On the exact controllability to trajectories of the nonlinear heat equation in unbounded domains

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MODELLING AND CONTROL OF NONLINEAR EVOLUTION EQUATIONS

Trieste, May 2011

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1. Introduction. Statement of the problem

Let $\Omega \subset \mathbb{R}^N$ be a domain (bounded or unbounded), $N \geq 1$, with boundary $\partial\Omega$ regular enough ($\Omega \in C^{0,1}$ uniformly). Let $\omega \subseteq \Omega$ be an open subset and let us fix $T > 0$.

We consider the **linear** and **nonlinear** problems for the **heat equation**:

(1)
$$\begin{cases} \partial_t y - \Delta y + ay = v 1_\omega & \text{in } Q = \Omega \times (0, T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

(2)
$$\begin{cases} \partial_t y - \Delta y + F(y) = v 1_\omega & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

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(2)
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In (1) and (2), 1_ω is the characteristic function of the set ω, $y(x, t)$ is the state, y_0 is the initial datum (given in an appropriate space), and v is the control function (which is localized in ω -**distributed control**-). In (1), $a \in L^\infty(Q)$ is given. We will assume that $F : \mathbb{R} \to \mathbb{R}$ is a given function.

1. Introduction. Statement of the problem

Remark

In this talk we are interested in studying the controllability properties of systems (1) and (2) (controllability to trajectories) when Ω is an **unbounded domain**.

BOUNDED DOMAINS

Linear Problem: For every ω and T system (1) is null controllable (equivalently exactly controllable to trajectories): For every $y_0 \in L^2(\Omega)$ there is $v \in L^2(Q)$ s.t. the solution y to (1) satisfies $y(T) \equiv 0$ in Ω.

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- H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.
- G. LEBEAU, L. ROBBIANO, *Contrôle exact de l'équation de la chaleur*, Comm. P.D.E. 20 (1995), no. 1-2, 335–356.
  $a \equiv 0$: $v \in C_0^\infty(\omega \times (0,T))$.
- O. YU. IMANUVILOV, *Controllability of parabolic equations*, (Russian) Mat. Sb. 186 (1995), no. 6, 109–132; translation in Sb. Math. 186 (1995), no. 6, 879–900.
  $a \in L^\infty(Q)$: $v \in L^2(Q)$.

1. Introduction. Statement of the problem

Nonlinear Problem (bounded domains): Under appropriate assumptions on the function F (which has a **superlinear growth** at infinity) system (2) is exactly controllable to trajectories at time T:

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- E. FERNÁNDEZ-CARA, *Null controllability of the semilinear heat equation*, ESAIM Control Optim. Calc. Var. 2 (1997), 87–103.

$$\boxed{F(s) \sim |s| \log(1 + |s|).}$$

- E. FERNÁNDEZ-CARA, E. ZUAZUA, *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (2000), no. 5, 583–616.

$$\boxed{F(s) \sim |s| \log^p(1 + |s|), \quad p \in [0, 3/2).}$$

- V. BARBU, *Exact controllability of the superlinear heat equation*, Appl. Math. Optim. 42 (2000), no. 1, 73–89.

  $F(s) \sim |s| \log^p(1 + |s|)$ ($p \in [0, 3/2)$), $1 \leq N < 6$ and a dissipativity condition on the nonlinearity: $sF(s) \geq -\mu_o |s|^2$ ($\mu_0 \geq 0$).

Nonlinear Problem (bounded domains):

- A. Doubova, E. Fernández-Cara, M. G.-B., E. Zuazua, *On the controllability of parabolic systems with a nonlinear term involving the state and the gradient*, SIAM J. Control Optim. 41 (2002), no. 3, 798–819.

