Some remarks on the exact controllability to trajectories for the nonlinear heat equation

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Workshop on Control and Inverse Problems

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1. Introduction. Statement of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial\Omega$ of class C^2. Let $\omega \subseteq \Omega$ be an open subset anf let us fix $T > 0$.

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We consider the **linear** and **nonlinear** problems for the **heat equation**:

$$(1) \qquad \begin{cases} \partial_t y - \Delta y + a y = v 1_\omega & \text{in } Q = \Omega \times (0, T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

$$(2) \qquad \begin{cases} \partial_t y - \Delta y + F(y) = v 1_\omega & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

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In (1) and (2), 1_ω represents the characteristic function of the set ω, $y(x,t)$ is the state, y_0 is the initial datum and is given in an appropriate space, and v is the control function (which is localized in ω -**distributed control**-). In (1), $a \in L^\infty(Q)$ is given. We will assume that $F : \mathbb{R} \to \mathbb{R}$ is a given function.

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Linear Problem: For every ω and T system (1) is null controllable (equivalently exactly controllable to trajectories): For every $y_0 \in L^2(\Omega)$ there is $v \in L^2(Q)$ s.t. the solution y to (1) satisfies $y(T) \equiv 0$ in Ω.

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1. H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.
2. G. LEBEAU, L. ROBBIANO, *Contrôle exact de l'équation de la chaleur*, Comm. P.D.E. 20 (1995), no. 1-2, 335–356.
   $a \equiv 0$: $v \in C_0^\infty(\omega \times (0, T))$.
3. O. YU. IMANUVILOV, *Controllability of parabolic equations*, (Russian) Mat. Sb. 186 (1995), no. 6, 109–132; translation in Sb. Math. 186 (1995), no. 6, 879–900.
   $a \in L^\infty(Q)$: $v \in L^2(Q)$.

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Nonlinear Problem: Under appropriate assumptions on the function F (which has a **superlinear growth** at infinity) system (2) is exactly controllable to trajectories at time T:

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1. E. FERNÁNDEZ-CARA, *Null controllability of the semilinear heat equation*, ESAIM Control Optim. Calc. Var. 2 (1997), 87–103.

$$\boxed{F(s) \sim |s| \log(1+|s|).}$$

2. E. FERNÁNDEZ-CARA, E. ZUAZUA, *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (2000), no. 5, 583–616.

$$\boxed{F(s) \sim |s| \log^p(1+|s|), \quad p \in [0, 3/2).}$$

3. V. BARBU, *Exact controllability of the superlinear heat equation*, Appl. Math. Optim. 42 (2000), no. 1, 73–89.

   $F(s) \sim |s| \log^p(1+|s|)$ ($p \in [0, 3/2)$), $1 \leq N < 6$ and a dissipativity condition on the the nonlinearity: $sF(s) \geq -\mu_0 |s|^2$ ($\mu_0 \geq 0$).

Remark

Common Point: *The linear problem* (1) *is solved with a control* v *in* $L^p(Q)$ *(*$p > \frac{N}{2} + 1$*) with estimates of its norm with respect to* T, $\|a\|_\infty$ *and* y_0. ■

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Common Point: *The linear problem* (1) *is solved with a control* v *in* $L^p(Q)$ *(*$p > \frac{N}{2} + 1$*) with estimates of its norm with respect to* T, $\|a\|_\infty$ *and* y_0. ■

DIFFERENT TECHNIQUES

GOAL:

Revisit the main known techniques which allow to prove the null controllability result for system (1) with a control $v \in L^p(Q)$, $p \in (2, \infty]$ (with estimates).

THREE BASIC REFERENCES

- O. YU. IMANUVILOV, *Controllability of parabolic equations*, (Russian) Mat. Sb. 186 (1995), no. 6, 109–132; translation in Sb. Math. 186 (1995), no. 6, 879–900.
- E. FERNÁNDEZ-CARA, E. ZUAZUA, *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **17**, No. 5, (2000), 583–616.
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ANOTHER REFERENCE

- O. BODART, M. G.-B., R. PÉREZ-GARCÍA, *Existence of insensitizing controls for a semilinear heat equation with a superlinear nonlinearity*, Comm. P.D.E 29 (2004), no. 7-8, 1017–1050.

2. Linear null controllability result with regular controls

We consider the distributed controllability problem for the linear system:

$$(1) \qquad \begin{cases} \partial_t y - \Delta y + a y = v 1_\omega & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

where $\omega \subset \Omega$ is an open subset, $v \in L^2(Q)$ is the control and y_0 is given in $L^2(\Omega)$.

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where $\omega \subset \Omega$ is an open subset, $v \in L^2(Q)$ is the control and y_0 is given in $L^2(\Omega)$.

