrak{G}}(\mathbf{1}_E)(x) > \gamma\}) \leqslant c^{\mu}_{\mathfrak{G},\gamma,\nu}\nu(E). \qquad (A^{\mu}_{\mathfrak{G},\gamma,\nu})$$

More precisely, the result that we present here gives a characterization of the boundedness of $M^{\mu}_{\mathfrak{G}}$ on $L^p(\nu)$, for sufficiently large $p > 1$, in terms of the Tauberian condition $(A^{\mu}_{\mathfrak{G},\gamma,\nu})$ whenever the measure μ is *doubling* with respect to $\mathfrak{G}$.

Theorem 3.1. *Let $\mathfrak{G}$ be a homothecy invariant basis consisting of rectangles and μ, ν be two non-negative measures on $\mathbb{R}^n$, finite on compact sets. Assume that μ is doubling with respect to $\mathfrak{G}$. The following are equivalent:*

(i) The measures μ, ν satisfy the Tauberian condition $(A^{\mu}_{\mathfrak{G},\gamma,\nu})$ with respect to some fixed level $\gamma \in (0, 1)$.

(ii) There exists $1 < p_o = p_o(c^{\mu}_{\mathfrak{G},\gamma,\nu}, \gamma, \mu) < +\infty$ such that $M^{\mu}_{\mathfrak{G}} : L^p(\nu) \to L^p(\nu)$ for all $p > p_o$.

The proof of Theorem 3.1 is presented in Section 3.3 and the notion of *doubling* is detailed in the following section. It is important to note that Theorem 3.1 has an interesting corollary whenever $\mu \equiv \nu$. In this special case our main theorem concerns the boundedness of the operator $M^{\mu}_{\mathfrak{G}}$ on $L^p(\mu)$ for sufficiently large $p > 1$ and μ doubling with respect to $\mathfrak{G}$. Indeed, we already know that for a doubling measure μ the operator M^{μ} is of weak type $(1, 1)$ and thus of strong type (p, p) for all $p > 1$. Thus both (i) and (ii) of this theorem are always satisfied for $\mathfrak{Q}$ and $\mu \equiv \nu$. However, for $\mathfrak{G} = \mathfrak{R}$ and μ *product doubling* we get a new characterization of the measures μ such that M^{μ}_{s} is bounded on $L^p(\mu)$ for sufficiently large $p > 1$. When $\mu \equiv \nu$ the mixed Tauberian condition becomes:

$$\mu(\{x \in \mathbb{R}^n : M^{\mu}_{\mathfrak{G}}(\mathbf{1}_E)(x) > \gamma\}) \leqslant c^{\mu}_{\mathfrak{G},\gamma,\mu}\mu(E). \qquad (A^{\mu}_{\mathfrak{G},\gamma,\mu})$$

We then have:

Corollary 3.2. *Let $\mathfrak{G}$ be a homothecy invariant basis consisting of rectangles and μ be a non-negative measure on $\mathbb{R}^n$, finite on compact sets. Assume that μ is doubling with respect to $\mathfrak{G}$. The following are equivalent:*

(i) The measure μ satisfies the Tauberian condition $(A^{\mu}_{\mathfrak{G},\gamma,\mu})$ with respect to some fixed level $\gamma \in (0, 1)$.

(ii) There exists $1 < p_o = p_o(c^{\mu}_{\mathfrak{G},\gamma,\nu}, \gamma, \mu) < +\infty$ such that $M^{\mu}_{\mathfrak{G}} : L^p(\mu) \to L^p(\mu)$ for all $p > p_o$.

Another important feature of Theorem 3.1 is that it is an essential step to study the problem for any homothecy invariant bases of convex sets. Indeed, we shall reduce this general case to a special case of rectangles $\mathfrak{G}_{\mathfrak{B}}$ *associated* to the basis $\mathfrak{B}$. In the next section we describe this *associated* basis and the extension of Theorem 3.1 to the more general context of homothecy invariant bases consisting of convex sets is presented in Section 3.4.

3.2 Doubling measures with respect to a basis of convex sets

In this section we discuss the properties of measures which are doubling with respect to a basis of convex sets. By a basis of convex sets we will always mean a homothecy invariant basis $\mathfrak{B}$ consisting of non-empty, bounded, open convex sets with non-empty interior. Definition 1.10 introduces the notion of doubling associated to the basis of cubes. Here we extend this definition to a more general context. Remember that for $\sigma \in \mathbb{R}^n$, $E \subset \mathbb{R}^n$ and $c > 0$ we write $\tau_\sigma E = \{\sigma + x : x \in E\}$ and $\mathrm{dil}_c E = \{cx : x \in E\}$.

Definition 3.3. Let μ be a non-negative measure which is finite on compact sets. We will say that μ is doubling with respect to $\mathfrak{B}$ if there is a constant $\Delta_{\mu,\mathfrak{B}} > 1$ such that, for every $B \in \mathfrak{B}$ and every $\sigma \in \mathbb{R}^n$ such that $B \subset \tau_\sigma \mathrm{dil}_2 B$ we have

$$\mu(\tau_\sigma \mathrm{dil}_2 B) \leqslant \Delta_{\mu,\mathfrak{B}}\, \mu(B).$$

We always assume $\Delta_{\mu,\mathfrak{B}}$ to be the smallest possible constant so that the previous inequality holds uniformly for all $B \in \mathfrak{B}$. When the underlying basis $\mathfrak{B}$ is clear from the context we will write Δ_μ for $\Delta_{\mu,\mathfrak{B}}$.

Remark 3.4. The previous definition of a doubling measure reduces to the usual doubling condition (up to changes in the doubling constant) if $\mathfrak{B} = \mathfrak{Q}$ or $\mathfrak{B} = \mathfrak{b}$. However, the doubling condition with respect to $\mathfrak{R}$ is quite different than the doubling condition with respect to cubes. In fact, if one wants to study the behavior of the operator M_S^μ with respect to a measure μ then the "natural" condition is that μ is doubling with respect to $\mathfrak{R}$. For example in [PWX11], measures that are doubling with respect to $\mathfrak{R}$ are called *product doubling* and we will adopt the same terminology here. The same notion of product doubling is discussed for example in [JT84]. Naturally, weights $w \in A_\infty^*$ give product doubling measures $w(x)dx$.

Observe also that for a general basis of convex sets $\mathfrak{B}$ there is in general no natural homothecy center as the convex sets in $\mathfrak{B}$ might not be symmetric with respect to some point. In order to avoid confusion in all these subtle issues we will always specify the basis according to which a measure is assumed to be doubling.

The John ellipsoid. One of the technical annoyances when dealing with general convex sets is the lack of a natural homothecy center as the convex sets we will consider will not in general be symmetric with respect to some point. In order to deal with this lack of symmetry and resulting technical issues, the classical lemma of F. John, [Joh48], will be very useful. See also [Bal97] for a very nice exposition of this and related topics.

Lemma 3.5 (F. John)**.** *Let B be a bounded convex set in $\mathbb{R}^n$. Then B contains a unique ellipsoid $\mathscr{E}_B$, of maximal volume. We will call $\mathscr{E}_B$ the* John ellipsoid *of B. The John ellipsoid of B is such that*

$$\mathscr{E}_B \subset B \subset n\mathscr{E}_B.$$

Here $c\mathscr{E}_B$ denotes the dilation of the ellipsoid $\mathscr{E}_B$ by a factor $c > 0$ with respect to its center.

Given a basis $\mathfrak{B}$ consisting of convex sets we will now construct an associated basis $\mathfrak{G}_\mathfrak{B}$ consisting of rectangles as in [HS09]. To this end let $B \in \mathfrak{B}$ and $\mathscr{E}_B$ be the John ellipsoid of B. Then there is a (not necessarily unique) rectangle $R \supset \mathscr{E}_B$ of minimal volume. It is elementary to check that for any ellipsoid $\mathscr{E}$, a rectangle R of minimal volume that contains $\mathscr{E}$ satisfies

$$\mathscr{E} \subset R \subset \sqrt{n}\mathscr{E}.$$

Given $B \in \mathscr{B}$, let R_B be a rectangle of minimal volume containing $n\mathscr{E}_B$. By the above observations and John's lemma we get for every $B \in \mathscr{B}$ that

$$B \subset n\mathscr{E}_B \subset R_B \subset n\sqrt{n}\mathscr{E}_B \subset n^{3/2}B.$$

Here the dilations cB are with respect to the center of the John ellipsoid associated to B. We now define the basis $\mathfrak{G}_{\mathfrak{B}}$ as

$$\mathfrak{G}_{\mathfrak{B}} := \{R_B : B \in \mathfrak{B}\}.$$

Since $\mathfrak{B}$ is homothecy invariant the rectangle R_B may be selected so that $\mathfrak{G}_{\mathfrak{B}}$ is homothecy invariant and we always assume that this is the case.

The following lemma is an immediate consequence of the above discussion. We omit the easy proof.

Lemma 3.6. *Let $\mathfrak{B}$ be a basis of convex sets and $\mathfrak{G}_{\mathfrak{B}}$ be the homothecy invariant basis of associated rectangles as constructed above. Suppose that μ is doubling with respect to $\mathfrak{B}$ with doubling constant $\Delta_{\mu,\mathfrak{B}}$. We have:*

(i) The measure μ is doubling with respect to $\mathfrak{G}_{\mathfrak{B}}$ with doubling constant

$$\Delta_{\mu,\mathfrak{G}_{\mathfrak{B}}} \leqslant \Delta_{\mu,\mathfrak{B}}^{1+\lceil \frac{3}{2}\log n\rceil}.$$

(ii) We have the pointwise equivalence

$$\frac{1}{c_n} M^{\mu}_{\mathfrak{G}_{\mathfrak{B}}} f(x) \leqslant M^{\mu}_{\mathfrak{B}} f(x) \leqslant c_n M^{\mu}_{\mathfrak{G}_{\mathfrak{B}}} f(x), \quad x \in \mathbb{R}^n,$$

where $c_n := \Delta_{\mu,\mathfrak{B}}^{\lceil \frac{3}{2}\log n\rceil}$.

(iii) If $B \in \mathfrak{B}$ and R_B is the associated rectangle of B with $B \subset R_B \subset n^{\frac{3}{2}} B$ then

$$\mu(B) \geqslant \rho\mu(R_B),$$

where $\rho := c_n^{-1}$ and c_n as defined in (ii).

Properties of general doubling measures. The doubling condition has some important consequences in that the measure is "homogeneously" distributed in the space. We summarize these properties in the proposition below. We note that these properties are classical and refer the reader to [Ste70, Chapter 8.6] for more details.

Proposition 3.7. *Let μ be a (not identically zero) locally finite, non-negative Borel measure. Assume that μ is doubling with respect to some family $\mathfrak{R}$ consisting of all the homothetic copies of a fixed rectangle. The following properties are satisfied.*

(i) We have $\mu(U) > 0$ for every open set $U \subset \mathbb{R}^n$.

(ii) Let $R \in \mathfrak{R}$ and $\mathscr{D}_R$ be the dyadic grid generated by R. There exists a constant $\gamma_\mu > 1$, depending only on the doubling constant of μ and the dimension n such that $\mu(R) \leqslant \gamma_\mu^{-m}\mu(R^{(m)})$, where $R^{(m)}$ is the ancestor of R, m generations higher. In particular $\mu(\mathbb{R}^n) = +\infty$.

(iii) The maximal operator $M^{\mu}_{\mathfrak{K}}$ *is of weak type* $(1,1)$ *and strong type* (p,p) *for all* $1 < p \leqslant \infty$*, with respect to* μ*, and the operator norms depend only on the doubling constant of the measure* μ*, the exponent* p *and the dimension* n*. Also the centered maximal operator* $M^{\mu}_{\mathfrak{K},c}$ *satisfies the same bounds.*

(iv) If B *is a convex set in* $\mathbb{R}^n$ *we have* $\mu(\partial B) = 0$ *where* $\partial B := \bar{B} \setminus B$ *is the boundary of* B*.*

Proof. We first introduce some notation related to dyadic rectangles that will be repeatedly used in the rest of this chapter. For a rectangle $R \in \mathfrak{R}$ we denote by $\mathscr{D}_R$ the mesh of "dyadic rectangles" associated to R. The "dyadic children" of R are produced by dividing each side of R into two equal parts while the dyadic parent of R is the rectangle $R^{(1)}$ whose sidelengths are double the corresponding sidelengths of R and shares exactly one corner with R. Thus every $R \subset \mathbb{R}^n$ has exactly 2^n dyadic children and is contained in a unique dyadic parent. For a dyadic rectangle R we write $R^{(1)}$ for the parent of R and $R^{(j)}$ for the ancestor of R which is j generations "before" R.

The proof of (i) can be found for example in [Ste70, Chapter 8]. For (ii) let $R^{(1)}$ be the dyadic parent of R and let $\{R_j\}_{j=1}^{2^n}$ denote the dyadic children of $R^{(1)}$ and suppose that $R = R_1$. Then

$$\mu(R^{(1)}) = \sum_{j=1}^{2^n} \mu(R_j) = \mu(R_1) + \sum_{j=2}^{2^n} \mu(R_j) \geqslant (1 + (2^n - 1)\delta_\mu^{-1})\mu(R_1),$$

where $\delta_\mu > 1$ is the doubling constant of μ. Let $\gamma_\mu = 1 + (2^n - 1)\delta_\mu^{-1} > 1$. Since R is m generations inside $R^{(m)}$ we iterate to get $\mu(R) \leqslant \gamma_\mu^{-m}\mu(R^{(m)})$ as desired.

For (iii) observe that $M^{\mu}_{\mathfrak{K}}$ is essentially the Hardy-Littlewood maximal operator with respect to a doubling measure and the result is classical. Since the measure μ is doubling the operators $M^{\mu}_{\mathfrak{K}}, M^{\mu}_{\mathfrak{K},c}$ are pointwise comparable and satisfy the same bounds.