  Nonlinearities $F(y, \nabla y)$ with

$$F(s,w) \sim |s| \log^p(1+|s|+|w|) + |w| \log^q(1+|s|+|w|),$$
$$p \in [0, 3/2), \ q \in [0, 1/2).$$

UNBOUNDED DOMAINS

Linear Problem:

- S. Micu, E. Zuazua, *On the lack of null-controllability of the heat equation on the half-line*, Trans. AMS 353 (2001), no. 4, 1635–1659.

$\Omega = (0, \infty)$, $a \equiv 0$, $\omega \subset (0, \infty)$ a bounded domain
(in fact, boundary control on $x = 0$) :

"Problem (1) is not null-controllable in finite time if y_0 belongs to a negative Sobolev space"

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(For a similar result for $\Omega = \mathbb{R}^N_+$, also see
S. MICU, E. ZUAZUA, *On the lack of null-controllability of the heat equation on the half-space*, Port. Math. 58 (2001), no. 4, 1–24).

1. Introduction. Statement of the problem

Linear Problem (unbounded domains):

- V. CABANILLAS, S. DE MENEZES, E. ZUAZUA, *Null controllability in unbounded domains for the semilinear heat equations with nonlinearities involving gradient terms*, J. Optim. Theory Appl. **110** (2001), no. 2, 245–264.

$a \in L^\infty(Q)$ and even **first order terms** BUT
$\Omega \subset \mathbb{R}^N$, an **unbounded domain** s.t. $\Omega \setminus \overline{\omega}$ is BOUNDED.

CLOSE TO THE BOUNDED CASE !!

Linear Problem (unbounded domains):

- P. CANNARSA, P. MARTINEZ, J. VANCOSTENABLE, *Null controllability of the heat equation in unbounded domains by finite measure control region*, ESAIM:COCV 10 (2004), 381–408.

$$\Omega = (0,\infty), \quad a \equiv 0, \quad \omega = \cup_{n\geq 1}(a_n, b_n) \quad \text{an unbounded open set}$$

BUT, $\Omega \setminus \overline{\omega}$ is also an unbounded open set.

Under technical assumptions on $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ the authors prove a null controllability result

$$y_0 \in L^2(\Omega;\rho_1) \subsetneq L^2(\Omega) \quad \text{and} \quad v \in L^2(Q), \quad \text{or}$$
$$y_0 \in L^2(\Omega) \quad \text{and} \quad v \in L^2(Q;\rho_2) \supsetneq L^2(Q),$$

with $\rho_1, \rho_2 : (0,\infty) \to (0,\infty)$ depending on the sequences. If $b_n - a_n \geq m > 0$ and $a_{n+1} - b_n \leq M$, then $\rho_1 \equiv \rho_2 \equiv 1$.

Linear Problem (unbounded domains):

- L. MILLER, *On the null controllability of the heat equation in unbounded domains*, Bull. Sci. Math. 129 (2005), no. 2, 175–185.

Positive and negative results for the null controllability of the heat equations ($a \equiv 0$) in domains $\widetilde{\Omega} = \Omega \times \mathcal{O}$:

"If the heat in Ω is null-controllable at time T with distributed controls supported in ω, then it is also null-controllable in $\Omega \times \mathcal{O}$ at time T with distributed controls supported in $\omega \times \mathcal{O}$. In addition,

$$\mathcal{C}_T(\Omega \times \mathcal{O}, \omega \times \mathcal{O}) \leq \mathcal{C}_T(\Omega, \omega)."$$

1. Introduction. Statement of the problem

Linear Problem (unbounded domains):

- M. G.-B., L. de Teresa, *Some results on controllability for linear and nonlinear heat equations in unbounded domains*, Adv. Diff. Eq. 12 (2007), no. 11, 1201–1240.

Global Carleman inequalities for the adjoint system under some geometrical assumptions on (Ω, ω) (more details later).
Consequence: Null controllability result for system (1) for every $a \in L^\infty(Q)$.

