Controllability of non-scalar parabolic systems: Some recent results and phenomena

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General objective:

Study some null controllability problems for **non-scalar parabolic systems**.

Non-scalar parabolic systems: arise in chemical reactions, when we model problems from the Biology and in a wide variety of physical situations.

In this course we will deal with non-scalar systems which in fact are **coupled parabolic scalar equations**. We do not present results relating to the controllability problems of systems which come from fluid mechanics as Stokes, Navier-Stokes, ...

GOAL:

1. Show the important differences between scalar and non-scalar problems.
2. Give necessary and sufficient conditions (**Kalman conditions**) which characterize the controllability properties of these systems.
3. Show some hyperbolic phenomena related to the controllability properties of these systems.

We will only deal with

1. **Linear systems**
2. In general, “simple” Parabolic Systems.

Contents

1. Introduction

1. Introduction

Let us fix $T > 0$ and let H and U be two separable Hilbert spaces. Let us consider the autonomous system:

$$(1) \qquad \begin{cases} y' = Ay + Bu & \text{on } (0,T), \\ y(0) = y_0 \in H. \end{cases}$$

A and B are "appropriate" operators, $y_0 \in H$ is the initial datum at $t = 0$ and $u \in L^2(0,T;U)$ is the control (exerted by means of the operator B).

Assume the problem is well-posed: $\forall (y_0, u)$ there exists a unique weak solution $y \in C^0([0,T];H)$ to (1) which depends continuously on the data.

Let us denote by $y(t; y_0, u) \in H$ the solution to the system at time $t \in [0,T]$.

Example

$H = \mathbb{R}^n$ $(n \geq 1)$, $U = \mathbb{R}^m$ $(m \geq 1)$, $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$: ordinary differential system with n variables and m controls.

1. Introduction

- **Exact Controllability:** System (1) is **exactly controllable** at time T if $\forall (y_0, y_1) \in H \times H$, there exists $u \in L^2(0,T;U)$ s.t. the solution y of (1) satisfies $y(T; y_0, u) = y_1$.
- **Controllability to trajectories:** System (1) is **controllable to trajectories** at time T if $\forall (y_0, \widehat{y}_0) \in H \times H$ and $\widehat{u} \in L^2(0,T;U)$, there exists $u \in L^2(0,T;U)$ s.t. the corresponding weak solution to (1) satisfies $y(T; y_0, u) = y(T; \widehat{y}_0, \widehat{u})$.
- **Null Controllability:** System (1) is **null controllable** at time T if $\forall y_0 \in H$ there exists $u \in L^2(0,T;U)$ s.t. $y(T; y_0, u) = 0$.
  Linear case: Controllability to trajectories and null controllability are equivalent.
- **Approximate Controllability:** System (1) is **approximately controllable** at time T if $\forall (y_0, y_1) \in H \times H$, and every $\varepsilon > 0$, there exists $u \in L^2(0,T;U)$ s.t.

$$\|y(T; y_0, u) - y_1\|_H \leq \varepsilon.$$

Remark

Problem (1) is **linear**. Then, System (1) is **null controllable** at time T **if and only if** the system is **exactly controllable to the trajectories** at time T. ■

Remark

We will deal with parabolic problems. So, due to the regularizing effect of these problems, it is well-known that the exact controllability result fails. Therefore, in this course we will study **null** or **approximate controllability** results for the system under consideration.

2. The parabolic scalar case

2. The parabolic scalar case

In this course we are going to deal with **time-dependent second order elliptic** operators. Thus, let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial\Omega$ of class C^2 and let us fix $T > 0$.

Notation: $Q_T = \Omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$ and, for $\mathcal{O} \subseteq \Omega$ or $\mathcal{O} \subseteq \partial\Omega$, $1_{\mathcal{O}}$ denotes the characteristic function of the set $\mathcal{O}$.

Let $L(t)$ be the operator given by:

$$(2) \qquad \boxed{L(t)y = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\alpha_{ij}(x,t)\frac{\partial y}{\partial x_j}\right) + D(x,t)\cdot\nabla y + c(x,t)y.}$$

The coefficients of L satisfy

$$(3) \qquad \begin{cases} \alpha_{ij} \in W^{1,\infty}(Q_T)\ (1 \leq i,j \leq N),\ D \in L^\infty(Q_T;\mathbb{R}^N),\ c \in L^\infty(Q_T), \\ \alpha_{ij}(x,t) = \alpha_{ji}(x,t) \quad \forall (x,t) \in Q_T, \end{cases}$$

and the **uniform elliptic condition**: there exists $a_0 > 0$ such that

$$(4) \qquad \sum_{i,j=1}^{N} \alpha_{ij}(x,t)\xi_i\xi_j \geq a_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad \forall (x,t) \in Q_T.$$

2. The parabolic scalar case

Let $\omega \subseteq \Omega$ be an open subset, $\Gamma_0 \subseteq \partial\Omega$ a relative open subset and let us fix $T > 0$.

We consider the **linear** problems for the **operator** $L(t)$:

(5)
$$\begin{cases} \partial_t y + L(t)y = v 1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

(6)
$$\begin{cases} \partial_t y + L(t)y = 0 & \text{in } Q_T, \\ y = h 1_{\Gamma_0} \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

In (5) and (6), $y(x,t)$ is the state, y_0 is the initial datum and v and h are the control functions (which are localized in ω -**distributed control**- or on Γ_0 -**boundary control**-).

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Question: Functional spaces for y_0, v and h?

CONTROL SPACES:

- **Distributed control problem**: We can take $L^2(Q_T)$ as control space and $L^2(\Omega)$ as initial datum space. The problem is well-posed: $\forall y_0 \in L^2(\Omega)$ and $v \in L^2(Q_T)$ there exists a unique weak solution to (5) $y \in C^0([0,T]; L^2(\Omega))$ which depends continuously on the data.

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- **Boundary control problem**:
  1. If in (2), $D \equiv 0$ in Q_T, we can take $L^2(\Sigma_T)$ as control space and $H^{-1}(\Omega)$ as initial datum space. Again, the problem is well-posed: $\forall y_0 \in H^{-1}(\Omega)$ and $h \in L^2(\Sigma_T)$ there exists a unique weak solution to (6) $y \in C^0([0,T];H^{-1}(\Omega))$ which depends continuously on the data. **Solution defined by transposition**.

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  2. In the general case, we can take $L^2(\Omega)$ as initial datum space and

$$X(\Gamma_0) = \{h : h = H|_{\Sigma_T} \text{ with } H \in L^2(0,T; H_0^1(\widetilde{\Omega})),\ H_t \in L^2(0,T; H^{-1}(\widetilde{\Omega}))\},$$

as control space, where $\widetilde{\Omega}$ is an open set s.t. $\Omega \subset \widetilde{\Omega}$, $\partial\Omega \cap \widetilde{\Omega} \subset\subset \Gamma_0$ and $\widetilde{\Omega} \setminus \overline{\Omega} \neq \emptyset$. The problem is well-posed and the solution depends continuously on the data.

Theorem

Let us fix $T > 0$. The following conditions are equivalent

1. *For any $\Omega \subset \mathbb{R}^N$, bounded open set with Ω having a C^2 boundary, any $\omega \subset \Omega$, nonempty open subset, and any coefficients α_{ij} $(1 \le i,j \le N)$, D and c, satisfying* (3) *and* (4)*, System* (5) *is null controllable in $L^2(\Omega)$ at time $T > 0$ with* ***distributed controls*** *$v \in L^2(Q_T)$.*
2. *For any $\Omega \subset \mathbb{R}^N$, bounded open set with Ω having a C^2 boundary, any $\Gamma_0 \subset \partial\Omega$, nonempty relative open subset, and any coefficients α_{ij} $(1 \le i,j \le N)$, D and c, satisfying* (3) *and* (4)*, System* (6) *is null controllable in $L^2(\Omega)$ at time $T > 0$ with* ***boundary controls*** *$h \in L^2(0,T;H^{1/2}(\partial\Omega))$.*

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Proof: We will use in a fundamental way that the problem under consideration is **scalar** (in fact, same number of equations and controls). We follow some ideas from [Bodart,G.-B.,Pérez-García] Comm. PDE (2004) and [G.-B.,Pérez-García] Asymp. Anal. (2006). · · ·

Remark (Regularizing effect)

The previous proof shows that if the distributed and boundary null controllability results for Systems (5) and (6) are valid with controls in $L^2(Q_T)$ and $L^2(0,T;H^{1/2}(\partial\Omega))$, then the previous systems are null controllable with controls in $L^\infty(Q_T)$ and $L^\infty(\Sigma_T)$ (and even better for regular coefficients). ■

Remark

In the proof of Theorem 1 we have strongly used that the operator $\partial_t + L(t)$ is scalar. We will see that the previous equivalence is not valid for non-scalar parabolic operators. ■

2. The parabolic scalar case

From now on, we will concentrate on the distributed control problem (5).
Let us introduce the **adjoint problem**

$$(7) \qquad \begin{cases} -\partial_t \varphi + L^*(t)\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_T & \text{in } \Omega, \end{cases}$$

where $\varphi_T \in L^2(\Omega)$ is given and $L^*(t)$ is the operator given by

$$L^*(t)\varphi = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\alpha_{ij}(x,t)\frac{\partial \varphi}{\partial x_j}\right) - \nabla \cdot (D\varphi) + c(x,t)\varphi \text{ a.e. in } Q_T.$$

This problem is also well-posed and the solution depends continuously on φ_T: there exists a constant $\widetilde{C} > 0$ such that $\forall \varphi_T \in L^2(\Omega)$ System (7) has only **one solution** $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$ and it satisfies

$$\|\varphi\|_{L^2(0,T;H_0^1(\Omega))} + \|\varphi\|_{C^0([0,T];L^2(\Omega))} \leq \widetilde{C}\|\varphi_T\|_{L^2(\Omega)}.$$

Theorem (Observability Inequality)

Under the previous assumptions, System (5) *is null controllable at time* $T > 0$ *if and only if there exists a constant* $C_T > 0$ *s.t.*

$$(8) \qquad \|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leq C_T \iint_{\omega\times(0,T)} |\varphi|^2 dxdt, \quad \forall \varphi_T \in L^2(\Omega),$$

where φ *is the solution of* (7) *associated to* φ_T.

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Remark

The **Observability Inequality** (8) in particular implies a better result: If (8) holds then, $\forall y_0 \in L^2(\Omega)$ there is a distributed control $v \in L^2(Q_T)$ s.t.

$$\|v\|^2_{L^2(Q_T)} \leq C_T\|y_0\|^2_{L^2(\Omega)} \quad \text{and} \quad y(\cdot,T) = 0,$$

being y the solution to (5) corresponding to y_0 and $C_T > 0$ the constant in (8).

Remark (Control cost)

The previous remark and inequality (8) provide an estimate of **the cost of the control** for system (5): If (8) holds at time $T > 0$, then

$$\mathcal{Z}_T(y_0) := \{v \in L^2(Q_T) : y(T; y_0, v) = 0\} \neq \emptyset, \quad \forall y_0 \in L^2(\Omega).$$

We can then define the control cost for system (5) at time T as

$$\mathcal{K}(T) = \sup_{\|y_0\|_{L^2(\Omega)}=1} \left(\inf_{v\in\mathcal{Z}_T(y_0)} \|v\|_{L^2(Q_T)} \right), \quad \forall T > 0.$$

Thus, $\mathcal{K}(T) \leq \sqrt{C_T}$. On the other hand, if $\mathcal{Z}_T(y_0) \neq \emptyset$, for any $y_0 \in L^2(\Omega)$, then, the observability inequality (8) for the adjoint system (7) holds with $C_T = \mathcal{K}(T)^2$. It is then clear that

$$\mathcal{K}(T) = \inf \left\{ \sqrt{C_T} : C_T > 0 \text{ is such that (8) holds} \right\}.$$

2. The parabolic scalar case

1. The one-dimensional case: The moment method

We follow [FATTORINI,RUSSELL] Arch. Rat. Mech. Anal. (1971).

Consider the boundary null controllability problem for the classical one-dimensional heat equation in $(0, \pi)$ (for simplicity):

$$(9) \qquad \begin{cases} y_t - y_{xx} = 0 & \text{in } Q_T = (0,\pi) \times (0,T), \\ y(0,\cdot) = v, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

with $y_0 \in H^{-1}(0,\pi)$ and $v \in L^2(0,T)$. The problem is **well-posed** and the solution (defined by transposition) depends continuously on the data y_0 and v. The operator $-\partial_{xx}$ on $(0,\pi)$ with homogenous Dirichlet boundary conditions admits a sequence of eigenvalues and normalized eigenfunctions given by

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0,\pi)$$

which is a Hilbert basis of $L^2(0,\pi)$. In the sequel, we will use the notation

$$y_k = (y, \phi_k)_{L^2(0,\pi)}, \quad \forall y \in L^2(0,\pi).$$

The idea of the **moment method** is simple: Given $y_0 \in H^{-1}(0,\pi)$, $\varphi_T \in H_0^1(0,\pi)$ and $v \in L^2(0,T)$, then

$$\langle y(\cdot,T), \varphi_T\rangle - \langle y_0, \varphi(\cdot,0)\rangle = \int_0^T v(t)\varphi_x(0,t)\,dt.$$

where y is the solution to (9) and φ is the solution to the **adjoint problem**

$$\begin{cases} -\varphi_t - \varphi_{xx} = 0 & \text{in } Q_T, \\ \varphi = 0 \text{ on } \{0,1\}\times(0,T), \quad \varphi(\cdot,T) = \varphi_T & \text{in } (0,\pi). \end{cases}$$

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Property

$v \in L^2(0,\pi)$ is a **null control** for system (9) (i.e., $v \in L^2(0,T)$ is a control s.t. the solution y to (9) satisfies $y(\cdot,T) = 0$ in $(0,\pi)$) if and only if

$$-\langle y_0, \varphi(\cdot,0) \rangle = \int_0^T v(t) \varphi_x(0,t)\, dt, \quad \forall \varphi_T \in H_0^1(0,\pi).$$

Given $y_0 \in H^{-1}(0,\pi)$, there exists a control $v \in L^2(0,T)$ such that the solution y to (9) satisfies $y(\cdot,T) = 0$ in $(0,\pi)$ if and only if there exists $\boxed{v \in L^2(0,T)}$ satisfying

$$\boxed{-\langle y_0, e^{-\lambda_k T}\phi_k\rangle = \int_0^T v(t) e^{-\lambda_k (T-t)} \phi_{k,x}(0)\, dt, \quad \forall k \geq 1,}$$

i.e., if and only if $\boxed{v \in L^2(0,T)}$ and

$$\boxed{\int_0^T e^{-\lambda_k t} v(T-t)\, dt = -\frac{1}{k}\sqrt{\frac{\pi}{2}} e^{-\lambda_k T} y_{0,k} \equiv c_k \quad \forall k \geq 1.}$$

This problem is called a **moment problem**.

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This problem is called a **moment problem**. We have the following result:

Given $y_0 \in H^{-1}(0,\pi)$, there exists a control $v \in L^2(0,T)$ such that the solution y to (9) satisfies $y(\cdot,T) = 0$ in $(0,\pi)$ if and only if there exists $\boxed{v \in L^2(0,T)}$ satisfying

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This problem is called a **moment problem**. We have the following result:

Theorem

*For any $y_0 \in H^{-1}(0,\pi)$ and $T > 0$, there exists $v \in L^2(0,T)$ solution to the previous **moment problem**. That is, v is a **null control** for equation (9).*

Proof: Biorthogonal Families: ([FATTORINI, RUSSELL] Arch. Rat. Mech. Anal. (1971)). There exists a family $\{q_k\}_{k\geq 1} \subset L^2(0,T)$ satisfying

1. $\displaystyle\int_0^T e^{-\lambda_k t} q_l(t)\, dt = \delta_{kl}, \quad \forall k,l \geq 1.$
2. $\forall \varepsilon > 0,\ \exists C(\varepsilon, T) > 0$ s.t. $\|q_k\|_{L^2(0,T)} \leq C(\varepsilon,T) e^{\varepsilon \lambda_k}$.

The control is obtained as a linear combination of $\{q_k\}_{k\geq 1}$, that is,

$$v(T-t) = \sum_{k\geq 1} c_k\, q_k(t) = -\sqrt{\frac{\pi}{2}} \sum_{k\geq 1} \frac{1}{k} e^{-\lambda_k T} y_{0,k}\, q_k(t)$$

and the previous bounds are used to prove that this combination converges in $L^2(0,T)$. ■

Two ingredients:

Existence and **bounds** of a biorthogonal family to real exponentials.

Remark

Theorem 2.2 is a consequence of the existence of a biorthogonal family in $L^2(0,T)$ to the sequence $\{e^{-\lambda_k t}\}_{k\geq 1}$ ($\lambda_k = k^2$), which satisfies appropriate **bounds**. In fact, in

1. LUXEMBURG, KOREVAAR, Trans. Amer. Math. Soc. 157 (1971),
2. FATTORINI, RUSSELL, Quart. Appl. Math. 32 (1974/75),
3. HANSEN, J. Math. Anal. Appl. 158 (1991), ...

it is proved a general result on existence of a **biorthogonal family** in $L^2(0,T)$ to $\{e^{-\Lambda_k t}\}_{k\geq 1}$ which satisfies appropriate **bounds** for sequences $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{R}_+$ such that

$$\sum_{k\geq 1} \frac{1}{\Lambda_k} < \infty \quad \text{and} \quad |\Lambda_k - \Lambda_l| \geq \rho |k-l|, \quad \forall k,l \geq 1.$$

for a constant $\rho > 0$. ■

Consequence:

The previous result is valid for any nonempty bounded interval (a, b) and for any second order operator **self-adjoint elliptic operator**

$$Ly = -\left(\alpha(x)y_x\right)_x + c(x)y,$$

with $\alpha \in C^1([a, b])$ and $\alpha > 0$ in (a, b), and $c \in C^0([a, b])$. **Then**, if we apply Theorem 1, we also get a **distributed controllability** result for the problem

$$\begin{cases} y_t + Ly = v1_\omega & \text{in } Q_T = (a, b) \times (0, T), \\ y(a, \cdot) = 0, \quad y(b, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (a, b), \end{cases}$$

with $y_0 \in L^2(0, \pi)$ and $\omega \subseteq (a, b)$, a nonempty open subset.

2. The parabolic scalar case

2. General case: Carleman Inequalities

We follow [FURSIKOV,IMANUVILOV] 1996 and [IMANUVILOV,YAMAMOTO] 2003.

We will consider the following parabolic equation:

$$(10) \qquad \begin{cases} -\partial_t z + L_0(t) z = F_0 + \sum_{i=1}^{N} \frac{\partial F_i}{\partial x_i} & \text{in } Q_T, \\ z = 0 \text{ on } \Sigma_T, \quad z(\cdot, T) = z_T & \text{in } \Omega, \end{cases}$$

with $z_T \in L^2(\Omega)$, $F_i \in L^2(Q_T)$, $i = 0, 1, \ldots, N$, and $L_0(t)$ the self-adjoint parabolic operator given by

$$L_0(t) y = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(\alpha_{ij}(x,t) \frac{\partial y}{\partial x_j} \right)$$

with coefficients α_{ij} satisfying (3) (regularity) and (4) (uniform elliptic condition).