Finally for (iv) let us fix the convex set B and $x \in \partial B$. Let H be a supporting hyperplane of B through x and let H^- be the open half-space defined by H so that $H^- \cap B = \emptyset$. Let $R \in \mathfrak{K}$, centered at x and sR be the rectangle with the same center as R and sides $s < 1$ times the corresponding sides of R. So sR is an homothetic copy of R. Consider the 4^n subrectangles $R_{s,j}$ produced by dividing each side of sR into four equal parts. Now at least one of these $R_{s,j}$'s is contained in the open half space H^-. Let us call this rectangle R' and observe that it is of the form $R' = z + \frac{1}{4}sR \subset sR$ and $R' \cap \overline{B} = \emptyset$. We can then estimate

$$\begin{aligned}
\mu(\partial B \cap sR) &= \mu(\partial B \cap sR \cap R') + \mu(\partial B \cap sR \setminus R') \\
&= \mu(\partial B \cap sR \setminus R') \leqslant \mu(sR) - \mu(R') \\
&\leqslant \mu(sR) - \frac{1}{\delta_\mu^2}\mu(sR) \leqslant c\mu(sR),
\end{aligned}$$

with $c < 1$. Applying (iii) for the centered operator $M^{\mu}_{\mathfrak{K},c}$ we see that

$$1 > c \geqslant \mu(\partial B \cap sR)/\mu(sR) \to \mathbf{1}_{\partial B},\ \mu\text{-almost everywhere as } s \to 0^+,$$

which implies that $\mu(\partial B) = 0$. □

3.3 Proof of the Theorem 3.1

In this section we give the details of the proof of Theorem 3.1. First of all observe that if $M^{\mu}_{\mathfrak{G}} : L^p(\nu) \to L^p(\nu)$ then trivially ($A^{\mu}_{\mathfrak{G},\gamma,\nu}$) is satisfied for every $\gamma \in (0,1)$. For the rest of this section we will thus assume that ($A^{\mu}_{\mathfrak{G},\gamma,\nu}$) holds for some $\gamma \in (0,1)$. Let $\beta \in (\gamma, 1)$. Any such choice of β will work equally well but for definitiveness we can take β to be the arithmetic mean of γ and 1. The hypothesis implies that

$$\nu(\{x \in \mathbb{R}^n : M^{\mu}_{\mathfrak{G}}(\mathbf{1}_E)(x) \geqslant \beta\}) \leqslant \mathbf{c}\nu(E) \quad \text{for all measurable sets} \quad E \subseteq \mathbb{R}^n. \tag{3.1}$$

Here $\mathbf{c} = c^{\mu}_{\mathfrak{G},\gamma,\nu}$ but we suppress these dependencies for the sake of simplicity. We will need the following notation introduced in [HS09]. For every measurable set $E \subset \mathbb{R}^n$ we define $\mathscr{H}^0_{\beta}(E) := E$ and for $k \geqslant 1$

$$\mathscr{H}^k_{\beta}(E) := \{x \in \mathbb{R}^n : M^{\mu}_{\mathfrak{G}}(\mathbf{1}_{\mathscr{H}^{k-1}_{\beta}(E)})(x) \geqslant \beta\}.$$

With these definitions at hand it is not difficult to check the following basic properties. Let $k, k' \geqslant 0$ be non-negative integers and A, B measurable subsets of $\mathbb{R}^n$. Then

$$\mathscr{H}^1_{\beta}(\mathscr{H}^k_{\beta}(A)) = \mathscr{H}^{k+1}_{\beta}(A), \tag{3.2}$$

$$A \subseteq B \Rightarrow \mathscr{H}^k_{\beta}(A) \subseteq H^k_{\beta}(B), \tag{3.3}$$

$$\text{If } k' \leqslant k \text{ then } \mathscr{H}^{k'}_{\beta}(A) \subseteq \mathscr{H}^k_{\beta}(A). \tag{3.4}$$

$$(A^{\mu}_{\mathfrak{G},\gamma,\mu}) \text{ implies (3.1) which in turn implies that } \nu(\mathscr{H}^k_{\beta}(A)) \leqslant \mathbf{c}^k\nu(A). \tag{3.5}$$

The properties above will be used in several parts of the proof with no particular mention.

The following lemma is the heart of the proof of Theorem 3.1.

Lemma 3.8. *Let μ be a doubling measure with respect to $\mathfrak{G}$, with doubling constant Δ_μ, and E be a measurable set in $\mathbb{R}^n$. Suppose that for some $\alpha \in (0,\beta)$ and $R \in \mathfrak{G}$ we have $\frac{1}{\mu(R)}\int_R 1_E d\mu = \alpha$. Then*

$$R \subset \mathscr{H}^{k_{\alpha,\beta}}_{\beta}(E) \quad \textit{where} \quad k_{\alpha,\beta} := \left\lceil \frac{-\log(\frac{\beta}{\alpha})}{\log\beta} \right\rceil \left\lceil 2 + \frac{\log^+(\beta\Delta_\mu))}{\log(1/\beta)} \right\rceil + 1.$$

Here we denote by $\lceil x \rceil$ the smallest positive integer which is no less than x.

Before giving the proof of the lemma let us see how we can use it to conclude the proof of Theorem 3.1. By restricted weak type interpolation it suffices to show that for every $0 < \lambda < 1$ and every measurable set $E \subset \mathbb{R}^n$ we have the estimate

$$\nu(\{x \in \mathbb{R}^n : M^{\mu}_{\mathfrak{G}}(\mathbf{1}_E)(x) > \lambda\}) \leqslant \frac{C}{\lambda^{p_o}}\nu(E) \tag{3.6}$$

for some $p_o > 1$ and some constant $C > 0$, independent of λ and E. Estimate (3.6) above is the claim that the sublinear operator $M^{\mu}_{\mathfrak{G}}$ is of restricted weak type (p_o, p_o) with respect to the measure ν, for some $p_o > 1$. Now we have

$$\begin{aligned}\nu(\{x \in \mathbb{R}^n : M^{\mu}_{\mathfrak{G}}(\mathbf{1}_E)(x) > \lambda\}) &\leqslant \nu(\{x \in \mathbb{R}^n : \lambda < M^{\mu}_{\mathfrak{G}}(\mathbf{1}_E)(x) < \beta\}) \\ &\quad + \nu(\{x \in \mathbb{R}^n : M^{\mu}_{\mathfrak{G}}(\mathbf{1}_E)(x) \geqslant \beta\}) \\ &\leqslant \nu(\{x \in \mathbb{R}^n : \lambda < M^{\mu}_{\mathfrak{G}}(\mathbf{1}_E)(x) < \beta\}) + \frac{\mathbf{c}}{\lambda^{p_o}}\nu(E),\end{aligned} \tag{3.7}$$

by (3.1), for all $p_o > 0$. In order to estimate the first summand let

$$E_{\lambda,\beta} \coloneqq \{\lambda < M^{\mu}_{\mathfrak{G}}(\mathbf{1}_E)(x) < \beta\}.$$

For every $x \in E_{\lambda,\beta}$ there exists $R_x \in \mathfrak{G}$ and $\lambda < \alpha < \beta$ with

$$R_x \ni x, \ \mu(R_x) > 0 \quad \text{and} \quad \frac{\mu(R_x \cap E)}{\mu(R_x)} = \alpha.$$

By Lemma 3.8 we get that $R_x \subset \mathscr{H}^{k_{\alpha,\beta}}_{\beta}(E)$. Now observe that $k_{\alpha,\beta}$ is a nonincreasing function of α. Thus for all $\alpha > \lambda$ we have that $k_{\alpha,\beta} \leqslant k_{\lambda,\beta}$ which by (3.4) implies that $\mathscr{H}^{k_{\alpha,\beta}}_{\beta}(E) \subseteq \mathscr{H}^{k_{\lambda,\beta}}_{\beta}(E)$. Combining these observations we get that

$$E_{\lambda,\beta} \subseteq \bigcup_{x \in E_{\lambda,\beta}} R_x \subseteq \mathscr{H}^{k_{\lambda,\beta}}_{\beta}(E).$$

Using (3.5) we now see that

$$\nu(E_{\lambda,\beta}) \leqslant \nu(\mathscr{H}^{k_{\lambda,\beta}}_{\beta}(E)) \leqslant \mathbf{c}^{k_{\lambda,\beta}}\nu(E).$$

By the explicit expression for $k_{\lambda,\beta}$ observe that we can write

$$k_{\lambda,\beta} \leqslant \frac{\log(\frac{\beta}{\lambda})}{\log\frac{1}{\beta}}\eta_{\beta,\mu} + 1$$

with $\eta_{\beta,\mu} \geqslant 2$, depending only on β and μ. Thus

$$\mathbf{c}^{k_{\lambda,\beta}} \leqslant \mathbf{c}\mathbf{c}^{\eta_{\beta,\mu}\log\frac{1}{\lambda}/\log\frac{1}{\beta}} \leqslant \frac{\mathbf{c}}{\lambda^{p_o}} = \frac{c^{\mu}_{\mathfrak{G},\gamma,\nu}}{\lambda^{p_o}},$$

with $p_o = \eta_{\beta,\mu}{}^{\log c^{\mu}_{\mathfrak{G},\gamma,\nu}/\log(1/\beta)} > 0$. Remember that β is completely determined by the level γ in hypothesis $(A^{\mu}_{\mathfrak{G},\gamma,\mu})$ so that $p_o = p_o(c^{\mu}_{\mathfrak{G},\gamma,\nu},\gamma,\mu)$. Together with (3.7) this completes the proof of (3.6) and thus of Theorem 3.1.

For the proof of Lemma 3.8 we will need an intermediate result. For this we introduce a final piece of notation. If $R \in \mathfrak{G}$ then there is a natural "dyadic system of rectangles" associated to R which we will denote by $\mathscr{D}_R$. This system has the properties

(i) We have that $R \in \mathscr{D}_R \subseteq \mathfrak{G}$.

(ii) Every $S \in \mathscr{D}_R$ has a unique dyadic parent $S^{(1)}$ and 2^n dyadic children. Furthermore, each corner of a rectangle $S \in \mathscr{D}_R$ is shared by S and exactly one of its dyadic children.

(iii) If $V, S \in \mathscr{D}_R$ then $V \cap S \in \{\emptyset, V, S\}$.

We leave the details of the dyadic construction above to the interested reader. In the case where the rectangles have sides parallel to the coordinate axes, this construction is described in detail in [Kor07, chapter 1]. We now define the dyadic weighted maximal function with respect to $\mathscr{D}_R$ and μ as

$$M^{\mu}_{\mathscr{D}_R} f(x) := \sup_{\substack{S\in\mathscr{D}_R\\ S\ni x\\ \mu(S)>0}} \frac{1}{\mu(S)}\int_S |f(y)|d\mu(y), \quad x\in\mathbb{R}^n.$$

The dyadic maximal function just defined satisfies all the desired bounds:

Proposition 3.9. *Let μ be a locally finite non-negative measure. We have that $M^{\mu}_{\mathscr{D}_R}$: $L^1(\mu)\to L^{1,\infty}(\mu)$. We conclude that the family $\{R : R\in\mathscr{D}_R, R\ni x, \mu(R)>0\}$ differentiates $L^1_{\mathrm{loc}}(\mu)$.*

Note that there is no doubling assumption on the measure μ in this proposition. Indeed, the proof amounts to selecting the maximal "dyadic rectangles" $S\in\mathscr{D}_R\cap[0,2^N)^n$ such that $\frac{1}{\mu(S)}\int_S |f(y)|d\mu(y)>\lambda$ and noting that they are disjoint. One then lets $N\to+\infty$. An identical argument works for "dyadic rectangles" contained in the other quadrants of $\mathbb{R}^n$. We omit the easy details of the proof.

Lemma 3.10. *Let μ, E and R be as in the hypothesis of Lemma 3.8 above. Then there exists a non-negative integer N such that*

$$\mu(R\cap\mathscr{H}^{N+2}_{\beta}(E)) \geqslant \frac{1}{\beta}\mu(E\cap R).$$

Proof. We perform a Calderón-Zygmund decomposition of $\mathbf{1}_{E\cap R}$ at level β with respect to the dyadic grid $\mathscr{D}_R$ as defined in Proposition 3.7. Namely, let $\{S_j\}_j\subset\mathscr{D}_R$ be the collection of "dyadic rectangles" which are maximal among the $S\in\mathscr{D}_R$ that satisfy

$$\frac{1}{\mu(S)}\int_S \mathbf{1}_{E\cap R}(y)d\mu(y)>\beta.$$

Observe that $\mu(S)>0$ for all rectangles S by Proposition 3.7. Furthermore $\mu(E\cap R)/\mu(R)<\beta$ so that every dyadic rectangle S as above is contained in a maximal dyadic rectangle. This selection algorithm together with the hypothesis $\mu(R\cap E)/\mu(R)=\alpha<\beta$ allows us to choose a μ-a.e. disjoint family $\{S_j\}_j\subset\mathscr{D}_R$ such that

$$\begin{aligned}
&\bigcup_j S_j\subseteq R, \quad S_j\neq R \text{ for all } j,\\
&\{x\in\mathbb{R}^n : M^{\mu}_{\mathscr{D}_R}(\mathbf{1}_{E\cap R})(x)>\beta\}=\bigcup_j S_j, && (3.8)\\
&\frac{1}{\mu(S_j)}\int_{S_j}\mathbf{1}_{E\cap R}d\mu>\beta,\\
&\mathbf{1}_{E\cap R}\leqslant\mathbf{1}_{\cup_j S_j} \quad \mu\text{-a.e. in} \quad R. && (3.9)
\end{aligned}$$

For any constant $c>1$ we let $c*S_j$ denote the rectangle containing S_j that has sidelength c times the sidelength of S_j and has a common corner with S_j and $S_j^{(1)}$. With this notation we have $S_j^{(1)} = 2 * S_j$ while the doubling hypothesis for μ implies that $\mu(S_j^{(1)}) \leqslant \Delta_\mu \mu(S_j)$ for every j.

