Some recent results on controllability of coupled parabolic systems: Towards a Kalman condition

Manuel González-Burgos,
UNIVERSIDAD DE SEVILLA

Thematic day on Control of Coupled Systems

Institut Henri Poincaré, Paris, November 2010

GOAL:

1. Show the important differences between scalar and non scalar problems.
2. Give necessary and sufficient conditions (*Kalman condition*) which characterize the controllability properties of these systems.

We will only deal with

“Simple" Parabolic Systems: **Coupling Matrices of Constant Coefficients**.

Contents

1. The parabolic scalar case: The heat equation
2. Finite-dimensional systems
3. Two simple examples
   - Distributed null controllability of a linear reaction-diffusion system
   - Boundary null controllability of a linear reaction-diffusion system
4. The Kalman condition for a class of parabolic systems. Distributed controls
5. The Kalman condition for a class of parabolic systems. Boundary controls
6. Comments and open problems

1. The parabolic scalar case: The heat equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial\Omega$ of class C^2. Let $\omega \subseteq \Omega$ be an open subset, $\gamma \subseteq \partial\Omega$ a relative open subset and let us fix $T > 0$.

1. The parabolic scalar case: The heat equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial\Omega$ of class C^2. Let $\omega \subseteq \Omega$ be an open subset, $\gamma \subseteq \partial\Omega$ a relative open subset and let us fix $T > 0$. We consider the **linear** problems for the **heat equation**:

$$(1) \qquad \begin{cases} \partial_t y - \Delta y = v1_\omega & \text{in } Q = \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases}$$

$$(2) \qquad \begin{cases} \partial_t y - \Delta y = 0 & \text{in } Q, \\ y = v1_\gamma \text{ on } \Sigma, \quad y(\cdot,0) = y_0 & \text{in } \Omega. \end{cases}$$

1. The parabolic scalar case: The heat equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial\Omega$ of class C^2. Let $\omega \subseteq \Omega$ be an open subset, $\gamma \subseteq \partial\Omega$ a relative open subset and let us fix $T > 0$. We consider the **linear** problems for the **heat equation**:

$$(1) \qquad \begin{cases} \partial_t y - \Delta y = v 1_\omega & \text{in } Q = \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

$$(2) \qquad \begin{cases} \partial_t y - \Delta y = 0 & \text{in } Q, \\ y = v 1_\gamma \text{ on } \Sigma, \quad y(\cdot,0) = y_0 & \text{in } \Omega. \end{cases}$$

In (1) and (2), 1_ω and 1_γ represent resp. the characteristic function of the sets ω and γ, $y(x,t)$ is the state, y_0 is the initial datum and is given in an appropriate space, and v is the control function (which is localized in ω -**distributed control**- or in γ -**boundary control**-).

1. The parabolic scalar case: The heat equation

Theorem (Distributed Controllability Results)

Fix $\omega \subseteq \Omega$ *and* $T > 0$. *Then,*

1. *System* (1) *is* ***approximately controllable*** *at time* T, *i.e., for any* $\varepsilon > 0$ *and* $y_0, y_d \in L^2(\Omega)$ *there is* $v \in L^2(Q)$ *s.t. the solution* y *to* (1) *satisfies*

$$\|y(\cdot, T) - y_d\|_{L^2(\Omega)} \leq \varepsilon.$$

2. *System* (1) *is* ***null controllable*** *at time* T, *i.e., for any* $y_0 \in L^2(\Omega)$ *there is* $v \in L^2(Q)$ *s.t. the solution* y *to* (1) *satisfies*

$$y(\cdot, T) \equiv 0 \text{ in } \Omega. \qquad \blacksquare$$

1. The parabolic scalar case: The heat equation

Theorem (Distributed Controllability Results)

Fix $\omega \subseteq \Omega$ *and* $T > 0$. *Then,*

1. *System* (1) *is* ***approximately controllable*** *at time* T, *i.e., for any* $\varepsilon > 0$ *and* $y_0, y_d \in L^2(\Omega)$ *there is* $v \in L^2(Q)$ *s.t. the solution* y *to* (1) *satisfies*

$$\|y(\cdot, T) - y_d\|_{L^2(\Omega)} \leq \varepsilon.$$

2. *System* (1) *is* ***null controllable*** *at time* T, *i.e., for any* $y_0 \in L^2(\Omega)$ *there is* $v \in L^2(Q)$ *s.t. the solution* y *to* (1) *satisfies*

$$y(\cdot, T) \equiv 0 \text{ in } \Omega. \qquad \blacksquare$$

Remark

System (1) *is* ***null controllable*** *at time* T ***if and only if*** *system* (1) *is* ***exactly controllable to the trajectories*** *at time* T*: for every trajectory* y^* *of* (1) *(a solution to* (1) *associated to* $y_0^* \in L^2(\Omega)$*) there exists* $v \in L^2(\mathbb{Q})$ *such that* $y(\cdot, T) \equiv y^*(\cdot, T)$ *in* Ω. $\blacksquare$

1. The parabolic scalar case: The heat equation

Adjoint Problem: Let us fix $\varphi_0 \in L^2(\Omega)$ and consider the *adjoint problem*

(3) $$\boxed{\begin{cases} \partial_t \varphi + \Delta \varphi = 0 & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(T) = \varphi_0 & \text{in } \Omega. \end{cases}}$$

1. The parabolic scalar case: The heat equation

Adjoint Problem: Let us fix $\varphi_0 \in L^2(\Omega)$ and consider the *adjoint problem*

$$(3) \qquad \boxed{\begin{cases} \partial_t \varphi + \Delta \varphi = 0 & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(T) = \varphi_0 & \text{in } \Omega. \end{cases}}$$

It is well known:

Theorem

System (1) is exactly controllable to trajectories at time T ***if and only if*** *there exists $C > 0$ s.t. (**observability inequality**)*

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C \iint_{\omega \times (0,T)} |\varphi(x,t)|^2 \, dx \, dt,$$

holds for every solution φ to the adjoint problem (3) associated to $\varphi_0 \in L^2(\Omega)$. ■

