New phenomena in the null controllability of coupled parabolic systems

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GOAL:

The general aim of this talk is to show some phenomenona which arise when we deal with the null controllability properties of **coupled parabolic** systems:

1. **First phenomenon:** **Boundary controllability** is not equivalent to **distributed controllability** for coupled parabolic systems.
2. **Second phenomenon:** The **null controllability** properties are not equivalent to the **approximated controllability** of these problems.
3. **Third phenomenon:** **Minimal time of controllability**. The null controllability only holds if is T is large enough.
4. **Fourth phenomenon:** The null controllability of parabolic system depends on the **position of the control open set** (de Teresa's talk).

Contents

1. Introduction. Statement of the problem

1 Introduction. Statement of the problem

Let us fix $T > 0$ and $\omega = (a, b) \subset (0, \pi)$. We consider the coupled parabolic systems:

$$(1) \quad \begin{cases} y_t - D y_{xx} + A_0 y = B u 1_\omega & \text{in } Q := (0, \pi) \times (0, T), \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

$$(2) \quad \begin{cases} y_t - D y_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0, \cdot) = B v, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

In (1) and (2), 1_ω is the characteristic function of the set ω, $y(x, t)$ is the state, $y_0 \in L^2(0, \pi; \mathbb{R}^2)$ (or $y_0 \in H^{-1}(0, \pi; \mathbb{R}^2)$) is the initial datum and

- $D = \operatorname{diag}(d_1, d_2) \in \mathcal{L}(\mathbb{R}^2)$, with $d_i > 0$, and $A_0 \in \mathcal{L}(\mathbb{R}^2)$ constant matrices; $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ constant vector of $\mathbb{R}^2$;
- $v \in L^2(0, T)$ and $u \in L^2(Q)$ are scalar control functions.

Remark

In this talk we are interested in studying the controllability properties of systems (2) and (1). Boundary and distributed control problems.

IMPORTANT

We have systems of two coupled heat equations and we want to control these systems (two states) only acting on the second equation.

Objective

We want to study the controllability properties of systems (1) and (2):

$$\begin{cases} y_t - Dy_{xx} + A_0 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

$$\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

under the assumption:

$$D = \operatorname{diag}(d_1, d_2).$$

We will consider the “simplest" case: $1-d$, two equations and

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2. First phenomenon: Boundary and distributed controllability

Let us consider the 2×2 linear reaction-diffusion system

$$(3) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_1 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

Here ω and T are as before, $y_0 \in L^2((0,\pi);\mathbb{R}^2)$ and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \quad A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

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Here ω and T are as before, $y_0 \in L^2((0,\pi);\mathbb{R}^2)$ and

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One has

Theorem

System (3) *is exactly controllable to trajectories at time* T *if and only if*

$$\det [B\,,\, A_1 B] \neq 0 \iff a_{12} \neq 0.$$

Proof: $\boxed{\Longrightarrow}$: If $a_{12} = 0$, then y_1 is independent of u.

$\boxed{\Longleftarrow}$: The controllability result for system (3) is equivalent to the observability inequality: $\exists C > 0$ such that

$$\|\varphi_1(\cdot,0)\|^2_{L^2} + \|\varphi_2(\cdot,0)\|^2_{L^2} \le C \iint_{\omega\times(0,T)} |\varphi_2(x,t)|^2 \, dx \, dt,$$

where φ is the solution associated to $\varphi_0 \in L^2(\Omega;\mathbb{R}^2)$ of the adjoint problem:

$$(4) \qquad \begin{cases} -\varphi_t - D\varphi_{xx} + A_1^*\varphi = 0 & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot,T) = \varphi_0 & \text{in } \Omega. \end{cases}$$

It is a consequence of well known global Carleman estimates for parabolic equations.

Lemma

There exist a positive regular function, α_0, and two positive constants C_0 and σ_0 (only depending on ω) s.t.

$$\begin{cases} \mathcal{I}(\phi) \equiv \iint_Q e^{-2s\alpha} [s\rho(t)]^{-1} \left(|\phi_t|^2 + |\phi_{xx}|^2\right) \\ + \iint_Q e^{-2s\alpha} [s\rho(t)] \, |\nabla\phi|^2 + \iint_Q e^{-2s\alpha} [s\rho(t)]^3 \, |\phi|^2 \\ \leq C_0 \left(\iint_{\omega\times(0,T)} e^{-2s\alpha} [s\rho(t)]^3 \, |\phi|^2 + \iint_Q e^{-2s\alpha} |\phi_t \pm \phi_{xx}|^2 \right), \end{cases}$$

$\forall s \geq s_0 = \sigma_0(\Omega,\omega)(T+T^2)$ *and* $\phi \in L^2(0,T;H_0^1(\Omega))$ *s.t.* $\phi_t \pm \phi_{xx} \in L^2(Q)$. *The functions* $\rho(t)$ *and* $\alpha = \alpha(x,t)$ *are given by*

$$\rho(t) = [t(T-t)]^{-1}, \quad \alpha(x,t) = \alpha_0(x)/t(T-t)\,. \qquad \blacksquare$$

Coming back to the adjoint problem for system (4), if we apply to $\phi = \varphi_1$ and $\phi = \varphi_2$ the previous inequality in $\omega_0 \subset\subset \omega$. After some computations we get

$$\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \leq C_1 s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-3} \left(|\varphi_1|^2 + |\varphi_2|^2 \right),$$

$\forall s \geq s_1 = \sigma_1(\Omega, \omega_0)(T + T^2)$.