For each j we set $S_{j,0} \coloneqq S_j$. Suppose we have defined $S_{j,0} \subset \cdots \subset S_{j,k}$ for some $k \geqslant 0$. We define $S_{j,k+1}$ to be a rectangle of the form $c_{j,k+1} * S_j$, where $c_{j,k+1} > 1$ is chosen so that $S_{j,k} \subset S_{j,k+1}$ and

$$\frac{\mu(S_{j,k+1})}{\mu(S_{j,k})} = \frac{1}{\beta} > 1. \tag{3.10}$$

Observe that such a choice is always possible since the function $f(c) \coloneqq \mu(c * S_{j,k})/\mu(S_{j,k})$ satisfies $f(1) = 1$, $f(c) \to +\infty$ as $c \to +\infty$ and by (iv) of Proposition 3.7 it is continuous on $[1, +\infty)$.

For $k \geqslant 0$ we now set

$$E_k \coloneqq \bigcup_j S_{j,k}.$$

Observe that for $k \geqslant 0$ we have

$$E_{k+1} \subset \{x \in \mathbb{R}^n : M^\mu_{\mathfrak{S}}(\mathbf{1}_{E_k})(x) \geqslant \beta\}. \tag{3.11}$$

Indeed if $x \in E_{k+1}$ then $x \in S_{j_0,k+1}$ for some j_0. We estimate

$$M^\mu_{\mathfrak{S}}(\mathbf{1}_{E_k})(x) = \sup_{\substack{S \in \mathfrak{S} \\ S \ni x}} \frac{\mu(S \cap E_k)}{\mu(S)} \geqslant \frac{\mu\big(S_{j_0,k+1} \cap \bigcup_j S_{j,k}\big)}{\mu(S_{j_0,k+1})} \geqslant \frac{\mu\big(S_{j_0,k}\big)}{\mu(S_{j_0,k+1})} = \beta,$$

by (3.10). Next we claim that for every $k \geqslant 0$ we have

$$E_k \subset \mathscr{H}_\beta^{k+1}(E). \tag{3.12}$$

For $k = 0$ this is an immediate consequence of (3.8) since

$$\begin{aligned} E_0 = \bigcup_j S_j &= \{x \in \mathbb{R}^n : M^\mu_{\mathscr{D}_R}(\mathbf{1}_{E \cap R})(x) > \beta\} \\ &\subseteq \{x \in \mathbb{R}^n : M^\mu_{\mathfrak{S}}(\mathbf{1}_E)(x) \geqslant \beta\} = \mathscr{H}^1_\beta(E). \end{aligned}$$

Assume now that (3.12) is valid for some $k \geqslant 0$. By (3.11), the inductive hypothesis and properties (3.2),(3.3) we get that

$$E_{k+1} \subset \{x \in \mathbb{R}^n : M^\mu_{\mathfrak{S}}(\mathbf{1}_{E_k})(x) \geqslant \beta\} = \mathscr{H}^1_\beta(E_k) \subseteq \mathscr{H}^1_\beta(\mathscr{H}^{k+1}_\beta(E)) = \mathscr{H}^{k+2}_\beta(E),$$

which proves the claim.

Now let N be the smallest non-negative integer such that $\beta^{-(N+1)} \geqslant \Delta_\mu$, where Δ_μ is the doubling constant of the measure μ. It follows that

$$S_j^{(1)} \subseteq S_{j,N+1} \tag{3.13}$$

for every j. Indeed, assume for the sake of contradiction that $S_{j,N+1} \subsetneq S_j^{(1)}$. Then the doubling property of μ implies that $\mu(S_{j,N+1}) < \mu(S_j^{(1)})$. Thus

$$\Delta_\mu \geqslant \frac{\mu(S_j^{(1)})}{\mu(S_j)} > \frac{\mu(S_{j,N+1})}{\mu(S_j)} = \beta^{-(N+1)}$$

which contradicts the choice of N.

Now (3.13) implies that for every j we have

$$\frac{\mu(S_{j,N})}{\mu(S_j^{(1)})} \geqslant \frac{\mu(S_{j,N})}{\mu(S_{j,N+1})} = \beta$$

and we can conclude that for every j

$$\frac{\mu(E_N \cap S_j^{(1)})}{\mu(S_j^{(1)})} = \frac{\mu\big(\bigcup_\nu S_{\nu,N} \cap S_j^{(1)}\big)}{\mu(S_j^{(1)})} \geqslant \frac{\mu\big(S_{j,N} \cap S_j^{(1)}\big)}{\mu(S_j^{(1)})} \geqslant \min\left(1, \frac{\mu(S_{j,N})}{\mu(S_j^{(1)})}\right) \geqslant \beta.$$

Hence

$$\bigcup_j S_j^{(1)} \subseteq \{x \in R : M^\mu_{\mathfrak{G}}(\mathbf{1}_{E_N})(x) \geqslant \beta\}. \tag{3.14}$$

Let $\{S_{j_k}^{(1)}\}_k$ denote the maximal elements of $\{S_j^{(1)}\}_j$. Then the sets $\{S_{j_k}^{(1)}\}_k$ are μ-a.e. pairwise disjoint and $\bigcup_k S_{j_k}^{(1)} = \bigcup_j S_j^{(1)}$. Note that all $S_{j_k}^{(1)}$'s are contained in R since for all j we have $S_j \subsetneqq R$. We also have that we have $S_{j_k}^{(1)} \neq S_m$ for any k, m. Indeed, if $S_{j_k}^{(1)} = S_m$ for some k, m then we would have $S_{j_k}^{(1)} \subsetneqq S_m^{(1)}$ which is impossible because of the maximality of the $S_{j_k}^{(1)}$'s among the $S_m^{(1)}$'s. Thus none of the $S_{j_k}^{(1)}$ were selected in the Calderón-Zygmund decomposition so that

$$\mu(S_{j_k}^{(1)} \cap E \cap R) \leqslant \beta\mu(S_{j_k}^{(1)})$$

and hence $\mu(S_{j_k}^{(1)} \cap E) \leqslant \beta\mu(S_{j_k}^{(1)})$ for all k since $S_{j_k}^{(1)} \subseteq R$ for all k. Using the last estimate and (3.14) we now have

$$\begin{aligned}
\mu(\{x \in R : M^\mu_{\mathfrak{G}}(\mathbf{1}_{E_N})(x) \geqslant \beta\}) &\geqslant \mu\Big(\bigcup_j S_j^{(1)}\Big) = \mu\Big(\bigcup_k S_{k_j}^{(1)}\Big) \\
&= \sum_k \mu(S_{k_j}^{(1)}) \geqslant \frac{1}{\beta} \sum_k \mu(E \cap S_{k_j}^{(1)}) \\
&= \frac{1}{\beta}\mu\Big(E \cap \bigcup_k S_{k_j}^{(1)}\Big) = \frac{1}{\beta}\mu\Big(E \cap \bigcup_j S_j^{(1)}\Big) \\
&\geqslant \frac{1}{\beta}\mu\Big(E \cap \bigcup_j S_j\Big).
\end{aligned}$$

Now (3.9) implies that $\mathbf{1}_{E \cap R} \leqslant \mathbf{1}_{R \cap \cup_j S_j}$ almost everywhere so that $\mu(E \cap R) \leqslant \mu(R \cap \cup_j S_j)$.

Thus the previous estimate reads

$$\mu(\{x \in R : M^{\mu}_{\mathfrak{G}}(\mathbf{1}_{E_N})(x) \geqslant \beta\}) \geqslant \frac{1}{\beta}\mu(E \cap R)$$

which by (3.12) implies that $\mu(R \cap \mathscr{H}_{\beta}^{N+2}(E)) \geqslant \beta^{-1}\mu(E \cap R)$ as desired. □

We can now conclude the proof of Lemma 3.8.

Proof of Lemma 3.8. By the hypothesis of the lemma there exists $\alpha \in (0, \beta)$ and $R \in \mathfrak{G}$ with $\mu(E \cap R)/\mu(R) = \alpha$. Let j_o be the smallest positive integer such that $\beta^{-j_o}\alpha \geqslant \beta$. Such an integer obviously exists since $\beta < 1$. There are two possibilities.

case 1: We have that $\mu(R \cap \mathscr{H}_{\beta}^{j(N+2)}(E)) < \beta\mu(R)$ for $j = 0, \ldots, j_o - 1$. Then we claim that we have

$$\mu(R \cap \mathscr{H}_{\beta}^{k(N+2)}(E)) \geqslant \frac{1}{\beta^k}\mu(R \cap E) \quad \text{for all} \quad k = 1, \ldots, j_o. \tag{3.15}$$

We will prove (3.15) by induction on k. Indeed, the case $k = 1$ is just Lemma 3.10. Assume that (3.15) is true for some $1 \leqslant k \leqslant j_o - 1$. Then, since $\mu(R \cap \mathscr{H}_{\beta}^{k(N+2)}(E)) < \beta\mu(R)$ we can apply Lemma 3.10 for the rectangle R and the set $H_{\beta}^{k(N+2)}(E)$ in place of E to conclude that

$$\begin{aligned}
\mu(R \cap \mathscr{H}_{\beta}^{N+2}(\mathscr{H}_{\beta}^{k(N+2)}(E))) &\geqslant \frac{1}{\beta}\mu\big(\mathscr{H}_{\beta}^{k(N+2)}(E) \cap R\big) \\
&\geqslant \frac{1}{\beta}\big(\frac{1}{\beta}\big)^k\mu(R \cap E) \\
&= \big(\frac{1}{\beta}\big)^{k+1}\mu(R \cap E).
\end{aligned}$$

However this is just (3.15) for $k+1$ since $\mathscr{H}_{\beta}^{N+2}(\mathscr{H}_{\beta}^{k(N+2)}(E)) = \mathscr{H}_{\beta}^{(k+1)(N+2)}(E)$.

Now by (3.15) for $k = j_o$ we get that

$$\frac{1}{\mu(R)}\mu\big(R \cap \mathscr{H}_{\beta}^{j_o(N+2)}(E)\big) \geqslant \big(\frac{1}{\beta}\big)^{j_o}\frac{\mu(R \cap E)}{\mu(R)} = \beta^{-j_o}\alpha \geqslant \beta$$

by the choice of j_o. This implies that $R \subseteq \mathscr{H}_{\beta}^{j_o(N+2)+1}(E)$.

case 2: We have that $\mu(R \cap \mathscr{H}_{\beta}^{j(N+2)}(E)) \geqslant \beta\mu(R)$ for some $j \in \{0, \ldots, j_o - 1\}$. In fact, by the hypothesis we necessarily have that $j \geqslant 1$ in this case. Then

$$\frac{1}{\mu(R)}\mu\big(R \cap \mathscr{H}_{\beta}^{j(N+2)}(E)\big) \geqslant \beta$$

which implies that $R \subseteq \{x \in \mathbb{R}^n : M^{\mu}_{\mathfrak{G}}(\mathbf{1}_{\mathscr{H}_{\beta}^{j(N+2)}(E)})(x) \geqslant \beta\} = \mathscr{H}_{\beta}^{j(N+2)+1}(E)$.

Observe that in either one of the complementary cases considered above we can conclude that $R \subseteq \mathscr{H}_{\beta}^{j_o(N+2)+1}(E)$. This proves the lemma with $k_{\alpha,\beta} = j_o(N+2)+1$. It remains to estimate $k_{\alpha,\beta}$. This can be easily done by going back to the way the integers N and

j_o were chosen. For N remember that it is the smallest non-negative integer such that $(1/\beta)^{N+1} \geqslant \Delta_\mu$. If $1/\beta \geqslant \Delta_\mu$ then the choice $N = 0$ will do. If $1/\beta < \Delta_\mu$ then we get that N is the smallest positive integer which is greater or equal to $\log(\beta\Delta_\mu)/\log(1/\beta)$. Thus the choice

$$N \coloneqq \left\lceil \frac{\log^+(\beta\Delta_\mu)}{\log(1/\beta)} \right\rceil$$

covers both cases. Likewise, j_o is the smallest integer such that $\beta^{-j_o} \geqslant \beta/\alpha$ or j_o is the smallest integer greater than $\log(\beta/\alpha)/\log(1/\beta)$. Thus we can choose

$$j_o \coloneqq \left\lceil \frac{\log(\frac{\beta}{\alpha})}{\log\frac{1}{\beta}} \right\rceil. \tag{3.16}$$

We set

$$\begin{aligned} k_{\alpha,\beta} \coloneqq j_o(N+2)+1 &= \left\lceil \frac{\log(\frac{\beta}{\alpha})}{\log\frac{1}{\beta}} \right\rceil \left(\left\lceil \frac{\log^+(\beta\Delta_\mu)}{\log(1/\beta)} \right\rceil + 2 \right) + 1 \\ &= \left\lceil \frac{\log(\frac{\beta}{\alpha})}{\log\frac{1}{\beta}} \right\rceil \left\lceil 2 + \frac{\log^+(\beta\Delta_\mu)}{\log(1/\beta)} \right\rceil + 1. \end{aligned}$$

Of course, any integer greater than the $k_{\alpha,\beta}$ above will also do since the sets $\mathscr{H}^k_\beta(E)$ are increasing in k. $\square$

3.4 Some remarks on Theorem 3.1

We now discuss some consequences of Theorem 3.1 and related issues.

Exentension of Theorem 3.1 to bases of convex sets Theorem 3.1 can be extended to the case that the Tauberian condition is given with respect to a homothecy invariant basis $\mathfrak{B}$ consisting of convex sets; that is,

$$\nu(\{x \in \mathbb{R}^n : M^\mu_{\mathfrak{B}}(\mathbf{1}_E)(x) > \gamma\}) \leqslant c^\mu_{\mathfrak{B},\gamma,\nu}\nu(E). \tag{$A^\mu_{\mathfrak{B},\gamma,\nu}$}$$

As in the case where the basis $\mathfrak{G}$ consisted of rectangles, we will need to assume the doubling property of the measure μ with respect to the basis $\mathfrak{B}$. In particular, the result that we obtain is the following.

Theorem 3.11. *Let $\mathfrak{B}$ be a homothecy invariant basis consisting of convex sets and μ, ν be two non-negative measures on $\mathbb{R}^n$, finite on compact sets. Assume that μ is doubling with respect to $\mathfrak{B}$. The following are equivalent:*

(i) The measures μ, ν satisfy the Tauberian condition ($A^\mu_{\mathfrak{B},\gamma,\nu}$) with respect to some fixed level $\gamma \in (0,1)$.