1. The parabolic scalar case: The heat equation

FOUR IMPORTANT REFERENCES

1. H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.
2. G. LEBEAU, L. ROBBIANO, *Contrôle exact de l'équation de la chaleur*, Comm. P.D.E. 20 (1995), no. 1-2, 335–356.
3. O. YU. IMANUVILOV, *Controllability of parabolic equations*, (Russian) Mat. Sb. 186 (1995), no. 6, 109–132; translation in Sb. Math. 186 (1995), no. 6, 879–900.
4. A. FURSIKOV, O. YU. IMANUVILOV, *Controllability of Evolution Equations*, Lecture Notes Series 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

1. The parabolic scalar case: The heat equation

Boundary Controllability Result:

Theorem

Let $\gamma \subseteq \partial\Omega$ *and* $T > 0$ *be given. Then, for any* $y_0 \in H^{-1}(\Omega)$ *there exists* $v \in L^2(\Sigma)$ *s.t. the solution* y *to* (2) *satisfies*

$$y(\cdot, T) \equiv 0 \text{ in } \Omega.$$

Proof: It is a consequence of the distributed controllability result. ■

1. The parabolic scalar case: The heat equation

Boundary Controllability Result:

Theorem

Let $\gamma \subseteq \partial\Omega$ and $T > 0$ be given. Then, for any $y_0 \in H^{-1}(\Omega)$ there exists $v \in L^2(\Sigma)$ s.t. the solution y to (2) satisfies

$$y(\cdot, T) \equiv 0 \text{ in } \Omega.$$

Proof: It is a consequence of the distributed controllability result. ∎

Important:

Distributed controllability result for system (1) **is equivalent** to the boundary controllability result for system (2).

1. The parabolic scalar case: The heat equation

Boundary Controllability Result:

Theorem

System (2) is exactly controllable to trajectories at time T ***if and only if*** *there exists $C > 0$ s.t. (****observability inequality****)*

$$\|\varphi(0)\|^2_{H^1_0(\Omega)} \le C \iint_{\gamma\times(0,T)} \left| \frac{\partial\varphi}{\partial n}(x,t) \right|^2 ,$$

holds for every solution φ to the adjoint problem (3) associated to $\varphi_0 \in H^1_0(\Omega)$ ($n = n(x)$ is the outward normal unit vector at $x \in \partial\Omega$). ■

1. The parabolic scalar case: The heat equation

Boundary Controllability Result:

Theorem

*System (2) is exactly controllable to trajectories at time T **if and only if** there exists $C > 0$ s.t. (**observability inequality**)*

$$\|\varphi(0)\|^2_{H^1_0(\Omega)} \leq C \iint_{\gamma \times (0,T)} \left| \frac{\partial \varphi}{\partial n}(x,t) \right|^2,$$

holds for every solution φ to the adjoint problem (3) associated to $\varphi_0 \in H^1_0(\Omega)$ ($n = n(x)$ is the outward normal unit vector at $x \in \partial\Omega$). ■

Summarizing:

- System (1) and system (2) are approximately controllable and exactly controllable to trajectories at time T.
- The controllability properties of both systems are equivalent. ■

2. Finite-dimensional systems

Let us consider the autonomous linear system

$$(4) \qquad y' = Ay + Bu \quad \text{in } [0,T], \quad y(0) = y_0,$$

where $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ are constant matrices, $y_0 \in \mathbb{C}^n$ and $u \in L^2(0,T;\mathbb{C}^m)$ is the control.

Problem: Given $y_0, y_d \in \mathbb{R}^n$, is there a control $u \in L^2(0,T;\mathbb{R}^m)$ such that the solution y to the problem satisfies

$$y(T) = y_d????$$

2. Finite-dimensional systems

Let us consider the autonomous linear system

$$(4) \qquad y' = Ay + Bu \quad \text{in } [0,T], \quad y(0) = y_0,$$

where $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ are constant matrices, $y_0 \in \mathbb{C}^n$ and $u \in L^2(0,T;\mathbb{C}^m)$ is the control.

Problem: Given $y_0, y_d \in \mathbb{R}^n$, is there a control $u \in L^2(0,T;\mathbb{R}^m)$ such that the solution y to the problem satisfies

$$y(T) = y_d????$$

Let us define (*controllability matrix*)

$$\boxed{[A\,|\,B] = [B\,|\,AB\,|\,A^2B\,|\,\cdots\,|\,A^{n-1}B] \in \mathcal{L}(\mathbb{C}^{nm};\mathbb{C}^n).}$$

2. Finite-dimensional systems

The following classical result can be found in

R. Kalman, Y.-Ch. Ho, K. Narendra, *Controllability of linear dynamical systems*, 1963

and gives a complete answer to the problem of controllability of finite dimensional autonomous linear systems:

Theorem

Under the previous assumptions, the following conditions are equivalent

1. *System* (4) *is exactly controllable at time* T, *for every* $T > 0$.
2. *There exists* $T > 0$ *such that system* (4) *is exactly controllable at time* T.
3. $rank\,[A \mid B] = n$ *(Kalman rank condition).*
4. $\ker[A \mid B]^* = \{0\}$. ∎

2. Finite-dimensional systems

Goal

We have a complete characterization of the controllability results for finite-dimensional linear differential systems (a Kalman condition). Is it possible to obtain similar results for PDE systems? We will focus on coupled linear parabolic systems.

What are the possible generalizations to Systems of Parabolic Equations?

Let us consider the 2×2 linear reaction-diffusion system

$$(5) \qquad \begin{cases} y_t - D\Delta y = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} y + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v 1_\omega & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \qquad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

Here Ω, ω and T are as before, $y_0 \in L^2(\Omega; \mathbb{R}^2)$ and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0 \quad \left(A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right).$$

Let us consider the 2×2 linear reaction-diffusion system

$$(5) \qquad \begin{cases} y_t - D\Delta y = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} y + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v 1_\omega & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \qquad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

Here Ω, ω and T are as before, $y_0 \in L^2(\Omega; \mathbb{R}^2)$ and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0 \quad \left(A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right).$$

One has

Theorem

System (5) is exactly controllable to trajectories at time T if and only if

$$\det [A \mid B] \neq 0 \iff a_{2,1} \neq 0.$$

Proof: $\boxed{\Longrightarrow}$: If $a_{2,1} = 0$, then y_2 is independent of v.