Lemma

*Let $\mathcal{B} \subset \Omega$ be a nonempty open subset and $d \in \mathbb{R}$. Then, $\exists\ \beta_0 \in C^2(\overline{\Omega})$ (**positive** and only depending on Ω and $\mathcal{B}$) and $\widetilde{C}_0, \widetilde{\sigma}_0 > 0$ (only depending on Ω, $\mathcal{B}$ and d) s.t. for every $z_T \in L^2(\Omega)$, the solution z to (10) satisfies*

(11)

$$\begin{cases} \mathcal{I}(d,z) \le \widetilde{C}_0 \left(s^d \iint_{\mathcal{B}\times(0,T)} e^{-2s\beta} \gamma(t)^d |z|^2 \right. \\ \left. + s^{d-3} \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-3} |F_0|^2 + s^{d-1} \sum_{i=1}^{N} \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-1} |F_i|^2 \right), \end{cases}$$

$\forall s \ge \widetilde{s}_0 = \widetilde{\sigma}_0 \, (T + T^2)$; $\boxed{\gamma(t) = t^{-1}(T-t)^{-1}}$, $\boxed{\beta(x,t) = \beta_0(x)/t(T-t)}$

and $\boxed{\mathcal{I}(d,z) \equiv s^{d-2} \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-2} |\nabla z|^2 + s^d \iint_{Q_T} e^{-2s\beta} \gamma(t)^d |z|^2}$.

Lemma

When $F_i \equiv 0$ *for* $1 \leq i \leq N$, $\exists\ \widetilde{C}_1$ *and* $\widetilde{\sigma}_1$ *(which only depend on* Ω, $\mathcal{B}$ *and* d*) s.t.,* $\forall z_T \in L^2(\Omega)$, *the solution* z *to* (10) *satisfies*
(12)

$$\mathcal{I}_1(d,z) \leq \widetilde{C}_1 \left(s^d \iint_{\mathcal{B}\times(0,T)} e^{-2s\beta}\gamma(t)^d |z|^2 + s^{d-3} \iint_{Q_T} e^{-2s\beta}\gamma(t)^{d-3} |F_0|^2 \right),$$

for all $s \geq \widetilde{s}_1 = \widetilde{\sigma}_1 \left(T + T^2\right)$ *where*

$$\mathcal{I}_1(d,z) \equiv s^{d-4} \iint_{Q_T} e^{-2s\beta}\gamma(t)^{d-4} \left(|\partial_t z|^2 + \sum_{i,j=1}^{N} \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 \right) + \mathcal{I}(d,z)\,.$$

Proof: See [FURSIKOV,IMANUVILOV] 1996; [IMANUVILOV,YAMAMOTO] (2003) and [FERNÁNDEZ-CARA,GUERRERO] SICON (2006).

Recall that our objective is to prove a **null controllability** result at time T for

$$(5)\qquad \boxed{\begin{cases} \partial_t y + L(t)y = v1_{\omega} & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases}}$$

with $L(t)$ given by:

$$\begin{cases} L(t)y = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\alpha_{ij}(x,t)\frac{\partial y}{\partial x_j}\right) + D(x,t)\cdot\nabla y + c(x,t)y \\ \qquad\quad = L_0(t)y + D(x,t)\cdot\nabla y + c(x,t)y, \end{cases}$$

with coefficients α_{ij} satisfying (3) and (4). We also know that this is equivalent to the **observability inequality** (8)

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \le C_T \iint_{\omega\times(0,T)} |\varphi|^2 dxdt, \quad \forall \varphi_T \in L^2(\Omega),$$

for the solutions to the **adjoint problem** (7).

Corollary

There exists a constant $C_0 = C_0(\Omega, \omega) > 0$ *such that* $\forall \varphi_T \in L^2(\Omega)$ *and* φ *the corresponding solution to* (7), *the* ***observability inequality*** (8) *holds with*

$$C_T = \exp\left(C_0 \left(1 + \frac{1}{T} + \|c\|_\infty^{2/3} + T\|c\|_\infty + (1+T)\|D\|_\infty^2 \right) \right).$$

Corollary

There exists a constant $C_0 = C_0(\Omega, \omega) > 0$ such that $\forall \varphi_T \in L^2(\Omega)$ and φ the corresponding solution to (7)*, the* ***observability inequality*** (8) *holds with*

$$C_T = \exp\left(C_0 \left(1 + \frac{1}{T} + \|c\|_\infty^{2/3} + T\|c\|_\infty + (1+T)\|D\|_\infty^2 \right) \right).$$

Proof: We follow [FERNÁNDEZ-CARA,ZUAZUA] Ann. IHP (2000) and [DOUBOVA,FERNÁNDEZ-CARA,MG-B,ZUAZUA] SICON (2002).
The Carleman inequality (11) applied to problem (7) implies ($\mathcal{B} \equiv \omega$, $d = 3$ and $-\partial_t \varphi + L_0(t)\varphi = \nabla \cdot (D\varphi) - c(x,t)\varphi$) that $\forall s \geq \widetilde{s}_0 = \widetilde{\sigma}_0\,(T + T^2)$:

$$\begin{aligned} s \iint_{Q_T} e^{-2s\beta}\gamma(t)|\nabla\varphi|^2 + s^3 \iint_{Q_T} e^{-2s\beta}\gamma(t)^3|\varphi|^2 \\ \leq \widetilde{C}_0 \left(s^3 \iint_{\omega\times(0,T)} e^{-2s\beta}\gamma(t)^3|\varphi|^2 \right. \\ \left. + \|c\|_\infty^2 \iint_{Q_T} e^{-2s\beta}|\varphi|^2 + s^2\|D\|_\infty^2 \iint_{Q_T} e^{-2s\beta}\gamma(t)^2|\varphi|^2 \right). \end{aligned}$$

As a consequence we can prove that for
$s \geq C_1(T + T^2 + T^2(\|c\|_\infty^{2/3} + \|D\|_\infty^2))$ ($C_1 = C_1(\Omega, \omega)$) one has

$$[s\gamma(t)]^3 - \widetilde{C}_0\|c\|_\infty^2 - \widetilde{C}_0[s\gamma(t)]^2\|D\|_\infty^2 \geq \frac{1}{2}[s\gamma(t)]^3 .$$

Consequently, for $s = C_1(T + T^2 + T^2(\|c\|_\infty^{2/3} + \|D\|_\infty^2))$ that

$$\iint_{Q_T} e^{-2s\beta} t^{-3}(T-t)^{-3}|\varphi|^2 \leq \widetilde{C}_1 \iint_{\omega\times(0,T)} e^{-2s\beta} t^{-3}(T-t)^{-3}|\varphi|^2$$

and therefore

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2 \leq e^{C(1+1/T+\|c\|_\infty^{2/3}+\|D\|_\infty^2)} \iint_{\omega\times(0,T)} |\varphi|^2.$$

This last inequality combined with **energy estimates** ($C = C(a_0) > 0$)

$$\frac{d}{dt}\left(e^{C(\|c\|_\infty+\|D\|_\infty^2)t}\int_\Omega |\varphi|^2(\cdot,t)\right) \geq 0 \quad \forall t \in [0,T]$$

implies (8) and the proof is complete. ■

Corollary

Let us fix $T > 0$, $\Omega \subset \mathbb{R}^N$, $\omega \subseteq \Omega$ *and* $\Gamma_0 \subseteq \partial\Omega$ *(arbitrary) as before. Then, there exist positive constants* $C_0 = C_0(\Omega, \omega)$ *and* $\widehat{C}_0 = \widehat{C}_0(\Omega, \Gamma_0)$ *s.t.*

1. $\forall y_0 \in L^2(\Omega)$ *there is a control* $v \in L^2(\Omega)$ *which satisfies*

$$\|v\|^2_{L^2(Q_T)} \leq e^{C_0\left(1+1/T+\|c\|_\infty^{2/3}+T\|c\|_\infty+(1+T)\|D\|_\infty^2\right)}\|y_0\|^2_{L^2(\Omega)},$$

and $y(\cdot, T) = 0$ *in* Ω, *(*y *is the solution to* (5) *associated to* y_0 *and* v*).*

2. $\forall y_0 \in L^2(\Omega)$ *there is a control* $h \in L^2(0, T; H^{1/2}(\Omega))$ *which satisfies*

$$\|h\|^2_{L^2(0,T;H^{1/2}(\Omega))} \leq e^{\widehat{C}_0\left(1+1/T+\|c\|_\infty^{2/3}+T\|c\|_\infty+(1+T)\|D\|_\infty^2\right)}\|y_0\|^2_{L^2(\Omega)},$$

and $y(\cdot, T) = 0$ *in* Ω, *(*y *is the solution to* (6) *associated to* y_0 *and* v *and, in fact,* $y \in L^2(0, T; H^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$*).*

Remark

It is important to point out that the **boundary null controllability** result for problem (6), when the coefficient D of $L(t)$ (see (2)) is regular enough, can be obtained from an appropriate boundary Carleman inequality for problem (10) with $F_i \equiv 0$, $1 \leq i \leq N$. This Carleman inequality is like (12) for an appropriate weight function $\widetilde{\beta}_0 \in C^2(\overline{\Omega})$ (which depends only on Ω and Γ_0) instead of β_0 and with the local term

$$s^{d-2} \iint_{\Gamma_0 \times (0,T)} e^{-2s\frac{\widetilde{\beta}_0}{t(T-t)}} \gamma(t)^{d-2} \left| \frac{\partial z}{\partial n} \right|^2$$

instead of the integral over $\mathcal{B} \times (0,T)$ in the right hand side of (12) (z is the solution to (10) associated to $z_T \in L^2(\Omega)$). ∎

2. The parabolic scalar case

3. Final comments in the scalar case

1. The null controllability property for the N-dimensional case was solved independently by G. Lebeau and L. Robbiano (for the heat equation) and by A. Fursikov and O. Imanuvilov (for a general parabolic equation). With a different approach, Lebeau-Robbiano obtained the distributed null controllability result for System (5)

$$\boxed{\begin{cases} \partial_t y + L_0 y = v 1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}}$$

when L_0 is a self-adjoint elliptic operator independent of t. For more details, see [LEBEAU,ROBBIANO] Comm. P.D.E. (1995).

2. Until now, we have only dealt with the **null controllability** problem for a scalar parabolic system with distributed and boundary controls. For the corresponding **approximate controllability** we can obtain similar results:

Approximate controllability

Proposition (Distributed control)

System (5) is ***approximately controllable*** *at time $T > 0$ if and only if the* ***adjoint problem*** *(7) satisfies the* ***unique continuation property****: "If φ is a solution to (7) and $\varphi = 0$ in $\omega \times (0, T)$, then $\varphi \equiv 0$ in Q_T".* ■

Remark (Boundary control)

In the case of System (6) we can get a similar result. In this case the **unique continuation property** for System (7) is: "If φ is a solution to (7) and $\partial_n \varphi = 0$ on $\Gamma_0 \times (0, T)$, then $\varphi \equiv 0$ in Q_T".

Theorem

System (5) (resp. System (6)) is ***approximately controllable*** *at time $T > 0$, for any ω and $T > 0$ (resp., for any Γ_0 and T).* ■

Remark

The **distributed controllability** result for System (5) **is equivalent** to the **boundary controllability** result for System (6).

Summarizing:

- System (5) and system (6) are approximately controllable and exactly controllable to trajectories at any time $T > 0$ for every geometrical data ω or Γ_0.
- The controllability properties of both systems are equivalent. ■

SOME REFERENCES

1. H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.
2. G. LEBEAU, L. ROBBIANO, *Contrôle exact de l'équation de la chaleur*, Comm. P.D.E. 20 (1995), no. 1-2, 335–356.
3. O. YU. IMANUVILOV, *Controllability of parabolic equations*, (Russian) Sb. Math. 186 (1995), no. 6, 879–900.
4. A. FURSIKOV, O. YU. IMANUVILOV, *Controllability of Evolution Equations*, Lecture Notes Series 34, Seoul National Univ., Seoul, 1996.
5. O. YU. IMANUVILOV, M. YAMAMOTO, *Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations*, Publ. Res. Inst. Math. Sci. **39** (2003), no. 2, 227–274.

3. Finite-dimensional systems

3. Finite-dimensional systems

Let us consider the autonomous linear system

$$(13) \qquad y' = Ay + Bu \quad \text{on } [0,T], \quad y(0) = y_0,$$

where $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ are constant matrices, $y_0 \in \mathbb{C}^n$ and $u \in L^2(0,T;\mathbb{C}^m)$ is the control.

Problem:

Given $y_0, y_d \in \mathbb{C}^n$, is there a control $u \in L^2(0,T;\mathbb{C}^m)$ such that the solution y to the problem satisfies

$$y(T) = y_d????$$

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Problem:

Given $y_0, y_d \in \mathbb{C}^n$, is there a control $u \in L^2(0,T;\mathbb{C}^m)$ such that the solution y to the problem satisfies

$$y(T) = y_d????$$

Let us define (*controllability matrix*)

$$\boxed{[A\,|\,B] = (B\,,\,AB\,,\,A^2B\,,\,\cdots\,,\,A^{n-1}B) \in \mathcal{L}(\mathbb{C}^{nm};\mathbb{C}^n).}$$

On the other hand, let $\{\theta_l\}_{1\le l\le \hat p} \subset \mathbb{C}$ be the set of distinct eigenvalues of A^*. For $l : 1 \le l \le \hat p$, we denote by m_l the geometric multiplicity of θ_l. The sequence $\{w_{l,j}\}_{1\le j\le m_l}$ will denote a basis of the eigenspace associated to θ_l.

3. Finite-dimensional systems

The following classical result can be found in

R. KALMAN, Y.-CH. HO, K. NARENDRA, *Controllability of linear dynamical systems*, 1963.

and gives a complete answer to the problem of controllability of finite dimensional autonomous linear systems:

Theorem

Under the previous assumptions, the following conditions are equivalent

1. *System* (13) *is exactly controllable at time T, for every $T > 0$.*
2. *There exists $T > 0$ such that system* (13) *is exactly controllable at time T.*
3. $\operatorname{rank}\,[A \mid B] = n$ *or* $\ker[A \mid B]^* = \{0\}$ *(Kalman rank condition).*
4. *Hautus test:* $\operatorname{rank} \begin{pmatrix} A^* - \theta_l I_n \\ B^* \end{pmatrix} = n, \quad \forall l : 1 \le l \le \hat{p}.$
5. $\operatorname{rank}\,[B^* w_{l,1}\,,\, B^* w_{l,2}\,,\, \cdots\,,\, B^* w_{l,m_l}] = m_l$*, for every* $l : 1 \le l \le \hat{p}$. ■

Remark

1. The four controllability concepts (exact, exact to trajectories, null and approximate controllability) for System (13) are equivalent (**finite-dimensional space**).
2. Observe that $\{B^* w_{l,1}, B^* w_{l,2}, \dots, B^* w_{l,m_l}\} \subset \mathbb{C}^m$. Condition 5 in Theorem 4 says this set is linearly independent for any $l : 1 \le l \le \hat{p}$. In particular, $\boxed{m_l \le m}$ $\forall l : 1 \le l \le \hat{p}$.
3. Given the o.d.s. (**adjoint problem**)

$$-\varphi' = A^* \varphi \quad \text{in } [0, T], \quad \varphi(T) = \varphi_T \in \mathbb{C}^n,$$

it is not difficult to prove the following result: "*System* (13) *is* ***exactly controllable*** *at time* T ***if and only if*** the following property for the **adjoint problem** holds (**unique continuation property**)

If $B^* \varphi(\cdot) = 0$ on $[0, T]$, then $\varphi_T \equiv 0$."

Goal

We have a complete characterization of the controllability results for finite-dimensional linear ordinary differential systems (a Kalman condition). Is it possible to obtain similar results for Partial Differentials Systems? We will focus on coupled linear parabolic systems.

What are the possible generalizations to Systems of Parabolic Equations?

4. Distributed controllability of 2×2 linear systems

4. Distributed controllability of 2 × 2 linear systems

Let us consider the 2×2 linear reaction-diffusion system ($Q_T = \Omega \times (0,T)$)

$$(14) \quad \begin{cases} \partial_t y_1 + L_0^1(t) y_1 + a_{11} y_1 + a_{12} y_2 = v 1_\omega & \text{in } Q_T, \\ \partial_t y_2 + L_0^2(t) y_2 + a_{21} y_1 + a_{22} y_2 = 0 & \text{in } Q_T, \\ y_i = 0 \text{ on } \Sigma_T = \partial\Omega \times (0,T), \quad y_i(\cdot,0) = y_0^i \text{ in } \Omega, & 1 \le i \le 2, \end{cases}$$

where Ω, ω and T are as before, $a_{ij} = a_{ij}(x,t) \in L^\infty(Q_T)$ $(1 \le i,j \le 2)$, $y_0^i \in L^2(\Omega)$ $(1 \le i \le 2)$ and $L_0^k(t)$ is, for every $1 \le k \le 2$, the second order operator $\boxed{L_0^k(t) y = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(\alpha_{ij}^k(x,t) \frac{\partial y}{\partial x_j}\right)}$ where α_{ij}^k satisfy (3) and (4).

Remark

System (14) is controlled by means of a scalar distributed control exerted on the right-hand side of the first equation. The second equation is indirectly controlled by the coupling term $\boxed{a_{21} y_1}$. **Necessary condition** $\boxed{a_{21} \not\equiv 0}$ ($a_{21} \in L^\infty(Q_T)$).

Equivalently, the previous system can be written as

$$(15)\qquad \begin{cases} \partial_t y + \widehat{L}(t)y + Ay = Bv1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases}$$

where $\widehat{L}(t)$ is the **matrix operator** given by $\widehat{L}(t) = \operatorname{diag}\left(L_0^1(t), L_0^2(t)\right)$, $y = (y_i)_{1\le i\le 2}$ is the state and where

$$\begin{cases} y_0 = (y_0^i)_{1\le i\le 2} \in L^2(\Omega;\mathbb{R}^n), \quad A(\cdot,\cdot) = (a_{ij}(\cdot,\cdot))_{1\le i,j\le 2} \in L^\infty(Q_T;\mathcal{L}(\mathbb{R}^n)), \\ \text{and } B \equiv e_1 = (1,0)^* \in \mathbb{R}^2 \end{cases}$$

are given. Let us observe that, for each $y_0 \in L^2(\Omega;\mathbb{R}^2)$ and $v \in L^2(Q_T)$, System (15) admits a **unique weak solution**

$$y \in L^2(0,T;H_0^1(\Omega;\mathbb{R}^2)) \cap C^0([0,T];L^2(\Omega;\mathbb{R}^2)).$$

Assumption

We assume that the coupling coefficient $a_{21} \in L^\infty(Q_T)$ satisfies

$$(16) \qquad \boxed{a_{21} \geq c_0 > 0} \text{ or } \boxed{-a_{21} \geq c_0 > 0} \text{ in } \omega_0 \times (0,T),$$

with $\omega_0 \subseteq \omega$ a new open subset.