Let us fix $\varphi_0 \in L^2(\Omega)$ and consider the *adjoint problem*

$$(3) \qquad \boxed{\begin{cases} -\partial_t \varphi - \Delta \varphi + a \varphi = 0 & \text{in } Q, \\ \varphi = 0 \text{ on } \quad \Sigma, \quad \varphi(T) = \varphi_0 & \text{in } \Omega. \end{cases}}$$

It is well known:

2. Linear null controllability result with regular controls

Theorem

The following conditions are equivalent:

1. *There exists* C *s.t.* $\forall y_0 \in L^2(\Omega)$, *there is* $v \in L^2(Q)$, *with*

$$\|v\|^2_{L^2(Q)} \leq C\|y_0\|^2_{L^2(\Omega)},$$

s.t. the solution y_v *to* (1) *associated to* y_0 *and* v *satisfies*

$$y_v(T) = 0 \text{ in } L^2(\Omega).$$

2. *There exists* $C > 0$ *s.t. (**observability inequality**)*

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C \iint_{\omega\times(0,T)} |\varphi(x,t)|^2 \, dx \, dt,$$

holds for every solution φ *to the adjoint problem* (3) *associated to* $\varphi_0 \in L^2(\Omega)$.

2. Linear null controllability result with regular controls

The observability inequality for the adjoint problem with an explicit expression of C with respect to the data can be obtained from a **global Carleman inequalities** for the linear parabolic problem:

$$(4) \qquad \begin{cases} -\partial_t \varphi - \Delta \varphi = F_0 & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } \Omega, \end{cases}$$

with $F_0 \in L^2(Q)$ and $\varphi_0 \in L^2(\Omega)$ are given.

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In

- V. A. FURSIKOV, O. YU. IMANUVILOV, *Controllability of Parabolic Equations*, Lecture Notes Series 34, Seoul National University, Research Institute of Mathematics, Seoul, 1996,
- E. FERNÁNDEZ-CARA, E. ZUAZUA, *The cost of approximate controllability for heat equations: the linear case*. Adv. Differential Equations 5 (2000), no. 4-6, 465–514,

it is proved:

Lemma

There exist a regular and strictly positive function, α_0, and two constants C_0 y σ_0 (only depending on Ω and ω) s.t.

$$\left\{ \begin{array}{c} \mathcal{I}(\varphi) \equiv s^{-1} \iint_Q e^{-2s\alpha} t(T-t) \left(|\partial_t \varphi|^2 + |\Delta\varphi|^2 \right) \\ + s \iint_Q e^{-2s\alpha} t^{-1}(T-t)^{-1} |\nabla\varphi|^2 + s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3} |\varphi|^2 \\ \leq C_0 \left(s^3 \iint_{\omega\times(0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2 + \iint_Q e^{-2s\alpha} |F_0|^2 \right), \end{array} \right.$$

$\forall s \geq s_0 = \sigma_0(\Omega, \omega)(T + T^2)$, *($\varphi$ is the solution to* (4) *associated to* $\varphi_0 \in L^2(\Omega)$*). The function* $\alpha = \alpha(x,t)$ *is given by*

$$\alpha(x,t) = \alpha_0(x)/t(T-t).$$

■

2. Linear null controllability result with regular controls

Coming back to the adjoint problem

$$(3) \qquad \begin{cases} -\partial_t \varphi - \Delta \varphi + a\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0 & \text{in} \quad \Omega. \end{cases}$$

2. Linear null controllability result with regular controls

Coming back to the adjoint problem

(3) $$\begin{cases} -\partial_t \varphi - \Delta \varphi + a\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0 & \text{in} \quad \Omega. \end{cases}$$

Lemma

There exist $C_1 > 0$ and $\sigma_1 > 0$ (only depending on Ω and ω) s.t.

(5)
$$\begin{cases} \mathcal{I}(\varphi) = s^{-1} \iint_Q e^{-2s\alpha} t(T-t) \left(|\partial_t \varphi|^2 + |\Delta \varphi|^2 \right) \\ + s \iint_Q e^{-2s\alpha} t^{-1}(T-t)^{-1} |\nabla \varphi|^2 + s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3} |\varphi|^2 \\ \leq C_1 s^3 \iint_{\omega \times (0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3} |\varphi|^2, \end{cases}$$

$$\forall s \geq s_1 = \sigma_1(\Omega, \omega) \left(T + T^2 + T^2 \|a\|_\infty^{2/3} \right).$$

2.1. First approach

We follow:

- E. FERNÁNDEZ-CARA, E. ZUAZUA, *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **17**, No. 5, (2000), 583–616.

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From the previous global Carleman inequality one has:

Theorem

For every $a \in L^\infty(Q)$ *and* $\varphi_0 \in L^2(\Omega)$ *one has (observability inequality)*

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq \exp\left[C\, M(T, \|a\|_\infty)\right] \iint_{\omega\times(0,T)} |\varphi|^2,$$

(φ *solution to* (3)*) with* $C = C(\Omega, \omega) > 0$ *and* M *given by:*

$$M(T, \|a\|_\infty) = 1 + \frac{1}{T} + T\|a\|_\infty + \|a\|^{2/3}_\infty.$$

Remark

This inequality shows the null controllability result for the linear system (1) *with a control* v *in* $L^2(Q)$ *(in fact,* $\mathrm{Supp}\, v \subset \omega \times (0,T)$*) and provides the following estimate for* $\|v\|_{L^2(Q)}$*:*

$$\|v\|^2_{L^2(Q)} \leq \exp\left[C\, M(T, \|a\|_\infty)\right] \|y_0\|^2,$$

with M given as before.