Proof: $\boxed{\Longrightarrow}$: If $a_{2,1} = 0$, then y_2 is independent of v.

$\boxed{\Longleftarrow}$: The controllability result for system (5) is equivalent to the observability inequality: $\exists C > 0$ such that

$$\|\varphi_1(\cdot, 0)\|_{L^2}^2 + \|\varphi_2(\cdot, 0)\|_{L^2}^2 \le C \iint_{\omega \times (0,T)} |\varphi_1(x, t)|^2 \, dx \, dt,$$

where φ is the solution associated to $\varphi_0 \in L^2(\Omega; \mathbb{R}^2)$ of the adjoint problem:

$$(6) \qquad \begin{cases} -\varphi_t - D\Delta\varphi = A^*\varphi & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } \Omega. \end{cases}$$

It is a consequence of well known global Carleman estimates for parabolic equations.

Lemma

There exist a positive regular function, α_0, and two positive constants C_0 and σ_0 (only depending on Ω and ω) s.t.

$$\begin{cases} \mathcal{I}(\phi) \equiv \displaystyle\iint_Q e^{-2s\alpha}\,[s\rho(t)]^{-1}\left(|\phi_t|^2 + |\Delta\phi|^2\right) \\ \quad + \displaystyle\iint_Q e^{-2s\alpha}\,[s\rho(t)]\,|\nabla\phi|^2 + \iint_Q e^{-2s\alpha}\,[s\rho(t)]^3\,|\phi|^2 \\ \leq C_0 \left(\displaystyle\iint_{\omega\times(0,T)} e^{-2s\alpha}\,[s\rho(t)]^3\,|\phi|^2 + \iint_Q e^{-2s\alpha}|\phi_t \pm \Delta\phi|^2 \right), \end{cases}$$

$\forall s \geq s_0 = \sigma_0(\Omega,\omega)(T+T^2)$ *and* $\phi \in L^2(0,T;H_0^1(\Omega))$ *s.t.* $\phi_t \pm \Delta\phi \in L^2(Q)$.
The functions $\rho(t)$ *and* $\alpha = \alpha(x,t)$ *are given by*

$$\rho(t) = [t(T-t)]^{-1}, \quad \alpha(x,t) = \alpha_0(x)/t(T-t)\,. \qquad \blacksquare$$

Coming back to the adjoint problem for system (6), if we apply to $\phi = \varphi_1$ and $\phi = \varphi_2$ the previous inequality in $\omega_0 \subset\subset \omega$. After some computations we get

$$\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \leq C_1 s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-3} \left(|\varphi_1|^2 + |\varphi_2|^2 \right),$$

$\forall s \geq s_1 = \sigma_1(\Omega, \omega_0)(T + T^2)$.

Coming back to the adjoint problem for system (6), if we apply to $\phi = \varphi_1$ and $\phi = \varphi_2$ the previous inequality in $\omega_0 \subset\subset \omega$. After some computations we get

$$\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \le C_1 s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-3} \left(|\varphi_1|^2 + |\varphi_2|^2 \right),$$

$\forall s \ge s_1 = \sigma_1(\Omega, \omega_0)(T + T^2)$.

We now use the first equation in (6), $\boxed{a_{2,1}\varphi_2 = -(\varphi_{1,t} + \Delta\varphi_1 + a_{1,1}\varphi_1)}$, to prove ($\varepsilon > 0$):

$$s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-3} |\varphi_2|^2 \le \varepsilon \mathcal{I}(\varphi_2) + \frac{C_2}{\varepsilon} s^7 \iint_{\omega \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-7} |\varphi_1|^2.$$

$\forall s \ge s_1 = \sigma_1(\Omega, \omega_0)(T + T^2)$.

From the two previous inequalities (global Carleman estimate)

$$\boxed{\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \le C_2 s^7 \iint_{\omega\times(0,T)} e^{-2s\alpha}[t(T-t)]^{-7}|\varphi_1|^2,}$$

$\forall s \ge s_1 = \sigma_1(\Omega, \omega_0)(T + T^2)$. Combining this inequality and energy estimates for system (6) we deduce the desired observability inequality.

Remark

- *System* (5) *is always controllable if we exert a control in each equation (two controls).*
- *The controllability result for system* (5) *is independent of the diffusion matrix D. We will see that the situation is more intricate if in the system a general control vector $B \in \mathbb{R}^2$ is considered.*
- *The same result can be obtained for the approximate controllability at time T. Therefore,* ***approximate*** *and* ***null controllability*** *are equivalent concepts.*

References

1. L. DE TERESA, *Insensitizing controls for a semilinear heat equation*, Comm. Partial Differential Equations 25 (2000), no. 1–2, 39–72.
2. F. AMMAR KHODJA, A. BENABDALLAH, C. DUPAIX ET I. KOSTIN, *Controllability to the trajectories of phase-field models by one control force*, SIAM J. Control Optim. 42 (2003), no. 5, 1661-1689.
3. M. G.-B., R. PÉREZ-GARCÍA, *Controllability results for some nonlinear coupled parabolic systems by one control force*, Asymptot. Anal. 46 (2006), no. 2, 123–162.
4. M. G.-B., L. DE TERESA, *Controllability results for cascade systems of m coupled parabolic PDEs by one control force*, Port. Math. 67 (2010), no. 1, 91–113.

Let us now consider the boundary controllability problem for the one-dimensional linear reaction-diffusion system:

$$(7)\qquad \begin{cases} y_t - D y_{xx} = A y & \text{in } Q = (0,\pi)\times(0,T), \\ y|_{x=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, \quad y|_{x=1} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

with $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, $v \in L^2(0,T)$ is the control and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0 \ \boxed{(d_1 \neq d_2)}, \text{ and } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Question

Are the controllability properties of system (7) independent of d_1 and d_2???
NO.