As in the scalar case, the controllability result for system (15) is equivalent to the observability inequality: $\exists C_T > 0$ such that

$$\|\varphi_1(\cdot,0)\|_{L^2}^2 + \|\varphi_2(\cdot,0)\|_{L^2}^2 \leq C_T \iint_{\omega \times (0,T)} |\varphi_1(x,t)|^2 \, dx \, dt,$$

where φ is the solution associated to $\varphi_0 \in L^2(\Omega;\mathbb{R}^2)$ of the adjoint problem:

$$(17) \qquad \begin{cases} -\varphi_t + \widehat{L}(t)\varphi + A^*\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot,T) = \varphi_0 & \text{in } \Omega. \end{cases}$$

4. Distributed controllability of 2×2 linear systems

Theorem

Under assumption (16), *there exist a positive function* $\alpha_0 \in C^2(\overline{\Omega})$ *(only depending on* Ω *and* ω_0*), two positive constants* C_0 *and* σ_0 *(only depending on* Ω*,* ω_0*,* c_0*,* $\|a_{21}\|_\infty$ *and* d*) such that, for every* $\varphi_T \in L^2(Q_T;\mathbb{R}^2)$*, the solution* φ *to the* ***adjoint problem*** (17) *satisfies*

$$\mathcal{I}_1(d+3,\varphi_1) + \mathcal{I}_1(d,\varphi_2) \leq C_0 s^{d+4} \iint_{\omega\times(0,T)} e^{-2s\alpha}\gamma(t)^{d+4}|\varphi_1|^2,$$

$\forall s \geq s_0 = \sigma_0 \left[T + T^2 + T^2\left(\|a_{11}\|_\infty^{2/3} + \|a_{12}\|_\infty^{1/3} + \|a_{22}\|_\infty^{2/3}\right)\right]$. *In the previous inequality,* $\boxed{\gamma(t) = t^{-1}(T-t)^{-1}}$, $\boxed{\alpha(x,t) = \alpha_0(x)/t(T-t)}$ *and* $\mathcal{I}_1(d,z)$ *is given in Lemmas 2.3 and 2.4 (with* α *instead of* β*).*

Proof: Given $\omega_0 \subset \omega$, we choose $\omega_1 \subset\subset \omega_0$. Let $\alpha_0 \in C^2(\overline{\Omega})$ be the function provided by Lemma 2.3 and associated to Ω and $\mathcal{B} \equiv \omega_1$. We will also consider $\boxed{\alpha(x,t) = \alpha_0(x)/t(T-t)}$ and $\boxed{\gamma(t) = t^{-1}(T-t)^{-1}}$. We will do the proof in two steps:

Step 1. Let φ be the solution to **adjoint system** associated to φ_T. Each component satisfies

$$-\partial_t \varphi_i + L_0^i(t)\varphi_i = \boxed{-a_{1i}\varphi_1 - a_{2i}\varphi_2}.$$

We begin applying inequality (12) with $\mathcal{B} = \omega_1$ to each function φ_i with $L_0 \equiv L_0^i$, $d = d + 3(2-i)$ and the corresponding right-hand side:

$$\mathcal{I}_1(d+3, \varphi_1) \leq \widetilde{C}_1 \left(\iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \left[s\gamma(t) \right]^{d+3} |\varphi_1|^2 \right.$$

$$\left. + \|a_{11}\|_\infty^2 \iint_{Q_T} e^{-2s\alpha} \left[s\gamma(t) \right]^{d} |\varphi_1|^2 + \|a_{21}\|_\infty^2 \iint_{Q_T} e^{-2s\alpha} \left[s\gamma(t) \right]^{d} |\varphi_2|^2 \right),$$

$$\mathcal{I}_1(d+3,\varphi_1) \le \widetilde{C}_1 \left(\iint_{\omega_1\times(0,T)} e^{-2s\alpha} \left[s\gamma(t)\right]^{d+3} |\varphi_1|^2 \right.$$
$$\left. + \|a_{11}\|_\infty^2 \iint_{Q_T} e^{-2s\alpha} \left[s\gamma(t)\right]^{d} |\varphi_1|^2 + \|a_{21}\|_\infty^2 \iint_{Q_T} e^{-2s\alpha} \left[s\gamma(t)\right]^{d} |\varphi_2|^2 \right),$$

and

$$\mathcal{I}_1(d,\varphi_2) \le \widetilde{C}_1 \left(\iint_{\omega_1\times(0,T)} e^{-2s\alpha} \left[s\gamma(t)\right]^{d} |\varphi_2|^2 \right.$$
$$\left. + \|a_{12}\|_\infty^2 \iint_{Q_T} e^{-2s\alpha} \left[s\gamma(t)\right]^{d-3} |\varphi_1|^2 + \|a_{22}\|_\infty^2 \iint_{Q_T} e^{-2s\alpha} \left[s\gamma(t)\right]^{d-3} |\varphi_2|^2 \right)$$

for all $s \ge \widetilde{s}_1 = \widetilde{\sigma}_1 \left(T + T^2\right)$.

Now if we take

$$s \geq s_1 = \sigma_1 \left[T + T^2 + T^2 \left(\|a_{11}\|_\infty^{2/3} + \|a_{12}\|_\infty^{1/3} + \|a_{22}\|_\infty^{2/3} \right) \right],$$

with $\sigma_1 = \sigma_1(\Omega, \omega_0, \|a_{21}\|_\infty) > 0$, we obtain the existence of a positive constants $C_1 = C_1(\Omega, \omega_0, \|a_{21}\|_\infty)$ such that if $s \geq s_1$, then

$$\mathcal{I}_1(d+3, \varphi_1) \leq C_1 \left(\iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \left[s\gamma(t) \right]^{d+3} |\varphi_1|^2 + \mathcal{I}_1(d, \varphi_2) \right)$$

and

$$\mathcal{I}_1(d, \varphi_2) \leq C_1 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \left[s\gamma(t) \right]^{d} |\varphi_2|^2 + \frac{1}{4C_1} \mathcal{I}_1(d+3, \varphi_1).$$

From these two previous inequalities we can also get

$$\mathcal{I}_1(d+3,\varphi_1)+\mathcal{I}_1(d,\varphi_2) \le C_2 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\left[s\gamma(t)\right]^{d+3}|\varphi_1|^2 + C_2 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\left[s\gamma(t)\right]^{d}|\varphi_2|^2,$$

$\forall s \ge s_1$, with $C_2 = C_2(\Omega, \omega_0, \|a_{21}\|_\infty)$ a new positive constant.
Step 2. Thanks to the assumption (16):

(16) $\boxed{a_{21} \ge c_0 > 0}$ or $\boxed{-a_{21} \ge c_0 > 0}$ in $\omega_0 \times (0,T)$,

with $\omega_0 \subseteq \omega$ an open subset, and the cascade structure

$$a_{21}\varphi_2 = \partial_t \varphi_1 - L_0^1(t)\varphi_1 - a_{11}\varphi_1 \text{ in } Q_T,$$

can eliminate the second local terms. In order to carry this process out, we will need the following result:

Lemma

Let us assume (16)*. Then, given* $\varepsilon > 0$*, there exist a constant* $\widetilde{C}_2$ *(only depending on* Ω*,* c_0 *and* $\|a_{21}\|_\infty$*), such that, if* $s \geq s_1$*, one has*

$$\iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \left[s\gamma(t)\right]^d |\varphi_2|^2 \leq \varepsilon \mathcal{I}_1(d, \varphi_2) + \widetilde{C}_2 \left(1 + \frac{1}{\varepsilon}\right) \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} \left[s\gamma(t)\right]^{d+4} |\varphi_1|^2.$$

The proof of Theorem 4.1 is a consequence of this Lemma and the inequality

$$\mathcal{I}_1(d+3, \varphi_1) + \mathcal{I}_1(d, \varphi_2) \leq C_2 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \left[s\gamma(t)\right]^{d+3} |\varphi_1|^2 + C_2 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \left[s\gamma(t)\right]^{d} |\varphi_2|^2.$$

This ends the proof. ■

Summarizing

We have proved that the solutions to the **adjoint system**

(17) $$\begin{cases} -\varphi_t + \widehat{L}(t)\varphi + A^*\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } \Omega. \end{cases}$$

satisfy the **Carleman inequality** $C_0 = C_0(\Omega, \omega_0, c_0, \|a_{21}\|_\infty, d)$

$$\mathcal{I}_1(d+3, \varphi_1) + \mathcal{I}_1(d, \varphi_2) \le C_0 s^{d+4} \iint_{\omega \times (0,T)} e^{-2s\alpha} \gamma(t)^{d+4} |\varphi_1|^2,$$

$\forall s \ge s_0 = \sigma_0 \left[T + T^2 + T^2 \left(\|a_{11}\|_\infty^{2/3} + \|a_{12}\|_\infty^{1/3} + \|a_{22}\|_\infty^{2/3} \right) \right]$.
($C_0 = C_0(\Omega, \omega_0, c_0, \|a_{21}\|_\infty, d)$ and $\sigma_0 = \sigma_0(\Omega, \omega_0, c_0, \|a_{21}\|_\infty, d)$ are positive constants).

4. Distributed controllability of 2×2 linear systems

As in the scalar case, combining the previous result and **energy inequalities** satisfied by the solutions of the **adjoint system** it is possible to prove an **observability inequality** for the **adjoint system** and deduce:

Corollary

Let us assume (16). *Then, there exists a positive constant C (only depending on Ω, ω, c_0 and $\|a_{21}\|_\infty$) such that for every $y_0 \in L^2(\Omega;\mathbb{R}^2)$ there is a control $v \in L^2(\Omega)$ which satisfies*

$$\|v\|\|^2_{L^2(Q_T)} \le e^{C\mathcal{H}}\|y_0\|^2_{L^2(\Omega;\mathbb{R}^2)},$$

and $y(\cdot,T) = 0$ in Ω, with y the solution to (15) *associated to y_0 and v. In the previous inequality, $\mathcal{H}$ is given by*

$$\mathcal{H} \equiv 1 + T + \frac{1}{T} + \|a_{11}\|^{2/3}_\infty + \|a_{12}\|^{1/3}_\infty + \|a_{22}\|^{2/3}_\infty + T \max_{1\le i,j\le 2}\|a_{ij}\|_\infty.$$

Remark

- System (14) is always controllable if we exert a control in each equation (**two controls**).
- The controllability result for system (14) is **independent** of the operators $L_0^1(t)$ and $L_0^2(t)$. We will see that the situation is more intricate if in the system a general control vector $B \in \mathbb{R}^2$ is considered.
- The same result can be obtained for the distributed approximate controllability at time T. Therefore, **approximate** and **null controllability** are equivalent concepts (distributed case).
- Using a different technique (**fictitious controls**), it is possible to prove a null controllability result as in the previous corollary when the coupling matrix $A \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^2))$ satisfies: There exist an open subset $\omega_0 \subset\subset \omega$ and a positive constant a_0 s.t.

$$\boxed{|a_{21}(x,t)| \geq a_0 > 0} \quad \text{in } \omega_0 \times (0,T). \qquad \blacksquare$$

References

1. L. DE TERESA, *Insensitizing controls for a semilinear heat equation*, Comm. Partial Differential Equations 25 (2000), no. 1–2, 39–72.
2. F. AMMAR KHODJA, A. BENABDALLAH, C. DUPAIX ET I. KOSTIN, *Controllability to the trajectories of phase-field models by one control force*, SIAM J. Control Optim. 42 (2003), no. 5, 1661-1689.
3. M. G.-B., R. PÉREZ-GARCÍA, *Controllability results for some nonlinear coupled parabolic systems by one control force*, Asymptot. Anal. 46 (2006), no. 2, 123–162.
4. M. G.-B., L. DE TERESA, *Controllability results for cascade systems of m coupled parabolic PDEs by one control force*, Port. Math. 67 (2010), no. 1, 91–113.

5. Boundary controllability of a 2×2 linear system

5. Boundary controllability of a 2×2 linear system

Let us now consider the boundary controllability problem for the one-dimensional linear reaction-diffusion system:

$$(18) \quad \begin{cases} y_t - D y_{xx} = A y & \text{in } Q_T = (0,\pi) \times (0,T), \\ y|_{x=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, \quad y|_{x=\pi} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

with $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, $v \in L^2(0,T)$ is the control and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \boxed{(d_1 \neq d_2)}, \text{ and } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Existence and uniqueness

For any $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$ and $v \in L^2(0,T)$, system (18) has a unique solution $y \in L^2(Q_T) \cap C^0([0,T];H^{-1}(0,\pi;\mathbb{R}^2))$ defined by transposition.

5. Boundary controllability of a 2×2 linear system

Let us now consider the boundary controllability problem for the one-dimensional linear reaction-diffusion system:

$$
(18) \qquad \begin{cases} y_t - D y_{xx} = A y & \text{in } Q_T = (0,\pi) \times (0,T), \\ y|_{x=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, \quad y|_{x=\pi} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}
$$

with $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, $v \in L^2(0,T)$ is the control and

$$
D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \ \boxed{(d_1 \neq d_2)}, \text{ and } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
$$

Question

Are the controllability properties of system (18) independent of d_1 and d_2???

NO.

As before, system (18) is null controllable at time T if and only if the observability inequality

$$\|\varphi_1(\cdot,0)\|^2_{H^1_0(0,\pi)} + \|\varphi_2(\cdot,0)\|^2_{H^1_0(0,\pi)} \leq C_T \int_0^T |\varphi_{1,x}(0,t)|^2\,dt,$$

holds. Again φ is the solution associated to $\varphi_0 \in H^1_0(0,\pi;\mathbb{R}^2)$ of the adjoint problem:

$$(19) \qquad \begin{cases} -\varphi_t - D\varphi_{xx} = A^*\varphi & \text{in } Q_T, \\ \varphi|_{x=0} = \varphi|_{x=\pi} = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi). \end{cases}$$

Let us see that, in general, this inequality fails (**even if** $a_{21} = 1 \neq 0$**!!!!!**).

A necessary condition:

Proposition

Assume that system (18) is null controllable at time T ($d_1 \neq d_2$). Then ($\lambda_k = k^2$),

$$d_1 \lambda_k \neq d_2 \lambda_j, \quad \forall k, j \geq 1 \quad (\Longleftrightarrow \sqrt{d_1/d_2} \notin \mathbb{Q}).$$

Proof: By contradiction, assume that $d_1 \lambda_k = d_2 \lambda_j$ for some k, j and take $K = \max\{k, j\}$. The idea is transforming system (19) into an o.d.s.
Recall that λ_k and ϕ_k are the eigenvalues and normalized eigenfunctions of $-\partial_{xx}$ on $(0, \pi)$ with homogenous Dirichlet boundary conditions:

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi).$$

Idea: Take $\varphi_0 \in X_K = \{\varphi_0 = \sum_{\ell=1}^{K} a_\ell \phi_\ell : a_\ell \in \mathbb{R}^2\} \subset H_0^1(0, \pi; \mathbb{R}^2)$.

Consider also

$$B_K = \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \in \mathbb{R}^{2K}, \quad \left(B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \quad \text{and}$$

$$\mathcal{L}_K^* = \text{diag}\,(-\lambda_1 D + A^*, -\lambda_2 D + A^*, \cdots, -\lambda_K D + A^*) \in \mathcal{L}(\mathbb{R}^{2K}).$$

Taking in (19) arbitrary initial data $\varphi_{0,K} = \sum_{\ell=1}^{K} a_\ell \phi_\ell \in H_0^1(0,\pi;\mathbb{R}^2)$ where $a_\ell \in \mathbb{R}^2$, it is not difficult to see that system (19) is equivalent to the o.d. system

$$(20) \qquad -Z' = \mathcal{L}_K^* Z \quad \text{on } [0,T], \quad Z(0) = Z_0 \in \mathbb{R}^{2K}.$$

From the observability inequality for system (19) we deduce the unique continuation property for the solutions to (20):

$$\boxed{B_K^* Z(\cdot) = 0 \quad \text{in } (0,T) \Longrightarrow Z \equiv 0.}$$

In particular system

$$Y' = \mathcal{L}_K Y + B_K v \quad \text{on } [0,T], \quad Y(0) = Y_0 \in \mathbb{R}^{2K}.$$

is exactly controllable at time T. Then $\boxed{\text{rank}\,[\mathcal{L}_K \mid B_K] = 2K}$.

We deduce that $\mathcal{L}_K^*$ cannot have eigenvalues with **geometric multiplicity** 2 or greater.

But $\theta = -d_1\lambda_k = -d_2\lambda_j$ is an eigenvalue of $\mathcal{L}_K^*$ with two linearly independent eigenvectors $V_1, V_2 \in \mathbb{R}^{2K}$ given by:

$$\begin{cases} V_1 = (V_{1,\ell})_{1\le\ell\le K}, & V_{1,k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } V_{1,\ell} = 0 \quad \forall \ell \ne k, \\ V_2 = (V_{2,\ell})_{1\le\ell\le K}, & V_{2,j} = \begin{pmatrix} \frac{1}{\lambda_j(d_1-d_2)} \\ 0 \end{pmatrix} \text{ and } V_{2,\ell} = 0 \quad \forall \ell \ne j. \blacksquare \end{cases}$$

The result has been proved in [Fernández-Cara, G.-B., de Teresa], J. Funct. Anal. (2010).

Conclusion: First difference with scalar problems

distributed controllability $\neq$ **boundary controllability**.

Even if System (14) is very close to System (18), their controllability properties are strongly different:

- System (14) (distributed control): We have obtained a complete characterization of the null controllability property in the constant case (and even, a distributed Carleman estimate for the **adjoint problem** (17)).
- System (18) (boundary control): The system is not null controllable if $d_1\lambda_k = d_2\lambda_j$ for some $k, j \geq 1$.

The same non-scalar parabolic problem can be controlled to zero with distributed controls supported on an interval ω *and, however, the* ***null controllability result*** *fails when the control acts on a part of the boundary.*

5. Boundary controllability of a 2×2 linear system

$$(18) \quad \begin{cases} y_t - Dy_{xx} = Ay & \text{in } Q_T, \\ y|_{x=0} = Bv, \quad y|_{x=\pi} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \ d_1, d_2 > 0, \ d_1 \neq d_2, \ A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Remark

- Again, System (18) is always null controllable at time T if we exert two independent controls at the same point. In this case, equivalence between distributed and boundary controllability (as in the **scalar case**; see Theorem 1).
- If $d_1 \neq d_2$, one has: "*System* (18) *is approximately controllable at time* T $\Longleftrightarrow$ $\boxed{\sqrt{d_1/d_2} \notin \mathbb{Q}}$". ■

5. Boundary controllability of a 2×2 linear system

$$(19) \qquad \begin{cases} -\varphi_t = D\varphi_{xx} + A^*\varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \{0,\pi\} \times (0,T), \quad \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi). \end{cases}$$

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \ d_1, d_2 > 0, \ d_1 \neq d_2, \text{ and } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Boundary approximate controllability

"*System* (18) *is approximately controllable at time* $T \iff \boxed{\sqrt{d_1/d_2} \notin \mathbb{Q}}$".
What does this condition mean???: The eigenvalues of the operator $\mathcal{R}^*\Phi = D\Phi_{xx} + A^*\Phi$ are

$$\left\{-d_1 k^2\right\}_{k\geq 1} \cup \left\{-d_2 i^2\right\}_{i\geq 1}.$$

Then, $\boxed{\sqrt{d_1/d_2} \notin \mathbb{Q}} \iff$ the eigenvalues of $\mathcal{R}^*$ are **simple**.