Is it possible to solve this problem with a control $v \in L^\infty(Q)$? **YES**.
The key point is a better observability inequality with a weaker norm on the right hand-side:

A refined observability inequality:

Proposition

There exists $C = C(\Omega, \omega) > 0$ *such that*

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq \exp\left[C\,\widetilde{M}(T, \|a\|_\infty)\right]\left(\iint_{\omega\times(0,T)} |\varphi|\right)^2,$$

with $C = C(\Omega, \omega) > 0$ *and* $\widetilde{M}$ *given by:*

$$\widetilde{M}(T, \|a\|_\infty) = 1 + \frac{1}{T} + T + \left(T^{1/2} + T\right)\|a\|_\infty + \|a\|_\infty^{2/3},$$

for any $\varphi_0 \in L^2(\Omega)$ *and* $T > 0$.

Sketch of the proof:

1. We fix $\omega_0 \subset\subset \omega$ and we apply the previous observability inequality with ω_0 and $[T/4, 3T/4]$ instead of ω and $[0, T]$. Using the energy inequality we get

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq \exp\left[C\, M(T, \|a\|_\infty)\right] \iint_{\omega_0 \times (T/4, 3T/4)} |\varphi|^2,$$

   for a new constant $C = C(\Omega, \omega_0)$ and M as before.

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   for a new constant $C = C(\Omega, \omega_0)$ and M as before.

2. We use the inequality

$$\boxed{\int_{\omega_0} \int_{T/4}^{3T/4} |\varphi|^2 \leq C T^\alpha (1 + T^{1/2}(1 + \|a\|_\infty))^\beta \left(\iint_{\omega \times (0,T)} |\varphi| \right)^2}$$

   valid for every solution φ to the adjoint problem (3) $(\alpha, \beta > 0)$.

Corollary

There exists $C = C(\Omega, \omega) > 0$ *s.t.* $\forall y_0 \in L^2(\Omega)$, *there is* $v \in L^\infty(Q)$, *with*

$$\|v\|^2_{L^\infty(Q)} \leq \exp\left[C\,\widetilde{M}(T, \|a\|_\infty)\right] \|y_0\|^2_{L^2(\Omega)},$$

s.t. the solution y_v *to* (1) *associated to* y_0 *and* v *satisfies*

$$y_v(T) = 0 \text{ in } L^2(\Omega).$$

($\widetilde{M}$ *is given by*

$$\widetilde{M}(T, \|a\|_\infty) = 1 + \frac{1}{T} + T + \left(T^{1/2} + T\right)\|a\|_\infty + \|a\|_\infty^{2/3}).$$

2.1. First approach

Remark

1. *The previous technique uses the* ***local regularizing effect*** *of the heat equation. The result is independent of the* ***initial condition*** y_0 *and the* ***boundary condition****.*
2. *This technique can be applied to linear parabolic problems with first order terms* $B \cdot \nabla y$*:*
   - A. DOUBOVA, E. FERNÁNDEZ-CARA, M. G.-B., E. ZUAZUA, *On the controllability of parabolic systems with a nonlinear term involving the state and the gradient, SIAM J. Control Optim. 41 (2002), no. 3, 798–819.*
3. *The existence of the bounded control is deduced from the* *observability inequality*: *"If system* (1) *is exactly controllable to trajectories at time* T *with controls in* $L^2(Q)$*, then system* (1) *is exactly controllable to trajectories at time* T *with controls in* $L^\infty(Q)$*".*

2.1. First approach

Remark

1. ***More regularity***??? *For example.,* $v \in L^2(0, T; H^2(\Omega) \times H_0^1(\Omega))$ *and* $\partial_t v \in L^2(Q)$ *or* $v \in C^\infty(Q)$ *when* $a \equiv 0$ *(as in the work of Lebeau-Robbiano).*
2. *What happens if* Ω *and* ω *are* ***unbounded*** *sets???*
3. *What happens if we consider coupled parabolic systems???*

2.1. First approach

Coupled parabolic systems: Let us consider a “simple” coupled parabolic system

$\begin{cases} \partial_t y - \Delta y = Ay + Bv1_\omega & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \; y(0) = y_0 & \text{in } \Omega, \end{cases}$	$\begin{cases} \partial_t \varphi + \Delta \varphi = -A^* \varphi & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \; \varphi(T) = \varphi_0 & \text{in } \Omega, \end{cases}$

with $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (**one control force**) and $y_0 \in L^2(\Omega)^2$.