UNBOUNDED DOMAINS

Nonlinear Problem:

1. V. CABANILLAS, S. DE MENEZES, E. ZUAZUA, *Null controllability in unbounded domains for the semilinear heat equations with nonlinearities involving gradient terms*, J. Optim. Theory Appl. **110** (2001), no. 2, 245–264.
2. M. G.-B., L. DE TERESA, *Some results on controllability for linear and nonlinear heat equations in unbounded domains*, Adv. Diff. Eq. 12 (2007), no. 11, 1201–1240.

(Ω, ω) such that $\Omega \setminus \overline{\omega}$ is **bounded**

1. In [1]: Globally Lipschitz-continuous nonlinearities $F = F(y, \nabla y)$ and distributed controls $v \in L^2(Q)$.
2. In [2]: Nonlinearities $F = F(y, \nabla y)$ with superlinear growth at infinity and distributed controls $v \in L^\infty(Q)$ (and more regular).

Some questions:

1. Given (Ω, ω), is system (1) null controllable at time T for any $a \in L^\infty(Q)$ and $y_0 \in L^2(\Omega)$ with controls v in $L^2(Q)$?
2. Is it possible to solve the null controllability problem for the linear system (1) with "**more regular controls**", for example, $v \in L^2(Q) \cap L^\infty(Q)$ or even $v \in L^2(Q) \cap C^{\alpha,\alpha/2}(\overline{Q})$ with $\alpha \in (0,1)$? (**Important** for dealing with null controllability of the nonlinear problem (2)).
3. Is it possible to extend the null controllability result to the nonlinear case (system (2))? (F sub-linear or super-linear nonlinearity). (**Difficulty:** The Sobolev **compact embeddings** fail when Ω is an unbounded open set).

2. Linear null controllability result with regular controls

ASSUMPTION (H1)

Given Ω, ω, T, $a \in L^\infty(Q)$ and $y_0 \in L^2(\Omega)$, there exists a control $\widetilde{v} \in L^2(Q)$ and $\omega_0 \subset \omega$ s. t. $d_0 = \text{dist}\,(\omega_0, \Omega \setminus \overline{\omega}) > 0$,
$\text{Supp}\,\widetilde{v} \subseteq \overline{\omega}_0 \times [0, T]$ and the solution $\widetilde{y}$ to (1) satisfies $\widetilde{y}(\cdot, T) \equiv 0$ in Ω.

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Theorem

Assume ***(H1)*** *and* $\partial\omega \cap \partial\Omega$ *is of class* C^2 *uniformly (if* $\partial\omega \cap \partial\Omega \neq \emptyset$*). Then, for any* $\alpha \in (0, 1)$*, there exist* $C_\alpha = C_\alpha(\Omega, \omega, d_0) > 0$ *and* $v \in L^2(Q) \cap C^{\alpha,\alpha/2}(\overline{Q})$ *such that* $\text{Supp}\, v \subseteq \overline{\omega} \times [0, T]$,

$$\|v\|_{L^2 \cap C^{\alpha,\alpha/2}} \leq e^{C_\alpha(1+T+T\|a\|_\infty)} \left(\|\widetilde{v}\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)} \right),$$

and the solution y *to* (1) *associated to* v *and* y_0 *satisfies*

$$y(\cdot, T) = 0 \quad in \quad \Omega.$$

2. Linear null controllability result with regular controls

Proof: Let us introduce two cut-off functions $\eta \in C^\infty([0,T])$ and $\theta \in C^\infty(\overline{\Omega})$ such that

$$\begin{cases} \eta \equiv 1 \text{ in } [0, \frac{T}{4}],\ \eta \equiv 0 \text{ in } [\frac{3T}{4}, T],\ 0 \le \eta \le 1 \text{ in } [0,T], |\eta'(t)| \le C/T,\ \forall t; \\ \theta \equiv 1 \text{ in } \overline{\omega}_0, \quad 0 \le \theta \le 1 \text{ in } \Omega \text{ and } \operatorname{dist}(\operatorname{Supp}\theta, \overline{\Omega}\setminus\omega) > 0. \end{cases}$$