5. Boundary controllability of a 2×2 linear system

$$(18) \quad \begin{cases} y_t - D y_{xx} = A y & \text{in } Q_T, \\ y|_{x=0} = B v, \quad y|_{x=\pi} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \; B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Second difference with scalar problems

Null controllability: Assume $\sqrt{d_1/d_2} \notin \mathbb{Q}$. Is System (18) null controllable at time T? i.e., are approximate controllability and null controllability equivalent for System (18)?

We will see that he answer is **negative**.

approximate controllability $\not\equiv$ **null controllability**.

(See also [Ammar-Khodja, Benabdallah, Dupaix, Kostine], ESAIM:COCV (2005) for some abstract non-scalar parabolic systems).

6. A generalization: Cascade systems

6. A generalization: Cascade systems

We consider the linear parabolic system

$$\begin{cases} \partial_t y_1 + L_0^1(t) y_1 + \sum_{j=1}^n C_{1j} \cdot \nabla y_j + \sum_{j=1}^n a_{1j} y_j = v 1_\omega & \text{in } Q_T = \Omega \times (0,T), \\ \partial_t y_2 + L_0^2(t) y_2 + \sum_{j=1}^n C_{2j} \cdot \nabla y_j + \sum_{j=1}^n a_{2j} y_j = 0 & \text{in } Q_T, \\ \dots & \\ \partial_t y_n + L_0^n(t) y_n + \sum_{j=1}^n C_{nj} \cdot \nabla y_j + \sum_{j=1}^n a_{nj} y_j = 0 & \text{in } Q_T, \\ y_i = 0 \text{ on } \Sigma_T = \partial\Omega \times (0,T), \quad y_i(\cdot, 0) = y_0^i \text{ in } \Omega, & 1 \le i \le n, \end{cases}$$

where $a_{ij} = a_{ij}(x,t) \in L^\infty(Q_T)$, $C_{ij} = C_{ij}(x,t) \in L^\infty(Q_T; \mathbb{R}^N)$ $(1 \le i,j \le n)$, $y_0^i \in L^2(\Omega)$ $(1 \le i \le n)$ and $L_0^k(t)$ is, for every $1 \le k \le n$, the second order operator

$$\boxed{L_0^k(t) y = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(\alpha_{ij}^k(x,t) \frac{\partial y}{\partial x_j} \right)}$$

where α_{ij}^k satisfy (3) and (4) for every k.

Objective

Controllability properties of the system: n equations controlled with a **unique** distributed control.

Equivalently, the previous system can be written as

$$(21) \qquad \begin{cases} \partial_t y + \widehat{L}(t)y + C \cdot \nabla y + Ay = Bv1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

where $\widehat{L}(t)$ is the **matrix operator** given by $\widehat{L}(t) = \text{diag}\,(L_0^1(t), \cdots, L_0^n(t))$, $y = (y_i)_{1 \le i \le n}$ is the state and $\nabla y = (\nabla y_i)_{1 \le i \le n}$, and where

$$\begin{cases} y_0 = (y_0^i)_{1 \le i \le n} \in L^2(\Omega; \mathbb{R}^n), \quad A(\cdot, \cdot) = (a_{ij}(\cdot, \cdot))_{1 \le i,j \le n} \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n)), \\ C(\cdot, \cdot) = (C_{ij}(\cdot, \cdot))_{1 \le i,j \le n} \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n; \mathbb{R}^{Nn})) \text{ and } B \equiv e_1 = (1, 0, \ldots, 0)^* \end{cases}$$

are given. Let us observe that, for each $y_0 \in L^2(\Omega; \mathbb{R}^n)$ and $v \in L^2(Q_T)$, System (21) admits a **unique weak solution**

$$y \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^n)) \cap C^0([0, T]; L^2(\Omega; \mathbb{R}^n)).$$

6. A generalization: Cascade systems

By **cascade system** we mean that matrices A and C have the following structure:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n,n-1} & a_{nn} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ 0 & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_{nn} \end{pmatrix}$$

with $a_{ij} \in L^\infty(Q_T)$ and $C_{ij} \in L^\infty(Q_T; \mathbb{R}^N)$ and the coefficients $a_{i,i-1}$ satisfy

$$a_{i,i-1} \geq c_0 > 0 \text{ or } -a_{i,i-1} \geq c_0 > 0 \text{ in } \omega_0 \times (0,T), \ \forall i : 2 \leq i \leq n,$$

with $\omega_0 \subseteq \omega$ a new open subset.

Remark

It is natural to assume that $a_{i,i-1} \not\equiv 0$ for any $i : 2 \leq i \leq n$. The previous assumption is **stronger** but will provide the controllability result. ■

6. A generalization: Cascade systems

In this case, the corresponding **adjoint problem** has the form

$$\begin{cases} -\partial_t \varphi_i + L_0^i(t)\varphi_i - \sum_{j=1}^{i} \left[\nabla\cdot(C_{ji}\varphi_j) - a_{ji}\varphi_j\right] = -a_{i+1,i}\varphi_{i+1} & \text{in } Q_T, \\ \cdots \qquad\qquad (1\le i\le n-1), & \\ -\partial_t \varphi_n + L_0^n(t)\varphi_n - \sum_{j=1}^{n} \left[\nabla\cdot(C_{jn}\varphi_j) - a_{jn}\varphi_j\right] = 0 & \text{in } Q_T, \\ \varphi_i = 0 \text{ on } \Sigma_T, \quad \varphi_i(\cdot,T) = \varphi_{i,T} \text{ in } \Omega, \quad 1\le i\le n, & \end{cases}$$

where $\varphi_{i,T}\in L^2(\Omega)$ $(1\le i\le n)$. Again, the null controllability of System (21) (with L^2-controls) at time T is equivalent to the existence of a constant $C_T>0$ such that the so-called **observability inequality**

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega;\mathbb{R}^n)} \le C_T \iint_{\omega\times(0,T)} |\varphi_1(x,t)|^2$$

holds for every solution $\varphi=(\varphi_1,\ldots,\varphi_n)^*$ to the **adjoint problem**.

6. A generalization: Cascade systems

Theorem

Under the previous assumptions, let $M_0 = \max_{2\le i\le n} \|a_{i,i-1}\|_\infty$. *Then, there exist a positive function* $\alpha_0 \in C^2(\overline{\Omega})$ *(only depending on* Ω *and* ω_0*), two positive constants* C_0 *and* σ_0 *(only depending on* Ω, ω_0, c_0, M_0 *and* d*) and* $l \ge 0$ *(only depending on* n*) such that, for every* $\varphi_T \in L^2(Q_T;\mathbb{R}^n)$*, the solution* φ *to the* ***adjoint problem*** *satisfies*

$$\sum_{i=1}^{n} \mathcal{I}(d+3(n-i),\varphi_i) \le C_0 s^{d+l} \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\gamma(t)^{d+l}|\varphi_1|^2,$$

$\forall s \ge s_0 = \sigma_0 \left[T + T^2 + T^2 \max_{i\le j}\left(\|a_{ij}\|_\infty^{\frac{2}{3(j-i)+3}} + \|C_{ij}\|_\infty^{\frac{2}{3(j-i)+1}}\right)\right]$. *In the previous inequality,* $\boxed{\gamma(t) = t^{-1}(T-t)^{-1}}$, $\boxed{\alpha(x,t) = \alpha_0(x)/t(T-t)}$ *and* $\mathcal{I}(d,z)$ *is given in Lemma 2.3 (with* α *instead of* β*).*

6. A generalization: Cascade systems

Combining the previous result and **energy inequalities** satisfied by the solutions of the **adjoint system** it is possible to prove an **observability inequality** for the **adjoint system** (as in the scalar case). Summarizing, we get

Corollary

Under assumptions of the previous result, there exists a positive constant C (only depending on Ω, ω, n, c_0 and M_0) such that for every $y_0 \in L^2(\Omega;\mathbb{R}^n)$ there is a control $v \in L^2(\Omega)$ which satisfies

$$\|v\|\|^2_{L^2(Q_T)} \leq e^{C\,\mathcal{H}} \|y_0\|^2_{L^2(\Omega;\mathbb{R}^n)},$$

and $y(\cdot,T) = 0$ in Ω, with y the solution to (21) associated to y_0 and v. In the previous inequality, $\mathcal{H}$ is given by

$$\mathcal{H} \equiv 1+T+\frac{1}{T}+\max_{i\leq j}\left(\|a_{ij}\|_\infty^{\frac{2}{3(j-i)+3}} + \|C_{ij}\|_\infty^{\frac{2}{3(j-i)+1}} + T\left(\|a_{ij}\|_\infty + \|C_{ij}\|^2_\infty\right)\right).$$

6. A generalization: Cascade systems

Sketch of the proof of Theorem 6.1: Given $\omega_0 \subset \omega$, we choose $\omega_1 \subset\subset \omega_0$. Let $\alpha_0 \in C^2(\overline{\Omega})$ be the function provided by Lemma 2.3 and associated to Ω and $\mathcal{B} \equiv \omega_1$. We will do the proof in two steps:

Step 1. Let φ be the solution to **adjoint system** associated to φ_T. Each component satisfies

$$-\partial_t \varphi_i + L_0^i(t)\varphi_i = \boxed{\sum_{j=1}^{i} \left[\nabla \cdot (C_{ji}\varphi_j) - a_{ji}\varphi_j \right] - a_{i+1,i}\varphi_{i+1}}.$$

We begin applying inequality (11) with $\mathcal{B} = \omega_1$ to each function φ_i with $L_0 \equiv L_0^i$, $d = d + 3(n-i)$ and the corresponding right-hand side. Now if we take

$$s \geq s_0 = \sigma_0 \left(T + T^2 + T^2 \max_{i \leq j} \left(\|a_{ij}\|_\infty^{\frac{2}{3(j-i)+3}} + \|C_{ij}\|_\infty^{\frac{2}{3(j-i)+1}} \right) \right),$$

with $\sigma_0 = \sigma_0(\Omega, \omega_0, c_0, M_0) > 0$, we obtain the existence of a positive constants $C_1 = C_1(\Omega, \omega_0, c_0, M_0)$ such that if $s \geq s_0$, then

$$\sum_{i=1}^{n} \mathcal{I}(d+3(n-i),\varphi_i) \leq C_1 \sum_{i=1}^{n} s^{s+3(n-i)} \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\gamma(t)^{s+3(n-i)}|\varphi_i|^2.$$

Step 2. Thanks to the assumption

$$a_{i,i-1} \geq c_0 > 0 \text{ or } -a_{i,i-1} \geq c_0 > 0 \text{ in } \omega_0 \times (0,T), \ \forall i : 2 \leq i \leq n,$$

with $\omega_0 \subseteq \omega$ an open subset, and the cascade structure

$$a_{i,i-1}\varphi_i = \partial_t \varphi_{i-1} - L_0^{i-1}(t)\varphi_{i-1} + \sum_{j=1}^{i-1} \left[\nabla\cdot(C_{j,i-1}\varphi_j) - a_{j,i-1}\varphi_{i-1}\right] \text{ in } Q_T,$$

can eliminate the local terms for $2 \leq i \leq n$. In order to carry this process out, we will need the following result:

6. A generalization: Cascade systems

Lemma

Under assumptions of Theorem 6.1 and given $l \in \mathbb{N}$, $\varepsilon > 0$, $k \in \{2, ..., n\}$ *and two open sets* $\mathcal{O}_0$ *and* $\mathcal{O}_1$ *such that* $\omega_1 \subset \mathcal{O}_1 \subset\subset \mathcal{O}_0 \subset \omega_0$, *there exist a constant* C_k *(only depending on* Ω, $\mathcal{O}_0$, $\mathcal{O}_1$, c_0 *and* M_0*) and* $l_{kj} \in \mathbb{N}$, $1 \leq j \leq k-1$ *(only depending on* l, n, k *and* j*), such that, if* $s \geq s_0$, *one has*

$$s^l \iint_{\mathcal{O}_1 \times (0,T)} e^{-2s\alpha} \gamma(t)^l |\varphi_k|^2 \leq \varepsilon \left[\mathcal{I}(d + 3(n-k), \varphi_k) + \mathcal{I}(d + 3(n-k-1), \varphi_{k+1}) \right] + C_k \left(1 + \frac{1}{\varepsilon}\right) \sum_{j=1}^{k-1} s^{l_{kj}} \iint_{\mathcal{O}_0 \times (0,T)} e^{-2s\alpha} \gamma(t)^{l_{kj}} |\varphi_j|^2.$$

(In this inequality we have taken $\varphi_{k+1} \equiv 0$ *when* $k = n$*).*

The proof of Theorem 6.1 is a consequence of this Lemma 6.3. For the details, see [DE TERESA], Comm. PDE (2000), [G.-B., PÉREZ-GARCÍA], Asymp. Anal. (2006) and [G.-B., DE TERESA], Port. Math. (2010).

6. A generalization: Cascade systems

Remark

1. **Cascade systems** appear in the context of existence of **insensitizing controls** for a scalar parabolic equation: Equivalent to a null controllability result for a 2×2 parabolic system ($n = 2$) with one equation forward in time and the other one backward. The coupling coefficient a_{21} is $1_{\mathcal{O}}$ with $\mathcal{O} \subseteq \Omega$ an open set and $\boxed{\mathcal{O} \cap \omega \neq \emptyset}$.
2. The previous proof uses the assumption

$$a_{i,i-1} \geq c_0 > 0 \text{ or } -a_{i,i-1} \geq c_0 > 0 \text{ in } \omega_0 \times (0,T), \ \forall i : 2 \leq i \leq n,$$

   in a crucial way. When $a_{i,i-1}$ are constant, this assumption is **necessary**. Is this condition **necessary** in the general case??? **No**.
3. Is it possible to provide a **necessary** and **sufficient** (**Kalman condition**) condition for the null controllability of non-scalar systems? **YES** in some **constant coefficient systems**.

6. A generalization: Cascade systems

Some additional references

1. L. MANIAR ET AL., Controllability results for degenerate parabolic cascade systems.
2. M. DUPREZ, P. LISSY, Controllability results for parabolic systems with first order coupling terms.

7. The Kalman condition for a class of parabolic systems. Distributed controls

7. The Kalman condition for a class of parabolic systems

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial\Omega$ of class C^2. Let $\omega \subseteq \Omega$ be an open subset and let us fix $T > 0$.

For $n, m \in \mathbb{N}$ we consider the following autonomous $n \times n$ parabolic system

$$(22) \qquad \begin{cases} \partial_t y + DL_0 y = Ay + Bv1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

where $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ and $D = \operatorname{diag}(d_1, d_2, \cdots, d_n) \in \mathcal{L}(\mathbb{R}^n)$ with $d_i > 0$. We assume that L_0 is the self-adjoint second order elliptic operator:

$$L_0 y = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\alpha_{ij}(x)\frac{\partial y}{\partial x_j}\right)$$

with coefficients satisfying (3) and (4). Finally, $y_0 \in L^2(\Omega; \mathbb{R}^n)$ is given and $v \in L^2(Q_T; \mathbb{R}^m)$ is the control (m **distributed controls**).

$$
(22) \qquad \begin{cases} \partial_t y + DL_0 y = Ay + Bv1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega. \end{cases}
$$

Remark

This problem is well posed: For any $y_0 \in L^2(\Omega; \mathbb{R}^n)$ and $v \in L^2(Q_T; \mathbb{R}^m)$, problem (22) has a unique solution

$$
y \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^n)) \cap C^0([0, T]; L^2(\Omega; \mathbb{R}^n)).
$$

Remark

We want to control the whole system (n equations) with m controls. The most interesting case is $m < n$ or even $m = 1$.

Difficulties:

1. In general $m < n$.
2. D is not the identity matrix. ■

The adjoint problem:

$$(23)\qquad \begin{cases} -\partial_t \varphi = (-DL_0 + A^*)\varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 \text{ in } \Omega, \end{cases}$$

where $\varphi_0 \in L^2(\Omega; \mathbb{R}^n)$. Then, the exact controllability to the trajectories of system (22) is equivalent to the existence of $C_T > 0$ such that, for every $\varphi_0 \in L^2(\Omega; \mathbb{R}^n)$, the solution $\varphi \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$ to the adjoint system (23) satisfies the **observability inequality**:

$$\|\varphi(\cdot, 0)\|^2_{L^2(\Omega)} \le C_T \iint_{\omega \times (0,T)} |B^* \varphi(x, t)|^2.$$

7. The Kalman condition for a class of parabolic systems

We come back to System (22):

$$(22)\qquad \begin{cases} \partial_t y + DL_0 y = Ay + Bv1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

where $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$ and $D = \operatorname{diag}(d_1, d_2, \cdots, d_n) \in \mathcal{L}(\mathbb{R}^n)$ with $d_i > 0$. Now we assume that L_0 is the self-adjoint second order elliptic operator:

$$L_0 y = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\alpha_{ij}(x)\frac{\partial y}{\partial x_j}\right)$$

with coefficients satisfying (3) and (4). Finally, $y_0 \in L^2(\Omega;\mathbb{R}^n)$ is given and $v \in L^2(Q_T;\mathbb{R}^m)$ is the control (m **distributed controls**).

7. The Kalman condition for a class of parabolic systems

Let us consider $\{\lambda_k\}_{k\geq 1}$ the sequence of eigenvalues for L_0 with homogeneous Dirichlet boundary conditions and $\{\phi_k\}_{k\geq 0}$ the corresponding normalized eigenfunctions.

Theorem (A Necessary Condition)

If system (22) is ***null controllable*** *at time T* ***then***

$$\text{rank}\,[-\lambda_k D + A \,|\, B] = n, \quad \forall k \geq 1. \tag{24}$$

where

$$[-\lambda_k D + A \,|\, B] = [B\,,\,(-\lambda_k D + A)B\,,\,(-\lambda_k D + A)^2 B\,,\,\cdots\,,\,(-\lambda_k D + A)^{n-1} B].$$

Proof: Reasoning by contradiction: $\exists k \geq 1$ such that $\text{rank}\,[-\lambda_k D + A \,|\, B] < n$. Then the o.d.s. $-Z' = (-\lambda_k D + A^*)Z \quad \text{in } (0,T)$, is not B^*-observable at time T.

