Partitioning Orthogonal Polygons by Extension of All Edges Incident to Reflex Vertices: lower and upper bounds on the number of pieces ⋆

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Abstract

Given an orthogonal polygon P, let $|\Pi(P)|$ be the number of rectangles that result when we partition P by extending the edges incident to reflex vertices towards $\mathrm{INT}(P)$. In [4] we showed that $|\Pi(P)| \leq 1 + r + r^2$, where r is the number of reflex vertices of P. We shall now give sharper bounds both for $\max_P |\Pi(P)|$ and $\min_P |\Pi(P)|$. Moreover, we characterize the structure of orthogonal polygons in general position for which these new bounds are exact.

Key words: Orthogonal Polygons, Decomposition, Rectilinear Cut, Square Grid.

1. Introduction

We shall call *simple polygon* P a region of a plane enclosed by a finite collection of straight line segments forming a simple cycle. This paper deals only with simple polygons, so that we call them just polygons, in the sequel. We will denote the interior of the polygon P by $\mathrm{INT}(P)$ and the boundary by $\mathrm{BND}(P)$. The boundary shall be considered part of the polygon, that is $P = \mathrm{INT}(P) \cup \mathrm{BND}(P)$. A vertex is called *convex* if the interior angle between its two incident edges is at most π; otherwise it is called *reflex* (or *concave*). We use r to represent the number of reflex vertices of P. A polygon is called *orthogonal* (or *rectilinear*) iff its edges meet at right angles. O'Rourke [2] has shown that $n = 2r + 4$ for every n-vertex orthogonal polygon (n-*ogon*, for short).

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Definition 1 *A rectilinear cut of an n-ogon P is obtained by extending each edge incident to a reflex vertex of P towards* $\mathrm{INT}(P)$ *until it hits* $\mathrm{BND}(P)$. *We denote this partition by* $\Pi(P)$ *and the number of its elements (pieces) by* $|\Pi(P)|$. *Each piece is a rectangle, so that we call it* r-piece.

Generic n-ogons may be obtained from a particular kind of n-ogons – the so-called *grid orthogonal polygons* [3], as illustrated in Fig. 1 (The reader may skip Definition 2 and Lemmas 3 and 4 if he/she has already read [3].)

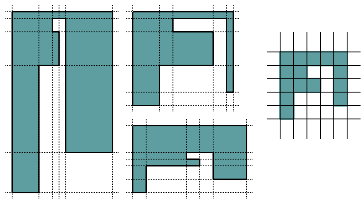


Fig. 1. Three 12-ogons mapped to the same grid 12-ogon.

Definition 2 *An n-ogon P is* in general position *iff P has no collinear edges. We call "grid n-ogon" each n-ogon in general position defined in a $\frac{n}{2} \times \frac{n}{2}$ square grid.*

Lemma 3 follows immediately from this definition.

Lemma 3 *Each grid n-ogon has exactly one edge in every line of the grid.*

Each n-ogon not in general position may be mapped to an n-ogon in general position by ϵ-perturbations, for a sufficiently small constant $\epsilon > 0$. Consequently, we shall first address n-ogons in general position.

Lemma 4 *Each n-ogon in general position is mapped to a unique grid n-ogon through top-to-bottom and left-to-right sweeping. And, reciprocally, given a grid n-ogon we may create an n-ogon that is an instance of its class by randomly spacing the grid lines in such a way that their relative order is kept.*

The number of classes may be further reduced if we group grid n-ogons that are symmetrically equivalent. In this way, the grid n-ogons in Fig. 2 represent the same class.

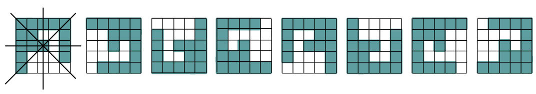


Fig. 2. Eight grid n-ogons that are symmetrically equivalent. From left to right, we see images by clockwise rotations of 90°, 180° and 270°, by flips wrt horizontal and vertical axes and flips wrt positive and negative diagonals.

Definition 5 *Given an n-ogon P in general position, GRID(P) denotes any grid n-ogon in the class that contains the grid n-ogon to which P is mapped by the sweep procedure described in Lemma 4*

The following result is a trivial consequence of the definition of GRID(P).

Lemma 6 *For all n-ogons P in general position,* $|\Pi(P)| = |\Pi(GRID(P))|$.