(ii) There exists $1 < p_o = p_o(c^\mu_{\mathfrak{B},\gamma,\nu}, n, \gamma, \mu) < +\infty$ such that $M^\mu_{\mathfrak{B}} : L^p(\nu) \to L^p(\nu)$ for all $p > p_o$.

Though we omit the proof of the theorem in this thesis, the general strategy that we pursue is the following. Assuming that ($A^{\mu}_{\mathfrak{B},\gamma,\nu}$) is satisfied for some level $\gamma \in (0,1)$ we show that the maximal operator $M^{\mu}_{\mathfrak{G}_{\mathfrak{B}}}$ also satisfies a Tauberian condition with respect to every level $\alpha \in (\gamma,1)$. This is the more technical and difficult part. We then use Theorem 3.1 to conclude that $M^{\mu}_{\mathfrak{G}_{\mathfrak{B}}}$ is bounded on some $L^p(\nu)$-space, for sufficiently large p. According to Lemma 3.6 the operators $M^{\mu}_{\mathfrak{G}_{\mathfrak{B}}}$ and $M^{\mu}_{\mathfrak{B}}$ are pointwise comparable so this completes the proof.

Weighted Tauberian conditions. Let w be a weight. When $\mu \equiv \nu \equiv w$ the mixed Tauberian condition ($A^{\mu}_{\mathfrak{G},\gamma,\nu}$) becomes the weighted Tauberian condition

$$w(\{x \in \mathbb{R}^n : M_{\mathfrak{B}}(\mathbf{1}_E)(x) > \gamma\}) \leqslant c_{\mathfrak{B},\gamma,w}\, w(E) \qquad (A_{\mathfrak{B},\gamma,w})$$

that was introduced in Definition 1.21. This condition has been considered many times in the literature, especially in the context of weighted inequalities for the strong maximal function and other Muckenhoupt bases. Indeed, it appears for example in [GLPT11], [Jaw86], [JT84], [LL], [Pér91] and [Pér93]. Condition ($A_{\mathfrak{B},\gamma,w}$) is typically presented in the literature as a presumably weaker substitute for the hypothesis $w \in A_{\infty,\mathfrak{B}}$ and is usually referred to as condition (A). For example, in [JT84] condition ($A_{\mathfrak{B},\gamma,w}$) is used as a hypothesis in order to prove $L^p(w)$-bounds for the weighted strong maximal function M^w_s. Likewise, in [Pér91] it is shown that if $\mathfrak{B}$ is a Muckenhoupt basis and $M_{\mathfrak{B}}$ satisfies ($A_{\mathfrak{B},\gamma,w}$) for a fixed $\gamma \in (0,1)$ then $M_{\mathfrak{B}}$ satisfies a Fefferman-Stein inequality, namely

$$\int_{\mathbb{R}^n} M_{\mathfrak{B}} f(x)^p w(x)dx \lesssim_{n,p,w} \int_{\mathbb{R}^n} |f(x)|^p M_{\mathfrak{B}} w(x)dx, \quad 1 < p < +\infty.$$

Finally, in [GLPT11] and [LL], the condition is used in order to deal with covering properties of rectangles which are relevant in the study of two weight problems for the strong maximal function. The following theorem shows however that condition ($A_{\mathfrak{B},\gamma,w}$) is just an equivalent characterization of $A_{\infty,\mathfrak{B}}$ for quite a large class of bases $\mathfrak{B}$. Observe first the next general result.

Theorem 3.12. *Let $\mathfrak{B}$ be a homothecy invariant basis consisting of convex sets. Let w be a non-negative, locally integrable function on $\mathbb{R}^n$. Then the following are equivalent:*

(i) Condition ($A_{\mathfrak{B},\gamma,w}$) is satisfied for the weight w and the basis $\mathfrak{B}$, for a fixed level $\gamma \in (0,1)$.

(ii) There exists $1 < p_o = p_o(c_{\mathfrak{B},\gamma,w}, \gamma, n) < +\infty$ such that $M_{\mathfrak{B}} : L^p(w) \to L^p(w)$ for all $p > p_o$.

Proof. It is trivial that (ii) implies (i). The converse implication for $w \equiv 1$ is essentially [HS09, Theorem 1]. Now a careful inspection of the proofs of the relevant results in [HS09] reveals that the arguments therein actually show that if $M_{\mathfrak{B}}$ and w satisfy ($A_{\mathfrak{B},\gamma,w}$) then $M_{\mathfrak{B}}$ is of restricted type (q,q) for some $q > 1$, with respect to the weight w. Marcinkiewicz interpolation now gives (ii) for any $p > q$. Alternatively, the proof follows from Theorem 3.11. □

It is important to note here that if the basis $\mathfrak{B}$ is additionally a Muckenhoupt basis Theorem 3.12 immediately gives Remark 1.27; namely, that conditions ($A_{\mathfrak{B},\gamma,w}$) and $A_{\infty,\mathfrak{B}}$ are actually equivalent whenever B is a Muckenhoupt basis.

Differentation theory A dual point of view for the investigations presented in this chapter can be given in the language of differentiation theory. Given a collection of convex sets in $\mathbb{R}^n$ which is invariant under dilations and translations we want to study when the corresponding maximal operator differentiates $L^\infty(\mathbb{R}^n)$. Theorem 3.11 shows that the boundedness properties of quite general maximal operators can also be characterized in terms of Tauberian conditions. In turn, it is a classical result of Busemann and Feller, [BF34], that a homothecy invariant basis $\mathfrak{B}$ consisting of open sets differentiates $L^\infty(\mathbb{R}^n)$ if and only if the corresponding maximal operator $M_{\mathfrak{B}}$ satisfies a Tauberian condition

$$|\{x \in \mathbb{R}^n : M_{\mathfrak{B}}(\mathbf{1}_E)(x) > \gamma\}| \leqslant c_\gamma |E|,$$

for *every* $\gamma \in (0,1)$ and for every measurable set E. This point of view is discussed in detail in [dG76] and taken up in [HS09]. In the last work it is shown that a homothecy invariant basis consisting of convex sets differentiates $L^\infty(\mathbb{R}^n)$ (with respect to the Lebesgue measure) if and only if it differentiates $L^p(\mathbb{R}^n)$ for some sufficiently large $p > 1$. Note here that, lacking the convexity hypothesis on the basis, one needs Tauberian conditions at all levels $\gamma \in (0,1)$. This should be contrasted to the results in [HS09] as well as in the current chapter where the convexity assumption allows us to only assume a Tauberian condition at a *fixed level*.

A direct consequence of Theorem 3.11, is the following corollary that is somehow a "weighted" version of the Busemann-Feller theorem.

Corollary 3.13. *Let $\mathfrak{B}$ be a homothecy invariant basis consisting of convex sets and μ, ν be locally finite, non-negative measures on $\mathbb{R}^n$. Assume in addition that μ is doubling with respect to $\mathfrak{B}$. If the condition ($A^{\mu}_{\mathfrak{B},\gamma,\nu}$) is satisfied then $\mathfrak{B}$ differentiates $L^\infty(\nu)$ with respect to the measure μ.*

3.5 References

The results that we present here are completely contained in [HLP]. My initial motivation to study this problem was to understand the so-called (A) condition described in Section 3.4. After studying the paper by Hagelstein and Stokolos [HS09], we realized that the (A) condition was equivalent to the Muckenhoupt condition $A_{\infty,\mathfrak{B}}$ in the case of $\mathfrak{B}$ being a Muckenhoupt basis. Then we tried to extend all the results described in [HS09] to the more general context of weights, in part to get a better sense of the boundedness of the weighted strong maximal function M_s^w.

Chapter 4

Optimal exponents in weighted estimates

In this chapter we study optimal quantitative estimates for the $L^p(w)$-norm of some of the operators T we have defined in Chapter 1. We first recall Problem 0.4 and we state Theorem 4.2, that points out the close connection between the weighted estimates and the behaviour of the unweighted operator norm at the endpoints $p = 1$ and $p = \infty$. As an application of this theorem, we derive the sharpness of certain known weighted inequalities without building any specific example. Then, we study in detail the case that T is the strong maximal function and we describe some partial results we have obtained.

4.1 Background of Problem 0.4

A main problem in modern Harmonic Analysis is the study of sharp norm bounds for an operator T on weighted Lebesgue spaces. The usual examples include the operators we have introduced in Section 1.2; that is, maximal functions, Calderón-Zygmund operators and fractional integral operators. In Section 1.3 we saw that, for these operators, the weighted norm inequality

$$\|Tf\|_{L^p(w)} \lesssim_{n,p,w,T} \|f\|_{L^p(w)}, \quad 1 < p < +\infty, \tag{4.1}$$

is characterized in terms of the A_p classes of weights. See Theorem 1.16, Theorem 1.17 and Theorem 1.18 for more details. The question that we address in Problem 0.4 is the following:

Which is the precise dependance on the weight in estimate (4.1) for each T considered so far?

We can answer this question in two steps:

i. The first step is to look for quantitative bounds for the strong norm $\|T\|_{L^p(w)}$ and the weak norm $\|T\|_{L^p(w)\to L^{p,\infty}(w)}$ in terms of the A_p constant of the weight. It is well known that the dependance on the A_p constant, for the main examples, should be of the form:

$$\|T\|_{L^p(w)} \lesssim_{n,p,T} [w]_{A_p}^{\beta} \quad w \in A_p, \tag{4.2}$$

and

$$\|T\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim_{n,p,T} [w]_{A_p}^{\beta'} \quad w \in A_p. \tag{4.3}$$

This is frequently *the more difficult* step.

ii. The second step is to find the sharp dependence, typically with respect to the power of $[w]_{A_p}$; that is, what are the smallest possible exponents β, β' so that (4.2) and (4.3) hold uniformly for all weights w? This step has been traditionally *the easier* one, with relatively straightforward counterexamples. However, as we will describe in Section 4.3 and 4.4, in some cases it is the hard part of the argument.

In recent years, the techniques employed to solve Problem 0.4 for some of the classical operators T has resulted a development of new tools and methods in harmonic analysis. We now focus our attention on the *strong* estimate (4.2). This kind of inequality was first obtained for $T = M$ by Buckley, [Buc93], who proved the following quantitative estimate

$$\|M\|_{L^p(w)} \lesssim_{n,p} [w]_{A_p}^{\frac{1}{p-1}}, \qquad w \in A_p, \tag{4.4}$$

and $1/(p-1)$ cannot be replaced with any ϵ, $0 < \epsilon < 1/(p-1)$. Afterwards, Petermichl [Pet07] proved the celebrated A_2 conjecture for $T = H$. In particular, she showed that the Hilbert transform satisfies

$$\|H\|_{L^p(w)} \lesssim_{n,p} [w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}, \qquad w \in A_p \tag{4.5}$$

and is sharp. In each of these cases, the optimality of the exponent is proven by exhibiting specific examples adapted to the operator under analysis. In the case of Hilbert transform, the examples are specific for the range $1 < p < 2$ and then, the sharpness for large p is obtained by duality.

Similar weighted estimates are known to be true for other classical operators, such as commutators $[b, T]$ of Calderón-Zygmund operators T and BMO functions b, the dyadic square function S_d, vector valued maximal operators $\overline{M}_q$ for $1 \leqslant p, q \leqslant \infty$, Bochner-Riesz multipliers B^λ and fractional integrals I_α. In all these cases, the sharpness of the exponent β (step (ii) above) is always obtained by constructing specific examples for each operator.

One of the main purposes of this chapter is to present a different approach to test sharpness of weighted estimates of the form (4.2). In particular, we show that the sharpness is intimately related to the asymptotic behaviour of the unweighted L^p-norm of T as $p \to 1$ and $p \to \infty$. To introduce this result, we need to present the next definition which captures the endpoint asymptotic behaviour of $\|T\|_{L^p(\mathbb{R}^n)}$.

Definition 4.1. Given a bounded operator T on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, we define α_T to be the "endpoint order" of T as follows:

$$\alpha_T =: \sup\{\alpha \geqslant 0 : \forall \varepsilon > 0, \limsup_{p\to 1} (p-1)^{\alpha-\varepsilon} \|T\|_{L^p(\mathbb{R}^n)} = \infty\}. \tag{4.6}$$

The analogue of (4.6) for p large is the following. Let γ_T be defined as

$$\gamma_T =: \sup\{\gamma \geqslant 0 : \forall \varepsilon > 0, \limsup_{p\to\infty} \frac{\|T\|_{L^p(\mathbb{R}^n)}}{p^{\gamma-\varepsilon}} = \infty\}. \tag{4.7}$$

To illustrate this definition, consider for example $T = H$. Then, it is known that the size of its kernel (see 1.10), implies that the unweighted L^p norm satisfies

$$\|H\|_{L^p(\mathbb{R}^n)} \simeq \frac{1}{p-1} \tag{4.8}$$

as $p \to 1$. Indeed, if we consider the indicator function in the interval $[0,1]$, a simple calculation shows

$$H(\mathbf{1}_{[0,1]})(x) = \frac{1}{\pi} \log \frac{|x|}{|x-1|}.$$

Note that $H(\mathbf{1}_{[0,1]})(x)$ blows up logarithmically in x near the points 1 and 0 and decays like $|x|^{-1}$ outside the interval $(0,1)$. Thus

$$\|H\|_{L^p(\mathbb{R}^n)} \gtrsim \int_1^\infty |x|^{-p} dx$$

and (4.8) follows immediately. Therefore $\alpha_H = 1$. Moreover by duality

$$\|H\|_{L^p(\mathbb{R}^n)} \simeq p \tag{4.9}$$

as $p \to \infty$. Then $\gamma_H = 1$. The computation of the orders α_T and β_T for a general Calderón-Zygmund operator T follows similarly. See Section 4.3 below for a more complete discussion on this subject. Once we have presented this definition, we can state the following result.