7. The Kalman condition for a class of parabolic systems

There exists $Z_0 \in \mathbb{R}^n$, $Z_0 \neq 0$, such that the solution Z to the previous system satisfies $B^*Z(\cdot) = 0$ on $(0, T)$. But $\varphi(x,t) = Z(t)\phi_k(x)$ is the solution to **adjoint problem**

$$\begin{cases} -\partial_t \varphi + DL_0\varphi = A^*\varphi \quad \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 \text{ in } \Omega, \end{cases}$$

associated to $\varphi_0(x) = Z_0\phi_k \not\equiv 0$ and $\boxed{B^*\varphi(\cdot,\cdot) \equiv 0}$ in Q_T. Then, the **observability inequality**

$$\|\varphi(\cdot, 0)\|^2_{L^2(\Omega)} \leq C_T \iint_{\omega \times (0,T)} |B^*\varphi(x,t)|^2,$$

fails and the system is not null controllable at time T. ■

Remark

If condition (24) is not satisfied, then system (22) is neither approximately controllable nor null controllable at time T (for any $T > 0$) even if $\omega \equiv \Omega$. ■

7. The Kalman condition for a class of parabolic systems

Question:

Is condition (24) rank $[-\lambda_k D + A \mid B] = n, \forall k \geq 1$, a **sufficient condition** for the null controllability of system (22)??? ■

Let us now introduce the unbounded matrix operator

7. The Kalman condition for a class of parabolic systems

Question:

Is condition (24) rank $[-\lambda_k D + A \,|\, B] = n$, $\forall k \geq 1$, a **sufficient condition** for the null controllability of system (22)??? ■

Let us now introduce the unbounded matrix operator

$$\mathcal{K} = [DL_0 + A \,|\, B] = [B\,,\ (-DL_0 + A)B\,,\ \cdots\,,\ (-DL_0 + A)^{n-1}B],$$

$$\begin{cases} \mathcal{K} : D(\mathcal{K}) \subset L^2(\Omega; \mathbb{R}^{nm}) \to L^2(\Omega; \mathbb{R}^n), \text{ with} \\ D(\mathcal{K}) := \{y \in L^2(\Omega; \mathbb{R}^{nm}) : \mathcal{K}y \in L^2(\Omega; \mathbb{R}^n)\}. \end{cases}$$

Then,

Proposition

$\ker \mathcal{K}^* = \{0\}$ ***if and only if*** *condition* (24), rank $[-\lambda_k D + A \,|\, B] = n$, $\forall k \geq 1$, *holds.* ■

$$(22)\qquad \begin{cases} \partial_t y + DL_0 y = Ay + Bv1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

Theorem (Kalman condition)

System (22) is ***exactly controllable to trajectories*** *at time T* ***if and only if***
System (22) is ***approximately controllable*** *at time T* ***if and only if***
$\boxed{\ker \mathcal{K}^* = \{0\}}$ *($\Longleftrightarrow$ $\text{rank}\,[-\lambda_k D + A \,|\, B] = n$, $\forall k \geq 1$).*

Remark

One can prove, either there exists $k_0 \geq 1$ such that

$$\boxed{\text{rank}\,[-\lambda_k D + A \,|\, B] = n, \quad \forall k \geq k_0}$$

or

$$\boxed{\text{rank}\,[-\lambda_k D + A \,|\, B] < n, \quad \forall k \geq 1}.$$

7. The Kalman condition for a class of parabolic systems

Controllability (outside a finite dimensional space) **if and only if** the algebraic Kalman condition $\boxed{\operatorname{rank}\,[-\lambda_k D + A \mid B] = n}$ is satisfied for one frequency $k \geq 1$.

Remark

System (22) can be exactly controlled to the trajectories with one control force ($m = 1$ and $B \in \mathbb{R}^n$) even if $\boxed{A \equiv 0}$. Indeed, let us assume that $B = (b_i)_{1 \leq i \leq n} \in \mathbb{R}^n$. Then,

$$[(-\lambda_k D + A) \mid B] = \begin{bmatrix} b_1 & (-\lambda_k d_1) b_1 & \cdots & (-\lambda_k d_1)^{n-1} b_1 \\ b_2 & (-\lambda_k d_2) b_2 & \cdots & (-\lambda_k d_2)^{n-1} b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & (-\lambda_k d_n) b_n & \cdots & (-\lambda_k d_n)^{n-1} b_n \end{bmatrix} \in \mathcal{L}(\mathbb{R}^n),$$

and (24) holds **if and only if** $b_i \neq 0$ for every i and d_i are **distinct**. ■

7. The Kalman condition for a class of parabolic systems

Idea of the proof: We have proved the **necessary condition**. Therefore, let us prove that $\boxed{\text{rank}\,[-\lambda_k D + A \,|\, B] = n}$, for any k, is a **sufficient condition** for the null controllability at time T of the system.

Then, the objective is to prove the **observability inequality**:

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leq C \iint_{\omega\times(0,T)} |B^*\varphi(x,t)|^2,$$

for the solutions to the **adjoint problem**.

To this end we use two arguments:

- Prove a global Carleman estimate for a scalar parabolic equation of order n in time.
- Prove a **coercivity** property for the Kalman operator $\mathcal{K}$.

7. The Kalman condition for a class of parabolic systems

Let us fix $\varphi_0 \in D(L_0^i)$, $\forall i \geq 0$ and consider φ the corresponding solution to the **adjoint system** (23)

$$\begin{cases} -\partial_t \varphi + DL_0 \varphi = A^* \varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 \text{ in } \Omega. \end{cases}$$

Let us take $\boxed{\Phi = \sum_{i=1}^{n} a_i \varphi_i}$, with $a_i \in \mathbb{R}$ $(1 \leq i \leq n)$. Then, Φ is a regular solution ($L_0^i \partial_t^j \Phi \in L^2(Q_T)$, $\forall i, j$) to the linear parabolic scalar equation of order n in time

$$\begin{cases} \det \left(I_d \partial_t - DL_0 + A^* \right) \Phi = 0 & \text{in } Q_T, \\ L_0^i \Phi = 0 \quad \text{on } \Sigma_T, \quad \forall i \geq 0. \end{cases}$$

The key point is to prove a Carleman inequality for the solutions to the previous problem. Fix $\omega_0 \subset\subset \omega$ a nonempty open subset. Recall Lemmas 2.3 and 2.4:

Lemma

There exist a $\alpha_0 \in C^2(\overline{\Omega})$ *(positive), and two constants* $C_0, \sigma_0 > 0$ *(only depending on* Ω*,* ω_0 *and d) s.t.*

$$\begin{cases} \mathcal{I}_1(d,\phi) \equiv \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d-4} \left(|\phi_t|^2 + |L_0\phi|^2\right) \\ \quad + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d-2} |\nabla\phi|^2 + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d} |\phi|^2 \\ \leq C_0 \left(\iint_{\omega_0\times(0,T)} e^{-2s\alpha} [s\gamma(t)]^{d} |\phi|^2 + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d-3} |\phi_t \pm L_0\phi|^2 \right), \end{cases}$$

$\forall s \geq s_0 = \sigma_0(\Omega,\omega)(T+T^2)$, $\forall \phi \in L^2(0,T;H_0^1(\Omega))$ s.t. $\phi_t \pm L_0\phi \in L^2(Q_T)$.

$\boxed{\gamma(t) = t^{-1}(T-t)^{-1}}$, $\boxed{\alpha(x,t) = \alpha_0(x)/t(T-t)}$.

7. The Kalman condition for a class of parabolic systems

Theorem

Let $n, k_1, k_2 \in \mathbb{N}$ *and* $d \in \mathbb{R}$. *There exist two constants* C *and* σ *(only depending on* Ω, ω, n, D, A, k_1, k_2 *and* d*), and* $r_0 = r_0(n) \in \mathbb{N}$ *such that*

$$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \mathcal{J}(d - 4(i+j), L_0^i \partial_t^j \Phi) \leq C \iint_{\omega \times (0,T)} e^{-2s\alpha} \left[s\gamma(t) \right]^{3+r_0} |\Phi|^2, \quad ,$$

$\forall s \geq s = \sigma(\Omega, \omega)(T + T^2)$, Φ *solution to the previous problem and*

$$\mathcal{J}(\tau, z) := \mathcal{I}_1(\tau + 3(n-1), z) + \sum_{i=1}^{n} \mathcal{I}_1(\tau + 3(n-2), P_i z) + \sum_{p=2}^{n-1} \sum_{1 \leq i_1 < \cdots < i_p \leq n} \mathcal{I}_1(\tau + 3(n-p-1), P_{i_p} \cdots P_{i_1} z).$$

$(P_i \equiv \partial_t - d_i L_0)$

Sketch of the proof: We will give the main ideas in the case $\boxed{k_1 = k_2 = 0}$. If we use the notation $P_i \equiv \partial_t - d_i L_0$ $(1 \le i \le n)$, one has:

$$\det\left(I_d \partial_t - DL_0 + A^*\right) \equiv P_n \cdots P_1 + \sum_{p=2}^{n-1} \sum_{1 \le i_1 < \cdots < i_p \le n} b_{i_1,\ldots,i_p} P_{i_1} \ldots P_{i_p}$$
$$+ \sum_{i=1}^{n} b_i P_i + b := P_n \cdots P_1 - F,$$

with $b_{i_1,\ldots,i_p}, b_i, b \in \mathbb{R}$ only depending on D and A.

We have a function Φ s.t. $L_0^i \partial_t^j \Phi \in L^2(Q_T)$, $\forall i, j$, and it is solution to

$$\begin{cases} \det\left(I_d \partial_t - DL_0 + A^*\right)\Phi = 0 & \text{in } Q_T, \\ L_0^i \Phi = 0 \quad \text{on } \Sigma, & \forall i \ge 0. \end{cases}$$

In particular, $P_n \cdots P_1 \Phi = F(\Phi)$ in Q_T.

In particular, $P_n \cdots P_1 \Phi = F(\Phi)$ in Q_T. We rewrite the order-n equation as a system performing the change of variables:

$$\begin{cases} \psi_1 := \Phi, \\ \psi_i := P_{i-1}\psi_{i-1} \equiv (\partial_t - d_{i-1})\psi_{i-1}, \quad 2 \leq i \leq n. \end{cases}$$

Then, $\Psi = (\psi_1, \psi_2, \ldots, \psi_n)^*$ satisfies the **cascade system**

$$\begin{cases} (\partial_t - d_1 L_0)\,\psi_1 = \psi_2 & \text{in } Q_T, \\ (\partial_t - d_2 L_0)\,\psi_2 = \psi_3 & \text{in } Q_T, \\ \qquad\qquad \vdots & \\ (\partial_t - d_n L_0)\,\psi_n = F(\Phi) & \text{in } Q_T, \\ \psi_i = 0 \text{ on } \Sigma_T, & \forall i : 1 \leq i \leq n. \end{cases}$$

We can apply Theorem 6.1 (cascade systems) and obtain:

7. The Kalman condition for a class of parabolic systems

We can apply Theorem 6.1 and obtain (cascade systems) ($d \in \mathbb{R}$ is given):

$$\sum_{i=1}^{n} \mathcal{I}_1(d+3(n-i),\psi_i) \le C_0 \bigg(\iint_{\omega\times(0,T)} e^{-2s\alpha}[s\gamma(t)]^{d+r_0}|\psi_1|^2 + \iint_{Q_T} e^{-2s\alpha}[s\gamma(t)]^{d}|F(\Phi)|^2 \bigg),$$

$\forall s \ge s_0 = \sigma_0 \left(T+T^2\right)$ with $r_0 = r_0(n)$ and

$$\mathcal{I}_1(d,z) \equiv \iint_{Q_T} e^{-2s\alpha}[s\gamma(t)]^{d}\{[s\gamma(t)]^{-4}(|\partial_t z|^2+|L_0 z|^2)+[s\gamma(t)]^{-2}|\nabla z|^2+|z|^2\}.$$

Coming to the original variables, one has

$$\mathcal{I}_1(d+3(n-1),\Phi) + \sum_{i=2}^{n} \mathcal{I}_1(d+3(n-i),P_{i-1}\cdots P_1\Phi)$$
$$\le C_0 \bigg(\iint_{\omega\times(0,T)} e^{-2s\alpha}[s\gamma(t)]^{d+r_0}|\Phi|^2 + \iint_{Q_T} e^{-2s\alpha}[s\gamma(t)]^{d}|F(\Phi)|^2 \bigg).$$

7. The Kalman condition for a class of parabolic systems

We can reproduce the previous argument for a general permutation Π of the set $\{1, 2, \ldots, n\}$, taking

$$\begin{cases} \psi_1 := \Phi, \\ \psi_i := P_{\Pi(i-1)}\psi_{i-1} \equiv (\partial_t - d_{\Pi(i-1)})\psi_{\Pi(i-1)}, \quad 2 \leq i \leq n. \end{cases}$$

Thus,

$$\mathcal{I}_1(d + 3(n-1), \Phi) + \sum_{i=2}^{n} \mathcal{I}_1(d + 3(n-i), P_{\Pi(i-1)} \cdots P_{\Pi(1)}\Phi)$$
$$\leq C_0 \left(\iint_{\omega\times(0,T)} e^{-2s\alpha}[s\gamma(t)]^{d+r_0}|\Phi|^2 + \iint_{Q_T} e^{-2s\alpha}[s\gamma(t)]^{d}|F(\Phi)|^2 \right),$$

$\forall s \geq s_0 = \sigma_0 \left(T + T^2\right)$. Adding all these inequalities (for any permutation Π) with $d = 3$, we get

Adding all these inequalities (for any permutation Π) with $d = 3$, we get

$$\mathcal{J}(d,\Phi) \leq C\left(\iint_{\omega\times(0,T)} e^{-2s\alpha}[s\gamma(t)]^{d+r_0}|\Phi|^2 + \iint_{Q_T} e^{-2s\alpha}[s\gamma(t)]^{d}|F(\Phi)|^2\right),$$

$\forall s \geq s_0 = \sigma_0\left(T+T^2\right)$ ($\mathcal{J}(\tau,z)$ given in the statement of Theorem 10 and

$$F(\Phi) = \sum_{p=2}^{n-1}\sum_{1\leq i_1<\cdots<i_p\leq n} b_{i_1,\ldots,i_p}P_{i_1}\ldots P_{i_p}\Phi + \sum_{i=1}^{n} b_iP_i\Phi + b\Phi).$$

From these expressions, it is possible to absorb the last term of the previous inequality and obtain

$$\mathcal{J}(d,\Phi) \leq C\iint_{\omega\times(0,T)} e^{-2s\alpha}[s\gamma(t)]^{d+r_0}|\Phi|^2,$$

for a new constant C, with $s \geq s = \sigma\left(T+T^2\right)$. This ends the proof in the case $k_1 = k_2 = 0$. ∎

Remark

Theorem 10 is, in fact, a Carleman inequality for the regular solutions Φ to the linear parabolic scalar equation of order n in time

$$\begin{cases} \det\left(I_d \partial_t - DL_0 + A^*\right)\Phi = 0 & \text{in } Q_T, \\ L_0^i \Phi = 0 \quad \text{on } \Sigma, & \forall i \geq 0. \end{cases}$$

Conclusion

If φ is a regular solution to the **adjoint problem**

$$\begin{cases} -\partial_t \varphi + DL_0 \varphi = A^* \varphi \quad \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 \text{ in } \Omega, \end{cases}$$

then, any linear combination $\Phi = \sum_{i=1}^{n} a_i \varphi_i$ satisfies Theorem 10. In particular any component of $B^* \varphi$. ∎

Conclusion

If φ is a regular solution to the **adjoint problem**

$$\begin{cases} -\partial_t \varphi + DL_0 \varphi = A^* \varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 \text{ in } \Omega, \end{cases}$$

then, any linear combination $\Phi = \sum_{i=1}^n a_i \varphi_i$ satisfies Theorem 10. In particular any component of $B^* \varphi$. ■

Recall $\mathcal{K} = [DL_0 + A \,|\, B] = [B\,,\, (-DL_0 + A)B\,,\, \cdots\,,\, (-DL_0 + A)^{n-1} B]$, then

$$\begin{aligned} \mathcal{K}^* \varphi(\cdot, t) &= [B^* \varphi\,,\, B^*(-DL_0 + A^*)\varphi\,,\, \cdots\,,\, B^*(-DL_0 + A^*)^{n-1} \varphi]^{tr}(\cdot, t) \\ &= [B^* \varphi\,,\, -\partial_t (B^* \varphi)\,,\, \cdots\,,\, (-1)^{n-1} \partial_t^{n-1} (B^* \varphi)]^{tr}(\cdot, t) \in \mathbb{R}^{nm}. \end{aligned}$$

We apply Theorem 10 with $k_1 = n - 1$ and $k_2 = k \geq 0$. Then, after some computations, we deduce ($d = 3$)

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Then, after some computations, we deduce ($d = 3$)

$$\int_0^T e^{\frac{-2sM_0}{t(T-t)}} [s\gamma(t)]^3 \, \|L_0^k \mathcal{K}^* \varphi\|^2_{L^2(\Omega)^{nm}} \leq C \iint_{\omega\times(0,T)} e^{-2s\alpha} [s\gamma(t)]^{3+r_0} |B^*\varphi|^2$$

for every $s \geq \sigma\left(T + T^2\right)$. In this inequality, $M_0 = \max_{\overline{\Omega}} \alpha_0$ and $r_0 \geq 0$ is an integer only depending on n.

Remark

The previous inequality is a partial observability estimate. It is valid even if the Kalman condition does not hold, i.e., even if $\ker \mathcal{K}^* \neq \{0\}$. ■

The **coercivity** property of $\mathcal{K}^*$:

Theorem

Assume that $\ker \mathcal{K}^* = \{0\}$ *and consider* $k = (n-1)(2n-1)$*. Then there exists* $C > 0$ *such that if* $z \in L^2(\Omega)^n$ *satisfies* $\mathcal{K}^* z \in D(L_0^k)^{nm}$*, one has*

$$\|z\|^2_{L^2(\Omega)^n} \leq C \|L_0^k \mathcal{K}^* z\|^2_{L^2(\Omega)^{nm}} .$$

So, from the previous inequality we get

$$\boxed{\int_0^T e^{\frac{-2sM_0}{t(T-t)}} \left[s\gamma(t)\right]^3 \|\varphi\|^2_{L^2(\Omega)^{nm}} \leq C \iint_{\omega\times(0,T)} e^{-2s\alpha} \left[s\gamma(t)\right]^{3+r_0} |B^*\varphi|^2}$$

and the **observability inequality**:

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leq C \iint_{\omega\times(0,T)} |B^*\varphi(x,t)|^2.$$

Summarizing

1. We have established a **Kalman condition**

$$\ker \mathcal{K}^* = \{0\}$$

which characterizes the controllability properties of system (22).

2. The **Kalman condition** for system (22) $\ker \mathcal{K}^* = \{0\}$ generalizes the **algebraic Kalman condition** $\ker[A \mid B]^* = \{0\}$ for o.d.s.

3. This **Kalman condition** is also equivalent to the **approximate controllability** of system (22) at time T. Again, **approximate** and **null controllability** are equivalent concepts for system (22).

7. The Kalman condition for a class of parabolic systems

References

1. F. AMMAR-KHODJA, A. BENABDALLAH, C. DUPAIX, M. G.-B., *A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems*, Differ. Equ. Appl. **1** (2009), no. 3, 139–151.
   $D = I_d,\ A = A(t)$ and $B = B(t)$.
2. F. AMMAR-KHODJA, A. BENABDALLAH, C. DUPAIX, M. G.-B., *A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems*, J. Evol. Equ. **9** (2009), no. 2, 267–291.
   D diagonal matrix, A and B constant matrices.
3. E. FERNÁNDEZ-CARA, M. G.-B, L. DE TERESA, *Controllability of linear and semilinear non-diagonalizable parabolic systems*, ESAIM Control Optim. Calc. Var. **21** (2015), no. 4, 1178–1204.
   D non-diagonalizable matrix with Jordan blocks of dimension ≤ 4,
   A and B constant matrices.

Open problems

- Null controllability properties of

$$(22) \qquad \begin{cases} \partial_t y + DL_0 y = A(t)y + B(t)v1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

  when $A(t)$ and $B(t)$ depend on t (for instance, $A \in C^\infty([0,T];\mathcal{L}(\mathbb{R}^n))$ and $B \in C^\infty([0,T];\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n))$) and $D = \operatorname{diag}(d_1, d_2, \cdots, d_n) \in \mathcal{L}(\mathbb{R}^n)$ with $d_i > 0$.