Coming back to the adjoint problem for system (4), if we apply to $\phi = \varphi_1$ and $\phi = \varphi_2$ the previous inequality in $\omega_0 \subset\subset \omega$. After some computations we get

$$\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \le C_1 s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-3} \left(|\varphi_1|^2 + |\varphi_2|^2 \right),$$

$\forall s \ge s_1 = \sigma_1(\Omega, \omega_0)(T + T^2)$.

We now use the second equation in (4), $\boxed{a_{12}\varphi_1 = \varphi_{2,t} + d_2 \varphi_{2,xx} - a_{22}\varphi_2}$, to prove ($\varepsilon > 0$):

$$s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-3} |\varphi_1|^2 \le \varepsilon \mathcal{I}(\varphi_1) + \frac{C_2}{\varepsilon} s^7 \iint_{\omega \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-7} |\varphi_2|^2.$$

$\forall s \ge s_1 = \sigma_1(\Omega, \omega_0)(T + T^2)$.

From the two previous inequalities (global Carleman estimate)

$$\boxed{\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \le C_2 s^7 \iint_{\omega \times (0,T)} e^{-2s\alpha} [t(T-t)]^{-7} |\varphi_2|^2,}$$

$\forall s \ge s_1 = \sigma_1(\Omega, \omega_0)(T + T^2)$. Combining this inequality and energy estimates for system (4) we deduce the desired observability inequality.

$$(3) \qquad \begin{cases} y_t - Dy_{xx} + A_1 y = Bu1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Remark

- *System* (3) *is always controllable if we exert a control in each equation (two controls).* ***Important****:* D *is a diagonal matrix.*
- *The controllability result for system* (3) *is independent of the diffusion matrix* D*. This positive controllability result is also valid in the* N*-dimensional case.*
- *The same result can be obtained for the approximate controllability at time* T*. Therefore,* ***approximate*** *and* ***null controllability*** *are equivalent concepts.*

References

- DE TERESA, *Insensitizing controls for a semilinear heat equation*, Comm. Partial Differential Equations 25 (2000).
- AMMAR KHODJA, BENABDALLAH, DUPAIX, KOSTIN, *Controllability to the trajectories of phase-field models by one control force*, SIAM J. Control Optim. 42 (2003).
- G.-B., PÉREZ-GARCÍA, *Controllability results for some nonlinear coupled parabolic systems by one control force*, Asymptot. Anal. 46 (2006).
- G.-B., DE TERESA, *Controllability results for cascade systems of m coupled parabolic PDEs by one control force*, Port. Math. 67 (2010).

Let us consider the problem

$$(5)\qquad \begin{cases} y_t - D\Delta y + Ay = Bv1_\omega & \text{in } Q = \Omega \times (0,T), \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T), & y(\cdot,0) = y_0 \text{ in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a nonempty smooth bounded connected open set, $\omega \subset \Omega$ a nonempty open subset, $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and $D \in \mathcal{L}(\mathbb{R}^n)$ (positive definite). The control $v \in L^2(Q;\mathbb{R}^m)$: m-controls.

$$(5) \qquad \begin{cases} y_t - D\Delta y + Ay = Bv1_\omega & \text{in } Q = \Omega \times (0,T), \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T), & y(\cdot,0) = y_0 \text{ in } \Omega, \end{cases}$$

The dimensions of the Jordan blocks of the canonical form of D are ≤ 4.

Theorem (Distributed control)

System (5) is ***null controllable*** *at time T* ***if and only if***

$$\operatorname{rank}\,[\lambda_k D + A \,|\, B] = n, \quad \forall k \geq 1.$$

where $\{\lambda_k\}_{k\geq 1}$ is the sequence of eigenvalues for $-\Delta$ with homogeneous Dirichlet boundary conditions and

$$[\lambda_k D + A \,|\, B] = [B\,,\ (\lambda_k D + A)B\,,\ (\lambda_k D + A)^2 B\,,\ \cdots\,,\ (\lambda_k D + A)^{n-1}B].$$

$$(5)\quad \begin{cases} y_t - D\Delta y + Ay = Bv1_\omega & \text{in } Q = \Omega \times (0,T), \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T), & y(\cdot,0) = y_0 \text{ in } \Omega, \end{cases}$$

References

- AMMAR KHODJA, BENABDALLAH, DUPAIX, G.-B., *A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems*, J. Evol. Equ. 9 (2009).
- FERNÁNDEZ-CARA,G.-B., DE TERESA, *Controllability of linear and semilinear non-diagonalizable parabolic systems*, to appear in ESAIM Control Optim. Calc. Var. (2015).

$$(6)\qquad \begin{cases} y_t - Dy_{xx} + A_1 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $\boxed{A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ $v \in L^2(0,T)$: scalar control.