As before, system (7) is null controllable at time T if and only if the observability inequality

$$\|\varphi_1(\cdot,0)\|^2_{H^1_0(0,\pi)} + \|\varphi_2(\cdot,0)\|^2_{H^1_0(0,\pi)} \leq C \int_0^T |\varphi_{1,x}(0,t)|^2 \, dt,$$

holds. Again φ is the solution associated to $\varphi_0 \in H^1_0(0,\pi;\mathbb{R}^2)$ of the adjoint problem:

$$(8) \qquad \begin{cases} -\varphi_t - D\varphi_{xx} = A^*\varphi & \text{in } Q, \\ \varphi|_{x=0} = \varphi|_{x=1} = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi). \end{cases}$$

Let us see that, in general, this inequality fails (**even if** $a_{2,1} = 1 \neq 0$**!!!!!**).

A necessary condition:

Proposition

Assume that system (7) is null controllable at time T. Then ($\lambda_k = k^2$),

$$d_1 \lambda_k \neq d_2 \lambda_j, \quad \forall k, j \geq 1 \quad (\Longleftrightarrow \sqrt{d_1/d_2} \notin \mathbb{Q}).$$

Proof: By contradiction, assume that $d_1 \lambda_k = d_2 \lambda_j$ for some k, j and take $K = \max\{k, j\}$. The idea is transforming system (8) into an o.d. system.

A necessary condition:

Proposition

Assume that system (7) is null controllable at time T. Then ($\lambda_k = k^2$),

$$d_1 \lambda_k \neq d_2 \lambda_j, \quad \forall k, j \geq 1 \quad (\Longleftrightarrow \sqrt{d_1/d_2} \notin \mathbb{Q}).$$

Proof: By contradiction, assume that $d_1 \lambda_k = d_2 \lambda_j$ for some k, j and take $K = \max\{k, j\}$. The idea is transforming system (8) into an o.d. system. Let us consider the sequence of eigenvalues and normalized eigenfunctions of $-\partial_{xx}$ on $(0, \pi)$ with homogenous Dirichlet boundary conditions:

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi).$$

Idea: Take $\varphi_0 \in X_K = \{\varphi_0 = \sum_{\ell=1}^{K} a_\ell \phi_\ell : a_\ell \in \mathbb{R}^2\} \subset H_0^1(0, \pi; \mathbb{R}^2)$.

Consider also

$$B_K = \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \in \mathbb{R}^{2K}, \quad \left(B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \quad \text{and}$$

$$\mathcal{L}_K^* = \text{diag}\ (-\lambda_1 D + A^*, -\lambda_2 D + A^*, \cdots, -\lambda_K D + A^*) \in \mathcal{L}(\mathbb{R}^{2K}).$$

Taking in (8) arbitrary initial data $\varphi_{0,K} = \sum_{\ell=1}^{K} a_\ell \phi_\ell \in H_0^1(0,\pi;\mathbb{R}^2)$ where $a_\ell \in \mathbb{R}^2$, it is not difficult to see that system (8) is equivalent to the o.d. system

$$(9) \qquad -Z' = \mathcal{L}_K^* Z \quad \text{on } [0,T], \quad Z(0) = Z_0 \in \mathbb{R}^{2K}.$$

From the observability inequality for system (8) we deduce the unique continuation property for the solutions to (9):

$$\boxed{B_K^* Z(\cdot) = 0 \quad \text{in } (0,T) \Longrightarrow Z \equiv 0.}$$

In particular system

$$Y' = \mathcal{L}_K Y + B_K v \quad \text{on } [0,T], \quad Y(0) = Y_0 \in \mathbb{R}^{2K}.$$

is exactly controllable at time T. Then

$$\text{rank}\,[\mathcal{L}_K \,|\, B_K] = 2K.$$

We deduce that $\mathcal{L}_K^*$ cannot have eigenvalues with **geometric multiplicity** 2 or greater.
But $\theta = -d_1\lambda_k = -d_2\lambda_j$ is an eigenvalue of $\mathcal{L}_K^*$ with two linearly independent eigenvectors $V_1, V_2 \in \mathbb{R}^{2K}$ given by:

$$\begin{cases} V_1 = (V_{1,\ell})_{1\le\ell\le K}, \quad V_{1,k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } V_{1,\ell} = 0 \quad \forall \ell \neq k, \\ V_2 = (V_{2,\ell})_{1\le\ell\le K}, \quad V_{2,j} = \begin{pmatrix} \frac{1}{\lambda_j(d_1-d_2)} \\ 0 \end{pmatrix} \text{ and } V_{2,\ell} = 0 \quad \forall \ell \neq j. \blacksquare \end{cases}$$

The result has been proved in

- E. FERNÁNDEZ-CARA, M. G.-B., L. DE TERESA, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. 259 (2010), no. 7, 1720–1758.

Remark

- *Again, the system is always null controllable at time T if we exert two controls.*
- *In fact, system (7) is approximately controllable at time T* $\iff$

$$\sqrt{d_1/d_2} \notin \mathbb{Q}.$$

4. The Kalman condition for a class of parabolic systems. Distributed controls

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial\Omega$ of class C^2. Let $\omega \subseteq \Omega$ be an open subset and let us fix $T > 0$.

For $n, m \in \mathbb{N}$ we consider the following $n \times n$ parabolic system

$$(10) \qquad \begin{cases} \partial_t y - D\Delta y = Ay + Bv1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

with $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ are constant matrices $y_0 \in L^2(\Omega; \mathbb{R}^n)$ and

$$D = diag\,(d_1, d_2, \cdots, d_n) \in \mathcal{L}(\mathbb{R}^n), \quad (d_i > 0, \ \forall i).$$

$v \in L^2(Q; \mathbb{R}^m)$ is the control (m components).