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with $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (**one control force**) and $y_0 \in L^2(\Omega)^2$.
The particular structure of A and B (**cascade system**) gives:

$$\|\varphi(0)\|^2_{L^2(\Omega)} \le C \iint_{\omega_0 \times (T/4, 3T/4)} |\varphi_1|^2,$$

for a constant $C > 0$. Then, there is $v \in L^2(Q)$ s.t. $y_v(T) = 0$ in Ω and $\|v\|^2_{L^2(\Omega)} \le C\|y_0\|^2_{L^2(\Omega)^2}$. **Control in $L^\infty(Q)$??**

2.1. First approach

Following this technique, does the following inequality

$$\iint_{\omega_0 \times (T/4, 3T/4)} |\varphi_1|^2 \le C \left(\iint_{\omega \times (0,T)} |\varphi_1| \right)^2$$

hold?? **NO**.

Remark

This first approach cannot be applied to the previous coupled system since the ***local regularizing effect*** *of the linear* ***adjoint problem*** *involves the functions φ_1 and φ_2 while the corresponding "refined" observability inequality should only involve φ_1 (recall that the control v only appears in first equation of the direct problem).*

2.2. Second approach

We follow

- V. BARBU, *Exact controllability of the superlinear heat equation*, Appl. Math. Optim. 42 (2000), no. 1, 73–89.

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- V. BARBU, *Exact controllability of the superlinear heat equation*, Appl. Math. Optim. 42 (2000), no. 1, 73–89.

We recall the **global Carleman inequality** ($\partial\Omega \in C^2$):

$$\begin{cases} \mathcal{I}(\varphi) = s^{-1} \iint_Q e^{-2s\alpha} t(T-t) \left(|\partial_t \varphi|^2 + |\Delta\varphi|^2 \right) \\ + s \iint_Q e^{-2s\alpha} t^{-1}(T-t)^{-1} |\nabla\varphi|^2 + s^3 \iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3} |\varphi|^2 \\ \leq C_1 s^3 \iint_{\omega\times(0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3} |\varphi|^2, \end{cases}$$

$\forall s \geq s_1 = \sigma_1(\Omega, \omega) \left(T + T^2 + T^2 \|a\|_\infty^{2/3} \right)$, where $C_1 = C_1(\Omega, \omega) > 0$ and φ the solution to

$$\text{(3)} \qquad \begin{cases} -\partial_t \varphi - \Delta\varphi + a\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0(\cdot) & \text{in} \quad \Omega. \end{cases}$$

2.2. Second approach

In this work, a control in $L^p(Q)$, with $p = p(N)$, is obtained from the previous global Carleman inequality (we fix

$$s = s_1 = \sigma_1(\Omega, \omega) \left(T + T^2 + T^2 \|a\|_\infty^{2/3} \right)).$$

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$$s = s_1 = \sigma_1(\Omega, \omega)\left(T + T^2 + T^2\|a\|_\infty^{2/3}\right)).$$

First Step:

Lemma

For every $a \in L^\infty(Q)$ and $\varphi_0 \in L^2(\Omega)$ one has (observability inequality)

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq \exp\left[C\, M(T, \|a\|_\infty)\right] \iint_{\omega\times(0,T)} e^{-2s_1\alpha} t^{-3}(T-t)^{-3}|\varphi|^2,$$

(φ solution to (3)) with $C = C(\Omega, \omega) > 0$ and M given by:

$$M(T, \|a\|_\infty) = 1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3}.$$

Second Step: From this **observability inequality** we deduce

Proposition

$\forall y_0 \in L^2(\Omega)$, there is $v \in L^{p(N)}(Q)$, with $p(N) < \infty$ if $N = 2$ and $p(N) = \dfrac{2(N+2)}{N-2}$ if $N \geq 3$, and

$$\|v\|^2_{L^{p(N)}(Q)} \leq e^{[C\, M(T,\|a\|_\infty)]} \|y_0\|^2_{L^2(\Omega)},$$

s.t. the solution y_v to (1) associated to y_0 and v satisfies

$$y_v(T) = 0 \text{ in } L^2(\Omega).$$

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$$\|v\|^2_{L^{p(N)}(Q)} \leq e^{[C\,M(T,\|a\|_\infty)]}\|y_0\|^2_{L^2(\Omega)},$$

s.t. the solution y_v to (1) associated to y_0 and v satisfies

$$y_v(T) = 0 \text{ in } L^2(\Omega).$$

Sketch of the proof: **1.-** We consider the optimal control problem

$$\min_{v \in L^2(Q)} \left(\frac{1}{2} \iint_Q e^{2s_1\alpha} t^3 (T-t)^3 |v(x,t)|^2 \, dx\, dt + \frac{1}{2\varepsilon} \|y_v(T)\|^2_{L^2(\Omega)} \right),$$

($y_v \in L^2(Q)^2$ is the solution of (1) associated to y_0 and v).