Theorem 4.2. *Let T be an operator (not necessarily linear). Suppose further that for some $1 < p_0 < \infty$*

$$\|T\|_{L^{p_0}(w)} \lesssim_{n,p_0,T} [w]^{\beta}_{A_{p_0}} \qquad w \in A_{p_0}. \tag{4.10}$$

Then

$$\beta \geqslant \max\left\{\gamma_T; \frac{\alpha_T}{p_0 - 1}\right\}.$$

Indeed, the computation of estimates like (4.8) and (4.9) has already played a role in analysis. In this sense, we mention Yano's extrapolation argument [Yan51] (see also [dG81, p. 61, Theorem 3.5.1]), where this kind of estimate allows one to prove local endpoint boundedness properties for the operator T in appropriate $L \log L$ spaces or $\exp L$. Although Theorem 4.2 and Yano's extrapolation theorem are quite different, both share the intention of *extract* more information, than the initial provided, of the operator T.

This theorem presents a *standardized* answer to the second step (ii) we have stated above. In particular, if we apply this result to known weighted inequalities like (4.4) and (4.5), we derive their sharpness without building any particular example for each operator. Furthermore, this approach provides some lower bounds on what one might expect in new situations, where the construction of suitable counterexamples may be more tricky.

4.2 Rubio de Francia algorithm and proof of Theorem 4.2

The key ingredient to prove Theorem 4.2 is an application of the so called Rubio de Francia algorithm. This is a basic but powerful method that was fruitful since it was first applied to factorization of weights and extrapolation. A classic reference for an explanation of this iteration algorithm is [GCRdF85, Lemma 5.1]. In the proof we introduce two versions of the algorithm, both can be found in [CUMP11, Proof of Theorem 1.4].

Proof of Theorem 4.2. We first prove the bound $\beta \geqslant \frac{\alpha_T}{p_0-1}$. The first step is to prove the following inequality, which can be seen as an unweighted Coifman-Fefferman type inequality that relates the operator T to some maximal operator, namely the Hardy–Littlewood maximal operator M. We have that

$$\|T\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p_0,T} \|M\|_{L^p(\mathbb{R}^n)}^{\beta(p_0-p)} \qquad 1<p<p_0. \tag{4.11}$$

Lets start by defining, for $1<p<p_0$, the operator R as follows:

$$R(h) := \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{M^k(h)}{\|M\|_{L^p(\mathbb{R}^n)}^k},$$

where for $k \geqslant 1$, $M^k := M \circ \cdots \circ M$ is k iterations of the maximal operator, and $M^0 h := |h|$. The operator R has the following properties:

(A) $h \leqslant R(h)$;

(B) $\|R(h)\|_{L^p(\mathbb{R}^n)} \leqslant 2\,\|h\|_{L^p(\mathbb{R}^n)}$;

(C) $R(h) \in A_1$ and $[R(h)]_{A_1} \leqslant 2\,\|M\|_{L^p(\mathbb{R}^n)}$.

To verify (4.11), consider $1<p<p_0$ and apply Holder's inequality to obtain

$$\begin{aligned}\|T(f)\|_{L^p(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} |Tf|^p\,(Rf)^{-(p_0-p)\frac{p}{p_0}}\,(Rf)^{(p_0-p)\frac{p}{p_0}}\,dx\right)^{1/p}\\ &\leqslant \left(\int_{\mathbb{R}^n} |Tf|^{p_0}\,(Rf)^{-(p_0-p)}\,dx\right)^{1/p_0}\left(\int_{\mathbb{R}^n} (Rf)^p\,dx\right)^{\frac{p_0-p}{pp_0}}.\end{aligned}$$

For clarity in the exposition, we denote $w := (Rf)^{-(p_0-p)}$. Then, by the key hypothesis (4.10) together with properties (A) and (B) of the Rubio de Francia's algorithm, we have that

$$\begin{aligned}\|T(f)\|_{L^p(\mathbb{R}^n)} &\lesssim_{n,p_0,T} [w]_{A_{p_0}}^{\beta}\left(\int_{\mathbb{R}^n}|f|^{p_0}\,w\,dx\right)^{1/p_0}\|f\|_{L^p(\mathbb{R}^n)}^{\frac{p_0-p}{p_0}}\\ &\lesssim_{n,p_0,T} [w]_{A_{p_0}}^{\beta}\left(\int_{\mathbb{R}^n}|f|^{p}\,dx\right)^{1/p_0}\|f\|_{L^p(\mathbb{R}^n)}^{1-\frac{p}{p_0}}\\ &\simeq_{n,p_0,T} [w]_{A_{p_0}}^{\beta}\|f\|_{L^p(\mathbb{R}^n)}\\ &\simeq_{n,p_0,T} [w^{1-p_0'}]_{A_{p_0'}}^{\beta(p_0-1)}\|f\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

since $[w]_{A_q} = [w^{1-q'}]_{A_{q'}}^{q-1}$. Now, since $\frac{p_0-p}{p_0-1} < 1$ we can use Jensen's inequality to compute the constant of the weight as follows

$$[w^{1-p_0'}]_{A_{p_0'}} = [(Rf)^{\frac{p_0-p}{p_0-1}}]_{A_{p_0'}} \leqslant [R(f)]_{A_{p_0'}}^{\frac{p_0-p}{p_0-1}} \leqslant [R(f)]_{A_1}^{\frac{p_0-p}{p_0-1}}$$

Finally, by making use of property (C), we conclude that

$$\|T(f)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p_0,T} \|M\|_{L^p(\mathbb{R}^n)}^{\beta(p_0-p)} \|f\|_{L^p(\mathbb{R}^n)},$$

which clearly implies (4.11). Once we have proved the key inequality (4.11), we can relate the exponent on the weighted estimate to the endpoint order of T. To that end, we will use the known asymptotic behaviour of the unweighted L^p norm of the maximal function. It is well known that

$$\|M\|_{L^p(\mathbb{R}^n)} \lesssim_n \frac{1}{p-1}. \tag{4.12}$$

when p is close to 1. Then, for p close to 1, we obtain

$$\|T\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p_0,T} (p-1)^{-\beta(p_0-p)} \lesssim_{n,p_0,T} (p-1)^{-\beta(p_0-1)} \tag{4.13}$$

Therefore, multiplying by $(p-1)^{\alpha_T-\varepsilon}$ (any $\varepsilon > 0$), using the definition of α_T and taking upper limits we have,

$$+\infty = \limsup_{p\to 1} (p-1)^{\alpha_T-\varepsilon} \|T\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p_0,T} \limsup_{p\to 1} (p-1)^{\alpha_T-\varepsilon-\beta(p_0-1)}.$$

This last inequality implies that $\beta \geqslant \frac{\alpha_T}{p_0-1}$, so we conclude the first part of the statement of the theorem.

For the proof of the other inequality, $\beta \geqslant \gamma_T$, we follow the same line of ideas, but with a twist involving the dual space $L^{p'}(\mathbb{R}^n)$. Fix p, $p > p_0$. We perform the same iteration technique as before changing p with p'. We repeat details for the sake of completeness.

$$R'(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{M^k(h)}{\|M\|_{L^{p'}(\mathbb{R}^n)}^k}.$$

Then we have

(A') $h \leqslant R'(h)$;

(B') $\|R'(h)\|_{L^{p'}(\mathbb{R}^n)} \leqslant 2\,\|h\|_{L^{p'}(\mathbb{R}^n)}$;

(C') $[R'(h)]_{A_1} \leqslant 2\,\|M\|_{L^{p'}(\mathbb{R}^n)}$.

Fix $f \in L^p(\mathbb{R}^n)$. By duality there exists a non-negative function $h \in L^{p'}(\mathbb{R}^n)$, $\|h\|_{L^{p'}(\mathbb{R}^n)} = 1$, such that,

$$\begin{aligned}
\|Tf\|_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |Tf(x)|h(x)\,dx \\
&\leqslant \int_{\mathbb{R}^n} |Tf|(R'h)^{\frac{p-p_0}{p_0(p-1)}}\, h^{\frac{p(p_0-1)}{p_0(p-1)}}\,dx \\
&\leqslant \left(\int_{\mathbb{R}^n} |Tf|^{p_0}(R'h)^{\frac{p-p_0}{p-1}}\,dx\right)^{1/p_0} \left(\int_{\mathbb{R}^n} h^{p'}\,dx\right)^{1/p_0'} \\
&= \left(\int_{\mathbb{R}^n} |Tf|^{p_0}(R'h)^{\frac{p-p_0}{p-1}}\,dx\right)^{1/p_0}.
\end{aligned}$$

Now we use the key hypothesis (4.10) and Hölder's inequality to obtain

$$\begin{aligned}
\|Tf\|_{L^p(\mathbb{R}^n)} &\lesssim_{n,p_0,T} [(R'h)^{\frac{p-p_0}{p-1}}]^{\beta}_{A_{p_0}} \left(\int_{\mathbb{R}^n} |f|^{p_0} (R'h)^{\frac{p-p_0}{p-1}}\, dx\right)^{1/p_0} \\
&\lesssim_{n,p_0,T} [(R'h)^{\frac{p-p_0}{p-1}}]^{\beta}_{A_{p_0}} \left(\int_{\mathbb{R}^n} |f|^{p}\, dx\right)^{1/p} \left(\int_{\mathbb{R}^n} (R'h)^{p'}\, dx\right)^{\frac{1}{p'}\frac{p-p_0}{p_0(p-1)}} \\
&\lesssim_{n,p_0,T} [(R'h)^{\frac{p-p_0}{p-1}}]^{\beta}_{A_{p_0}} \left(\int_{\mathbb{R}^n} |f|^{p}\, dx\right)^{1/p} \qquad \text{by (B')} . \\
&\lesssim_{n,p_0,T} [R'h]^{\beta\frac{p-p_0}{p-1}}_{A_{p_0}} \left(\int_{\mathbb{R}^n} |f|^{p}\, dx\right)^{1/p} \qquad \text{by Jensen's} \\
&\lesssim_{n,p_0,T} \|M\|^{\beta\frac{p-p_0}{p-1}}_{L^{p'}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f|^{p}\, dx\right)^{1/p} \qquad \text{by (C')}.
\end{aligned}$$

Hence,

$$\|T\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p_0,T} \|M\|^{\beta\frac{p-p_0}{p-1}}_{L^{p'}(\mathbb{R}^n)} \qquad p > p_0. \tag{4.14}$$

This estimate is dual to (4.11). To finish the proof we recall that, for large p, namely $p > p_0$, we have the asymptotic estimate, $\|M\|_{L^{p'}(\mathbb{R}^n)} \simeq \frac{1}{p'-1} \leqslant p$. Therefore, we have that

$$\|T\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p_0,T} p^{\beta\frac{p-p_0}{p-1}} \lesssim_{n,p_0,T} p^{\beta}$$

since $p > p_0 > 1$. As before, dividing by $p^{\gamma_T - \varepsilon}$ and taking upper limits, we obtain

$$+\infty = \limsup_{p\to\infty} \frac{\|T\|_{L^p(\mathbb{R}^n)}}{p^{\gamma_T-\varepsilon}} \lesssim_{n,p_0,T} \limsup_{p\to\infty} p^{\beta-\gamma_T+\varepsilon}.$$

This last inequality implies that $\beta \geqslant \gamma_T$, so we conclude the proof of the theorem.

□

4.3 Application of Theorem 4.2

In this section we first show how to derive the sharpness of several known weighted inequalities from our general result in Theorem 4.2. Then we will provide additional *information* that can be deduced from the theorem. In particular, we prove a new weighted estimate for a certain class of Orlicz maximal operators which we show is sharp as a consequence of Theorem 4.2. In the cases where it is not known a sharp weighted estimate, we provide a lower bound for the exponent β of the A_p constant. This is the case for the Bochner-Riesz multipliers and the maximal operator $M_{\mathfrak{B}}$, where $\mathfrak{B}$ is any Muckenhoupt basis.

4.3.1 Optimality without examples

Next results will follow directly from Theorem 4.2 if we check the appropriate *endpoint* values α_T and γ_T that were introduced in Definition 4.1. Although this technique requires a precise computation of $\|T\|_{L^p(\mathbb{R}^n)}$, it avoids the construction of counterexamples that involve weights and functions. More precisely, we answer to the second step of the question presented at the beginning of the chapter without constructing specific examples.

Operators with large kernel and commutators Consider any Calderón-Zygmund operator whose kernel K satisfies

$$|K(x,y)| \geqslant \frac{c}{|x-y|^n}. \tag{4.15}$$

for some $c>0$ and if $x \neq y$ (we can consider the Hilbert transform H as a model example of this class in $\mathbb{R}$ and the Riesz transforms for $\mathbb{R}^n$, $n \geqslant 2$). Then, it is true (see [Ste93, p. 42]) that, for $p \to 1^+$,

$$\|T\|_{L^p(\mathbb{R}^n)} \simeq \frac{1}{p-1}, \tag{4.16}$$

which clearly implies that $\alpha_T = 1$. By duality we can see that $\gamma_T = 1$.

For the Calderón-Zygmund operators T that satisfy (4.15), we consider the commutator

$$[b,T]f := bT(f) - T(bf),$$

where b is a BMO function. In order to calculate its corresponding endpoint values, we use the example from [Pér97, Section 5, p. 755]. There, for the choices $b(x) = \log(|x|)$ and $T = H$ the Hilbert transform, it is shown that

$$\|[b,H]\|_{L^p(\mathbb{R}^n)} \gtrsim \frac{1}{(p-1)^2},$$

which implies that $\alpha_{[b,H]} = 2$. More generally, the k-th iteration of the commutator which is defined recursively by

$$T_b^k := [b, T_b^{k-1}], \qquad k \in \mathbb{N},$$

satisfies $\alpha_{H_b^k} = \gamma_{H_b^k} = k$. The value for $\gamma_{H_b^k}$ follows by duality as in the case of Calderón-Zygmund operators.

We then obtain, as an immediate consequence of Theorem 4.2, that the following known weighted inequalities are sharp:

$$\|T\|_{L^p(w)} \lesssim_{n,p,T} [w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}, \qquad w \in A_p,$$

$$\|[b,T]\|_{L^p(w)} \lesssim_{n,p,T} \|b\|_{BMO}\, [w]_{A_p}^{2\max\{1,\frac{1}{p-1}\}}, \qquad w \in A_p,$$

$$\|T_b^k\|_{L^p(w)} \lesssim_{n,p,T} \|b\|_{BMO}\, [w]_{A_p}^{(k+1)\max\{1,\frac{1}{p-1}\}}, \qquad w \in A_p.$$

For proof of the upper bounds of these estimates, see [Hyt12] for the case of Calderón-Zygmund operators and [CPP12] for the case of commutators.