Let Y be the solution to system (1) corresponding to $v \equiv 0$:

$$(3) \qquad \begin{cases} \partial_t Y - \Delta Y + aY = 0 & \text{in } Q, \\ Y = 0 \text{ on } \Sigma, \quad Y(\cdot,0) = y_0(\cdot) & \text{in } \Omega, \end{cases}$$

We now take $y = (1-\theta)\widetilde{y} + \eta\theta Y$ in Q and

$$\boxed{v = (\partial_t - \Delta + a)y = 2\nabla\theta\cdot\nabla\widetilde{y} + (\Delta\theta)\widetilde{y} + (\partial_t - \Delta + a)(\eta\theta Y).}$$

It is clear that $\operatorname{Supp} v(\cdot,t) \subseteq \operatorname{Supp}\theta$ (and $\operatorname{Supp} v(\cdot,t) \cap (\overline{\Omega}\setminus\omega) = \emptyset$), y is the solution to (1) corresponding to the control v and, taking into account that $\widetilde{y}(T) \equiv 0$ in Ω, we get $y(\cdot,T) \equiv 0$ in Ω.

In fact v is a regular control and its regularity properties are independent of y_0 and $\widetilde{v}$. Indeed, we can express y and v as

$$y \equiv (1-\theta)q + \eta(t)Y, \quad v \equiv \theta\eta' Y + 2\nabla\theta\cdot\nabla q + (\Delta\theta)q,$$

where q is given by $q = \widetilde{y} - \eta Y$ and, therefore, satisfies

$$\begin{cases} \partial_t q - \Delta q + aq = \widetilde{v}1_\omega - \eta' Y \text{ in } Q, \\ q = 0 \text{ on } \Sigma, \quad q(\cdot,0) = 0 \text{ in } \Omega. \end{cases}$$

Let us fix $\delta \in (0, T/4)$, $p \in [2,\infty)$ and $\mathcal{O}_0, \mathcal{O}_1 \subset \omega$ such that $\operatorname{dist}(\overline{\mathcal{O}}_i, \overline{\Omega}\setminus\omega) > 0$ $(i = 0,1)$ and $\operatorname{dist}(\overline{\omega}_0, \mathcal{O}_1) > 0$ (and, in particular, $\overline{\mathcal{O}}_1 \cap \mathrm{Supp}\,\widetilde{v} = \emptyset$). If we denote by

$$\begin{cases} X_0^p = \{y \in L^p(\delta, T; W^{2,p}(\mathcal{O}_0)) : \partial_t y \in L^p(\mathcal{O}_0 \times (\delta, T))\}, \\ X_1^p = \{y \in L^p(0, T; W^{2,p}(\mathcal{O}_1)) : \partial_t y \in L^p(\mathcal{O}_1 \times (0, T))\} \end{cases}$$

then, $Y \in X_0^p$ (see (3)), $q \in X_1^p$ and $v \in L^p(0,T; W^{1,p}(\Omega))$.

2. Linear null controllability result with regular controls

In fact, we can obtain something better: if $p > N + 2$, one has $X_0^p \hookrightarrow C^{1+\alpha,(1+\alpha)/2}(\overline{\mathcal{O}}_0 \times [\delta, T])$ and $X_1^p \hookrightarrow C^{1+\alpha,(1+\alpha)/2}(\overline{\mathcal{O}}_1 \times [0, T])$ with $\alpha = 1 - (N + 2)/p$. Thus, $v \in C^{\alpha,\alpha/2}(\overline{Q})$ and

$$\|v\|_{C^{\alpha,\alpha/2}} \leq e^{C_\alpha(1+T+T\|a\|_\infty)}\|\widetilde{y}\|_{W(0,T)}$$

with $C_\alpha = C_\alpha(\Omega, T) > 0$ and

$$W(0, T) = \{y \in L^2(0, T; H_0^1(\Omega)) : \partial_t y \in L^2(0, T; H^{-1}(\Omega))\}.$$

2. Linear null controllability result with regular controls

1. The previous regularity result for v is independent of the regularity of the initial datum y_0, the control $\widetilde{v}$ and the boundary $\partial\Omega \setminus \partial\omega$. We have only used **local regularity** properties of the operator $L \equiv \partial_t - \Delta + a$. In the case in which $a \equiv 0$, we obtain $v \in C^\infty(\overline{Q})$ (as in the bounded case; see paper of Lebeau-Robbiano).