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where $A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $v \in L^2(0,T)$: scalar control.

Theorem (Fernández-Cara, M.G.-B., de Teresa, (2010))

Assume $d_1 = d_2 > 0$. *Assume* μ_1, μ_2 *are the eigenvalues of* A_1. *Then system* (6) *is null controllable at time* T *if and only* $\det\,[B\,,\,A_1 B] = a_{12} \neq 0$ *and*

$$\pi^{-2}(\mu_1 - \mu_2) \neq j^2 - k^2 \qquad \forall k, j \in \mathbb{N} \text{ with } k \neq j.$$

- FERNÁNDEZ-CARA, G.-B., DE TERESA, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. 259 (2010).

$$(6) \qquad \begin{cases} y_t - Dy_{xx} + A_1 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

First phenomenon

The boundary and distributed controllability properties of the system

$$y_t - Dy_{xx} + A_1 y$$

are different and not equivalent.

- AMMAR KHODJA, BENABDALLAH, G.-B., DE TERESA, *The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials*, J. Math. Pures Appl. (2011).

$$
(6) \qquad \begin{cases} y_t - Dy_{xx} + A_1 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}
$$

Remark

The same result can be obtained for the approximate controllability at time T. Therefore, ***approximate*** *and* ***null controllability*** *are equivalent concepts.*

3. Second phenomenon: Approximate and null controllability

3. Second phenomenon

$$(2) \quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $D = \operatorname{diag}(d_1, d_2)$, $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We will assume that $d_1 \neq d_2$ and, for instance, $d_1 = 1$, $d_2 = d \neq 1$.

GOAL

Given $T > 0$, does there exist $v \in L^2(0,T)$ s.t. $y(T) = 0$?

Remark

Recall that the parabolic system $y_t - Dy_{xx} + A_0 y = u 1_\omega$ is approximate and null controllable at time T for any $T > 0$.

$$(2)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Approximate controllability:

Theorem (Fernández-Cara, M.G.-B., de Teresa, (2010))

Assume $d \neq 1$. *Then system* (2) *is approximately controllable at time* $T > 0$ *if and only if* $\boxed{\sqrt{d} \notin \mathbb{Q}}$.

3. Second phenomenon

(2)
$$\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Approximate controllability:

Theorem (Fernández-Cara, M.G.-B., de Teresa, (2010))

Assume $d \neq 1$. Then system (2) is approximately controllable at time $T > 0$ if and only if $\boxed{\sqrt{d} \notin \mathbb{Q}}$.

Is this problem null controllable when $\boxed{\sqrt{d} \notin \mathbb{Q}}$??? No:

3. Second phenomenon

$$(2) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Theorem (Luca, de Teresa, (2012))

There exists $d > 0$ *with* $\boxed{\sqrt{d} \notin \mathbb{Q}}$ *such that system* (2) *is not null controllable at any time* $T > 0$.

- LUCA, DE TERESA, *Control of coupled parabolic systems and Diophantine approximations*, SeMA J. 61 (2013).

Second phenomenon

For system (2): Approximate controllability $\boxed{\not\Leftrightarrow}$ null controllability.

4. Third phenomenon: Minimal time of controllability

4. Third phenomenon

$$(2) \quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $D = \operatorname{diag}(d_1, d_2)$, $\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

Assumption

In the sequel, $\boxed{D = \operatorname{diag}(1, d)}$ with $\boxed{d \neq 1}$ and $\boxed{\sqrt{d} \notin \mathbb{Q}}$.

Goal

Analyze the null controllability properties at time $T > 0$ of system (2).

4. Third phenomenon

$$(2) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Let φ be a solution of the adjoint problem:

$$\begin{cases} -\varphi_t - D\varphi_{xx} + A_0^*\varphi = 0 & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 \in H_0^1(0,\pi)^2 & \text{in } (0,\pi). \end{cases}$$

If y is a solution of the direct problem, then

$$\langle y(T), \varphi_0 \rangle - \langle y_0, \varphi(0) \rangle = \int_0^T v(t) B^* D \varphi_x(0,t)\, dt$$

4. Third phenomenon

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If y is a solution of the direct problem, then

$$\langle y(T), \varphi_0 \rangle - \langle y_0, \varphi(0) \rangle = \int_0^T v(t) B^* D \varphi_x(0,t)\, dt$$

Thus $\boxed{y(T) = 0} \iff \exists v \in L^2(0,T)$ such that

$$\boxed{\int_0^T v(t) B^* D \varphi_x(0,t)\, dt = -\langle y_0, \varphi(0) \rangle, \quad \forall \varphi_0 \in H_0^1(0,\pi)^2}$$