2. Lower and Upper bounds on $|\Pi(P)|$

In [4] we showed that $\Pi(P)$ has at most $1+r+r^2$ pieces. Later we noted that this upper bound is not sufficiently tightened. Actually, for small values of r, namely $r = 3, 4, 5, 6, 7$, we experimentally found that the difference between $1 + r + r^2$ and $\max|\Pi(P)|$ was 1, 2, 4, 6 and 9, respectively.

Definition 7 *A grid n-ogon Q is called* FAT *iff* $|\Pi(Q)| \geq |\Pi(P)|$, *for all grid n-ogons P. Similarly, a grid n-ogon Q is called* THIN *iff* $|\Pi(Q)| \leq |\Pi(P)|$, *for all grid n-ogons P.*

The experimental results supported our conjecture that there was a single FAT n-ogon (except for symmetries of the grid) and that it had the form illustrated in Fig. 3.

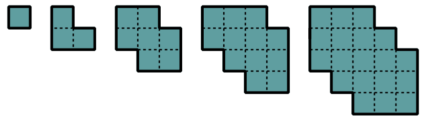


Fig. 3. The unique FAT n-ogons (symmetries excluded), for $n = 4, 6, 8, 10, 12$.

Clearly, each piece r-piece is defined by four vertices. Each vertex is either in INT(P) (*internal vertex*) or is in BND(P) (*boundary vertex*). Similar definitions hold for edges. An edge e of r-piece R is called an *internal edge* if $e \cap \mathrm{INT}(P) \neq \emptyset$, and it is called *boundary edge* otherwise.

Lemma 8 *The total number $|V_i|$ of internal vertices in $\Pi(P)$, when the grid n-ogon P is as illustrated in Fig. 3 is given by (1)*

$$|V_i| = \begin{cases} \dfrac{3r^2 - 2r}{4}, & \text{for } r \text{ even} \\ \dfrac{(3r+1)(r-1)}{4}, & \text{for } r \text{ odd} \end{cases} \tag{1}$$

where r is the number of reflex vertices of P.

PROOF. In case r is even,

$$|V_i| = 2\sum_{k=1}^{\frac{r}{2}}(r-k)$$

and in case r is odd

$$|V_i| = (r - \frac{r+1}{2}) + 2\sum_{k=1}^{\frac{r-1}{2}}(r-k)$$

□

Proposition 9 *Every n-vertex orthogonal polygon P such that the number of internal vertices of $\Pi(P)$ is given by (1) has at most a single reflex vertex in each horizontal and vertical line.*

PROOF. We shall suppose first that P is a grid n-ogon. Then, let $v_{L_1} = (x_{L_1}, y_{L_1})$ and $v_{R_1} = (x_{R_1}, y_{R_1})$ be leftmost and rightmost reflex vertices of P, respectively. The horizontal chord with origin at v_{L_1} can intersect at most $x_{R_1} - x_{L_1}$ vertical chords, since we shall not count the intersection with the vertical chord defined by v_{L_1}. The same may be said about the the horizontal chord with origin at v_{R_1}. There are exactly r vertical and r horizontal chords, and thus $x_{R_1} - x_{L_1} \leq r - 1$. If there were c vertical edges such that both extreme points are reflex vertices then $x_{R_1} - x_{L_1} \leq r - 1 - c$.

This would imply that the number of internal vertices of $\Pi(P)$ would be strictly smaller than the value defined by (1). Indeed, we could proceed to consider the second leftmost vertex (for $x > x_{L_1}$), say v_{L_2}, then the second rightmost vertex (with $x < x_{R_1}$) and so forth. The horizontal chord that v_{L_2} defines either intersects only the vertical chord defined by v_{L_1} or it does not intersect it at all. So, it intersects at most $r-2-c$ vertical chords. In sum, c should be null, and by symmetry, we would conclude that there is exactly a reflex vertex in each vertical grid line (for $x > 1$ and $x < \frac{n}{2} = r+2$).

Now, if P is not a grid n-ogon but is in general position, then $\Pi(P)$ has the same combinatorial structure as $\Pi(GRID(P))$, so that we do not have to prove anything more.