Maximal operators and square functions We consider first the k-th iteration of the maximal function; that is, $M^k := M(M^{k-1})$, where $k \in \mathbb{N}$. In this case we have that $\alpha_{M^k} = k$. The case $k = 1$ corresponds to estimate (4.35) and the case $k > 1$ is obtained by iteration of Buckley's result (4.4). The fact that $\gamma_{M^k} = 0$ is trivial. Then the following weighted inequality is sharp.

$$\|M^k\|_{L^p(w)} \leqslant c\,[w]_{A_p}^{\frac{k}{p-1}}, \qquad w \in A_p.$$

We now consider the vector-valued extension of the Hardy-Littlewood maximal function. Let $1 < q < \infty$ and $1 < p < \infty$, then this operator is defined as:

$$\overline{M}_q f(x) := \Big(\sum_{j=1}^{\infty} (Mf_j(x))^q \Big)^{1/q},$$

where $f := \{f_j\}_{j=1}^{\infty}$ is a vector-valued function. The fact that $\alpha_{\overline{M}_q} = 1$ can be verified in the same way as in the case $q = 1$. In particular, the computation of α_M follows the same ideas described for the Hilbert transform in Section 4.1. For $\gamma_{\overline{M}_q}$, we can find an example of a vector-valued function satisfying $\|\overline{M}_q f\|_{L^p(\mathbb{R}^n)} \gtrsim_n p^{1/q} \|\overline{f}_q\|_{L^p(\mathbb{R}^n)}$ which implies that $\gamma_{\overline{M}_q} = 1/q$. This is already known; see [Ste93, p.75] for the classic proof. Then the following inequality is sharp.

$$\|\overline{M}_q f\|_{L^p(w)} \lesssim_{n,p,q} [w]^{\max\{\frac{1}{q}, \frac{1}{p-1}\}} \|\overline{f}_q\|_{L^p(w)}, \qquad w \in A_p. \tag{4.17}$$

We finally include here the case of the dyadic square function S_d. We first note that $\alpha_{S_d} = 1$ by testing against the indicator function of the unit cube (as in the case of the Hilbert transform). The value of $\gamma_{S_d} = \frac{1}{2}$ was previously known, see for instance [CUMP12, p. 434]. As before, we conclude that the following inequality is sharp.

$$\|S_d f\|_{L^p(w)} \lesssim_{n,p} [w]_{A_p}^{\max\{\frac{1}{2}, \frac{1}{p-1}\}} \|f\|_{L^p(w)} \qquad w \in A_p. \tag{4.18}$$

The proof of inequalities (4.18) and (4.17) can be found in [CUMP12].

Fractional integral operators We first consider the maximal fractional operator M_α introduced in Definition 1.12. Note that

$$\|M_\alpha\|^q_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \gtrsim \frac{1}{q - \frac{n}{n-\alpha}}.$$

This can be seen again by considering the indicator of the unit cube. Now we can use an off-diagonal version of the extrapolation theorem for $A_{p,q}$ classes from [Duo11, Theorem 5.1]. Then we obtain, by the same line of ideas as in Theorem 4.2, that the following inequality is sharp.

$$\|M_\alpha\|_{L^p(w^p) \to L^q(w^q)} \lesssim_{n,p,\alpha} [w]_{A_{p,q}}^{\frac{p'}{q}(1-\frac{\alpha}{n})}, \tag{4.19}$$

for $0 \leqslant \alpha < n$, $1 < p < n/\alpha$ and q is defined by the equation (1.12).

For the case of the fractional integral I_α we can easily compute the following estimate:

$$\|I_\alpha\|^q_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \geqslant \frac{1}{q - \frac{n}{n-\alpha}}.$$

Then, arguing as above we conclude that the following weighted inequality is also sharp.

$$\|I_\alpha\|_{L^p(w^p) \to L^q(w^q)} \lesssim_{n,p,q} [w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max\{1, \frac{p'}{q}\}}. \tag{4.20}$$

The proof of inequalities (4.19) and (4.20) can be found in [LMPT10].

4.3.2 New further results

On Orlicz maximal operators. We are interested here in the operator M_Φ that we introduced in Section 2.3.1 (see equation (2.12)). As we have seen these operators plays an important role in applications. In particular, we will study those associated with the young functions $\Phi_\lambda(t) := t\log^\lambda(e+t)$, $\lambda \in [0,\infty)$. Note that the case $\lambda = 0$ corresponds to M. The case $\lambda = k \in \mathbb{N}$ corresponds to $M_{L(\log L)^k}$, which is pointwise comparable to M^{k+1} (see, for example, [Pér95a]).

We have seen that the sharp exponent in weighted estimates for these operators is $1/(p-1)$ for $\lambda = 0$ and $k/(p-1)$ for $\lambda = k \in \mathbb{N}$. The following theorem provides a sharp bound for these intermediate exponents in $\mathbb{R}_+ \setminus \mathbb{N}$. This theorem is a *mixed* $A_p - A_\infty$ result involving the Fujii-Wilson A_∞'s constant defined as

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w)dx.$$

Theorem 4.3. *Let $\lambda > 0$, $1 < p < \infty$ and $w \in A_p$. Then*

$$\|M_{\Phi_\lambda}\|_{L^p(w)} \lesssim_{n,p,\lambda} [w]_{A_p}^{\frac{1}{p}} [\sigma]_{A_\infty}^{\frac{1}{p}+\lambda} \tag{4.21}$$

where $\sigma = w^{1-p'}$. As a consequence we have

$$\|M_{\Phi_\lambda}\|_{L^p(w)} \lesssim_{n,p,\lambda} [w]_{A_p}^{\frac{1+\lambda}{p-1}}.$$

Furthermore, the exponent is sharp.

Mixed estimates like (4.21) were proved for first time in [HP] (See Section 4.5 for more information on this subject).

Proof. We start with the following variant of the classical Fefferman-Stein inequality which holds for any weight w. For $t > 0$ and any nonnegative function f, we have that

$$w\left(\{x \in \mathbb{R}^n : M_{\Phi_\lambda} f(x) > t\}\right) \lesssim_{n,\lambda} \int_{\mathbb{R}^n} \Phi_\lambda\left(\frac{f(x)}{t}\right) Mw(x)\, dx, \tag{4.22}$$

where M is the usual Hardy–Littlewood maximal operator. The result can be obtained by using a Calderón–Zygmund decomposition adapted to M_{Φ_λ} as in Lemma 4.1 from [Pér95b].

Now, if the weight w is in A_1, then inequality (4.22) yields the linear dependence on $[w]_{A_1}$,

$$w\left(\{x \in \mathbb{R}^n : M_{\Phi_\lambda} f(x) > t\}\right) \lesssim_{n,\lambda} [w]_{A_1} \int_{\mathbb{R}^n} \Phi_\lambda\left(\frac{f(x)}{t}\right) w(x)\, dx.$$

From this estimate and by using an extrapolation type argument as in [Pér, Section 4.1], we derive easily that, for any $w \in A_p$

$$w\left(\{x \in \mathbb{R}^n : M_{\Phi_\lambda} f(x) > t\}\right) \lesssim_{n,\lambda} [w]_{A_p} \int_{\mathbb{R}^n} \Phi_\lambda\left(\frac{f(x)}{t}\right)^p w(x)\, dx. \tag{4.23}$$

Now, we follow the same ideas from [HPR12, Theorem 1.3]. We write the L^p norm as

$$\|M_{\Phi_\lambda} f\|_{L^p(w)}^p \lesssim_p \int_0^\infty t^p w\{x \in \mathbb{R}^n : M_{\Phi_\lambda} f_t(x) > t\} \frac{dt}{t}$$

where $f_t := f\chi_{f>t}$. Since $w \in A_p$, then by the precise open property of A_p classes (see [HPR12, Theorem 1.2]), we have that $w \in A_{p-\varepsilon}$ where $\varepsilon \sim \frac{1}{[\sigma]_{A_\infty}}$. Moreover, the constants satisfy that $[w]_{A_{p-\varepsilon}} \leqslant c[w]_{A_p}$. We apply (4.23) with $p-\varepsilon$ instead of p to obtain after a change of variable

$$\begin{aligned}
\|M_{\Phi_\lambda} f\|_{L^p(w)}^p \quad & \lesssim_{n,p,\lambda} [w]_{A_p} \int_{\mathbb{R}^n} f^p \int_1^\infty \frac{\Phi_\lambda(t)^{p-\varepsilon}}{t^p} \frac{dt}{t}\, w\, dx \\
& \lesssim_{n,p,\lambda} [w]_{A_p} \int_1^\infty \frac{(\log(e+t))^{p\lambda}}{t^\varepsilon} \frac{dt}{t} \|f\|_{L^p(w)}^p \\
& \lesssim_{n,p,\lambda} [w]_{A_p} \left(\frac{1}{\varepsilon}\right)^{\lambda p+1} \|f\|_{L^p(w)}^p \\
& \lesssim_{n,p,\lambda} [w]_{A_p} [\sigma]_{A_\infty}^{\lambda p+1} \|f\|_{L^p(w)}^p.
\end{aligned}$$

Taking p-roots we obtain the desired estimate (4.21).

Regarding the sharpness, we will prove now that the exponent in the term on the right hand side of (4.21) cannot be improved. This follows from Theorem 4.2 since it is easy to verify (again by testing against the indicator of the unit cube) that

$$\|M_{\Phi_\lambda}\|_{L^p(\mathbb{R}^n)} \simeq \frac{1}{(p-1)^{1+\lambda}}.$$

From this estimate we conclude that the endpoint order verifies $\alpha_T = 1+\lambda$ for $T = M_{\Phi_\lambda}$. Then Theorem 4.2 provides the desired conclusion. □

Remark 4.4. Given a Young function Φ and a weight $w \in A_p$, we cannot say in general that M_Φ is bounded on $L^p(w)$. We first need to restrict to those functions $\Phi \in B_p$, because only in that case M_Φ is a bounded operator on $L^p(\mathbb{R}^n)$ (recall the statement 2.13). But even in that case, $w \in A_p$ is not a sufficient condition for M_Φ to be bounded on $L^p(w)$. Consider, in particular, the young function $\Phi(t) = t^q$, $1 < q < p$, then $\Phi \in B_p$ and $M_\Phi : L^p(w) \to L^p(w)$ if and only if $w \in A_{p/q} \subsetneq A_p$. Therefore, Theorem 4.3 just covers the particular logarithmic class of Young functions Φ_λ.

On Bochner-Riesz multipliers. For $\lambda > 0$ and $R > 0$, the Bochner-Riesz multiplier is defined as follows

$$(B_R^\lambda f)(x) := \int_{\mathbb{R}^n} \left(1 - (|\xi|/R)^2\right)_+^\lambda \hat{f}(\xi) e^{2\pi i \xi x}\, d\xi,$$

where $\hat{f}$ denotes the Fourier transform of $f \in \mathscr{S}(\mathbb{R}^n)$. For $R = 1$ we write simply B^λ. It is a known fact that this operator has a kernel $K_\lambda(x)$ defined by

$$K_\lambda(x) = \frac{\Gamma(\lambda+1)}{\pi^\lambda} \frac{J_{n/2+\lambda}(2\pi|x|)}{|x|^{n/2+\lambda}}, \tag{4.24}$$

where Γ is the Gamma function and J_η is the Bessel function of integral order η (see [Gra09, p. 352]). The following result is a consequence of Theorem 4.2.

Corollary 4.5. *Let $1 < p < \infty$. Suppose further that the following estimate holds*

$$\|B^{(n-1)/2}\|_{L^p(w)} \lesssim_{n,p} [w]_{A_p}^{\beta}, \tag{4.25}$$

for any $w \in A_p$. Then $\beta \geqslant \max\left\{1; \frac{1}{p-1}\right\}$.

Proof. We use the known asymptotics for Bessel functions, namely

$$J_\eta(r) = cr^{-1/2}\cos(r-\tau) + O(r^{-3/2})$$

for some constants $c, \tau > 0$, $\tau = \tau_\eta$, and $r > r_0 \gg 1$ (see [Ste93, p.338, Example 1.4.1, eq. (14)]). Combining this with (4.24), we obtain that

$$K_{(n-1)/2}(x) \simeq \frac{\cos(|x|-\tau) + \varphi(|x|)}{|x|^n}. \tag{4.26}$$

for some $\varphi : \mathbb{R} \to \mathbb{R}$ such that $|\varphi(r)| \lesssim r^{-1}$. We see that this kernel does not satisfy the size condition (4.15). However, (4.26) is sufficient to conclude that $\alpha_{B^{(n-1)/2}} = \gamma_{B^{(n-1)/2}} = 1$. Testing on the indicator function of the unit cube (we use again [Ste93, p. 42]) we obtain, after a change of variables and for some $r_1 \geqslant r_0$, that

$$\|B^{(n-1)/2}\|_{L^p(\mathbb{R}^n)}^p \gtrsim \int_{r>r_1} \frac{|\cos(r-\tau)+\varphi(r)|^p}{r^p}\, dr.$$

We choose $r_2 \geqslant r_1$ large enough such that $|\varphi(r)| < 1/4$ and consider the set $A = \{r \in R : r > r_2, |\cos(r-\tau)| > 1/2\}$. We obtain that

$$\|B^{(n-1)/2}\|_{L^p(\mathbb{R}^n)}^p \gtrsim \int_A \frac{1}{r^p}\, dr \gtrsim \int_{r>1} \frac{1}{r^p}\, dr \gtrsim \frac{1}{p-1}$$

for p close to 1. The estimate in the middle follows by the monotonicity of the function $t \mapsto t^{-p}$ and taking into account that we can find the exact description of the set A as a union of intervals. The value for $\gamma_{B^{(n-1)/2}} = 1$ follows by duality. ☐

In particular, this result shows that the claimed weighted norm inequality for the maximal Bochner-Riesz operator from [LS12] cannot hold (see also [LS13]).