Fattorini-Russell Method

Material at our disposal

- $\sigma(-D\partial_{xx}^2 + A_0^*) = \bigcup_{k\geq 1} \{k^2, dk^2\} := \bigcup_{k\geq 1} \{\lambda_{k,1}, \lambda_{k,2}\}$
- $V_{k,1}$ and $V_{k,2}$: eigenvectors of the matrix $(k^2 D + A_0^*)$ associated to the eigenvalues k^2, dk^2.
- $\Phi_{k,i} = V_{k,i} \sin kx$, $i = 1, 2$: eigenfunctions of $(-D\partial_{xx}^2 + A_0^*)$.
- $\{\Phi_{k,i}\}$ is a (Riesz) basis of $H_0^1(0,\pi)^2$. Let $\{\Psi_{k,i}\}$ be the associated biorthogonal family (for the duality $\langle \cdot, \cdot \rangle_{((H_0^1)^2, (H^{-1})^2)}$)

$$f \in H_0^1(0,\pi)^2 \Longleftrightarrow f = \sum_{k\geq 1, i=1,2} \langle f, \Psi_{k,i} \rangle \Phi_{k,i}$$

$$\|f\|^2_{(H_0^1)^2} \sim \sum_{k\geq 1, i=1,2} |\langle f, \Psi_{k,i} \rangle|^2$$

4. Third phenomenon

$$(2) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Objective: Existence of $v \in L^2(0,T)$ s.t.

$$\int_0^T v(t) B^* D\varphi_x(0,t)\,dt = -\langle y_0, \varphi(0)\rangle\,, \quad \forall \varphi_0 \in H_0^1(0,\pi)^2$$

$$(2)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Objective: Existence of $v \in L^2(0,T)$ s.t.

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- Choosing $\varphi_0 = \Phi_{k,i}$, we have $\varphi(\cdot,t) = e^{-\lambda_{k,i}(T-t)} \Phi_{k,i}$ and

$$\varphi(x,0) = e^{-\lambda_{k,i} T} \Phi_{k,i}(x), \quad \varphi_x(0,t) = k e^{-\lambda_{k,i}(T-t)} V_{k,i}$$

- The identity connecting y and φ writes (**moment problem**)

$$k B^* D V_{k,i} \int_0^T v(T-t) e^{-\lambda_{k,i} t}\, dt = -e^{-\lambda_{k,i} T} \langle y_0, \Phi_{k,i} \rangle, \quad \forall (k,i)$$

4. Third phenomenon

$$(2) \quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Approximate controllability: a necessary condition (I)

- $$\boxed{\boxed{kB^*DV_{k,i}} \int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = -e^{-\lambda_{k,i}T}\langle y_0, \Phi_{k,i}\rangle, \quad \forall (k,i)}$$

(2)
$$\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Approximate controllability: a necessary condition (I)

- $$\boxed{kB^*DV_{k,i}} \int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = -e^{-\lambda_{k,i}T}\langle y_0, \Phi_{k,i}\rangle, \quad \forall (k,i)$$
- A necessary condition: $B^*DV_{k,i} \neq 0$ for all $k \geq 1, i = 1,2$
- Recall $\boxed{d \neq 1}$,

$$B^* = (0,1), \quad V_{k,1} = \begin{pmatrix} 1 \\ \frac{1}{(d-1)k^2} \end{pmatrix}, \quad V_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \forall k \geq 1.$$

So, here $\boxed{B^*DV_{k,i} \neq 0, \quad \forall k \geq 1, i = 1,2}$

$$(2) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Approximate controllability: a necessary condition (II)

$$\lambda_{k,1} = \lambda_{j,2} = \lambda \Rightarrow \begin{cases} kB^*DV_{k,1} \int_0^T v(T-t)e^{-\lambda t}\,dt = -e^{-\lambda T}\langle y_0, \Phi_{k,1}\rangle \\ jB^*DV_{j,2} \int_0^T v(T-t)e^{-\lambda t\,dt} = -e^{-\lambda T}\langle y_0, \Phi_{j,2}\rangle \end{cases}$$

So it is necessary to have $\lambda_{k,1} \neq \lambda_{j,2}$. This leads to

$$k^2 \neq dj^2, \quad \forall k \neq j \geq 1 \iff \boxed{\sqrt{d} \notin \mathbb{Q}}$$

4. Third phenomenon

$$(2) \quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

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$$k^2 \neq dj^2, \quad \forall k \neq j \geq 1 \iff \boxed{\sqrt{d} \notin \mathbb{Q}}$$

In the sequel, we will assume $\sqrt{d} \notin \mathbb{Q}$, i.e., the eigenvalues of $-D\partial_{xx}^2 + A_0^*$ with Dirichlet boundary conditions are pairwise distinct.

$$(2)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$$kB^*DV_{k,i} \int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = -e^{-\lambda_{k,i}T}\left\langle y_0, \Phi_{k,i}\right\rangle, \quad \forall (k,i)$$