This problem has a unique solution $v_\varepsilon \in L^2(Q)$ and, using the optimality system, it is characterized: $\boxed{v_\varepsilon = e^{-2s_1\alpha} t^{-3}(T-t)^{-3}\varphi_\varepsilon 1_\omega}$ and

$$\begin{cases} \partial_t y_\varepsilon - \Delta y_\varepsilon + a y_\varepsilon = v_\varepsilon 1_\omega & \text{in } Q, \\ y_\varepsilon = 0 \text{ sobre } \Sigma, \quad y_\varepsilon(\cdot,0) = y_0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} -\partial_t \varphi_\varepsilon - \Delta \varphi_\varepsilon + a\varphi_\varepsilon = 0 & \text{in } Q, \\ \varphi_\varepsilon = 0 \text{ on } \Sigma, \quad \varphi_\varepsilon(\cdot,T) = -\dfrac{1}{\varepsilon} y_\varepsilon(\cdot,T) & \text{in } \Omega. \end{cases}$$

The previous **observability inequality** (Lemma 7) gives:

$$\iint_{\omega\times(0,T)} e^{-2s_1\alpha} t^{-3}(T-t)^{-3}|\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}\|y_\varepsilon(T)\|^2_{L^2(\Omega)} \le e^{[C\,M(T,\|a\|_\infty)]}\|y_0\|^2_{L^2(\Omega)}$$

2.- Combining the last inequality and the **global Carleman inequality** (5) we get

$$\iint_Q e^{-2s_1\alpha} t(T-t)\left(|\partial_t\varphi_\varepsilon|^2 + |\Delta\varphi_\varepsilon|^2\right) \leq e^{[C\,M(T,\|a\|_\infty)]}\|y_0\|^2_{L^2(\Omega)}.$$

Taking into account the expresion $v_\varepsilon = e^{-2s_1\alpha}t^{-3}(T-t)^{-3}\varphi_\varepsilon 1_\omega$, we deduce

$$\begin{cases} v_\varepsilon \in H^{2,1}(Q) = \{q : q \in L^2(0,T;D(-\Delta)),\ \partial_t q \in L^2(Q)\}, \\ \|v_\varepsilon\|^2_{H^{1,2}(Q)} + \dfrac{1}{\varepsilon}\|y_\varepsilon(T)\|^2_{L^2(\Omega)} \leq e^{[C\,M(T,\|a\|_\infty)]}\|y_0\|^2_{L^2(\Omega)}, \end{cases}$$

for a new constant $C(\Omega,\omega) > 0$. Thus, $\{v_\varepsilon\}_{\varepsilon>0}$ is bounded in $H^{2,1}(Q)$. We can extract a subsequence that converges to v weakly in $H^{2,1}(Q)$. Clearly,

$$\|v\|^2_{H^{1,2}(Q)} \leq e^{[C\,M(T,\|a\|_\infty)]}\|y_0\|^2_{L^2(\Omega)} \text{ and } y_v(T) = 0 \text{ in } \Omega.$$

2.2. Second approach

3.- Finally, using the continuous embedding

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Remark

Observe that the previous control $v \in L^{p(N)}(Q)$ provides a solution $y_v \in W^{2,1,p(N)} = \{q \in L^{p(N)}(0,T; W^{2,p(N)}(\Omega) \cap W_0^{1,p(N)}(\Omega)) : \partial_t q \in L^{p(N)}(Q)\}$. Thus, using again the continuous embedding of this space, if $p(N) > N/2 + 1$, i.e., if $\boxed{1 \leq N < 6}$, the solution $y_v \in L^\infty(Q)$. In Barbu's work, the nonlinear null controllability problem is treated with this constraint on the dimension N.

2.2. Second approach. Remarks I

1. This technique uses the **global regularizing effect** of the heat equation. Then, the result depends on the **boundary conditions** but is independent of the **initial condition** y_0.
2. This technique cannot be directly applied if we consider a linear parabolic problem with a first order term $B \cdot \nabla y$. Observe that in the **global Carleman inequality** for the corresponding **adjoint system** the terms $\partial_t \varphi$ and $\Delta \varphi$ do not appear.
3. In fact, the control v provided by this approach lies in $H^{2,1}(Q)$, but, **more regularity**??? For example., when $a \equiv 0$, $v \in C^\infty(Q)$ (as in the work of Lebeau-Robbiano)??.

2.2. Second approach. Remarks II

4. The existence of a control v in $L^{p(N)}$ is deduced from the global Carleman inequality satisfied by the **adjoint system**. When Ω and ω are unbounded open sets and under some geometric conditions on (Ω, ω), it is possible to establish a **global Carleman inequality** for the **adjoint system**:
L. DE TERESA, M. G.-B., *Some results on controllability for linear and nonlinear heat equations in unbounded domains*,
Adv. Diff. Eq. 12 (2007), no. 11, 1201–1240.
In this situation it is possible to obtain a control v whith the same regularity as before.

2.2. Second approach. Remarks III

5. This approach also works in the case of coupled parabolic systems (if we have proved a global Carleman inequality for the corresponding adjoint system).