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2. This technique can be applied if we consider a linear parabolic problem with a first order term $B \cdot \nabla y$ ($B \in L^\infty(Q)^N$) obtaining the same regularity result.

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2. This technique can be applied if we consider a linear parabolic problem with a first order term $B \cdot \nabla y$ ($B \in L^\infty(Q)^N$) obtaining the same regularity result.
3. This approach also works in the case of systems of two coupled parabolic equations.

2. Linear null controllability result with regular controls

The previous result has been proved in

L. DE TERESA, M. G.-B., *Some results on controllability for linear and nonlinear heat equations in unbounded domains*, Adv. Diff. Eq. 12 (2007), no. 11, 1201–1240.

3. Controllability of the nonlinear problem

Now, let us see a null controllability result for the nonlinear problem

$$(2) \qquad \begin{cases} \partial_t y - \Delta y + F(y) = v 1_\omega & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

where $y_0 \in L^2(\Omega)$ and $F : \mathbb{R} \to \mathbb{R}$ is a given function.

ASSUMPTION (H2)

Asume that (Ω, ω) and $T > 0$ satisfy: for any $a \in L^\infty(Q)$ and $y_0 \in L^2(\Omega)$, there exists a control $\widetilde{v} \in L^2(Q)$ s.t. the solution $\widetilde{y}$ to (1) satisfies $\widetilde{y}(\cdot, T) \equiv 0$ in Ω and

$$\|\widetilde{v}\|_{L^2(Q)} \leq C(\Omega, \omega, T, \|a\|_\infty) \|y_0\|_{L^2(\Omega)},$$

with $C(\Omega, \omega, T, \cdot)$ an increasing function with respect to its last argument.

3. Controllability of the nonlinear problem

Theorem

Let us assume ***(H2)****. Let* $F \in C^1(\mathbb{R})$ *be a globally Lipschitz-continuous function such that* $F(0) = 0$*. Then, system* (2) *is null controllable at time* T*.*

3. Controllability of the nonlinear problem

Theorem

*Let us assume **(H2)**. Let $F \in C^1(\mathbb{R})$ be a globally Lipschitz-continuous function such that $F(0) = 0$. Then, system (2) is null controllable at time T.*

Proof: As usual, we are going to perform a **fixed point argument**: Let us fix $y_0 \in L^2(\Omega)$. We introduce a set-valued mapping as follows:

- We take $G(s) = F(s)/s$ if $s \neq 0$ and $G(0) = F'(0)$. Then $G \in C^0(\mathbb{R})$ and $G \in L^\infty(\mathbb{R})$ ($M = \|G\|_\infty$).
- If $z \in L^2(Q)$, we consider the linear null controllability problem

$$(4) \qquad \boxed{\begin{cases} \partial_t y - \Delta y + G(z) y = v 1_\omega & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}}$$

$$\mathcal{U}(z) = \{ v \in L^2(Q) : y_v(T) = 0 \text{ and } \|v\|_{L^2(Q)} \leq C(\Omega, \omega, T, M) \|y_0\|_{L^2(\Omega)} \}.$$

(Assumption (H2) implies $\mathcal{U}(z) \neq \emptyset$ for any $z \in L^2(Q)$).