Summarizing

Let $m_{k,i} = -\left\langle y_0, \Phi_{k,i}\right\rangle$, $b_{k,i} = kB^*DV_{k,i}$ (for any $\varepsilon > 0$, $|m_{k,i}| \leq C_\varepsilon e^{\varepsilon\lambda_{k,i}}$ and $|b_{k,i}| \geq C_\varepsilon e^{-\varepsilon\lambda_{k,i}}$),

$$\exists\,?\, v \in L^2(0,T) \;:\; \int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = \frac{m_{k,i}}{b_{k,i}} e^{-\lambda_{k,i}T}, \quad \forall k \geq 1,\ i = 1,2$$

Let $\Lambda = \{\lambda_k\}_{k\geq 1} \subset (0,\infty)$ be a sequence with **pairwise distinct elements**:

$$\boxed{\sum_{k\geq 1} \frac{1}{|\lambda_k|} < \infty}$$

Goal: Given $\{m_k\}_{k\geq 1}, \{b_k\}_{k\geq 1} \subset \mathbb{R}$ satisfying $\boxed{|m_k| \leq C_\varepsilon e^{\varepsilon\lambda_k}}$ and $\boxed{|b_k| \geq C_\varepsilon e^{-\varepsilon\lambda_k}}$, find $v \in L^2(0,T)$ s.t.

$$\int_0^T v(T-t)e^{-\lambda_k t}\,dt = \frac{m_k}{b_k} e^{-\lambda_k T}, \quad \forall k \geq 1.$$

Theorem

Under the previous assumptions, $\{e^{-\lambda_k t}\}_{k\geq 1} \subset L^2(0,T)$ *admits a* ***biorthogonal family*** $\{q_k\}_{k\geq 1}$ *in* $L^2(0,T)$*, i.e.:*

$$\int_0^T e^{-\lambda_k t} q_l(t)\, dt = \delta_{kl}, \quad \forall k, l \geq 1$$

A formal solution to

$$\int_0^T v(T-t)e^{-\lambda_k t}\,dt = \frac{m_k}{b_k}e^{-\lambda_k T}, \quad \forall k \geq 1,$$

is v given by: $$v(T-t) = \sum_{k\geq 1} \frac{m_k}{b_k} e^{-\lambda_k T} q_k(t),$$

A formal solution to

$$\boxed{\int_0^T v(T-t)e^{-\lambda_k t}\,dt = \frac{m_k}{b_k}e^{-\lambda_k T}, \quad \forall k \geq 1,}$$

is v given by: $\boxed{v(T-t) = \sum_{k\geq 1} \frac{m_k}{b_k}e^{-\lambda_k T} q_k(t)}$,

Question: $v \in L^2(0,T)$?, i.e., is the series $\boxed{\sum_{k\geq 1} \frac{m_k}{b_k}e^{-\lambda_k T} q_k(t)}$ convergent in $L^2(0,T)$?

But this question itself amounts to:

$$\|q_k\|_{L^2(0,T)} \underset{k\to\infty}{\sim} ?$$

Theorem

Assume

$$\sum_{k\geq 1} \frac{1}{|\lambda_k|} < \infty.$$

Then, for any $\varepsilon > 0$ *one has*

$$C_{1,\varepsilon} \frac{e^{-\varepsilon\lambda_k}}{|E'(\lambda_k)|} \leq \|q_k\|_{L^2(0,T)} \leq C_{2,\varepsilon} \frac{e^{\varepsilon\lambda_k}}{|E'(\lambda_k)|}, \quad \forall k \geq 1,$$

where $E(z)$ *is the interpolating function:*

$$E(z) = \prod_{k=1}^{\infty} (1 - \frac{z^2}{\lambda_k^2}), \quad \boxed{E'(\lambda_k) = -\frac{2}{\lambda_k} \prod_{j\neq k}^{\infty} \left(1 - \frac{\lambda_k^2}{\lambda_j^2}\right)}$$

Definition

The **condensation index** of $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ is:

$$c(\Lambda) = \limsup_{k\to\infty} \frac{-\ln |E'(\lambda_k)|}{\Re(\lambda_k)} \in [0, +\infty] .$$

Corollary

For any $\varepsilon > 0$ *one has*

$$\|q_k\|_{L^2(0,T;)} \leq C_\varepsilon e^{(c(\Lambda)+\varepsilon)\lambda_k}, \quad \forall k \geq 1.$$

Recall that we had m_k s.t. $|m_k| \le C_\varepsilon e^{\varepsilon \lambda_k}$, $|b_k| \ge C_\varepsilon e^{-\varepsilon \lambda_k}$, for any $\varepsilon > 0$, and we wanted to solve: $v \in L^2(0,T)$ and

$$\boxed{\int_0^T v(T-t) e^{-\lambda_k t}\, dt = \frac{m_k}{b_k} e^{-\lambda_k T}, \quad \forall k},$$

We took $v(T-t) = \sum_{k \ge 1} \frac{m_k}{b_k} e^{-\lambda_k T} q_k(t)$.