Following the same approach, it is possible to solve the null controllability result for system (1) with controls in $W_p^{2,1}(Q)$, for every $p \in [1, \infty)$: Last section.

2.3. Third approach

We follow

- O. Bodart, M. G.-B., R. Pérez-García, *Existence of insensitizing controls for a semilinear heat equation with a superlinear nonlinearity*, Comm. P.D.E 29 (2004), no. 7-8, 1017–1050.

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ASSUMPTION

Given $y_0 \in L^2(\Omega)$, there is $\widetilde{v} \in L^2(Q)$, with $\mathrm{Supp}\, \widetilde{v} \subset \omega_0$ and $\omega_0 \subset\subset \omega$, such that the solution to (1) $\widetilde{y}$ satisfies $\widetilde{y}(\cdot, T) \equiv 0$ in Ω.

One has $\widetilde{y} \in W(0,T) = \{y \in L^2(0,T; H_0^1(\Omega)) : \partial_t y \in L^2(0,T; H^{-1}(\Omega))\}$ and a explicit estimate $\|\widetilde{y}\|_{W(0,T)} \leq \exp\left(C(1+T)\|a\|_\infty\right)\left(\|y_0\|_2 + \|\widetilde{v}\|_2\right)$.

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The function $\widetilde{y}$ is regular except near $t = 0$ and near ω_0. The idea is eliminate these irregular parts of $\widetilde{y}$.

2.3. Third approach

Let us now introduce two cut-off functions $\eta \in C^\infty([0,T])$ and $\theta \in C^\infty(\overline{\Omega})$ such that

$$\begin{cases} \eta \equiv 1 \text{ in } [0, \frac{T}{4}],\ \eta \equiv 0 \text{ in } [\frac{3T}{4}, T],\ 0 \le \eta \le 1 \text{ in } [0,T], |\eta'(t)| \le C/T,\ \forall t; \\ \theta \equiv 1 \text{ in } \overline{\omega}_0, \quad 0 \le \theta \le 1 \text{ in } \Omega \text{ and } \mathrm{Supp}\,\theta \subset \omega. \end{cases}$$

Let Y be the solution to system (1) corresponding to $v \equiv 0$:

$$\begin{cases} \partial_t Y - \Delta Y + aY = 0 & \text{in } Q, \\ Y = 0 \text{ on } \Sigma, \quad Y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega, \end{cases}$$

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We now take

$$\begin{cases} y = (1-\theta)\widetilde{y} + \eta\theta Y \text{ in } Q, \\ v = (\partial_t - \Delta + a) y. \end{cases}$$

It is clear that $\mathrm{Supp}\, v(\cdot, t) \subseteq \mathrm{Supp}\, \theta \subset \omega$, y is the solution to (1) corresponding to the control v and, taking into account that $\widetilde{y}(T) \equiv 0$ in Ω, we get $y(\cdot, T) \equiv 0$ in Ω.

2.3. Third approach

In fact v is a regular control and its regularity properties are independent of y_0 and $\widetilde{v}$. Indeed, we can express y and v as

$$y \equiv (1-\theta)q + \eta(t)Y, \quad v \equiv \theta\eta' Y + 2\nabla\theta \cdot \nabla q + (\Delta\theta)q,$$

where q is given by $q = \widetilde{y} - \eta Y$ and, therefore, satisfies

$$\begin{cases} \partial_t q - \Delta q + aq = \widetilde{v}1_\omega - \eta' Y \text{ in } Q, \\ q = 0 \text{ on } \Sigma, \quad q(\cdot, 0) = 0 \text{ in } \Omega. \end{cases}$$

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Let us fix $\delta \in (0, T/4)$, $p \in [2,\infty)$ and $\mathcal{O}_0, \mathcal{O}_1 \subset\subset \Omega$ such that $\mathcal{O}_1 \subset\subset \Omega \setminus \overline{\omega}_0$ (and, in particular, $\overline{\mathcal{O}}_1 \cap \operatorname{Supp} \widetilde{v} = \emptyset$). If we denote by

$$\begin{cases} X_0^p = \{y \in L^p(\delta, T; W^{2,p}(\mathcal{O}_0)) : \partial_t y \in L^p(\mathcal{O}_0 \times (\delta, T))\}, \\ X_1^p = \{y \in L^p(0, T; W^{2,p}(\mathcal{O}_1)) : \partial_t y \in L^p(\mathcal{O}_i \times (0, T))\} \end{cases}$$

then, $Y \in X_0^p$, $q \in X_1^p$ and $v \in L^p(0, T; W_0^{1,p}(\Omega))$.