3. Controllability of the nonlinear problem

Thus, we introduce the set-valued mapping

$$\Lambda : z \in L^2(Q) \mapsto \Lambda(z) \subset L^2(Q),$$

where

$$\Lambda(z) = \{y_v \in L^2(Q) : y_v \text{ is the solution to (4) associated to } v \in \mathcal{U}(z)\}.$$

Does the mapping Λ admit a **fixed point** ??? Λ must be **upper semicontinuous** and **compact** in $L^2(Q)$.

Difficulty

The open set Ω and the uncontrolled open set $\Omega \setminus \overline{\omega}$ could be unbounded: Lack of compactness in the Sobolev embeddings.

Compactness of Λ??

3. Controllability of the nonlinear problem

We apply the Kakutani Fixed Point Theorem:

Theorem

Let K be a compact convex set of a locally convex space X and $\Lambda : K \to K$ an upper semicontinuous set-valued map such that $\Lambda(x)$ is a nonempty, compact and convex set for every $x \in K$. Then Λ has a ***fixed point*** *$x_* \in K$, i.e., there exists $x_* \in K$ such that $x_* \in \Lambda(x_*)$.*

We can apply this result to $X = L^2(Q)$ with the **weak topology**, and $K = \overline{\text{conv}}\ [\Lambda(L^2(Q))]$ (which is a bounded set and then, a compact set with respect to the **weak topology** of $L^2(Q)$).

Technical difficulty: Λ is upper semicontinuous in K with respect to the **weak topology** of $L^2(Q)$. ■

3. Controllability of the nonlinear problem

ASSUMPTION (H3)

Assume that (Ω, ω) and $T > 0$ satisfy the property: There is $\omega_0 \subset \omega$ with $d_0 = \text{dist}\,(\omega_0, \Omega \setminus \overline{\omega}) > 0$ such that for any $a \in L^\infty(Q)$ and $y_0 \in L^2(\Omega)$, there exists a control $\widetilde{v} \in L^2(Q)$ satisfying:

1. $\operatorname{Supp} \widetilde{v} \subseteq \overline{\omega}_0 \times [0, T]$ and
$$\|\widetilde{v}\|_{L^2(Q)} \leq C(\Omega, \omega, T, \|a\|_\infty)\|y_0\|_{L^2(\Omega)},$$
with $C(\Omega, \omega, T, \cdot)$ an increasing function with respect to its last argument.
2. the solution $\widetilde{y}$ to (1) satisfies $\widetilde{y}(\cdot, T) \equiv 0$ in Ω

Of course, **(H3)** implies **(H1)** and **(H2)**.

3. Controllability of the nonlinear problem

As a consequence we get a local null controllability result for system (2) for general nonlinearities F:

Corollary

*Let us assume **(H3)** and $\partial\omega \cap \partial\Omega$ is of class C^2 uniformly (if $\partial\omega \cap \partial\Omega \neq \emptyset$). Let $F \in C^1(\mathbb{R})$ be a function s. t. $F(0) = 0$. Then, there exists $\varepsilon > 0$ s.t. for any $y_0 \in L^2(\Omega) \cap L^\infty(\Omega)$ satisfying*

$$\|y_0\|_{L^2 \cap L^\infty} \leq \varepsilon,$$

there is $v \in L^2(Q) \cap L^\infty(Q)$ such that the solution y to (2) satisfies $y(\cdot, T) = 0$ in Ω.

M. G.-B., *Some remarks on the exact controllability to trajectories for the nonlinear heat equations in unbounded domains*, In preparation.

4. Global Carleman Inequalities. Some Examples

Goal: Carleman Inequalities in unbounded domains for the **adjoint problem**:

(5)
$$\begin{cases} -\partial_t \varphi - \Delta \varphi + a\varphi = 0 \ \text{ in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(x,T) = \varphi_0(x) \text{ in } \Omega. \end{cases}$$

ASSUMPTION (H4)