- Null controllability properties of

$$(22) \qquad \begin{cases} \partial_t y + DL_0 y = Ay + Bv1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

  when A and B are constant matrices and D is a general non-diagonalizable matrix (definite positive).

8. The Kalman condition for a class of parabolic systems. Boundary controls

[Ammar-Khodja, Benabdallah, G.-B., de Teresa], J. Math. Pures Appl. (2011).

8. The Kalman condition for a class of parabolic systems. Boundary controls

Let us consider the **boundary controllability problem**:

$$(25) \qquad \begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T = (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)$ are two given matrices and $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$ is the initial datum. In system (25), $v \in L^2(0,T;\mathbb{C}^m)$ is the control function (to be determined).

Simpler problem: One-dimensional case and $D = Id$.

This problem has been studied in the case $n = 2$:

- E. Fernández-Cara, M. G.-B., L. de Teresa, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. 259 (2010), no. 7, 1720–1758.

8. The Kalman condition for a class of parabolic systems. Boundary controls

We consider again $\{\lambda_k\}_{k\geq 1}$ the sequence of eigenvalues for $-\partial_{xx}$ in $(0,\pi)$ with homogenuous Dirichlet boundary conditions and $\{\phi_k\}_{k\geq 0}$ the corresponding normalized eigenfunctions:

$$\boxed{\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}}\sin kx, \quad k \geq 1, \quad x \in (0,\pi).}$$

Theorem ($n = 2$, $m = 1$)

Let $A \in \mathcal{L}(\mathbb{C}^2)$ and $B \in \mathbb{C}^2$ be given and let us denote by μ_1 and μ_2 the eigenvalues of A^*. Then (25) is ***exactly controllable to the trajectories*** *at any time $T > 0$* ***if and only if*** $\boxed{\operatorname{rank}\,[A \mid B] = 2}$ *and*

$$\boxed{\lambda_k - \lambda_j \neq \mu_1 - \mu_2 \quad \forall k, j \in \mathbb{N} \text{ with } k \neq j.}$$

∎

Distributed controllability and boundary controllability

1. We proved that system

$$\begin{cases} y_t = y_{xx} + Ay + Bv1_\omega & \text{in } Q_T = (0,\pi)\times(0,T), \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

is null controllable at time $T > 0$ **if and only if** $\boxed{\text{rank}\,[A \mid B] = 2}$.

2. System

$$\begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T = (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

is null controllable at time $T > 0$ **if and only if** $\boxed{\text{rank}\,[A \mid B] = 2}$ and $\boxed{\lambda_k - \lambda_j \neq \mu_1 - \mu_2}$.

Remark ($n = 2$, $m = 1$)

For the previous **boundary controllability problem**, one has

1. A complete characterization of the **exact controllability to trajectories** at time T: **Kalman condition**.
2. **Boundary controllability** and **distributed controllability** are not equivalent
3. **Approximate controllability** $\Longleftrightarrow$ **null controllability**.

What happens if $n > 2$??

As we saw before, we will work in the following finite-dimensional space:

$$X_k = \{\varphi_0 = \sum_{\ell=1}^{k} a_\ell \phi_\ell : a_\ell \in \mathbb{C}^n\} \subset H_0^1(0, \pi; \mathbb{C}^n).$$

8. The Kalman condition for a class of parabolic systems. Boundary controls

Adjoint Problem:

$$(26) \qquad \begin{cases} -\varphi_t = \varphi_{xx} + A^*\varphi & \text{in } Q_T, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi), \end{cases}$$

with $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$. Then, system (25) is exactly controllable to trajectories at time $T \iff$ for a constant $C > 0$ one has (**observability inequality**)

$$\|\varphi(\cdot,0)\|^2_{H_0^1(0,\pi;\mathbb{C}^n)} \le C \int_0^T |B^*\varphi_x(0,t)|^2 \, dt.$$

Taking initial data in X_k, we deduce that an appropriate o.d. system in $\mathbb{C}^{nk}$ also satisfies an **observability inequality**. Let us analyze this finite-dimensional system.

8. The Kalman condition for a class of parabolic systems. Boundary controls

Notation

For $k \geq 1$, we introduce $L_k = -\lambda_k I_d + A \in \mathcal{L}(\mathbb{C}^n)$ and the matrices

$$B_k = \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^{nk}), \quad \mathcal{L}_k = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & L_k \end{pmatrix} \in \mathcal{L}(\mathbb{C}^{nk}),$$

and let us write the Kalman matrix associated with the pair $(\mathcal{L}_k, B_k)$:

$$\mathcal{K}_k = [\mathcal{L}_k \,|\, B_k] = [B_k \,,\, \mathcal{L}_k B_k \,,\, \mathcal{L}_k^2 B_k \,,\, \cdots \,,\, \mathcal{L}_k^{nk-1} B_k] \in \mathcal{L}(\mathbb{C}^{mnk}, \mathbb{C}^{nk}).$$

With this notation, the o.d. system associated to the **adjoint system** (26) for $\varphi_0 \in X_k$ is $\boxed{-Z' = \mathcal{L}_k^* Z \text{ on } (0,T), Z(T) = Z_0 \in \mathbb{C}^{nk}}$, and the solutions must be B_k^*-observable, i.e., rank $\mathcal{K}_k = nk$: **necessary condition**. One has:

8. The Kalman condition for a class of parabolic systems. Boundary controls

Theorem

Let us fix $A \in \mathcal{L}(\mathbb{C}^n)$ *and* $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$. *Then, system* (25) *is* ***exactly controllable to trajectories*** *at time* T *if and only if*

$$\text{rank}\, \mathcal{K}_k = nk, \quad \forall k \geq 1. \tag{27}$$

Remark

1. This result gives a complete characterization of the **exact controllability to trajectories** at time T: **Kalman condition**.
2. If for $k \geq 1$ one has rank $\mathcal{K}_k = nk$, then rank $[A \mid B] = n$ and system
$$\begin{cases} \partial_t y - \Delta y = Ay + Bv1_\omega \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$
is **exactly controllable to trajectories** at time T. But rank $[A \mid B] = n$ does not imply condition (27). So **boundary controllability** and **distributed controllability** are not equivalent.

Remark

Condition (27) is also a **necessary** and **sufficient condition** for the boundary approximate controllability of system (25). Then

$$\textbf{Approximate controllability} \iff \textbf{null controllability}.$$

Remark (n controls)

If $\boxed{\operatorname{rank} B = n}$ (and thus $m \geq n$), then the pair (A, B) fulfills condition (27) and the system is **exactly controllable to trajectories** at time T. ■

8. The Kalman condition for a class of parabolic systems. Boundary controls

Remark (One control, $m = 1$)

When $m = 1$, the **Kalman condition** (27) is equivalent to $\boxed{\text{rank}\,[A \mid B] = n}$ and $\boxed{\lambda_k - \lambda_l \neq \mu_i - \mu_j}$ for any $k, l \in \mathbb{N}$ and $1 \leq i, j \leq p$ with $(k, i) \neq (l, j)$, where $\{\mu_i\}_{1 \leq i \leq p} \subset \mathbb{C}$ is the set of distinct eigenvalues of A^*. We generalize the results of [Fernández-Cara, G.-B., de Teresa], J. Funct. Anal. (2010). ■

One control, $m = 1$

We have imposed two conditions:

1. $\text{rank}\,[A \mid B] = n$: System (25) is not decoupled.
2. $\lambda_k - \lambda_l \neq \mu_i - \mu_j$: The adjoint system can be written ($\mathcal{R}_0 = I_d \partial_{xx} + A^*$)

$$(26) \qquad \begin{cases} -\varphi_t = \mathcal{R}_0 \varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } (0, \pi), \end{cases}$$

and the eigenvalues of $\mathcal{R}_0$ are simple. ■

8. The Kalman condition for a class of parabolic systems. Boundary controls

Before proving the result, let us analyze the **Kalman condition** (27) rank $\mathcal{K}_k = nk$, $\forall k \geq 1$:

Proposition

Let us denote by $\{\mu_i\}_{1 \leq i \leq p} \subset \mathbb{C}$ *the set of distinct eigenvalues of* A^*. *Then,*

1. *There exists an integer* $k_0 = k_0(A) \in \mathbb{N}$, *only depending on* A, *such that,*

$$\boxed{\lambda_k - \lambda_l \neq \mu_i - \mu_j}, \quad \forall k > k_0,\ l \geq 1,\ k \neq l,\ \text{and } 1 \leq i, j \leq p.$$

2. *The following conditions are equivalent:*
   (a) rank $\mathcal{K}_k = nk$ *for every* $k \geq 1$.
   (b) rank $\mathcal{K}_k = nk$ *for every* $k : 1 \leq k \leq k_0$.
   (c) rank $\mathcal{K}_{k_0} = nk_0$. ∎

8. The Kalman condition for a class of parabolic systems. Boundary controls

Necessary implication. We reason as before: if rank $\mathcal{K}_k < nk$, for some $k \geq 1$, then the o.d.s.

$$-Z' = \mathcal{L}_k^* Z \quad \text{on } (0,T), \quad Z(T) = Z_0 \in \mathbb{C}^{nk}$$

is not B_k^*-observable on $(0,T)$, i.e., there exists $Z_0 \neq 0$ s.t. $B_k^* Z(t) = 0$ for every $t \in (0,T)$. From Z_0 it is possible to construct $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$ with $\varphi_0 \not\equiv 0$ such that the corresponding solution to the adjoint problem (27) satisfies

$$B^* \varphi_x(0,t) = 0 \quad \forall t \in (0,T).$$

As a consequence: The **unique continuation property** and the previous **observability inequality** for the adjoint problem fail:

Neither approximate nor null controllability at any T for system (25).

8. The Kalman condition for a class of parabolic systems. Boundary controls

Sufficient implication. For the proof we follow the ideas from

- H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.

Two "big" steps:

(I) We reformulate the null controllability problem for system (25) as a **vector moment problem**.

(II) Existence and bounds of a family **biorthogonal** to appropriate complex matrix exponentials.

8. The Kalman condition for a class of parabolic systems. Boundary controls

(I) The **vector moment problem**: As in the scalar case, $v \in L^2(0,T;\mathbb{C}^m)$ is a **null control** for system

$$(25) \qquad \begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

(i.e., the solution y to (25) satisfies $y(\cdot,T) = 0$ in $(0,\pi)$) $\iff$ v satisfies

$$\boxed{-\langle y_0, \varphi(\cdot,0)\rangle = \int_0^T (v(t)\,,\, B^*\varphi_x(0,t))_{\mathbb{C}^m}\,dt, \quad \forall \varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n),}$$

where φ is the solution to the adjoint problem

$$(26) \qquad \begin{cases} -\varphi_t = \varphi_{xx} + A^*\varphi & \text{in } Q_T, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi). \end{cases}$$

8. The Kalman condition for a class of parabolic systems. Boundary controls

(I) The **vector moment problem**:

Thus, the idea is to take firstly $\varphi_0 \in X_{k_0}$,
($X_{k_0} = \{\varphi_0 : \varphi_0 = \sum_{i=1}^{k_0} a_i \phi_i$ with $a_i \in \mathbb{C}^n\}$) and then $\varphi_0 = a\phi_k$, with $k > k_0$ and $a \in \mathbb{C}^n$. Therefore, we want $v \in L^2(0,T;\mathbb{C}^m)$ s.t.

$$\begin{cases} \displaystyle\int_0^T \left(v(T-t)\,,\, B_{k_0}^* e^{\mathcal{L}_{k_0}^* t}\Phi_0\right)_{\mathbb{C}^m} dt = \boxed{F(Y_0,\Phi_0)}, & \forall \Phi_0 \in \mathbb{C}^{nk_0}, \\ \displaystyle\int_0^T \left(v(T-t)\,,\, B^* e^{(-\lambda_k I_d + A^*)t} a\right)_{\mathbb{C}^m} dt = \boxed{f_k(y_0,a)}, & \forall a \in \mathbb{C}^n,\ \forall k > k_0, \end{cases}$$

In some sense, v has to solve an infinite number of null controllability problems for appropriate o.d. systems:

$$\begin{cases} \boxed{Y' = \mathcal{L}_{k_0} Y + B_{k_0} v \text{ on } (0,T), \quad Y(0) = Y_0}; \\ \boxed{Z' = (-\lambda_k I_d + A)Z + Bv \text{ on } (0,T), \quad Z(0) = y_{0k} := (y_0,\phi_k)}, \quad \forall k > k_0. \end{cases}$$

8. The Kalman condition for a class of parabolic systems. Boundary controls

(II) **Biorthogonal families** to appropriate complex matrix exponentials.
From the previous step, we have obtained the **complex matrix exponentials**

$$e^{\mathcal{L}_{k_0}^* t} \quad \text{and} \quad \{e^{(-\lambda_k I_d + A^*)t}\}_{k>k_0}.$$

Let us denote $\{\gamma_\ell\}_{1\le\ell\le\widetilde{p}} \subset \mathbb{C}$ the set of distinct eigenvalues of $\mathcal{L}_{k_0}^*$ and recall that $\{\mu_i\}_{1\le i\le p} \subset \mathbb{C}$ is the set of distinct eigenvalues of A^*. Then, the set $\Lambda = \{\gamma_\ell\}_{1\le\ell\le\widetilde{p}} \cup \{-\lambda_k + \mu_i\}_{k>k_0, 1\le i\le p}$ is the set of eigenvalues of the operator $\partial_{xx} Id + A^*$. Thus, our next purpose is:

Objective

As in the scalar case, construction of a **biorthogonal family** in $L^2(0,T;\mathbb{C})$ to

$$\left\{ t^j e^{\gamma_\ell t}, t^j e^{(-\lambda_k+\mu_i)t} : 1 \le \ell \le \widetilde{p},\ 1 \le i \le p,\ 0 \le j \le \eta - 1,\ k > k_0 \right\},$$

which satisfies appropriate bounds (see (22)). In the previous expression, η is the maximal dimension of the Jordan blocks associated to γ_ℓ and μ_i.

(II) **Biorthogonal families** to appropriate complex matrix exponentials.
Let us fix $\eta \geq 1$, an integer, $T \in (0, \infty]$ and $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}_+$ a sequence s.t.

$$\Lambda_k \neq \Lambda_j, \quad \forall k, j \geq \text{ with } k \neq j.$$

Let us recall that the family $\{q_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset L^2(0,T;\mathbb{C})$ is **biorthogonal** to $\{t^j e^{-\Lambda_k t}\}_{k\geq 1, 0\leq j\leq \eta-1}$ if one has

$$\int_0^T t^j e^{-\Lambda_k t} q^*_{l,i}(t)\, dt = \delta_{kl}\delta_{ij}, \quad \forall (k,j), (l,i) : k, l \geq 1,\ 0 \leq i, j \leq \eta - 1.$$

In addition, we want the family $\{q_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset L^2(0,T;\mathbb{C})$ to satisfy the property:

For any $\varepsilon > 0$, there is $C(\varepsilon, T) > 0$ s.t. $\boxed{\|q_{k,j}\|_{L^2(0,T;\mathbb{C})} \leq C(\varepsilon, T) e^{\varepsilon \Re \Lambda_k}}$, $\forall k \geq 1$ and $0 \leq j \leq \eta - 1$.

8. The Kalman condition for a class of parabolic systems. Boundary controls

(II) **Biorthogonal families** to appropriate complex matrix exponentials.

Theorem

Let us fix $\boxed{T \in (0, \infty]}$ *and assume that for two positive constants* δ *and* ρ *one has*

$$\begin{cases} \Re\Lambda_k \geq \delta|\Lambda_k|, \quad |\Lambda_k - \Lambda_l| \geq \rho|k-l|, \quad \forall k, l \geq 1, \\ \displaystyle\sum_{k\geq 1} \frac{1}{|\Lambda_k|} < \infty. \end{cases}$$

Then, $\exists \{q_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1}$ ***biorthogonal*** *to* $\left\{t^j e^{-\Lambda_k t}\right\}_{k\geq 1, 0\leq j\leq \eta-1}$ *such that, for every* $\varepsilon > 0$*, there exists* $C(\varepsilon, T) > 0$ *satisfying*

$$\|q_{k,j}\|_{L^2(0,T;\mathbb{C})} \leq C(\varepsilon, T) e^{\varepsilon \Re\Lambda_k}, \quad \forall (k,j) : k \geq 1,\ 0 \leq j \leq \eta - 1.$$

(II) **Biorthogonal families** to appropriate complex matrix exponentials.

Proof:

The proof of this result is very technical. It can be found in [AMMAR-KHODJA,BENABDALLAH,G.-B.,DE TERESA], *The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials*, J. Math. Pures Appl. (2011).

(II.1) **Biorthogonal families**: **EXISTENCE**.

Lemma

Assume that $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}_+$, *with* $\Lambda_k \neq \Lambda_j$ $\forall k, j \geq$ *with* $k \neq j$, *and*

$$\Re\Lambda_k \geq \delta|\Lambda_k| \quad and \quad \sum_{k\geq 1} \frac{1}{|\Lambda_k|} < \infty,$$

Then, there exists a biorthogonal family $\{q_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset L^2(0,\infty;\mathbb{C})$ *to* $\left\{t^j e^{-\Lambda_k t}\right\}_{k\geq 1, 0\leq j\leq \eta-1}$ *such that*

$$\|q_{k,j}\|_{L^2} \leq C(\Re\Lambda_k)^{\eta(\eta-j)}|1+\Lambda_k|^{2\eta(\eta-j)}\mathcal{P}_k^{\eta(\eta-j)},$$

with $C = C(\eta) > 0$, *a constant, and* $\mathcal{P}_k := \prod_{\substack{\ell\geq 1\\ \ell\neq k}} \left|\frac{1+\Lambda_k/\Lambda_\ell^*}{1-\Lambda_k/\Lambda_\ell}\right|$.

(II.1) **Biorthogonal families**: **EXISTENCE**.

Remark

Observe that the assumptions

$$\Re\Lambda_k \geq \delta|\Lambda_k| \quad \text{and} \quad \sum_{k\geq 1} \frac{1}{|\Lambda_k|} < \infty,$$

imply the existence of the **biorthogonal family** $\{q_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1}$ to $\left\{t^j e^{-\Lambda_k t}\right\}_{k\geq 1, 0\leq j\leq \eta-1}$ in $L^2(0,\infty;\mathbb{C})$. In addition, the norm $\|q_{k,j}\|_{L^2}$ is bound with respect to the Blaschke product

$$\mathcal{P}_k = \prod_{\substack{\ell\geq 1\\ \ell\neq k}} \left| \frac{1+\Lambda_k/\Lambda_\ell^*}{1-\Lambda_k/\Lambda_\ell} \right|.$$

(II.2) **Biorthogonal families**: **BOUNDS**.

Proposition

Let $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}_+$ *be a sequence satisfying*

$$\Re\Lambda_k \geq \delta|\Lambda_k|, \quad \boxed{|\Lambda_k - \Lambda_l| \geq \rho|k-l|}, \ \forall k, l \geq 1, \ \text{and} \ \sum_{k\geq 1} \frac{1}{|\Lambda_k|} < \infty,$$

for $\delta, \rho > 0$. *Then, for every* $\varepsilon > 0$ *there exists a constant* $C(\varepsilon) > 0$ *such that*

$$\mathcal{P}_k := \prod_{\ell\geq 1, \ell\neq k} \left| \frac{1 + \Lambda_k/\Lambda_\ell^*}{1 - \Lambda_k/\Lambda_\ell} \right| \leq C(\varepsilon) e^{\varepsilon \Re\Lambda_k}, \quad \forall k \geq 1. \qquad \blacksquare$$

For a proof of this result: [FATTORINI,RUSSELL] Quart. Appl. Math. (1974/75) (real case) or [FERNÁNDEZ-CARA,G.-B.,DE TERESA], J. Funct. Anal. (2010) (general case).