On Muckenhoupt basis As a final application of Theorem 4.2, we can derive a lower bound for the optimal exponent that one could expect in a weighted estimate for a maximal operator associated to a generic Muckenhoupt basis $M_{\mathfrak{B}}$. We note that it is not even possible to have an example working for a general basis. The only requirement on the operator $M_{\mathfrak{B}}$ is that its L^p norm must blow up when p goes to 1 (no matter the ratio of blow up). Precisely, we have the following theorem.

Theorem 4.6. *Let $\mathfrak{B}$ be a Muckenhoupt basis. Suppose in addition that the associated maximal operator $M_{\mathfrak{B}}$ satisfies the following weighted estimate:*

$$\|M_{\mathfrak{B}}\|_{L^{p_0}(w)} \lesssim_{\mathfrak{B},p_0} [w]_{A_{p_0,\mathfrak{B}}}^{\beta} \tag{4.27}$$

for some $1 < p_0 < \infty$*. If* $\limsup_{p\to 1^+} \|M_{\mathfrak{B}}\|_{L^p(\mathbb{R}^n)} = +\infty$*, then* $\beta \geqslant \frac{1}{p_0-1}$.

Proof of Theorem 4.6. The idea is to perform the iteration technique from Theorem 4.2 but with $M_{\mathfrak{B}}$ instead of the standard H–L maximal operator. Then we obtain, for $1 < p < p_0$, that

$$\|M_{\mathfrak{B}}\|_{L^p(\mathbb{R}^n)} \lesssim_{\mathfrak{B},p_0} \|M_{\mathfrak{B}}\|_{L^p(\mathbb{R}^n)}^{\beta(p_0-p)} \lesssim_{\mathfrak{B},p_0} \|M_{\mathfrak{B}}\|_{L^p(\mathbb{R}^n)}^{\beta(p_0-1)}. \tag{4.28}$$

The last inequality holds since $\|M_{\mathfrak{B}}\|_{L^p(\mathbb{R}^n)} \geqslant 1$. We remark here that, since we are comparing $M_{\mathfrak{B}}$ to itself, it is irrelevant to know the precise quantitative behaviour of its L^p-norm for p close to 1. In fact, we cannot use any estimate like (4.12) since we are dealing with a generic basis. Just knowing that the L^p norm blows up when p goes to 1, allows us to conclude that $\beta \geqslant \frac{1}{p_0-1}$. $\square$

As an application, we can show that the result for Calderón weights from [DMRO13a] is sharp. Precisely, for the basis $\mathfrak{B}_0$ of open sets in $\mathbb{R}$ of the form $(0,b)$, $b>0$, the authors prove that the associated maximal operator N defined as

$$Nf(t) = \sup_{b>t} \frac{1}{b}\int_0^b |f(x)|\,dx$$

is bounded on $L^p(w)$ if and only if $w \in A_{p,\mathfrak{B}_0}$ and, moreover, that

$$\|N\|_{L^p(w)} \lesssim_p [w]_{A_{p,\mathfrak{B}_0}}^{\frac{1}{p-1}}.$$

By the preceding result, this inequality is sharp with respect to the exponent on the characteristic of the weight.

4.4 Quantitative multiparameter theory

All the results mentioned so far concern the classical or one-parameter theory, where the operators under study commute with one-parameter dilations of $\mathbb{R}^n$. A natural starting point for a quantitative multi-parameter weighted theory would be the analogue of Buckley's estimate (4.4) for the strong maximal function, namely, a sharp estimate on $\|M_s\|_{L^p(w)}$ in terms of the A_p^*-constant of the weight. This problem together with the corresponding weak one are addressed in this section.

Let $w \in A_p^*$. Then Lemma 1.22 assures that $w_{\bar{x}_j} \in A_p$, $1 \leqslant j \leqslant n$ (see Subsection 1.3.2 for the definition of $w_{\bar{x}_j}$). Thus, using the pointwise inequality (1.14), we can estimate $\|M_s\|_{L^p(w)}$ by applying the one dimensional sharp weighted inequality of Buckley (4.4) for every weight $w_{\bar{x}_j}$, $1 \leqslant j \leqslant n$; that is,

$$\|M^j\|_{L^p(w_{\bar{x}_j})} \lesssim_p \sup_{\bar{x}_j} [w_{\bar{x}_j}]_{A_p}^{\frac{1}{p-1}}. \tag{4.29}$$

Then, we obtain:

$$\begin{aligned} \|M_s f\|_{L^p(w)} &\lesssim_{n,p} \sup_{\bar{x}_1} [w_{\bar{x}_1}]_{A_p}^{\frac{1}{p-1}} \cdots \sup_{\bar{x}_j} [w_{\bar{x}_n}]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)} \\ &\lesssim_{n,p} [w]_{A_{p^*}}^{\frac{n}{p-1}} \|f\|_{L^p(w)}, \end{aligned} \tag{4.30}$$

where in (4.29) we have used the relation (1.19).

In order to estimate $\|M_s\|_{L^p(w)\to L^{p,\infty}(w)}$, we can follow exactly the same iteration argument. We will use again the sharp one-dimensional weighted inequality (4.29) together with its weak-L^p analogue

$$\|M^j\|_{L^p(w^j)\to L^{p,\infty}(w^j)} \lesssim \sup_{\bar{x}_j}[w_{\bar{x}_j}]_{A_p}^{\frac{1}{p}}, \tag{4.31}$$

which is due to Muckenhoupt [Muc72]. Now for any $1 \leqslant j \leqslant n$ we have

$$\begin{aligned}\|M_s f\|_{L^{p,\infty}(w)} &\lesssim \sup_{\bar{x}_j}[w_{\bar{x}_j}]_{A_p}^{\frac{1}{p}} \|M^1 \circ \cdots \circ M^{j-1} \circ M^{j+1} \circ \cdots \circ M^n f\|_{L^p(w)} \\ &\lesssim_{n,p} \sup_{\bar{x}_j}[w_{\bar{x}_j}]_{A_p}^{\frac{1}{p}} \prod_{\substack{k=1\\k\neq j}}^{n} \sup_{\bar{x}_k}[w_{\bar{x}_k}]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)}.\end{aligned}$$

Then, using again the relation (1.19), we have

$$\|M_s f\|_{L^{p,\infty}(w)} \lesssim_{n,p} [w]_{A_p^*}^{\frac{1}{p}+\frac{n-1}{p-1}} \|f\|_{L^p(w)}. \tag{4.32}$$

The question now is whether estimates (4.30) and (4.32) are sharp. Estimate (1.14) has *produced* unweighted sharp results, like the end-point estimate (1.8) obtained in [JMZ35]. However, we do not have enough evidence to even conjecture which are the sharp estimates in the weighted framework.

The general strategy to test sharpness described in Theorem 4.2 gives trivial information for the case of rectangles. Observe that since the basis $\mathfrak{R}$ is a Muckenhoupt basis, we just get that a lower bound for β in estimate (4.2) for $T = M_s$ is $1/(p-1)$ (see Theorem 4.6). The main obstacle in the multiparameter case is the failure of the *classical* covering arguments (Vitali or Besicovitch) for rectangles with arbitrary eccentricities. Indeed, it is an essential fact underlaying the sharp quantitative estimate (4.4), that M_c^w, defined with respect to a general measure, is always bounded independently of the measure. In this sense, see the elegant proof by Lerner [Ler08] of Buckley's result. However, this fails for the strong maximal function and it is just another manifestation of the failure of the covering arguments. These facts were explained in Section 1.2.

For these reasons, the only cases with complete answers are those concerning product weights and power weights. As we will see now, estimates (4.30) and (4.32) are not sharp in these cases.

4.4.1 Power weights

Throughout this subsection, let $w(x) = |x|^\alpha$. It is a classical result (see for example [Kur80, p.236]) that this power weight w is in A_p^* if and only if $-1 < \alpha < p-1$, $1 < p < \infty$. In the one parameter case potential weights have been very useful for testing the optimality of quantitative estimates. For example, Buckley [Buc93] proved that estimate (4.4) was sharp by constructing a counterexample with power weights. However, in the multilinear case these weights, in some sense, *behave* like one dimensional weights.

Proposition 4.7. *Let $1 < p < \infty$. We have the following estimates for $w(x) = |x|^\alpha$, $-1 < \alpha < p-1$.*

(i) $\|M_s f\|_{L^p(w)} \lesssim_{n,p} [w]_{A_p^*}^{\frac{1}{p-1}} \|f\|_{L^p(w)}$.

(ii) $\|M_s f\|_{L^{p,\infty}(w)} \lesssim_{n,p} [w]_{A_p^*}^{\frac{1}{p}} \|f\|_{L^p(w)}$.

Proof. We just give the proof of (ii). The proof of (i) follows the same lines. Observe also that an argument similar to the one presented in the proof of Theorem 4.3 also proves (i) from (ii). In this case, we would need the sharp reverse Hölder for rectangles described in the last section of this chapter.

There are two cases:

case $\alpha \geqslant 0$. We estimate

$$\begin{aligned}
w(\{x \in \mathbb{R}^n : M_s f > \lambda\}) &= \int_{\{x\in\mathbb{R}^n : M_s f > \lambda\}} |x|^\alpha dx \leqslant n^{\frac{\alpha}{2}} \int_{\{x\in\mathbb{R}^n : M_s f>\lambda\}} \sum_{j=1}^n |x_j|^\alpha dx \\
&\lesssim_n \sum_{j=1}^n \int_{\{x\in\mathbb{R}^n : M_s f>\lambda\}} |x_j|^\alpha dx \\
&\lesssim_{n,p} \sum_{j=1}^n \frac{1}{\lambda^p} [|x_j|^\alpha]_{A_p} \int_{\mathbb{R}^n} |f|^p |x_j|^\alpha dx,
\end{aligned}$$

where in the last step we have used estimate (4.31). Observe that, by using the relation (1.19) in Lemma 1.22, $[|x_j|^\alpha]_{A_p} \leqslant [w]_{A_p*}$, $1 \leqslant j \leqslant n$. Thus we obtain the required estimate after taking the p-th roots. Moreover, for this particular weight we also have $[w]_{A_p^*} \leqslant [|x_j|^\alpha]_{A_p}$ uniformly in $\bar{x}_j \in \mathbb{R}^{n-1}$, for all $1 \leqslant j \leqslant n$. Thus the weight w has the same A_p constant in every direction, which is also the same as the A_p^* constant of w. Then Buckley's example proves the sharpness of this result.

case $\alpha < 0$. We denote by $S_j := \{x \in \mathbb{R}^n : |x_j| = \max_{1\leqslant k\leqslant n} |x_k|\}$.

$$\begin{aligned}
w(\{x \in \mathbb{R}^n : M_s f > \lambda\}) &= \int_{\{x\in\mathbb{R}^n : M_s f>\lambda\}} |x|^\alpha dx \leqslant \sum_{j=1}^n \int_{\{x\in\mathbb{R}^n : M_s(f\mathbf{1}_{S_j})>\frac{\lambda}{n}\}} |x|^\alpha dx \\
&\lesssim_n \sum_{j=1}^n \int_{\{x\in\mathbb{R}^n : M_s(f\mathbf{1}_{S_j})>\frac{\lambda}{n}\}} |x_j|^\alpha dx \\
&\lesssim_{n,p} \sum_{j=1}^n \frac{1}{\lambda^p} [|x_j|^\alpha]_{A_p} \int_{\mathbb{R}^n} |f\mathbf{1}_{S_j}|^p |x_j|^\alpha dx \\
&\lesssim_{n,p} \frac{1}{\lambda^p} [w]_{A_p*} \int_{\mathbb{R}^n} |f|^p w dx.
\end{aligned}$$

□

4.4.2 Product weights

We say that w is a product weight if $w(x) = w_1(x_1)\cdots w_n(x_n)$. Note that in this case $w_{\bar{x}_j} := w_j$ for every $1 \leqslant j \leqslant n$. Then by Lemma 1.22 a product weight is in A_p^* if and only if $w_j \in A_p$.

Theorem 4.8. *Let $w \in A_p^*$ be a product weight, $1 < p < \infty$. Then the following quantitative weighted inequalities hold and are sharp:*

(i) $\|M_s f\|_{L^p(w)} \lesssim_{n,p} [w]_{A_p^*}^{\frac{1}{p-1}} \|f\|_{L^p(w)}$.

(ii) $\|M_s f\|_{L^{p,\infty}(w)} \lesssim_{n,p} [w]_{A_p^*}^{\frac{1}{p-1}(1-\frac{1}{np})} \|f\|_{L^p(w)}$.

Observe that this theorem assures that the behaviour of $\|M_s\|_{L^p(w)\to L^{p,\infty}(w)}$ is worse than the case of the Hardy-Littlewood maximal function.

Proof of Theorem 4.8. The proof of (i) follows by noticing that in the product case $[w]_{A_p^*} = [w_1]_{A_p} \cdots [w_n]_{A_p}$. Then the desired estimate follows directly from (4.30). To see that this is sharp we use the example of Buckley from [Buc93]. In particular we set

$$w_j(x_j) := |x_j|^{(p-1)(1-\delta)}, \quad 1 \leqslant j \leqslant n, \tag{4.33}$$

where $0 < \delta < 1$ and

$$f_R(x) := \prod_{j=1}^{n} |x_j|^{(\delta-1)} \mathbf{1}_{[0,R]^n}(x) \tag{4.34}$$

where $R > 0$. Then it is not hard to see that $[w]_{A_p^*} \sim \frac{1}{\delta^{n(p-1)}}$ and $M_s f_R(x) \geqslant \frac{1}{\delta^n} f_R(x)$ for any $x \in [0,R]^n$. Thus an estimate of the type

$$\|M_s f\|_{L^p(w)} \lesssim_{n,p} [w]_{A_p^*}^{\beta} \|f\|_{L^p(w)}$$

for some $\beta > 0$ implies that

$$\frac{1}{\delta^n} \|f_R\|_{L^p(w)} \lesssim_{n,p} [w]_{A_p^*}^{\beta} \|f_R\|_{L^p(w)} \Rightarrow \frac{1}{\delta^n} \lesssim_{n,p} \frac{1}{\delta^{n(p-1)\beta}}.$$

Letting $\delta \to 0^+$ we see that necessarily $\beta \geqslant \frac{1}{p-1}$.