Recall that we had m_k s.t. $|m_k| \le C_\varepsilon e^{\varepsilon \lambda_k}$, $|b_k| \ge C_\varepsilon e^{-\varepsilon \lambda_k}$, for any $\varepsilon > 0$, and we wanted to solve: $v \in L^2(0,T)$ and

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We took $v(T-t) = \sum_{k \ge 1} \frac{m_k}{b_k} e^{-\lambda_k T} q_k(t)$.

From the previous result: Given $\varepsilon > 0$:

$$\left| \frac{m_k}{b_k} \right| e^{-\lambda_k T} \| q_k \|_{L^2(0,T)} \le C_\varepsilon e^{-\lambda_{k,i}(T - c(\Lambda) - \varepsilon)}$$

Then

$$T > c(\Lambda) \Longrightarrow v(T-t) = \sum_{k \ge 1} \frac{m_k}{b_k} e^{-\lambda_k T} q_k(t) \in L^2(0,T).$$

4. Third phenomenon

$$(2) \quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

In our case,

$$\Lambda_d := \{\lambda_k\}_{k\geq 1} = \left\{j^2, dj^2\right\}_{j\geq 1}.$$

Then

If $T > c(\Lambda_d)$, system (2) is null controllable at time T, where $c(\Lambda_d)$ is the **condensation index** of the sequence Λ_d.

$$
(2) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}
$$

$$
\Lambda_d = \{k^2, dk^2\}_{k\geq 1}, \qquad \sqrt{d} \notin \mathbb{Q}.
$$

We have proved:

Theorem

There exists $T_0 = c(\Lambda_d) \in [0, +\infty]$ such that if $\boxed{T > T_0}$ then system (2) is null controllable at time T

$$(2) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

$$\Lambda_d = \{k^2, dk^2\}_{k\geq 1}, \quad \sqrt{d} \notin \mathbb{Q}.$$

We have proved:

Theorem

There exists $T_0 = c(\Lambda_d) \in [0, +\infty]$ *such that if* $\boxed{T > T_0}$ *then system* (2) *is null controllable at time* T

$\boxed{T > c(\Lambda_d)}$ is a sufficient condition for the null controllability of system (2) at time T. But,

what happens if $\boxed{T < c(\Lambda_d)}$?

$$(2) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

The null controllability property at time T of system (2) is equivalent to the **observability inequality**:

$$\|\varphi(\cdot,0)\|^2_{(H_0^1)^2} \leq C_T \int_0^T \left|B^* D \partial_x \varphi(0,t)\right|^2 \, dt,$$

for the solutions to **the adjoint problem**

$$\begin{cases} -\varphi_t - D\varphi_{xx} + A_0^* \varphi = 0 & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \end{cases}$$

$$\begin{cases} -\varphi_t - D\varphi_{xx} + A_0^*\varphi = 0 & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \end{cases}$$

- $\sigma(-D\partial_{xx}^2 + A_0^*) = \bigcup_{k\geq 1} \{k^2, dk^2\} := \bigcup_{k\geq 1} \{\lambda_{k,1}, \lambda_{k,2}\}$
- $V_{k,1}$ and $V_{k,2}$: eigenvectors of the matrix $(k^2 D + A_0^*)$ associated to the eigenvalues k^2, dk^2.
- $\Phi_{k,i} = V_{k,i} \sin kx$, $i = 1,2$: eigenfunctions of $(-D\partial_{xx}^2 + A_0^*)$.
- $\{\Phi_{k,i}\}$ is a (Riesz) basis of $H_0^1(0,\pi)^2$. Let $\{\Psi_{k,i}\}$ be the associated biorthogonal family (for the duality $\langle\cdot,\cdot\rangle_{((H_0^1)^2,(H^{-1})^2)}$)

$$f \in H_0^1(0,\pi)^2 \Longleftrightarrow f = \sum_{k\geq 1, i=1,2} \langle f, \Psi_{k,i}\rangle \Phi_{k,i}$$

$$\|f\|^2_{(H_0^1)^2} = \sum_{k\geq 1, i=1,2} |\langle f, \Psi_{k,i}\rangle|^2$$

$$\begin{cases} -\varphi_t - D\varphi_{xx} + A_0^*\varphi = 0 & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \end{cases}$$

Thus, the **observability inequality** for the adjoint system writes

$$\sum_{n,i} e^{-2\lambda_{n,i}T} |a_{n,i}|^2 \leq C_T \int_0^T \left| \sum_{n,i} nB^* D V_{n,i} e^{-\lambda_{n,i}t} a_{n,i} \right|^2 dt,$$

$\forall \{a_{n,i}\}_{n,i} \in \ell^2$.

$$\sum_{n,i} e^{-2\lambda_{n,i}T}|a_{n,i}|^2 \leq C_T \int_0^T \left| \sum_{n,i} nB^* DV_{n,i} e^{-\lambda_{n,i}t} a_{n,i} \right|^2 dt,$$

Assume $T \in (0, c(\Lambda_d))$.