8. The Kalman condition for a class of parabolic systems. Boundary controls

Summarizing

For the problem

$$(25) \quad \begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T = (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

($A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)$) we know:

"System (25) is approximate controllable at time T $\iff$ System (25) is null controllable at time T $\iff$ the ***Kalman condition*** rank $\mathcal{K}_k = nk, \quad \forall k \geq 1$".

ESSENTIAL ASSUMPTION: Diffusion matrix $D = I_d$

What happens if $D \neq I_d$???

8. The Kalman condition for a class of parabolic systems. Boundary controls

Some references

1. F. AMMAR-KHODJA, A. BENABDALLAH, M. G-B, L. DE TERESA, *The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials*, J. Math. Pures Appl. (9), **96** (2011), no. 6, 555–590.
2. G. OLIVE, *Null-controllability for some linear parabolic systems with controls acting on different parts of the domain and its boundary*, Math. Control Signals Systems **23** (2012), no. 4, 257–280.
3. A. BENABDALLAH, F. BOYER, M. G-B, G. OLIVE, *Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null controllability in cylindrical domains*, SIAM J. Control Optim. **52** (2014), no. 5, 2970–3001.

9. New phenomena: Minimal time of controllability

9. New phenomena: Minimal time of controllability

We are going to revisited problem (18). With a slightly change of notations, this problem is:

$$(18)\qquad \left\{\begin{array}{ll} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T = (0,\pi)\times(0,T),\\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T),\\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{array}\right.$$

where $D = \operatorname{diag}(1, d)$, $\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$. When $d = 1$ (i.e., $D = Id$), we saw

Theorem ($d = 1$)

Let $A_0 \in \mathcal{L}(\mathbb{C}^2)$ and $B \in \mathbb{C}^2$ be given and let us denote by μ_1 and μ_2 the eigenvalues of A_0^*. Then (18) is ***approximate and null controllable*** *at any time $T > 0$* ***if and only if*** $\boxed{\operatorname{rank}\,[A \mid B] = 2}$ *and ($\lambda_k = k^2$)*

$$\boxed{\lambda_k - \lambda_j \neq \mu_1 - \mu_2 \quad \forall k, j \in \mathbb{N} \text{ with } k \neq j.}$$

■

9. New phenomena: Minimal time of controllability

$$(18)\quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T = (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $D = \text{diag}\,(1, d)$, $\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$.

Theorem ($d \neq 1$)

Under the previous assumptions, system (18) is ***approximate controllable*** *at time $T > 0$* ***if and only if*** $\boxed{\sqrt{d} \notin \mathbb{Q}}$.

Therefore:

1. If $d = 1$, (18) is **approximate and null controllable** at any $T > 0$.
2. If $d \neq 1$, we only know that system (18) is **approximate controllable** at time $T > 0$ **if and only if** $\boxed{\sqrt{d} \notin \mathbb{Q}}$.

$$(18)\quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $D = \operatorname{diag}(1,d)$, $\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

Assumption

In the sequel, $\boxed{D = \operatorname{diag}(1,d)}$ with $\boxed{d \neq 1}$ and $\boxed{\sqrt{d} \notin \mathbb{Q}}$.

Goal

Analyze the **null controllability** properties at time $T > 0$ of system (18).

9. New phenomena: Minimal time of controllability

$$(18)\quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Let φ be a solution of the adjoint problem:

$$\begin{cases} -\varphi_t - D\varphi_{xx} + A_0^*\varphi = 0 & \text{in } Q_T, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 \ \in H_0^1(0,\pi)^2 & \text{in } (0,\pi). \end{cases}$$

If y is a solution of the direct problem, then

$$\langle y(T),\varphi_0\rangle - \langle y_0,\varphi(0)\rangle = \int_0^T v(t) B^* D\varphi_x(0,t)\,dt$$

9. New phenomena: Minimal time of controllability

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If y is a solution of the direct problem, then

$$\langle y(T), \varphi_0\rangle - \langle y_0, \varphi(0)\rangle = \int_0^T v(t) B^* D \varphi_x(0,t)\, dt$$

Thus $\boxed{y(T) = 0} \iff \exists v \in L^2(0,T)$ such that

$$\boxed{\int_0^T v(t) B^* D\varphi_x(0,t)\, dt = -\langle y_0, \varphi(0)\rangle\,, \quad \forall \varphi_0 \in H_0^1(0,\pi;\mathbb{R}^2)}$$

Fattorini-Russell Method

Fattorini-Russell Method

- $\sigma(-D\partial_{xx}^2 + A_0^*) = \bigcup_{k\geq 1} \left\{k^2, dk^2\right\} := \bigcup_{k\geq 1} \{\lambda_{k,1}, \lambda_{k,2}\}$.
- $\{\Phi_{k,i}\}$ a (Riesz) basis of $H_0^1(0,\pi)^2$, where $\Phi_{k,i} = V_{k,i} \sin kx$, $i = 1, 2$ are eigenfunctions of the operator $\boxed{-D\partial_{xx}^2 + A_0^*}$.
- $V_{k,1}$ and $V_{k,2}$: eigenvectors of the matrix $\boxed{k^2 D + A_0^*}$ associated to the eigenvalues k^2, dk^2.

9. New phenomena: Minimal time of controllability

$$(18)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Objective: Existence of $v \in L^2(0,T)$ s.t.

$$\int_0^T v(t) B^* D \varphi_x(0,t)\, dt = -\langle y_0, \varphi(0) \rangle, \quad \forall \varphi_0 \in H_0^1(0,\pi;\mathbb{R}^2)$$

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Objective: Existence of $v \in L^2(0,T)$ s.t.

$$\int_0^T v(t) B^* D \varphi_x(0,t)\, dt = -\langle y_0, \varphi(0)\rangle, \quad \forall \varphi_0 \in H_0^1(0,\pi;\mathbb{R}^2)$$

- Choosing $\varphi_0 = \Phi_{k,i}$, we have $\varphi(\cdot,t) = e^{-\lambda_{k,i}(T-t)}\Phi_{k,i}$ and

$$\varphi(x,0) = e^{-\lambda_{k,i}T}\Phi_{k,i}(x), \quad \varphi_x(0,t) = k e^{-\lambda_{k,i}(T-t)} V_{k,i}$$

- The identity connecting y and φ writes (**moment problem**)

$$kB^* D V_{k,i} \int_0^T v(T-t) e^{-\lambda_{k,i}t}\, dt = -e^{-\lambda_{k,i}T}\langle y_0, \Phi_{k,i}\rangle, \quad \forall (k,i)$$

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$$

Approximate controllability: a necessary condition (I)

- $$\boxed{ \boxed{kB^*DV_{k,i}} \int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = -e^{-\lambda_{k,i}T}\left\langle y_0, \Phi_{k,i}\right\rangle, \quad \forall (k,i) }$$

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- A necessary condition: $B^* DV_{k,i} \neq 0$ for all $k \geq 1, i = 1,2$
- Recall $\boxed{d \neq 1}$,

$$B^* = (0,1), \quad V_{k,1} = \begin{pmatrix} 1 \\ \frac{1}{(d-1)k^2} \end{pmatrix}, \quad V_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \forall k \geq 1.$$

So, here $\boxed{B^* DV_{k,i} \neq 0, \quad \forall k \geq 1, i = 1,2}$ (**algebraic Kalman condition**)

9. New phenomena: Minimal time of controllability

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Approximate controllability: a necessary condition (II)

$$\lambda_{k,1} = \lambda_{j,2} = \lambda \Rightarrow \begin{cases} kB^*DV_{k,1} \int_0^T v(T-t)e^{-\lambda t}\,dt = -e^{-\lambda T}\langle y_0, \Phi_{k,1}\rangle \\ jB^*DV_{j,2} \int_0^T v(T-t)e^{-\lambda t\,dt} = -e^{-\lambda T}\langle y_0, \Phi_{j,2}\rangle \end{cases}$$

So it is necessary to have $\lambda_{k,1} \neq \lambda_{j,2}$. This leads to

$$k^2 \neq dj^2, \quad \forall k \neq j \geq 1 \iff \boxed{\sqrt{d} \notin \mathbb{Q}}$$

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So it is necessary to have $\lambda_{k,1} \neq \lambda_{j,2}$. This leads to

$$k^2 \neq dj^2, \quad \forall k \neq j \geq 1 \Longleftrightarrow \boxed{\sqrt{d} \notin \mathbb{Q}}$$

In the sequel, we will assume $\sqrt{d} \notin \mathbb{Q}$, i.e., the eigenvalues of $-D\partial_{xx}^2 + A_0^*$ with Dirichlet boundary conditions are pairwise distinct.

$$(18)\quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$$\boxed{kB^*DV_{k,i}\int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = -e^{-\lambda_{k,i}T}\langle y_0, \Phi_{k,i}\rangle, \quad \forall (k,i)}$$

Summarizing

Let $m_{k,i} = -\langle y_0, \Phi_{k,i}\rangle$, $b_{k,i} = kB^*DV_{k,i}$ (for any $\varepsilon > 0$, $\boxed{|m_{k,i}| \le C_\varepsilon e^{\varepsilon\lambda_{k,i}}}$ and $\boxed{|b_{k,i}| \ge C_\varepsilon e^{-\varepsilon\lambda_{k,i}}}$),

$$\exists\,?\, v \in L^2(0,T) \,:\, \int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = \frac{m_{k,i}}{b_{k,i}}e^{-\lambda_{k,i}T}, \quad \forall k \ge 1,\ i = 1,2$$

Let $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset (0,\infty)$ be a sequence with **pairwise distinct elements**:

$$\boxed{\sum_{k\geq 1} \frac{1}{\Lambda_k} < \infty}$$

Goal: Given $\{m_k\}_{k\geq 1}, \{b_k\}_{k\geq 1} \subset \mathbb{R}$ satisfying $\boxed{|m_k| \leq C_\varepsilon e^{\varepsilon \Lambda_k}}$ and $\boxed{|b_k| \geq C_\varepsilon e^{-\varepsilon \Lambda_k}}$, find $v \in L^2(0,T)$ s.t.

$$\int_0^T v(T-t) e^{-\Lambda_k t}\, dt = \frac{m_k}{b_k} e^{-\Lambda_k T}, \quad \forall k \geq 1.$$

Recall that the assumption

$$\boxed{\sum_{k\geq 1} \frac{1}{\Lambda_k} < \infty}$$

implies:

Theorem

Under the previous assumptions, $\{e^{-\Lambda_k t}\}_{k\geq 1} \subset L^2(0,T)$ *admits a* ***biorthogonal family*** $\{q_k\}_{k\geq 1}$ *in* $L^2(0,T)$, *i.e.:*

$$\int_0^T e^{-\Lambda_k t} q_l(t)\,dt = \delta_{kl}, \quad \forall k, l \geq 1$$

A formal solution to

$$\int_0^T v(T-t)e^{-\Lambda_k t}\,dt = \frac{m_k}{b_k}e^{-\Lambda_k T}, \quad \forall k \geq 1,$$

is v given by: $v(T-t) = \sum_{k\geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t)$,

Question: $v \in L^2(0,T)$?, i.e., is the series $\sum_{k\geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t)$ convergent in $L^2(0,T)$?

But this question itself amounts to:

$$\|q_k\|_{L^2(0,T)} \underset{k\to\infty}{\sim} ?$$

Theorem

Assume that $\sum_{k\geq 1} \frac{1}{\Lambda_k} < \infty$ *and (**gap condition**)*

$$\boxed{\exists \rho > 0 : |\Lambda_k - \Lambda_j| \geq \rho |k - j|, \quad \forall k, j}.$$

Then, for any $\varepsilon > 0$ *one has*

$$\|q_k\|_{L^2(0,T)} \leq C_\varepsilon e^{\varepsilon \Lambda_k}, \quad \forall k \geq 1,$$

and, for $T > 0$*, the control* $v(T-t) = \sum_{k\geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t) \in L^2(0,T)$.

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Recall that in our case $\Lambda = \{\Lambda_k\}_{k\geq 1} = \{j^2, dj^2\}_{j\geq 1}$, and the property

$$\boxed{\exists \rho > 0 : |\Lambda_k - \Lambda_j| \geq \rho |k-j|, \quad \forall k,j},$$

does not hold.

How does this fact affect our problem??

Theorem

Assume $\boxed{\sum_{k\geq 1} \frac{1}{|\Lambda_k|} < \infty}$. *Then, for any* $\varepsilon > 0$ *one has*

$$C_{1,\varepsilon} \frac{e^{-\varepsilon \Lambda_k}}{|W'(\Lambda_k)|} \leq \|q_k\|_{L^2(0,T)} \leq C_{2,\varepsilon} \frac{e^{\varepsilon \Lambda_k}}{|W'(\Lambda_k)|}, \quad \forall k \geq 1,$$

where $W(z)$ *is the Blaschke product:*

$$W(z) = \prod_{k=1}^{\infty} \frac{1 - z/\Lambda_k}{1 + z/\Lambda_k}, \qquad \boxed{W'(\Lambda_k) = -\frac{1}{2\Lambda_k} \prod_{j\neq k}^{\infty} \frac{1 - \Lambda_k/\Lambda_j}{1 + \Lambda_k/\Lambda_j}}$$

Definition

The **condensation index** of $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ is:

$$c(\Lambda) = \limsup_{k\to\infty} \frac{-\log |W'(\Lambda_k)|}{\Re(\Lambda_k)} \in [0, +\infty] \,.$$

Corollary

For any $\varepsilon > 0$ *one has*

$$\|q_k\|_{L^2(0,T)} \leq C_\varepsilon e^{(c(\Lambda)+\varepsilon)\Lambda_k}, \quad \forall k \geq 1.$$

Recall that we had m_k s.t. $|m_k| \leq C_\varepsilon e^{\varepsilon \Lambda_k}$, $|b_k| \geq C_\varepsilon e^{-\varepsilon \Lambda_k}$, for any $\varepsilon > 0$, and we wanted to solve: $v \in L^2(0,T)$ and

$$\boxed{\int_0^T v(T-t) e^{-\Lambda_k t}\, dt = \frac{m_k}{b_k} e^{-\Lambda_k T}, \quad \forall k},$$

We took $v(T-t) = \sum_{k \geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t)$.

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We took $v(T-t) = \sum_{k\ge1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t)$.

From the previous result: Given $\varepsilon > 0$:

$$\left| \frac{m_k}{b_k} \right| e^{-\Lambda_k T} \|q_k\|_{L^2(0,T)} \le C_\varepsilon e^{-\Lambda_k (T - c(\Lambda) - \varepsilon)}$$

Then

$$T > c(\Lambda) \Longrightarrow v(T-t) = \sum_{k\ge1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t) \in L^2(0,T).$$

9. New phenomena: Minimal time of controllability

$$(18) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

In our case,

$$\Lambda_d := \{\Lambda_k\}_{k\geq 1} = \left\{j^2, dj^2\right\}_{j\geq 1}.$$

Then

If $T > c(\Lambda_d)$, system (18) is null controllable at time T, where $c(\Lambda_d)$ is the **condensation index** of the sequence Λ_d.

$$(18)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

- The **index of condensation** of a sequence $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ is a real number $c(\Lambda) \in [0,+\infty]$ associated with this sequence and which "measures" the condensation at infinity.

$$c(\Lambda) = \limsup_{k\to\infty} \frac{-\log|W'(\Lambda_k)|}{\Re(\Lambda_k)} \in [0,+\infty], \quad W'(\Lambda_k) = \frac{-1}{2\Lambda_k}\prod_{j\neq k}^{\infty} \frac{1-\frac{\Lambda_k}{\Lambda_j}}{1+\frac{\Lambda_k}{\Lambda_j}}.$$

- This notion has been :
  - introduced by V.I. Bernstein in 1933: Leçons sur les progrès récents de la théorie des séries de Dirichlet for real sequences,
  - extended by J. R. Shackell in 1967 for complex sequences.

1. **Gap property**: $\exists \rho > 0 \ : \ |\Lambda_k - \Lambda_l| \geq \rho \, |k - l| \Rightarrow \boxed{c(\Lambda) = 0}$.

   In particular: for the scalar Dirichlet-Laplacien operator: $\Lambda_k = k^2$, $|\Lambda_k - \Lambda_l| = |k^2 - l^2| \geq |k - l|$. So

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2. $\alpha > 1$, $\beta > 0$ and $\Lambda = \{\Lambda_k\}_{k \geq 1}$ with $\Lambda_{2k} = k^\alpha$, $\Lambda_{2k+1} = k^\alpha + e^{-k^\beta}$
   $$c(\Lambda) = \begin{cases} 0 & \beta < \alpha \\ 1 & \beta = \alpha \\ +\infty & \beta > \alpha \end{cases} \qquad \text{(Note that } \boxed{\liminf |\Lambda_{k+1} - \Lambda_k| = 0}\text{)}$$

1. **Gap property**: $\exists \rho > 0 \ : \ |\Lambda_k - \Lambda_l| \geq \rho \, |k - l| \Rightarrow \boxed{c(\Lambda) = 0}$.
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3. $\Lambda = \{\Lambda_k\}_{k \geq 1}$ with
   $$\Lambda_{k^2+n} = k^2 + ne^{-k^2}, \quad n \in \{0, \cdots, 2k\}, \quad k \geq 1$$
   $$\boxed{c(\Lambda) = +\infty}$$

$$
(18) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}
$$

$$
D = \operatorname{diag}(1, d), \quad \Lambda_d = \{k^2, dk^2\}_{k\geq 1}, \quad \sqrt{d} \notin \mathbb{Q}.
$$

We have proved:

Theorem

There exists $T_0 = c(\Lambda_d) \in [0, +\infty]$ such that if $\boxed{T > T_0}$ then system (18) is null controllable at time T

$$(18)\qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

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We have proved:

Theorem

There exists $T_0 = c(\Lambda_d) \in [0, +\infty]$ *such that if* $\boxed{T > T_0}$ *then system* (18) *is null controllable at time* T

$\boxed{T > c(\Lambda_d)}$ is a sufficient condition for the null controllability of system (18) at time T. But,

what happens if $\boxed{T < c(\Lambda_d)}$?

One can prove:

Theorem

Let us take

$$\boxed{T_0 = c(\Lambda_d) \in [0, +\infty]}.$$

Then, if $\boxed{T < T_0}$, *system* (18) *is not null controllable at time* T.