Remark

This problem is well posed: For any $y_0 \in L^2(\Omega; \mathbb{R}^n)$ and $v \in L^2(Q; \mathbb{R}^m)$, problem (10) has a unique solution $y \in L^2(0, T; H_0^1) \cap C^0([0, T]; L^2)$. ■

4. The Kalman condition for a class of parabolic systems. Distributed controls

$$(10) \qquad \begin{cases} \partial_t y - D\Delta y = Ay + Bv1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega. \end{cases}$$

Remark

We want to control the whole system (n equations) with m controls. The most interesting case is $m < n$ or even $m = 1$.

Difficulties:

1. *In general $m < n$.*
2. *D is not the identity matrix.*

4. The Kalman condition for a class of parabolic systems. Distributed controls

The adjoint problem:

(11)
$$\begin{cases} -\partial_t \varphi = (D\Delta + A^*)\varphi \quad \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0 \text{ in } \Omega, \end{cases}$$

where $\varphi_0 \in L^2(\Omega; \mathbb{R}^n)$. Then, the exact controllability to the trajectories of system (10) is equivalent to the existence of $C > 0$ such that, for every $\varphi_0 \in L^2(\Omega; \mathbb{R}^n)$, the solution $\varphi \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$ to the adjoint system (11) satisfies the **observability inequality**:

(12)
$$||\varphi(\cdot, 0)||^2_{L^2(\Omega)} \leq C \iint_{\omega \times (0,T)} |B^* \varphi(x, t)|^2,$$

4. The Kalman condition for a class of parabolic systems. Distributed controls

Let us consider $\{\lambda_k\}_{k\geq 1}$ the sequence of eigenvalues for $-\Delta$ with homogenuous Dirichlet boundary conditions and $\{\phi_k\}_{k\geq 0}$ the corresponding normalized eigenfunctions.

Theorem (A Necessary Condition)

If system (10) *is **null controllable** at time T **then***

(13) $$rank\,[-\lambda_k D + A \,|\, B] = n, \quad \forall k \geq 1.$$

where

$$[-\lambda_k D + A \,|\, B] = [B \,|\, (-\lambda_k D + A)B \,|\, (-\lambda_k D + A)^2 B \,|\, \cdots \,|\, (-\lambda_k D + A)^{n-1} B].$$

4. The Kalman condition for a class of parabolic systems. Distributed controls

Let us consider $\{\lambda_k\}_{k\geq 1}$ the sequence of eigenvalues for $-\Delta$ with homogenuous Dirichlet boundary conditions and $\{\phi_k\}_{k\geq 0}$ the corresponding normalized eigenfunctions.

Theorem (A Necessary Condition)

If system (10) *is **null controllable** at time* T ***then***

$$rank\,[-\lambda_k D + A \mid B] = n, \quad \forall k \geq 1. \tag{13}$$

where

$$[-\lambda_k D + A \mid B] = [B \mid (-\lambda_k D + A)B \mid (-\lambda_k D + A)^2 B \mid \cdots \mid (-\lambda_k D + A)^{n-1} B].$$

Proof: Reasoning by contradiction: $\exists k \geq 1$ such that rank $[-\lambda_k D + A \mid B] < n$. Then the o.d.s.

$$-Z' = (-\lambda_k D + A^*)Z \quad \text{in } (0,T),$$

is not B^*-observable at time T.

4. The Kalman condition for a class of parabolic systems. Distributed controls

There exists $Z_0 \in \mathbb{R}^n$, $Z_0 \neq 0$, such that the solution Z to the previous system satisfies $Z(\cdot) = 0$ on $(0, T)$. But

$$\varphi(x,t) = Z(t)\phi_k(x)$$

is the solution to **adjoint problem** (11) associated to $\varphi_0(x) = Z_0\phi_k$ and

$$B^*\varphi(x,t) = 0, \quad \forall (x,t) \in \Omega \times (0,T).$$

Then, the **observability inequality** (12) fails and system (10) is not null controllable at time T. ∎

Remark

Observe that, if condition (13) is not satisfied, then system (10) is neither approximately controllable nor null controllable at time T (for any $T > 0$) even if $\omega = \Omega$. ∎

4. The Kalman condition for a class of parabolic systems. Distributed controls

Question:

Is condition (13) a **sufficient condition** for the null controllability of system (10)??? ■

4. The Kalman condition for a class of parabolic systems. Distributed controls

Question:

Is condition (13) a **sufficient condition** for the null controllability of system (10)??? ■

Let us now introduce the unbounded matrix operator

$$\mathcal{K} = [D\Delta + A \mid B] = [B \mid (D\Delta + A)B \mid \cdots \mid (D\Delta + A)^{n-1}B],$$

$$\begin{cases} \mathcal{K} : D(\mathcal{K}) \subset L^2(\Omega; \mathbb{R}^{nm}) \to L^2(\Omega; \mathbb{R}^n), \text{ with} \\ D(\mathcal{K}) := \{y \in L^2(\Omega; \mathbb{R}^{nm}) : \mathcal{K}y \in L^2(\Omega; \mathbb{R}^n)\}. \end{cases}$$

Then,

Proposition

$\ker \mathcal{K}^* = \{0\}$ ***if and only if*** *condition* (13) *holds.* ■

4. The Kalman condition for a class of parabolic systems. Distributed controls

Theorem (Kalman condition)

System (10) is ***exactly controllable to trajectories*** *(resp.,* ***approximately controllable****) at time T* ***if and only if***

$$\ker \mathcal{K}^* = \{0\} \iff rank\,[-\lambda_k D + A \,|\, B] = n, \quad \forall k \geq 1.$$

Remark

One can prove, either there exists $k_0 \geq 1$ such that

$$rank\,[-\lambda_k D + A \,|\, B] = n, \quad \forall k \geq k_0$$

or

$$rank\,[-\lambda_k D + A \,|\, B] < n, \quad \forall k \geq 1.$$

4. The Kalman condition for a class of parabolic systems. Distributed controls

Controllability (outside a finite dimensional space) **if and only if** the algebraic Kalman condition $\boxed{\text{rank}\,[-\lambda_k D + A \,|\, B] = n}$ is satisfied for one frequency $k \geq 1$.