By contradiction: Assume the **observability inequality** holds for $C_T > 0$

Construction of a suitable sequence of initial data

The idea is to construct sequences $\{a_{n,i}^{(k)}\}_{n,i} \in \ell^2$ such that

$$\int_0^T \left| \sum_{n,i} nB^* DV_{n,i} e^{-\lambda_{n,i}t} a_{n,i}^{(k)} \right|^2 \to 0, \quad \sum_{n,i} e^{-2\lambda_{n,i}T}|a_{n,i}^{(k)}|^2 \geq \delta > 0.$$

Argument: Use the overconvergence of Dirichlet series

Theorem

Suppose that the sequence $\Lambda = \{\lambda_n\}_{n\geq 1}$ has condensation index $c(\Lambda)$. We can choose a sequence of finite sets $N_k \subset \mathbb{N}$, a sequence $\{\alpha_n\}_{n\geq 1} \subset \mathbb{C}$, such that there exists $R \geq 0$ such that

1. *the series $\sum_{n\geq 1} \alpha_n e^{-\lambda_n z}$ converges in the region $\Re z > R$*
2. *the series $\sum_{n\geq 1} \alpha_n e^{-\lambda_n z}$ diverges in the region $\Re z < R$*
3. *the series $\sum_{k\geq 1}(\sum_{n\in N_k} \alpha_n e^{-\lambda_n z})$ converges in the region $\Re z > R - c(\Lambda)$*

- One can construct $\{\alpha_n\}_{n\geq 1}$ such that $R = c(\Lambda)$.
- The construction of the sequence $\{\alpha_n\}_{n\geq 1}$ is explicit.

- $\Lambda_d = \{\lambda_n\}_{n\geq 1} = \{k^2, dk^2\}_{k\geq 1}$. We construct $\{a_n^{(k)}\}_{n\geq 1} \in \ell^2$:

$$a_n^{(k)} = \begin{cases} \frac{\alpha_n}{b_n} & n \in N_k \\ 0 & n \notin N_k \end{cases}$$

$$b_n = n \left| B^* D V_n \right|$$

- $\{a_n^{(k)}\}_{n\geq 1} \in \ell^2$ (recall that the sets N_k are finite).
- The **observability inequality** is

$$\boxed{\sum_{n\in N_k} e^{-2\lambda_n T} \left|a_n^{(k)}\right|^2 \leq C_T \int_0^T \left| \sum_{n\in N_k} e^{-\lambda_n t} \alpha_n \right|^2 dt,}$$

$$\sigma_1^{(k)} := \sum_{n\in N_k} e^{-2\lambda_n T}|a_n^{(k)}|^2 \le C_T \int_0^T \left|\sum_{n\in N_k} e^{-\lambda_n t}\alpha_n\right|^2 dt := \sigma_2^{(k)},$$

- The convergence of the series $\sum_{k\ge 1}(\sum_{n\in N_k}\alpha_n e^{-\lambda_n t})$ for all $t>0$ (recall that $R = c(\Lambda_d)$ and then $R - c(\Lambda_d) = 0$) implies:

$$\lim_{k\to +\infty} \sum_{n\in N_k} \alpha_n e^{-\lambda_n t} = 0, \quad \forall t > 0$$

$$\sigma_1^{(k)} := \sum_{n\in N_k} e^{-2\lambda_n T}|a_n^{(k)}|^2 \leq C_T \int_0^T \left|\sum_{n\in N_k} e^{-\lambda_n t}\alpha_n\right|^2 dt := \sigma_2^{(k)},$$

- The convergence of the series $\sum_{k\geq 1}(\sum_{n\in N_k}\alpha_n e^{-\lambda_n t})$ for all $t>0$ (recall that $R = c(\Lambda_d)$ and then $R - c(\Lambda_d) = 0$) implies:

$$\lim_{k\to+\infty}\sum_{n\in N_k}\alpha_n e^{-\lambda_n t} = 0, \quad \forall t>0$$

- Moreover, one can prove there exist $C_1, C_2 > 0$ such that

$$\left|\sum_{n\in N_k}\alpha_n e^{-\lambda_n t}\right| \leq C_1 e^{-C_2 t}.$$

- Thus, from Lebesgue's dominated convergence theorem, we obtain $\sigma_2^{(k)} \to 0$.

$$\sigma_1^{(k)} := \sum_{n\in N_k} e^{-2\lambda_n T} |a_n^{(k)}|^2 \leq C_T \int_0^T \left| \sum_{n\in N_k} e^{-\lambda_n t} \alpha_n \right|^2 dt := \sigma_2^{(k)},$$

- By construction the sequence $\{\alpha_n\}_{n\geq 1}$ satisfies that for all $k \geq 1$ there exists $n_k \in N_k$ such that

$$\left| a_{n_k}^{(k)} \right| = \left| \frac{\alpha_{n_k}}{b_{n_k}} \right| \geq C_\varepsilon e^{\Re(\lambda_{n_k})(c(\Lambda_d)-\varepsilon)}$$

- One gets:

$$\sigma_1^{(k)} \geq e^{-2\lambda_{n_k} T} \left| a_{n_k}^{(k)} \right|^2 \geq C_\varepsilon e^{2\Re(\lambda_{n_k})(c(\Lambda_d)-T-\varepsilon)} \underset{T<c(\Lambda_d)}{\longrightarrow} +\infty .$$