We now give the proof of the statement in (ii). We will use again the sharp one-dimensional weighted inequality (4.29) together with its weak-L^p analogue (4.31). Now for any $1 \leqslant j \leqslant n$ we have

$$\begin{aligned} w(\{x \in \mathbb{R}^n : M_s f > \lambda\}) &\lesssim \frac{1}{\lambda^p} [w_j]_{A_p} \int_{\mathbb{R}^n} [M^1 \circ \cdots \circ M^{j-1} \circ M^{j+1} \circ \cdots \circ M^n f]^p w \\ &\lesssim_{n,p} [w_j]_{A_p} \prod_{\substack{k=1 \\ k\neq j}}^{n} [w_k]_{A_p}^{\frac{p}{p-1}} \frac{1}{\lambda^p} \int_{\mathbb{R}^n} |f|^p w \\ &\simeq_{n,p} \frac{[w]_{A_p^*}^{\frac{p}{p-1}}}{[w_j]_{A_p}^{\frac{1}{p-1}}} \frac{1}{\lambda^p} \|f\|_{L^p(w)}^p. \end{aligned}$$

We have thus proved that

$$\|M_s f\|_{L^{p,\infty}(w)} \lesssim_{n,p} [w]_{A_p^*}^{\frac{1}{p-1}} \max_{1\leqslant j\leqslant n} [w_j]_{A_p}^{-\frac{1}{p(p-1)}} \|f\|_{L^p(w)}.$$

Observe that $[w]_{A_p^*} \leqslant (\max_{1\leqslant j\leqslant n}[w_j]_{A_p})^n$ so we get

$$\|M_s f\|_{L^{p,\infty}(w)} \lesssim_{n,p} [w]_{A_p^*}^{\frac{1}{p-1}-\frac{1}{np(p-1)}}\|f\|_{L^p(w)},$$

which is the desired estimate.

Now we show that the power $[w]_{A_p^*}^{\frac{1}{p-1}(1-\frac{1}{np})}$ cannot be replaced by any smaller power of $[w]_{A_p^*}$. For this we consider the function f_R and the weight w from (4.34) and (4.33), respectively. Since $M_s f_R \geqslant \frac{1}{\delta^n} f_R$ on $[0,R]^n$ we have

$$\begin{aligned} w(\{x\in\mathbb{R}^n : M_s f_R(x) > \delta^{-n}\}) &\geqslant w(\{x\in[0,R]^n : f_R(x) > 1\}) \\ &= w(\{x\in[0,R]^n : x_1\cdots x_n < 1\}) \\ &=: w(E_R). \end{aligned}$$

Now we estimate for $R>1$ and $n\geqslant 2$

$$\begin{aligned} w(E_R) &\geqslant \int_{R^{-\frac{1}{n-1}}}^R x_1^{(p-1)(1-\delta)}\cdots\int_{R^{-\frac{1}{n-1}}}^R x_{n-1}^{(p-1)(1-\delta)}\left(\int_0^{\frac{1}{x_1\cdots x_{n-1}}} x_n^{(p-1)(1-\delta)}dx_n\right)dx_{n-1}\cdots dx_1 \\ &= \frac{1}{(p-1)(1-\delta)+1}\prod_{j=1}^{n-1}\int_{R^{-\frac{1}{n-1}}}^R \frac{dx_j}{x_j} \\ &\gtrsim_p (\log R)^{n-1}. \end{aligned}$$

An easy calculation also shows that $\|f_R\|_{L^p(w)} = (R^\delta/\delta)^{\frac{n}{p}}$. Now we assume that we have an estimate of the form

$$\lambda w(x\in\mathbb{R}^n : M_s f(x) > \lambda\})^{\frac{1}{p}} \lesssim_{n,p} [w]_{A_p^*}^\beta \|f\|_{L^p(w)}$$

for all $\lambda>0$, $f\in L^p(w)$ and some $\beta\geqslant 0$. Plugging in our choices of f_R and w, choosing $\lambda := \delta^{-n}$ and raising to the power p we get that we should have

$$\frac{(\log R)^{n-1}}{R^{n\delta}} \lesssim_{n,p} \frac{1}{\delta^{\beta np(p-1)}}\delta^{np}\delta^{-n}$$

for all $\delta\in(0,1)$ and $R>1$. The value $\log R := (n-1)(n\delta)^{-1}$ gives $R>1$ for δ small and maximizes the left hand side and gives

$$\frac{1}{\delta^{n-1}} \lesssim_{n,p} \frac{1}{\delta^{\beta np(p-1)+n-np}}.$$

Letting $\delta\to 0^+$ we get $\beta np(p-1)+n-np\geqslant n-1 \Rightarrow \beta\geqslant\frac{1}{p-1}(1-\frac{1}{np})$. $\square$

4.5 Notes and references

References

Results concerning Theorem 4.2 and its application are contained in [LPRa]. This result extends the ideas of [FP97] used to prove sharp weighted estimates for the Hilbert transform with weights in A_1 and the range $p\geqslant 2$. Furthermore, we remark that the first part of the proof, namely the proof of inequality (4.11) can be obtained as an immediate consequence of the extrapolation result from Duoandikoetxea [Duo11]. Indeed, this inequality is equivalent to the first inequality of (3.3) in Theorem 3.1 of his mentioned paper. We describe in detail the proof for the sake of completeness. In turn, estimate (4.14) coincides with the second inequality of (3.3) in Theorem 3.1. The proof of this estimate is slightly different to the one presented in [Duo11] avoiding the factorization lemma.

The quantitative multiparameter results presented in Section 4.4 are a joint work with I. Parissis and it is still in progress.

Improving the sharp bounds

In Theorem 4.2 we studied sharpness with respect to the power of the A_p constant of the weight. Although the results we have presented are sharp, there are several further improvements that can be made in the following sense:

- **Beyond power functions.** The techniques used in the proof of Theorem 4.2 actually allow us to deduce sharper results for some particular cases. For the H–L maximal function M, by considering the indicator function of the unit cube, it is easy to conclude that
$$\|M\|_{L^p(\mathbb{R}^n)} \simeq (p-1)^{-1}, \tag{4.35}$$
for p close to 1. This precise endpoint behavior allows us to prove that we cannot replace in the weighted inequality (4.4) the function $t \mapsto t^{(p-1)^{-1}}$ by any other *smaller* growth function φ. To be more precise, the following inequality fails
$$\|M\|_{L^p(w)} \lesssim \varphi([w]_{A_p})$$
for any non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ such that
$$\lim_{t\to\infty} \frac{\varphi(t)}{t^{\frac{1}{p-1}}} = 0.$$
The proof follows the same ideas of Theorem 4.2. A similar argument can be used to derive an analogous result for a generic operator T if the asymptotic behaviour of $\|T\|_{L^p(\mathbb{R}^n)}$ as $p \to 1$ and $p \to \infty$ is known.

- **Mixed A_p-A_∞ bounds.** This kind of estimates were first introduced in [HP]. More precisely, it was shown that the maximal function satisfies
$$\|M\|_{L^p(w)} \leqslant c\,[w]_{A_p}^{\frac{1}{p}} [\sigma]_{A_\infty}^{\frac{1}{p}} \qquad w \in A_p. \tag{4.36}$$
where $\sigma = w^{1-p'}$ and where
$$[\sigma]_{A_\infty} := \sup_Q \frac{1}{\sigma(Q)} \int_Q M(\chi_Q \sigma)dx$$
is the Fujii-Wilson A_∞'s constant which is smaller than the usual one (1.17). Estimate (4.36) was used in [HP] to improve the A_2 theorem from [Hyt12]. The idea behind the mixed estimates is that one only needs the weight to be in A_p for part of the estimates, while for the other part something weaker, like $w \in A_r$ for $r > p$, is

enough. Here we just want to note that we can formulate Theorem 4.2 in terms of mixed constants. In particular, if we replace (4.10) by

$$\|T\|_{L^{p_0}(w)} \lesssim_{n,p,T} [w]_{A_{p_0}}^{\beta_1}[\sigma]_{A_\infty}^{\beta_2}.$$

Then, trivially, $\beta_1+\frac{\beta_2}{p_0-1} \geqslant \max\left\{\gamma_T;\frac{\alpha_T}{p_0-1}\right\}$. It would be interesting to give a lower bound just for β_1, in order to quantify the role of the A_p in the estimate above.

Quantitative Reverse Hölder inequality for rectangles

The sharp reverse Hölder property for the A_p classes of weights has a very important role in the study of quantitative estimates. We show next a result for the strong A_p weights. As far as the author knows this proof has not appeared in the literature before. It follows the same lines as in the case of cubes, but we need to deal with a different covering argument.

Lemma 4.9. *Let $w \in A_p^*$, $1<p<\infty$ and let $r(w)=1+\frac{1}{2^{2p+1}[w]_{A_p^*}}$. Then*

$$\left(\frac{1}{|R|}\int_R w^{r(w)}\right)^{1/r(w)} \leqslant \frac{2}{|R|}\int_R w.$$

Proof. Let $w_R=\frac{1}{|R|}\int_R w$. Then for arbitrary positive δ we have

$$\begin{aligned}\frac{1}{|R|}\int_R w(x)^\delta w(x)dx \quad &= \frac{\delta}{|R|}\int_0^\infty \lambda^\delta w(\{x\in R: w(x)>\lambda\})\frac{d\lambda}{\lambda}\\ &= \frac{\delta}{|R|}\int_0^{w_R}+\frac{\delta}{|R|}\int_{w_R}^\infty \cdots\\ &= I+II.\end{aligned}$$

Observe that $I \leqslant (w_R)^{\delta+1}$.

For II we need to make some further calculations. First, for any rectangle R we define

$$E_R=\{x\in R: w(x)\leqslant \frac{1}{2^{p-1}[w]_{A_p^*}}w_R\}.$$

Using Hölder's inequality with p and its conjugate p', we have that for every rectangle R and every $f \geqslant 0$

$$\left(\frac{1}{|R|}\int_R f dx\right)^p w(R) \leqslant [w]_{A_p^*}\int_R f^p w dx.$$

In particular, for any measurable set $E\subset R$ we can rewrite the last inequality for $f\equiv \mathbf{1}_E$

$$\left(\frac{|E|}{|R|}\right)^p \leqslant [w]_{A_p^*}\frac{w(E)}{w(R)} \tag{4.37}$$

Hence, as E_R is a mesurable subset of R, we have

$$\left(\frac{|E_R|}{|R|}\right)^p \leqslant [w]_{A_p^*}\frac{w(E_R)}{w(R)} \leqslant [w]_{A_p^*}\frac{w_R}{w(R)}|E_R|\frac{1}{2^{p-1}[w]_{A_p^*}} = \frac{1}{2^{p-1}}\frac{|E_R|}{|R|}.$$

Then,

$$|E_R| \leqslant \frac{1}{2}|R|. \tag{4.38}$$

Second, we make the following claim: for every $\lambda > w_R$

$$w(\{x \in \mathbb{R}^n : w(x) > \lambda\}) \leqslant 2\lambda|\{x \in R : w(x) > \frac{\lambda}{2^{p-1}[w]_{A_p^*}} w_R\}| \tag{4.39}$$

To prove this claim we consider a multidimensional analogue of Riesz's lemma (see [KLS05]). As $w \in L^1_{loc}(\mathbb{R}^n)$ and we assume $w_R < \lambda$, then this lemma assures the existence of a countable set of pairwise disjoint rectangles $R_j \subset R$ satisfying

$$\frac{1}{|R_j|}\int_{R_j} w dx = \lambda$$

for each j, and $w(x) \leqslant \lambda$ for almost all points $x \in R \setminus \left(\bigcup_{j \geqslant 1} R_j\right)$. This decomposition together with (4.38) yields

$$\begin{aligned} w(\{x \in R : w(x) > \lambda\}) &\leqslant w(\bigcup_{j \geqslant 1} R_j) \leqslant \sum_j w(R_j) = \lambda \sum_j |R_j| \\ &\leqslant 2\lambda \sum_j |\{x \in R_j : w(x) > \frac{1}{2^{p-1}[w]_{A_p^*}} w_{R_j}\}| \\ &\leqslant 2\lambda |\{x \in R : w(x) > \frac{1}{2^{p-1}[w]_{A_p^*}} \lambda\}|, \end{aligned}$$

since $w_{R_j} = \lambda$. This proves claim (4.39). Now we can estimate II

$$\begin{aligned} \mathrm{II} &= \frac{\delta}{|R|}\int_{w_R}^{\infty} \lambda^{\delta} w(\{x \in R : w(x) > \lambda\})\frac{d\lambda}{\lambda} \\ &\leqslant \frac{2\delta}{|R|}\int_{w_R}^{\infty} \lambda^{\delta+1}|\{x \in R : w(x) > \frac{1}{2^{p-1}[w]_{A_p^*}}\lambda\}|\frac{d\lambda}{\lambda} \\ &= (2^{p-1}[w]_{A_p^*})^{\delta+1}\frac{2\delta}{|R|}\int_{\frac{w_R}{2^{p-1}[w]_{A_p^*}}}^{\infty} \lambda^{\delta+1}|\{x \in R : w(x) > \lambda\}|\frac{d\lambda}{\lambda} \\ &\leqslant (2^{p-1}[w]_{A_p^*})^{\delta+1} 2\frac{\delta}{1+\delta}\frac{1}{|R|}\int_R w^{1+\delta} dx. \end{aligned}$$

Setting $\delta = \frac{1}{2^{2p+1}[w]_{A_p^*}}$, we obtain using $t^{1/t} \leqslant 2, t \geqslant 1$

$$\frac{1}{|R|}\int_R w^{\delta+1} dx \leqslant 2(w_R)^{\delta+1}.$$

which concludes the proof. □

This result and its extension to $p = \infty$ is contained in the manuscript [LPRb].

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