Remark

System (10) *can be exactly controlled to the trajectories with one control force (*$m = 1$ *and* $B \in \mathbb{R}^n$*) even if* $\boxed{A \equiv 0}$. *Indeed, let us assume that* $B = (b_i)_{1 \leq i \leq n} \in \mathbb{R}^n$. *Then,*

$$[(-\lambda_k D + A) \,|\, B] = \begin{bmatrix} b_1 & (-\lambda_k d_1) b_1 & \cdots & (-\lambda_k d_1)^{n-1} b_1 \\ b_2 & (-\lambda_k d_2) b_2 & \cdots & (-\lambda_k d_2)^{n-1} b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & (-\lambda_k d_n) b_n & \cdots & (-\lambda_k d_n)^{n-1} b_n \end{bmatrix} \in \mathcal{L}(\mathbb{R}^n),$$

and (13) *holds* ***if and only if*** $b_i \neq 0$ *for every* i *and* d_i *are* ***distinct***. ■

Idea of the proof: The objective is to prove the **observability inequality** (12):

$$||\varphi(\cdot, 0)||^2_{L^2(\Omega)} \leq C \iint_{\omega \times (0,T)} |B^* \varphi(x,t)|^2. \tag{12}$$

To this end we use two arguments:

- Prove a Carleman type observability estimate for a scalar equation of order n in time,
- Prove a **coercivity** property for the Kalman operator $\mathcal{K}$.

4. The Kalman condition for a class of parabolic systems. Distributed controls

Let us consider φ a regular solution of the **adjoint system** (11) and take

$$\Phi = \sum_{i=1}^{n} \alpha_i \varphi_i, \quad \text{with } \alpha_i \in \mathbb{R} \ \forall i : 1 \leq i \leq n.$$

Then, Φ is a regular solution to the linear scalar equation of order n in time

$$\begin{cases} \det \left(I_d \partial_t + D\Delta + A^* \right) \Phi = 0 & \text{in } Q, \\ \Delta^j \Phi = 0 \quad \text{on } \Sigma, & \forall j \geq 1. \end{cases}$$

4. The Kalman condition for a class of parabolic systems. Distributed controls

The key point is to prove a Carleman inequality for the solutions to the previous problem:

Theorem

Let $n, k_1, k_2 \in \mathbb{N}$. There exist two constants r_0 and C (only depending on Ω, ω, n, D, A, k_1 and k_2) such that

$$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \mathcal{J}(3 - 4(i+j), \Delta^i \partial_t^j \Phi) \leq C \iint_{\omega \times (0,T)} e^{-2s\alpha} \left[s\rho(t)\right]^{3+r_0} |\Phi|^2, \quad ,$$

$\forall s \geq s_0 = \sigma_0(\Omega, \omega)(T + T^2)$ (see Lemma 6) and Φ solution to the previous problem. ■

4. The Kalman condition for a class of parabolic systems. Distributed controls

From this result and after some operations, one deduces

$$\int_0^T e^{\frac{-2sM_0}{t(T-t)}} [s\rho(t)]^3 \|\Delta^k \mathcal{K}^* \varphi\|^2_{L^2(\Omega)^{nm}} \leq C \iint_{\omega \times (0,T)} e^{-2s\alpha} [s\rho(t)]^{3+r} |B^* \varphi|^2$$

for every $s \geq \sigma_0 \left(T + T^2\right)$. In this inequality, ρ and α are as in Lemma 6, $M_0 = \max_{\overline{\Omega}} \alpha_0$ and $r \geq 0$ is an integer only depending on n.

Remark

The previous inequality is a partial observability estimate. It is valid even if the Kalman condition does not hold, i.e., even if $\ker \mathcal{K}^* \neq \{0\}$. ■

4. The Kalman condition for a class of parabolic systems. Distributed controls

The **coercivity** property of $\mathcal{K}$:

Theorem

Assume that $\ker \mathcal{K}^* = \{0\}$ *and consider* $k = (n-1)(2n-1)$. *Then there exists* $C > 0$ *such that if* $z \in L^2(\Omega)^n$ *satisfies* $\mathcal{K}^* z \in D(\Delta^k)^{nm}$, *one has*

$$\|z\|^2_{L^2(\Omega)^n} \leq C \|\Delta^k \mathcal{K}^* z\|^2_{L^2(\Omega)^{nm}} .$$

So, from the previous inequality we get

$$\boxed{\int_0^T e^{\frac{-2sM_0}{t(T-t)}} [s\rho(t)]^3 \|\varphi\|^2_{L^2(\Omega)^{nm}} \leq C \iint_{\omega\times(0,T)} e^{-2s\alpha} [s\rho(t)]^{3+r} |B^*\varphi|^2}$$

and the **observability inequality** (12):

$$\text{(12)} \qquad ||\varphi(\cdot,0)||^2_{L^2(\Omega)} \leq C \iint_{\omega\times(0,T)} |B^*\varphi(x,t)|^2.$$

4. The Kalman condition for a class of parabolic systems. Distributed controls

Summarizing

1. We have established a **Kalman condition**

$$\ker \mathcal{K}^* = \{0\}$$

which characterizes the controllability properties of system (10).

2. The **Kalman condition** for system (10) $\ker \mathcal{K}^* = \{0\}$ generalizes the **algebraic Kalman condition** $\ker[A \mid B]^* = \{0\}$ for o.d.s.

3. This **Kalman condition** is also equivalent to the **approximate controllability** of system (10) at time T. Again, **approximate** and **null controllability** are equivalent concepts for system (10).

4. The Kalman condition for a class of parabolic systems. Distributed controls

A special case: $D = Id$.