Assume (Ω, ω) s.t. $D_\Omega(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, $\exists \omega_1 \subset \omega$ with $d_1 = \operatorname{dist}(\omega_1, \Omega \setminus \overline{\omega}) > 0$, and there exist η^0 and $\mathcal{C}_0, \mathcal{C}_1 > 0$ such that

$$\begin{cases} \eta^0 \in C^2(\mathbb{R}^N), \quad \eta^0 \geq 0 \text{ in } \Omega, \\ |\nabla \eta^0| \geq \mathcal{C}_0 > 0 \quad \text{in} \quad \overline{\Omega} \setminus \omega_0, \\ \frac{\partial \eta^0}{\partial n} \leq 0 \quad \text{on} \quad \partial\Omega, \\ |\eta^0| + |\nabla \eta^0| + \sum_{i,j} |\frac{\partial^2 \eta^0}{\partial x_i \partial x_j}| \leq \mathcal{C}_1 \quad \text{in} \quad \Omega. \end{cases}$$

4. Global Carleman Inequalities. Some Examples

$$\alpha(x,t) = \frac{e^{2\lambda m||\eta^0||_\infty} - e^{\lambda(m||\eta^0||_\infty + \eta^0(x))}}{t(T-t)}, \quad \xi(x,t) = \frac{e^{\lambda(m||\eta^0||_\infty + \eta^0(x))}}{t(T-t)},$$

$s, \lambda > 0, \quad (m > 4$ is fixed).

Theorem

*Assume **(H4)**. Then, there exist positive constants σ_1, λ_1 and C_1, only depending on $\mathcal{C}_0$, $\mathcal{C}_1$ and d_1, such that*

$$\mathcal{I}(\varphi) \leq C_1 s^3 \lambda^4 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} \xi^3 |\varphi|^2,$$

$\forall s \geq s_1 = \sigma_1 (T + T^2 + T^2 ||a||_\infty^{2/3})$, $\lambda \geq \lambda_1$, *with* φ *solution to* (5) *and*

$$\mathcal{I}(\varphi) \equiv s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} [|\partial_t \varphi|^2 + |\Delta\varphi|^2] + s\lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla\varphi|^2 + s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |\varphi|^2.$$

$$\omega_0 = \{x \in \omega : \text{dist}\,(x, \omega_1) < d_1/2\}.$$

4. Global Carleman Inequalities. Some Examples

The proof is given in

L. DE TERESA, M. G.-B., *Some results on controllability for linear and nonlinear heat equations in unbounded domains*, Adv. Diff. Eq. 12 (2007), no. 11, 1201–1240.

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Corollary

(H4) *implies* ***(H3)*** *(p. 25) for the previous* ω_0, $d_0 = d_1/2$ *and*

$$C(\Omega, \omega, T, \|a\|_\infty) \equiv \exp\left\{ C(1 + 1/T + T||a||_\infty + ||a||_\infty^{2/3}) \right\}$$

with $C = C(\mathcal{C}_0, \mathcal{C}_1, d_1) > 0$.

We have an **explicit dependence** of the constant C with respect to $||a||_\infty$, then, the proof of Corollary 3.3 also gives a global controllability result for system (2) when the nonlinearity F satisfies some growth assumption:

Corollary

Let us assume **(H4)** *and* $\partial\omega \cap \partial\Omega$ *is of class* C^2 *uniformly (if* $\partial\omega \cap \partial\Omega \neq \emptyset$*). Let* $F \in C^1(\mathbb{R})$ *be a function s.t.* $F(0) = 0$ *and*

$$\lim_{|s|\to\infty} \frac{|F(s)|}{|s|\log^{3/2}(1+|s|)} = 0.$$

Then, for any $y_0 \in L^2(\Omega) \cap L^\infty(\Omega)$ *there exists* $v \in L^2(Q) \cap L^\infty(Q)$ *such that the solution* y *to* (2) *satisfies* $y(\cdot, T) = 0$ *in* Ω.

Question

Is it possible to provide open sets Ω and ω which fulfill assumption **(H4)**??? YES.