2.3. Third approach

In fact, we can obtain something better: if $p > N + 2$, one has $X_0^p \hookrightarrow C^{1+\alpha,(1+\alpha)/2}(\overline{\mathcal{O}}_0 \times [\delta, T])$ and $X_1^p \hookrightarrow C^{1+\alpha,(1+\alpha)/2}(\overline{\mathcal{O}}_1 \times [0, T])$ with $\alpha = 1 - (N+2)/p$. Thus, $v \in C_0^0(\overline{Q})$ and

$$\|v\|_{C^0} \leq e^{C(1+T+T\|a\|_\infty)} \|\widetilde{y}\|_{W(0,T)}$$

with $C = C(\Omega, T) > 0$.

2.2. Third approach. Remarks I

1. The previous regularity result for v is independent of the initial datum y_0, the control $\widetilde{v}$ and the regularity of the boundary $\partial\Omega$. We have only used the **local regularity** properties of the operator $L \equiv \partial_t - \Delta + a$. In the case in which $a \equiv 0$, we obtain $v \in C^\infty(\overline{Q})$ (as in the paper of Lebeau-Robbiano).
2. In fact we have proved: *"Let us fix $y_0 \in L^2(\Omega)$ and assume that there exists $\widetilde{v} \in L^2(Q)$ such that the solution $\widetilde{y}$ to the linear problem* (1) *satisfies $\widetilde{y}(T) \equiv 0$ in Ω. Then, there exists $\widetilde{v} \in C_0^0(\overline{Q})$ s.t. the solution y_v of* (1) *also satisfies $y_v(T) \equiv 0$ in Ω".*
3. This technique can be applied if we consider a linear parabolic problem with a first order term $B \cdot \nabla y$ obtaining the same regularity result.

2.2. Third approach. Remarks II

4. When Ω and ω are unbounded open sets we can obtain the same result:
   L. DE TERESA, M. G.-B., *Some results on controllability for linear and nonlinear heat equations in unbounded domains*, Adv. Diff. Eq. 12 (2007), no. 11, 1201–1240.
5. This approach also works in the case of systems of two coupled parabolic equations.

3. The “best” null control

We consider once again the linear problem

$$(1) \qquad \begin{cases} \partial_t y - \Delta y + ay = v 1_\omega & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega. \end{cases}$$

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Question

Fix $p \in [1, \infty)$. Given $y_0 \in L^2(\Omega)$, does there exist $v \in W_p^{2,1}(Q)$ s.t. the solution to (1) satisfies $y(T) = 0$ in Ω??? Estimates of v???

$$W_p^{2,1}(Q) = \{u \in L^p(0, T; W^{2,p}(\Omega)) : \partial_t u \in L^p(Q)\}.$$

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Idea

We are going to add "better" terms on the left hand-side of the **global Carleman inequality** for the **adjoint problem** and then apply again the approach of Barbu.

3. The “best” null control

The adjoint problem:

$$(3) \qquad \begin{cases} -\partial_t\varphi - \Delta\varphi + a\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0(\cdot) & \text{in} \quad \Omega. \end{cases}$$

From the Carleman inequality, we deduce,

$$\begin{cases} s^{-1} \displaystyle\iint_Q e^{-2s\alpha} t(T-t) \left(|\partial_t\varphi|^2 + |\Delta\varphi|^2 \right) \\ \leq C_1 s^3 \displaystyle\iint_{\omega\times(0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2, \end{cases}$$

$\forall s \geq s_1 = \sigma_1(\Omega, \omega) \left(T + T^2 + T^2\|a\|_\infty^{2/3} \right)$, where $C_1 = C_1(\Omega, \omega) > 0$.
We take:

$$\alpha_0^* = \max_{x\in\overline{\Omega}} \alpha_0(x), \quad \alpha^*(t) = \frac{\alpha_0^*}{t(T-t)}.$$

3. The "best" null control

Remark

The function α_0 is given by

$$\alpha_0(x) = e^{2Cm\|\eta_0\|_\infty} - e^{C(m\|\eta_0\|_\infty + \eta_0(x))},$$

with $m > 1$ an arbitrary constant, η_0, a function only depending on Ω and ω, and $C = C(\Omega, \omega) > 0$. The construction of $\eta_0 = \eta_0(x)$ is given in [FURSIKOV-IMANUVILOV]. This function satisfies:

$$\eta_0 \in C^2(\overline{\Omega}), \quad \eta_0 \geq 0 \text{ in } \Omega, \quad \frac{\partial \eta_0}{\partial n} \leq \text{ on } \partial\Omega \quad \text{and} \quad \nabla \eta_0 \neq 0 \text{ in } \overline{\Omega} \setminus \omega.$$

($n = n(x)$: the outward unit normal to Ω at point $x \in \partial\Omega$). ■

3. The "best" null control

We take

$$\boxed{\psi = s^{-5/2} e^{-s\alpha^*(t)} t^{5/2} (T-t)^{5/2} \varphi = \rho_0(t) \varphi.}$$

Then,

$$\begin{cases} \partial_t \psi + \Delta \psi = a \rho_0(t) \varphi + \partial_t \rho_0(t) \varphi & \text{in} \quad Q, \\ \psi = 0 \text{ on } \Sigma, \quad \psi(\cdot, T) = 0 & \text{in} \quad \Omega. \end{cases}$$