It is possible to get better results when $D = Id$. In this case system (10) is given by

(14) $$\begin{cases} \partial_t y - \Delta y = Ay + Bv1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega, \end{cases}$$

where again $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ are constant matrices and $y_0 \in L^2(\Omega; \mathbb{R}^n)$ is given. In this case, $\ker \mathcal{K}^* = \{0\}$ **is equivalent** to the **algebraic Kalman condition**

$$\text{rank}\,[A \,|\, B] = \text{rank}\,[B \,|\, AB \,|\, A^2B \,|\, \cdots \,|\, A^{n-1}B] = n.$$

In this case we can obtain a better Carleman inequality for the adjoint system

(15) $$\begin{cases} -\partial_t \varphi - \Delta \varphi = A^* \varphi \text{ in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0(\cdot) \text{ in } \Omega. \end{cases}$$

4. The Kalman condition for a class of parabolic systems. Distributed controls

Theorem

There exist a positive function $\alpha_0 \in C^2(\overline{\Omega})$ *(only depending on* Ω *and* ω*), positive constants* C_0 *and* σ_0 *(only depending on* Ω, ω, n, m, A *and* B*) and a positive integer* $\ell \geq 3$ *(only depending on* n *and* m*) such that, if rank* $[A \mid B] = n$*, for every* $\varphi_0 \in L^2(Q;\mathbb{R}^n)$*, the solution* φ *to* (15) *satisfies*

$$\mathcal{I}(\varphi) \leq C_0 \left(s^\ell \iint_{\omega \times (0,T)} e^{-2s\alpha} \rho(t)^\ell |B^* \varphi|^2 \right),$$

$\forall s \geq s_0 = \sigma_0 \left(T + T^2\right)$*. In this inequality,* $\alpha(x,t)$*,* $\rho(t)$ *and* $\mathcal{I}(z)$ *are as in Lemma 6.* ■

4. The Kalman condition for a class of parabolic systems. Distributed controls

References

1. F. Ammar-Khodja, A. Benabdallah, C. Dupaix, M. G.-B., *A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems*, Differ. Equ. Appl. 1 (2009), no. 3, 139–151.

$$\boxed{D = I_d, \; A = A(t) \text{ and } B = B(t).}$$

2. F. Ammar-Khodja, A. Benabdallah, C. Dupaix, M. G.-B., *A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems*, J. Evol. Equ. 9 (2009), no. 2, 267–291.

$$\boxed{D \text{ diagonal matrix, } A \text{ and } B \text{ constant matrices.}}$$

5. The Kalman condition for a class of parabolic systems. Boundary controls

Let us consider the **boundary controllability problem**:

$$(16) \qquad \begin{cases} y_t = y_{xx} + Ay & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n)$ are two given matrices and $y_0 \in H^{-1}(0,\pi;\mathbb{C}^n)$ is the initial datum. In system (16), $v \in L^2(0,T;\mathbb{C}^m)$ is the control function (to be determined).

Simpler problem: One-dimensional case and $D = Id$.

This problem has been studied in the case $n = 2$:

- E. Fernández-Cara, M. G.-B., L. de Teresa, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. 259 (2010), no. 7, 1720–1758.

5. The Kalman condition for a class of parabolic systems. Boundary controls

Theorem

$n = 2$, $m = 1$. Let $A \in \mathcal{L}(\mathbb{C}^2)$ and $B \in \mathbb{C}^2$ be given and let us denote by μ_1 and μ_2 the eigenvalues of A^*. Then (16) is ***exactly controllable to the trajectories*** *at any time $T > 0$* ***if and only if*** $\boxed{\mathrm{rank}\,[A \mid B] = 2}$ *and*

$$\boxed{\lambda_k - \lambda_j \neq \mu_1 - \mu_2 \quad \forall k, j \in \mathbb{N} \text{ with } k \neq j.}$$

■

Remark ($n = 2$, $m = 1$)

For the previous ***boundary controllability problem****, one has*

1. *A complete characterization of the* ***exact controllability to trajectories*** *at time T.*
2. ***Boundary controllability*** *and* ***distributed controllability*** *are not equivalent*
3. ***Approximate controllability*** $\Longleftrightarrow$ ***null controllability***.

5. The Kalman condition for a class of parabolic systems. Boundary controls

What does happen if $n > 2$??

We consider again $\{\lambda_k\}_{k\geq 1}$ the sequence of eigenvalues for $-\partial_{xx}$ in $(0, \pi)$ with homogenuous Dirichlet boundary conditions and $\{\phi_k\}_{k\geq 0}$ the corresponding normalized eigenfunctions:

$$\boxed{\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi).}$$

5. The Kalman condition for a class of parabolic systems. Boundary controls

Notation

For $k \geq 1$, we introduce $L_k = -\lambda_k I_d + A \in \mathcal{L}(\mathbb{C}^n)$ and the matrices

$$B_k = \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^{nk}), \quad \mathcal{L}_k = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & L_k \end{pmatrix} \in \mathcal{L}(\mathbb{C}^{nk}),$$

and let us write the Kalman matrix associated with the pair $(\mathcal{L}_k, B_k)$:

$$\mathcal{K}_k = [\mathcal{L}_k \mid B_k] = [B_k \mid \mathcal{L}_k B_k \mid \mathcal{L}_k^2 B_k \mid \cdots \mid \mathcal{L}_k^{nk-1} B_k] \in \mathcal{L}(\mathbb{C}^{mnk}, \mathbb{C}^{nk}).$$

5. The Kalman condition for a class of parabolic systems. Boundary controls

Theorem

*Let us fix $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$. Then, system (16) is **exactly controllable to trajectories** at time T if and only if*

$$\text{rank}\, \mathcal{K}_k = nk, \quad \forall k \geq 1. \tag{17}$$

Remark

1. *This result gives a complete characterization of the **exact controllability to trajectories** at time T: **Kalman condition**.*
2. *If for $k \geq 1$ one has rank $\mathcal{K}_k = nk$, then rank $[A \mid B] = n$ and system (14) is **exactly controllable to trajectories** at time T. But rank $[A \mid B] = n$ does not imply condition (17). So **boundary controllability** and **distributed controllability** are not equivalent.*

5. The Kalman condition for a class of parabolic systems. Boundary controls

Remark

Condition (17) *is also a* ***necessary*** *and* ***sufficient condition*** *for the boundary approximate controllability of system* (16)*. Then*

Approximate controllability $\iff$ ***null controllability***.