4. Global Carleman Inequalities. Some Examples

Corollary

Let us assume ***(H4)*** *and $\partial\omega \cap \partial\Omega$ is of class C^2 uniformly (if $\partial\omega \cap \partial\Omega \neq \emptyset$). Let $F \in C^1(\mathbb{R})$ be a function s.t. $F(0) = 0$ and*

$$\lim_{|s|\to\infty} \frac{|F(s)|}{|s| \log^{3/2}(1 + |s|)} = 0.$$

Then, for any $y_0 \in L^2(\Omega) \cap L^\infty(\Omega)$ there exists $v \in L^2(Q) \cap L^\infty(Q)$ such that the solution y to (2) satisfies $y(\cdot, T) = 0$ in Ω.

Question

Is it possible to provide open sets Ω and ω which fulfill assumption **(H4)**??? YES.

1. $\Omega \subset \mathbb{R}^N$ a **BOUNDED** open set with $\Omega \in C^2$ and $\omega \subset\subset \Omega$.

NONTRIVIAL EXAMPLES

2. $\Omega \subset \mathbb{R}^N$ an UNBOUNDED open set with $\partial\Omega \in C^2$ uniformly and $\omega \subset \Omega$ such that $\Omega \setminus \overline{\omega}$ is BOUNDED.

3. $\Omega = (0, \infty)$ and

$$\omega = \bigcup_{n \geq 0} (a_n, b_n),$$

with $0 < a_n < b_n < a_{n+1}$, $\lim a_n = \lim b_n = \infty$, $b_n - a_n \geq m > 0$ and $a_{n+1} - b_n \leq M < \infty$.

η_0 is an oscillating function,

$$d_1 = \frac{m}{8}, \quad \mathcal{C}_0 = \frac{1}{M+m} \quad \text{and} \quad \mathcal{C}_1 = C\left(1 + \frac{1}{m^2}\right)$$

4. $\Omega \subset \mathbb{R}^N$ unbounded open set of class C^2 uniformly and

$$\omega = \{x \in \Omega : \mathrm{dist}\,(x, \partial\Omega) > \delta\}$$

with $\delta = \delta(\Omega) > 0$ a constant.

- $\delta(\Omega) \sim$ radius such that Ω satisfies a uniform interior sphere condition,
- $\eta_0(x) \sim \mathrm{dist}\,(x, \partial\Omega)$ near $\partial\Omega$,
- $d_1 \sim \delta(\Omega)$, $\mathcal{C}_0 = 1$ and $\mathcal{C}_1 \sim \dfrac{1}{\delta(\Omega)^2}$.

4. Global Carleman Inequalities. Some Examples

5. $\Omega = \Omega_0 \times \mathcal{O}$ and $\omega = \omega_0 \times \mathcal{O}$ with $\omega_0 \subset \Omega_0 \subset \mathbb{R}^N$ satisfying **(H4)** and $\mathcal{O} \subset \mathbb{R}^M$ an arbitrary open set such that $D_\Omega(-\Delta) = H^2(\Omega) \times H_0^1(\Omega)$. The constants appearing in Theorem 4.1 and Corollary 4.2 are independent of $\mathcal{O}$.

$$\eta_0^{\omega}(x,y) = \eta_0^{\omega_0}(x), \quad \forall (x,y) \in \Omega_0 \times \mathcal{O}.$$

6. $\Omega = \Omega_0 \times \Omega_1$ and $\omega = \omega_0 \times \omega_1$ with $\omega_0 \subset \Omega_0 \subset \mathbb{R}^N$ and $\omega_1 \subset \Omega_1 \subset \mathbb{R}^M$ satisfying **(H4)** and $D_\Omega(-\Delta) = H^2(\Omega) \times H_0^1(\Omega)$.

$$\eta_0^{\omega}(x,y) = \eta_0^{\omega_0}(x)\eta_0^{\omega_1}(y), \quad \forall (x,y) \in \Omega_0 \times \Omega_1.$$

Thank you for your attention!!