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If $s \geq s_1 = \sigma_1 \left(T + T^2 + T^2 \|a\|_\infty^{2/3} \right)$, we have $\partial_t \rho_0(t)\varphi \in H^{2,1}(Q)$ and

$$\|\partial_t \rho_0(t)\varphi\|^2_{H^{2,1}} \leq C s^{-1} \iint_Q e^{-2s\alpha} t(T-t) \left(|\partial_t \varphi|^2 + |\Delta \varphi|^2 \right).$$

But, $H^{2,1}(Q) \hookrightarrow L^{p(N)}(Q)$ with $p(N) = \frac{2(N+2)}{N-2}$. Thus,

$$\|\partial_t \rho_0(t)\varphi\|_{L^{p(N)}(Q)} \leq C \|\partial_t \rho_0(t)\varphi\|_{H^{2,1}}$$

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If $s \geq s_1 = \sigma_1 \left(T + T^2 + T^2 \|a\|_\infty^{2/3} \right)$, we have $\partial_t \rho_0(t)\varphi \in H^{2,1}(Q)$ and

$$\|\partial_t \rho_0(t)\varphi\|^2_{H^{2,1}} \leq C s^{-1} \iint_Q e^{-2s\alpha} t(T-t) \left(|\partial_t \varphi|^2 + |\Delta \varphi|^2 \right).$$

But, $H^{2,1}(Q) \hookrightarrow L^{p(N)}(Q)$ with $p(N) = \frac{2(N+2)}{N-2}$. Thus,

$$\|\partial_t \rho_0(t)\varphi\|_{L^{p(N)}(Q)} \leq C \|\partial_t \rho_0(t)\varphi\|_{H^{2,1}}$$

We can also prove that $a\rho_0(t)\varphi \in L^{p(N)}(Q)$ and

$$\|a\rho_0(t)\varphi\|^2_{L^{p(N)}(Q)} \leq C s^{-1} \iint_Q e^{-2s\alpha} t(T-t) \left(|\partial_t \varphi|^2 + |\Delta \varphi|^2 \right).$$

3. The “best” null control

The maximal parabolic regularity for the heat equation ($\partial\Omega \in C^2$) gives $\psi = s^{-5/2} e^{-s\alpha^*(t)} t^{5/2} (T-t)^{5/2} \varphi \in W^{2,1}_{p(N)}(Q)$ and

$$\begin{cases} \|\psi\|^2_{W^{2,1}_{p(N)}(Q)} \leq C s^{-1} \iint_Q e^{-2s\alpha} t(T-t) \left(|\partial_t \varphi|^2 + |\Delta\varphi|^2 \right) \\ \qquad\qquad\qquad \leq C_2 s^3 \iint_{\omega\times(0,T)} e^{-2s\alpha} t^{-3} (T-t)^{-3} |\varphi|^2. \end{cases}$$

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Conclusion

We have obtained a new Carleman inequality for the problem (3)

$$\begin{cases} \|s^{-5/2} e^{-s\alpha^*(t)} t^{5/2} (T-t)^{5/2} \varphi\|^2_{W^{2,1}_{p(N)}(Q)} + \mathcal{I}(\varphi) \\ \le C_2 s^3 \iint_{\omega\times(0,T)} e^{-2s\alpha} t^{-3} (T-t)^{-3} |\varphi|^2, \\ \forall s \ge s_1 = \sigma_1 \left(T + T^2 + T^2 \|a\|_\infty^{2/3} \right). \end{cases}$$

3. The "best" null control

Corollary

$\forall y_0 \in L^2(\Omega)$, there is $v \in W^{2,1}_{p(N)}(Q)$, with $p(N) < \infty$ if $N = 2$ and $p(N) = \dfrac{2(N+2)}{N-2}$ if $N \geq 3$, and

$$\|v\|^2_{W^{2,1}_{p(N)}} \leq e^{[C\, M(T, \|a\|_\infty)]} \|y_0\|^2_{L^2(\Omega)},$$

s.t. the solution y_v to (1) associated to y_0 and v satisfies

$$y_v(T) = 0 \text{ in } L^2(\Omega).$$

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$\forall y_0 \in L^2(\Omega)$, there is $v \in W^{2,1}_{p(N)}(Q)$, with $p(N) < \infty$ if $N = 2$ and $p(N) = \dfrac{2(N+2)}{N-2}$ if $N \geq 3$, and

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s.t. the solution y_v to (1) associated to y_0 and v satisfies

$$y_v(T) = 0 \text{ in } L^2(\Omega).$$

Remark

We can apply a boot-strap argument and deduce that the previous result is valid for every $p \in [2, \infty)$. In this case the constant C also depends on p.

3. The “best” null control

Reference

- V. Barbu, *Controllability of parabolic and Navier-Stokes equations*, Sci. Math. Jpn. 56 (2002), no. 1, 143–211.

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