Adjoint Problem:

$$(18) \qquad \begin{cases} -\varphi_t = \varphi_{xx} + A^*\varphi & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi), \end{cases}$$

with $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$. Then, system (16) is exactly controllable to trajectories at time $T \iff$ for a constant $C > 0$ one has

$$\|\varphi(\cdot,0)\|^2_{H_0^1(0,\pi;\mathbb{C}^n)} \le C \int_0^T |B^*\varphi_x(0,t)|^2 \, dt.$$

5. The Kalman condition for a class of parabolic systems. Boundary controls

Necessary implication. We reason as before: if rank $\mathcal{K}_k < nk$, for some $k \geq 1$, then the o.d.s.

$$-Z' = \mathcal{L}_k^* Z \quad \text{on } (0,T), \quad Z(T) = Z_0 \in \mathbb{C}^{nk}$$

is not B_k^*-observable on $(0,T)$, i.e., there exists $Z_0 \neq 0$ s.t. $B_k^* Z(t) = 0$ for every $t \in (0,T)$. From Z_0 it is possible to construct $\varphi_0 \in H_0^1(0,\pi;\mathbb{C}^n)$ with $\varphi_0 \not\equiv 0$ such that the corresponding solution to the adjoint problem (17) satisfies

$$B^* \varphi_x(0,t) = 0 \quad \forall t \in (0,T).$$

As a consequence: The **unique continuation property** and the previous **observability inequality** for the adjoint problem fail:

Neither approximate nor null controllability at any T for system (14).

5. The Kalman condition for a class of parabolic systems. Boundary controls

Sufficient implication. For the proof we follow the ideas from

- H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.

Two "big" steps:

1. We reformulate the null controllability problem for system (16) as a **vector moment problem**.
2. Existence and bounds of a family **biorthogonal** to appropriate complex matrix exponentials.

5. The Kalman condition for a class of parabolic systems. Boundary controls

Sufficient implication. For the proof we follow the ideas from

- H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.

Two "big" steps:

1. We reformulate the null controllability problem for system (16) as a **vector moment problem**.
2. Existence and bounds of a family **biorthogonal** to appropriate complex matrix exponentials.

Let us fix $\eta \geq 1$, an integer, $T \in (0, \infty]$ and $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ a sequence. Let us recall that the family $\{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1} \subset L^2(0,T;\mathbb{C})$ is **biorthogonal** to $\{t^j e^{-\Lambda_k t}\}_{k\geq 1, 0\leq j\leq \eta-1}$ if one has

$$\int_0^T t^j e^{-\Lambda_k t} \varphi_{l,i}^*(t)\, dt = \delta_{kl}\delta_{ij}, \quad \forall (k,j), (l,i) : k,l \geq 1,\ 0 \leq i,j \leq \eta - 1.$$

5. The Kalman condition for a class of parabolic systems. Boundary controls

Theorem

Assume that for two positive constants δ and ρ one has

$$\begin{cases} \Re\Lambda_k \geq \delta|\Lambda_k|, \quad |\Lambda_k - \Lambda_l| \geq \rho|k-l|, \quad \forall k,l \geq 1, \\ \displaystyle\sum_{k\geq 1} \frac{1}{|\Lambda_k|} < \infty. \end{cases}$$

Then, $\exists \{\varphi_{k,j}\}_{k\geq 1, 0\leq j\leq \eta-1}$ ***biorthogonal*** *to* $\left\{t^j e^{-\Lambda_k t}\right\}_{k\geq 1, 0\leq j\leq \eta-1}$ *such that, for every* $\varepsilon > 0$, *there exists* $C(\varepsilon, T) > 0$ *satisfying*

$$\|\varphi_{k,j}\|_{L^2(0,T;\mathbb{C})} \leq C(\varepsilon, T) e^{\varepsilon \Re\Lambda_k}, \quad \forall (k,j) : k \geq 1, \ 0 \leq j \leq \eta - 1.$$

5. The Kalman condition for a class of parabolic systems. Boundary controls

Reference

F. AMMAR-KHODJA, A. BENABDALLAH, M. G.-B., L. DE TERESA, *The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials*, submitted.

6. Comments and open problems

Most of the controllability results for parabolic systems are open.

6. Comments and open problems

Most of the controllability results for parabolic systems are open.

Two "simple" open problems

A.- Let us consider the distributed controllability problem

$$(10) \qquad \begin{cases} \partial_t y - D\Delta y = Ay + I_d v 1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) \text{ in } \Omega. \end{cases}$$

with $A \in \mathcal{L}(\mathbb{R}^n)$ (as before), $\boxed{B = I_d}$ and with $D \in \mathcal{L}(\mathbb{R}^n)$ a non-symmetric matrix such that the Jordan canonical form J is real and positive definite, i.e., $J \in \mathcal{L}(\mathbb{R}^n)$ and

$$\xi J \xi^* > 0, \quad \forall \xi \in \mathbb{R}^n, \xi \neq 0.$$

Some partial results by
E. Fernández-Cara, M. G.-B., L. de Teresa, in preparation.

6. Comments and open problems

B.- Consider again the boundary controllability problem

$$\begin{cases} y_t - Dy_{xx} = Ay & \text{in } Q = (0,\pi)\times(0,T), \\ y|_{x=0} = Bv, \quad y|_{x=1} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

with $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, $v \in L^2(0,T)$ is the control and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \; d_1, d_2 > 0, \; B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We know:

- $d_1 = d_2$: Approximate and null controllability at time $T > 0$. **Kalman condition** for general $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^2)$ (the only interesting case is $m = 1$).
- $d_1 \neq d_2$: Approximate controllability at time $T > 0 \iff \sqrt{d_1/d_2} \notin \mathbb{Q}$.
- $d_1 \neq d_2$: There exist d_1, d_2 such that the **null controllability** property fails at any time T: F. LUCA, L. DE TERESA, 2011.

6. Comments and open problems

C.- Kalman condition: Only in the cases presented here.

Other situations ?

Thanks for your attention !

¡ Gracias por vuestra atención !