- So, one has proved

$$\sigma_1^{(k)} \to +\infty, \quad \sigma_2^{(k)} \to 0$$

$$
(2) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}
$$

The controllability result

1. $\forall T > 0$: **Approximate controllability** if and only if $\sqrt{d} \notin \mathbb{Q}$
2. Assume $\sqrt{d} \notin \mathbb{Q}$, $\exists T_0 = c(\Lambda_d) \in [0, +\infty]$ such that
   1. the system is null controllable at time T if $\boxed{T > T_0}$
   2. Even if $\sqrt{d} \notin \mathbb{Q}$, if $\boxed{T < T_0}$ the system is **not null controllable** at time T!

$$(2) \qquad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

In fact, the good minimal time is

$$T_0 = \limsup_{k\to\infty} \frac{-\left(\ln|b_k| + \ln|E'(\lambda_k)|\right)}{\Re(\lambda_k)} \in [0,\infty]$$

4. Third phenomenon

$$
(2) \qquad \begin{cases} y_t - D y_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}
$$

$T_0 > 0$?

Is it possible to have a minimal time of control > 0? I.e., for $\Lambda_d = \{k^2, dk^2\}_{k\geq 1}$ with $\sqrt{d} \notin \mathbb{Q}$, is it possible that $c(\Lambda_d) > 0$?

4. Third phenomenon

$$(2) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$T_0 > 0$?

Is it possible to have a minimal time of control > 0? I.e., for $\Lambda_d = \{k^2, dk^2\}_{k\geq 1}$ with $\sqrt{d} \notin \mathbb{Q}$, is it possible that $c(\Lambda_d) > 0$?

Theorem

For any $\tau \in [0,+\infty]$, there exists $\sqrt{d} \notin \mathbb{Q}$ such that $c(\Lambda_d) = \tau$.

Remark

- There exists $\sqrt{d} \notin \mathbb{Q}$ such that $c(\Lambda_d) = +\infty$ (LUCA, DE TERESA).
- $c(\Lambda_d) = 0$ for almost $d \in (0,\infty)$ such that $\sqrt{d} \notin \mathbb{Q}$.
- For any $\tau \in [0,+\infty]$, the set $\{d \in (0,\infty) : c(\Lambda_d) = \tau\}$ is dense in $(0,+\infty)$.

4. Third phenomenon

Remark

This minimal time also arises in other parabolic problems (degenerated problems):
BEAUCHARD, CANNARSA, GUGLIELMI, *Null controllability of Grushin-type operators in dimension two. J. Eur. Math. Soc. (JEMS) (2014).*

Reference

F. AMMAR KHODJA, A. BENABDALLAH, M.G.-B., L. DE TERESA, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences*, J. Funct. Anal. **267** (2014).

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Summarizing

Scalar case versus systems (parabolic problems)

	SCALAR CASE	SYSTEMS
boundary $\Leftrightarrow$ distributed control	Yes	No
approximate $\Leftrightarrow$ null controllability	Yes	No
minimal time for controling	No	Yes
geometrical conditions	No	Yes

Some references

$$(2) \quad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

Existing results: $\boxed{d_1 = d_2}$: Approximate and null controllability.

- L. Rosier , L. de Teresa, C. R. Math. Acad. Sci. Paris (2011), 2×2 systems, 1-d, cascade systems, sing conditions, sufficient conditions.
- F. Alabau-Boussouira, M. Léautaud, J. Math. Pures Appl. (2012): 2×2 systems, N-d, particular matrices depending on x, sing conditions, sufficient conditions, geometric control condition.
- F. Alabau-Boussouira, Math. Control Signals Systems (2014): 2×2 systems, N-d, cascade systems, sing conditions, sufficient conditions, geometric control condition.

Some references

$$(2) \quad \boxed{\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

Existing results: $\boxed{d_1 = d_2}$: Approximate and null controllability.

- A. Benabdallah, F. Boyer, M.G.-B., G. Olive, *Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null-controllability in cylindrical domains*, SIAM J. Control and Optim. (2014).

Some references

$$\begin{cases} \partial_t y_1 - d_1\partial_x^2 y_1 + a_{11}y_1 + \boxed{a_{12}}\, y_2 = 0 & \text{in } Q, \\ \partial_t y_2 - d_2\partial_x^2 y_2 + a_{22}y_2 + a_{21}y_1 = u1_\omega & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Existing results: a_{12} is a PD operator of order ≤ 2 with $\boxed{\omega \cap \text{Supp}\, a_{12} \neq \emptyset}$ and a_{12} is "invertible": Approximate and null controllability.

- S. Guerrero, SIAM J. Control Optim. **25** (2007).
- A. Benabdallah, M. Cristofol, P. Gaitan, L. de Teresa, Math. Control Relat. Fields (2014).
- K. Mauffrey, J. Math. Pures Appl. (2013).

Different diffusion coefficients, any space dimension.

Thank you for your attention!!