Idea of the proof

By contradiction:

- The null controllability at time T is equivalent to: $\exists C_T > 0$ s.t.

$$\sum_{n,i} e^{-2\Lambda_{n,i}T} |a_{n,i}|^2 \leq C_T \int_0^T \left| \sum_{n,i} nB^* DV_{n,i} e^{-\Lambda_{n,i}t} a_{n,i} \right|^2 dt, \ \forall \{a_{n,i}\}_{n,i} \in \ell^2.$$

- Argument: Use the overconvergence of Dirichlet series.

$$(18)\qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

The controllability result

1. $\forall T > 0$: **Approximate controllability** at time T if and only if $\boxed{\sqrt{d} \notin \mathbb{Q}}$.
2. Assume $\boxed{\sqrt{d} \notin \mathbb{Q}}$, $\exists T_0 = c(\Lambda_d) \in [0, +\infty]$ such that
   1. the system is null controllable at time T if $\boxed{T > T_0}$
   2. Even if $\sqrt{d} \notin \mathbb{Q}$, if $\boxed{T < T_0}$ the system is **not null controllable** at time T!

$$(18)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

In fact, the good minimal time is

$$T_0 = \limsup_{k\to\infty} \frac{-\left(\log|b_k| + \log|W'(\Lambda_k)|\right)}{\Re(\Lambda_k)} \in [0,\infty]$$

$$
(18) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}
$$

$T_0 > 0$?

Is it possible to have a minimal time of control > 0? I.e., for $\Lambda_d = \{k^2, dk^2\}_{k\geq 1}$ with $\sqrt{d} \notin \mathbb{Q}$, is it possible that $c(\Lambda_d) > 0$?

9. New phenomena: Minimal time of controllability

$$(18)\quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$T_0 > 0$?

Is it possible to have a minimal time of control > 0? I.e., for $\Lambda_d = \{k^2, dk^2\}_{k\geq 1}$ with $\sqrt{d} \notin \mathbb{Q}$, is it possible that $c(\Lambda_d) > 0$?

Theorem

For any $\tau \in [0,+\infty]$, there exists $\sqrt{d} \notin \mathbb{Q}$ such that $c(\Lambda_d) = \tau$.

Remark

- There exists $\sqrt{d} \notin \mathbb{Q}$ such that $c(\Lambda_d) = +\infty$ (Luca, de Teresa).
- $c(\Lambda_d) = 0$ for almost $d \in (0,\infty)$ such that $\sqrt{d} \notin \mathbb{Q}$.
- For any $\tau \in [0,+\infty]$, the set $\{d \in (0,\infty) : c(\Lambda_d) = \tau\}$ is dense in $(0,+\infty)$.

$$(18)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $D = \text{diag}\,(1, d)$, $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Third phenomenon

For system (18): If $\sqrt{d} \notin \mathbb{Q}$, then,

1. **Approximate controllability**: System (18) is approximately controllable at any time $T > 0$.
2. **Null controllability**: System (18) is null controllable is $T > T_0 = c(\Lambda_d)$ and is not if $T < T_0 = c(\Lambda_d)$.

Remark

This minimal time also arises in other parabolic problems (degenerated problems):
BEAUCHARD, CANNARSA, GUGLIELMI, *Null controllability of Grushin-type operators in dimension two. J. Eur. Math. Soc. (JEMS) (2014).*
BEAUCHARD, MILLER, MORANCEY, *2d Grushin-type equations: Minimal time and null controllable data, J. Differential Equations 259 (2015), no. 11*

Reference

F. AMMAR KHODJA, A. BENABDALLAH, M.G.-B., L. DE TERESA, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences*, J. Funct. Anal. **267** (2014).

http://personal.us.es/manoloburgos

10. New phenomena: Dependence on the position of the control set

10. New phenomena: Geometrical dependence

Let us fix $T > 0$ and $\omega = (a, b) \subset (0, \pi)$. We consider the coupled parabolic systems:

$$(28) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T := (0,\pi) \times (0,T), \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

In (28), 1_ω is the characteristic function of the set ω, $y(x,t)$ is the state, $y_0 \in L^2(0,\pi;\mathbb{R}^2)$ is the initial datum and

- $q \in L^\infty(0,\pi)$ is a given function, $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^2)$ is a constant matrix and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a constant vector of $\mathbb{R}^2$;
- $u \in L^2(Q_T)$ is a scalar control function.

$$(28)\quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Remark

If $q \in L^\infty(0,\pi)$ satisfies: There exist an open subset $\omega_0 \subseteq \omega$ and a constant $\delta > 0$ s.t.

$$\boxed{q \geq \delta > 0 \text{ a.e. } \omega_0} \quad \text{or} \quad \boxed{q \leq -\delta < 0 \text{ a.e. } \omega_0}$$

$\left(\Longrightarrow \boxed{\operatorname{Supp} q \cap \omega \neq \emptyset} \right)$, then it is possible to repeat the arguments of section 2 and prove:

Theorem

Under the previous assumption, system (28) is approximately and exactly controllable to zero at any time $T > 0$.

10. New phenomena: Geometrical dependence

Let us consider the 2×2 linear reaction-diffusion system

$$(28) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

where $q \in L^\infty(Q_T)$, $y_0 \in L^2(0,\pi;\mathbb{R}^2)$,

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$\omega = (a,b) \subset (0,\pi)$ and $u \in L^2(Q_T)$ is a scalar control function.

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where $q \in L^\infty(Q_T)$, $y_0 \in L^2(0,\pi;\mathbb{R}^2)$,

$$\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},}$$

$\boxed{\omega = (a,b)} \subset (0,\pi)$ and $u \in L^2(Q_T)$ is a scalar control function.

No sign conditions on q.

$$\boxed{\omega \cap \operatorname{Supp} q = \emptyset}$$

(28) $$\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Theorem (Ammar Khodja, Benabdallah, G-B, de Teresa (2011))

Assume $I_k(q) \neq 0$ *for any* $k \geq 1$*, where*

(29) $$I_k(q) := \int_0^\pi q(x)|\sin(kx)|^2 \, dx,$$

and

$$\int_0^\pi q(x)\, dx \neq 0.$$

Then, for any $T > 0$*, system* (28) *is* ***null controllable*** *at time* T.

$$(28)\qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Null controllability properties of system (28) when

$$\int_0^\pi q(x)\,dx = 0?$$

In order to simplify the problem, we will assume the **geometrical assumption**:

Assumption (A1)

The function q satisfies $\operatorname{Supp} q \subset [0,a]$ or $\operatorname{Supp} q \subset [b,\pi]$ ($\omega = (a,b)$).

10. New phenomena: Geometrical dependence

$$(28) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Proposition (Boyer and Olive (2014))

Under the geometrical assumption (A1), system (28) *is* ***approximately controllable*** *at time* $T > 0$ *if and only if*

$$I_k(q) \neq 0, \quad \forall k \geq 1.$$

10. New phenomena: Geometrical dependence

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$$I_k(q) \neq 0, \quad \forall k \geq 1.$$

Remarks

1. The approximate controllability of system (28) does not depend on T.
2. Again, condition

$$I_k(q) \neq 0, \quad \forall k \geq 1.$$

is necessary for the null controllability of system (28) at time $T > 0$

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

We have a Riesz basis $\mathcal{B} := \left\{ \Phi^*_{k,1}, \Phi^*_{k,2} \right\}_{k \geq 1}$ of eigenfunctions and generalized eigenfunctions of the operator $L^* := -\frac{d^2}{dx^2} + q(x)A_0^*$ associated to the eigenvalue k^2 (**simple**).

Idea:

We will work with controls $u(x,t) = f(x)v(t)$ with $v \in L^2(0,T)$ and $f \in L^2(0,\pi)$ (appropriate) satisfies $\mathrm{Supp} f \subset \omega$.

Objective

Apply Fattorini-Russell method: **moment problem**

(28) $$\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

The moment problem

Find $v \in L^2(0,T)$ s.t.

$$\begin{cases} \displaystyle\int_0^T v(T-t)\boxed{e^{-k^2 t}}\,dt = \frac{m_{k,1}}{f_k} e^{-k^2 T}, \quad \forall k \geq 1, \\ \displaystyle\int_0^T v(T-t)\boxed{t e^{-k^2 t}}\,dt = \frac{m_{k,2}}{I_k(q) f_k} e^{-k^2 T}, \quad \forall k \geq 1, \end{cases}$$

where $\boxed{|m_{k,i}| \leq C_\varepsilon e^{\varepsilon \lambda_k}}$ and $\boxed{|f_k| \sim k^{-3} \geq C_\varepsilon e^{-\varepsilon \lambda_k}}$ $(i = 1, 2)$.

$$(28)\qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

The moment problem

Find $v \in L^2(0,T)$ s.t.

$$\begin{cases} \displaystyle\int_0^T v(T-t)e^{-k^2 t}\,dt = \frac{m_{k,1}}{f_k} e^{-k^2 T}, & \forall k \geq 1, \\ \displaystyle\int_0^T v(T-t)te^{-k^2 t}\,dt = \frac{m_{k,2}}{\boxed{I_k(q)}\, f_k} \boxed{e^{-k^2 T}}, & \forall k \geq 1, \end{cases}$$

where $\boxed{|m_{k,i}| \leq C_\varepsilon e^{\varepsilon \lambda_k}}$ and $\boxed{|f_k| \sim k^{-3} \geq C_\varepsilon e^{-\varepsilon \lambda_k}}$ $(i = 1, 2)$.

$$(28)\quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Conclusion

We can obtain the positive controllability result if
$T > \widetilde{T}_0(q) = \limsup \frac{-\log |I_k(q)|}{k^2}$,

Theorem

Assume $I_k(q) \neq 0$ for all $k \geq 1$. Then, if $T > \widetilde{T}_0(q)$, system (28) is null-controllable at time T.

Does the minimal time depend on the choice $u(x,t) = f(x)v(t)$?

What happens if $\boxed{T < \widetilde{T}_0(q)}$?

$$
(28) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}
$$

As before, the null controllability property for system (28) is equivalent to the **observability inequality**:

$$
\|\varphi(\cdot,0)\|^2_{(L^2)^2} \le C_T \int_0^T \int_\omega |\varphi_2(x,t)|^2 \, dx\, dt,
$$

for the solutions to **the adjoint problem**

$$
\begin{cases} -\varphi_t - \varphi_{xx} + q(x)A_0^*\varphi = 0 & \text{in } Q_T, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \end{cases}
$$

$$\|\varphi(\cdot,0)\|^2_{(L^2)^2} \le C_T \int_0^T \int_\omega |\varphi_2(x,t)|^2 \, dx\, dt,$$

If $T < \widetilde{T}_0(q)$, we can prove that the inequality does not hold **reasoning by contradiction**: Then system

(28) $$\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

is not null controllable at time T.

$$(28) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

$$\omega \cap \operatorname{Supp} q = \emptyset$$

Theorem

Assume $I_k(q) \neq 0$ *for all* $k \geq 1$ *and let:*

$$\widetilde{T}_0(q) := \limsup \frac{-\log |I_k(q)|}{k^2} \in [0, +\infty]$$

Then,

1. *If* $T > \widetilde{T}_0(q)$, *then system (28) is null-controllable at time* T.
2. *If* $\operatorname{Supp} q \subset [0,a]$ *or* $\operatorname{Supp} q \subset [b,\pi]$, *for any* $T < \widetilde{T}_0(q)$, *the system is not null-controllable at time* T.

$$(28) \qquad \begin{cases} y_t - y_{xx} + q(x) A_0 y = B u 1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Remarks

1. The previous results cannot be obtained using Carleman inequalities.
2. Due to the geometrical assumption

   The function q satisfies $\operatorname{Supp} q \subset [0,a]$ or $\operatorname{Supp} q \subset [b,\pi]$ $(\omega = (a,b))$

   the boundary and distributed null controllability results coincide.

$$
\text{(28)} \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}
$$

General case

$$\omega = (a,b) \subset (0,\pi) \text{ and } \mathrm{Supp}\, q \cap \omega = \emptyset.$$

The condition $I_k(q) \neq 0$ is no longer necessary:

$$I_{1,k}(q) := \int_0^a q(x)|\sin(kx)|^2\,dx; \quad I_{2,k}(q) := \int_b^1 q(x)|\sin(kx)|^2\,dx$$

$$I_k(q) = I_{1,k}(q) + I_{2,k}(q) = \int_0^\pi q(x)|\sin(kx)|^2\,dx;$$

$$
(28) \qquad \begin{cases} y_t - y_{xx} + q(x) A_0 y = B u 1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}
$$

Proposition (Boyer and Olive (2014))

If $\omega = (a,b)$, system (28) is ***approximately controllable*** *at time $T > 0$ if and only if*

$$
|I_k(q)| + |I_{1,k}(q)| \neq 0, \quad \forall k \geq 1.
$$

The proof uses the independence of the functions $\sin(kx)$ and $\cos(kx)$ in ω.

$$
\text{(28)} \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}
$$

Remarks

1. The approximate controllability of system (28) does not depend on T.
2. Again, condition

$$
|I_k(q)| + |I_{1,k}(q)| \neq 0, \quad \forall k \geq 1.
$$

is necessary for the null controllability of system (28) at time $T > 0$.

Null controllability of system (28)???

$$
(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}
$$

In this case we can have $I_k(q) = 0$, and then,

$$
L := -\frac{d^2}{dx^2} + q(x)A_0 : L^2(0,\pi;\mathbb{R}^2) \longrightarrow L^2(0,\pi;\mathbb{R}^2)
$$

has eigenvalues (k^2) of multiplicity 2.

Idea

Apply Fattorini-Russell's method with control under the form:

$$
\boxed{u(x,t) = f_1(x)v_1(t) + f_2(t)v_2(t)}
$$

with $\operatorname{Supp} f_1, \operatorname{Supp} f_2 \subset (a,b)$

$$(28)\qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Theorem

Let $\omega = (a,b) \subset (0,\pi)$ *and* $q \in L^\infty(Q_T)$ *satisfying* $\omega \cap \operatorname{Supp} q = \emptyset$,

$$|I_{1,k}(q)|^2 + |I_{2,k}(q)|^2 \neq 0 \; (\iff |I_{1,k}(q)|^2 + |I_k(q)|^2 \neq 0).$$

and

$$T_0(q) = \limsup \frac{\min\left[-\log\left|I_{1,k}(q)\right|, -\log\left|I_k(q)\right|\right]}{k^2}$$

Then,

1. *If* $T > T_0(q)$, *then system* (28) *is null-controllable at time* T.
2. *For any* $T < T_0(q)$, *the system is not null-controllable at time* T.

$$
(28) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}
$$

Remark

If

$$
|I_{1,k}(q)|^2 + |I_{2,k}(q)|^2 \neq 0
$$

and

$$
\int_0^a q(x)\,dx \neq 0 \quad \text{or} \quad \int_b^\pi q(x)\,dx \neq 0 \quad \text{or} \quad \int_0^\pi q(x)\,dx \neq 0,
$$

Then $T_0(q) = 0$ (Null controllability of system (28) for every $T > 0$).

Idea of the proof:

1. The reasoning for $T < T_0(q)$ is by contradiction.
2. For proving the positive controllability result for $T > T_0(q)$ we have to "mesure" the linear independence of $\boxed{B^*\Phi_{k,1}^* := \psi_k}$ and $\boxed{B^*\Phi_{k,2}^* := \sin(kx)}$ in ω ($\Phi_{k,1}^*$ and $\Phi_{k,2}^*$ are the eigenfunctions or the eigenfunction and the generalized eigenfunction of $L^* := -\frac{d^2}{dx^2} + q(x)A_0^*$ associated to k^2). Thanks to the assumption $\omega \cap \operatorname{Supp} q = \emptyset$ and the expression of ψ_k in ω this amounts to prove

$$\det \begin{pmatrix} f_{1,k} & f_{2,k} \\ \widetilde{f}_{1,k} & \widetilde{f}_{2,k} \end{pmatrix} \geq \frac{C}{k^m} \frac{I_{1,k}(q)}{I_k(q)}, \text{ when } I_{1,k}(q) \neq 0 \text{ and } I_k(q) \neq 0$$

where $C > 0$, $m \geq 1$, $f_{i,k}$ is the Fourier coefficient of f_i and

$$\widetilde{f}_{i,k} = \int_0^\pi f_i(x)\psi_k(x)\,dx, \quad k \geq 1, \quad i = 1, 2.$$

Example

$$q(x) = \begin{cases} 1 & \text{si } x \in (a_1, a_1 + \ell) \\ -1 & \text{si } x \in (a_2, a_2 + \ell), \end{cases}$$

$a_1 > 0$, $a_1 + \ell < a_2$, $a_2 + \ell < \pi$, $\ell > 0$ and $\omega = (a, b)$.

1. $\omega \cap \operatorname{Supp} q \neq \emptyset$ or $\omega \subseteq (a_1 + \ell, a_2)$: $T_0(q) = 0$. **Null controllability** $\forall T > 0$.

Example

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$a_1 > 0$, $a_1 + \ell < a_2$, $a_2 + \ell < \pi$, $\ell > 0$ and $\omega = (a, b)$.

1. $\omega \cap \operatorname{Supp} q \neq \emptyset$ or $\omega \subseteq (a_1 + \ell, a_2)$: $T_0(q) = 0$. **Null controllability** $\forall T > 0$.
2. $\omega = (a, b) \subseteq (0, a_1)$: $I_{1,k}(q) = \int_0^a q(x)\, dx = 0$, $\forall k$,

$$I_{2,k}(q) = -\frac{2}{k\pi} \sin\left(k\left(a_1 + a_2 + \ell\right)\right) \sin\left(k(a_2 - a_1)\right) \sin\left(k\ell\right)$$

Example

$$q(x) = \begin{cases} 1 & \text{si } x \in (a_1, a_1 + \ell) \\ -1 & \text{si } x \in (a_2, a_2 + \ell), \end{cases}$$

$a_1 > 0$, $a_1 + \ell < a_2$, $a_2 + \ell < \pi$, $\ell > 0$ and $\omega = (a, b)$.

1. $\omega \cap \operatorname{Supp} q \neq \emptyset$ or $\omega \subseteq (a_1 + \ell, a_2)$: $T_0(q) = 0$. **Null controllability** $\forall T > 0$.
2. $\omega = (a, b) \subseteq (0, a_1)$: $I_{1,k}(q) = \int_0^a q(x)\, dx = 0$, $\forall k$,

$$I_{2,k}(q) = -\frac{2}{k\pi} \sin\left(k\left(a_1 + a_2 + \ell\right)\right) \sin\left(k(a_2 - a_1)\right) \sin\left(k\ell\right)$$

- **Aprox. Contr.** $T > 0 \iff \boxed{(a_1 + a_2 + \ell)/\pi}, \boxed{(a_2 - a_1)/\pi}, \boxed{\ell/\pi} \notin \mathbb{Q}$.
- Given $\tau \in [0, \infty]$, $\exists a_1$, a_2 y ℓ satisfying the previous property s.t. $\boxed{T_0(q) = \tau}$. **Minimal time** of null controllability which could be $\boxed{T_0(q) = \infty}$.

$$
(28) \qquad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}
$$

Fourth phenomenon

For system (28): $\omega = (a,b) \subset (0,\pi)$ and $\omega \cap \operatorname{Supp} q = \emptyset$, then,

1. The **approximate controllability** is not equivalent to the **null controllability**.
2. **Null controllability**: The controllability result depends on the relative position of ω with respect to $\operatorname{Supp} q$.

Summarizing

Scalar case versus systems (parabolic problems)

	SCALAR CASE	SYSTEMS
boundary $\Leftrightarrow$ distributed control	Yes	No
approximate $\Leftrightarrow$ null controllability	Yes	No
minimal time for controling	No	Yes
geometrical conditions	No	Yes

Thank you for your attention!!