If P is not in general position, then let we render it in general position by a sufficiently small ϵ-perturbation, so that the partition of this latter polygon would not have less internal vertices than $\Pi(P)$. □

Corollary 10 *For all grid n-ogons P, the number of internal vertices of $\Pi(P)$ is less than or equal to the value established by (1).*

PROOF. It results from the proof of Proposition 9. □

Theorem 11 *Let P be a grid n-ogon, $r = \frac{n-4}{2}$ the number of its reflex vertices. If P is* FAT *then*

$$|\Pi(P)| = \begin{cases} \dfrac{3r^2+6r+4}{4}, & \text{for } r \text{ even} \\ \dfrac{3(r+1)^2}{4}, & \text{for } r \text{ odd} \end{cases}$$

and if P is THIN then $|\Pi(P)| = 2r+1$.

PROOF. Suppose that P is a grid n-ogon. Let V, E and F be the sets of all vertices, edges and faces of $\Pi(P)$, respectively. Let us denote by V_i and V_b the sets of all internal and boundary vertices of the pieces of $\Pi(P)$. Similarly, E_i and E_b represent the sets of all internal and boundary edges of such pieces. Then, $V = V_i \cup V_b$ and $E = E_i \cup E_b$. Being P in general position, each chord we draw to form $\Pi(P)$ hits BND(P) in the interior of an edge and no two chords hit BND(P) in the same point. Hence, using O'Rourke's formula [2] we obtain $|E_b| = |V_b| = (2r+4)+2r = 4r+4$. It is easily seen that to obtain a FAT n-ogon we must maximize the number of internal vertices.

By Corollary 10,

$$\max_P |V_i| = \begin{cases} \dfrac{3r^2-2r}{4}, & \text{for } r \text{ even} \\ \dfrac{(3r+1)(r-1)}{4}, & \text{for } r \text{ odd} \end{cases}$$

and, therefore, $\max_P |V| = \max_P(|V_i| + |V_b|)$ is given by

$$\max_P |V| = \begin{cases} \dfrac{3r^2+14r+16}{4}, & \text{for } r \text{ even} \\ \dfrac{3r^2+14r+15}{4}, & \text{for } r \text{ odd} \end{cases}$$

From Graph Theory [1] we know that the sum of the degrees of vertices in a graph is twice the number of its edges, that is, $\sum_{v\in V} \delta(v) = 2|E|$. Using the definitions of grid n-ogon and of $\Pi(P)$, we may partition V as

$$V = V_c \cup V_r \cup (V_b \setminus (V_c \cup V_r)) \cup V_i$$

V_r and V_c representing the sets of reflex and of convex vertices of P, respectively. Moreover, we may conclude that $\delta(v) = 4$ for all $v \in V_r \cup V_i$, $\delta(v) = 3$ for all $v \in V_b \setminus (V_c \cup V_r)$ and $\delta(v) = 2$ for all $v \in V_c$. Hence,

$$\begin{aligned} 2|E| &= \sum_{v\in V_r\cup V_i} \delta(v) + \sum_{v\in V_c} \delta(v) + \sum_{v\in V_b\setminus(V_c\cup V_r)} \delta(v) \\ &= 4|V_i| + 4|V_r| + 2|V_c| + 3(|V_b| - |V_r| - |V_c|) \\ &= 4|V_i| + 12r + 8 \end{aligned}$$

and, consequently, $|E| = 2|V_i| + 6r + 4$.

Similarly, to obtain THIN n-ogons we must minimize the number of internal vertices of the arrangement. For all n, there are grid n-ogons such that $|V_i| = 0$. Thus, for THIN n-ogons $|V| = 4r+4$.

Finally, to conclude the proof, we have to deduce the expression of the upper and lower bound of the number of faces of $\Pi(P)$, that is of $|\Pi(P)|$. Using Euler's formula $|F| = 1 + |E| - |V|$, and the expressions deduced above, we have $\max_P |F| = 1 + 2(\max_P |V_i|) + 6r + 4 - \max_P |V|$. That is, $\max_P |F| = \max_P |V_i| + 6r + 5$, so that

$$\max_P |F| = \begin{cases} \dfrac{3r^2+6r+4}{4}, & \text{for } r \text{ even} \\ \dfrac{3(r+1)^2}{4}, & \text{for } r \text{ odd} \end{cases}$$

and

$$\min_P |F| = 1 + 2(\min_P |V_i|) + 6r + 4 - \min_P |V|$$
$$= 1 + 6r + 4 - 4r - 4 = 2r + 1$$

The existence of FAT grid n-ogons and THIN grid n-ogons for all n (such that n is even and $n \geq 4$) follows from Lemma 8 and the construction indicated in Fig. 5, respectively. □

Figs. 4 and 5 show some THIN n-ogons.

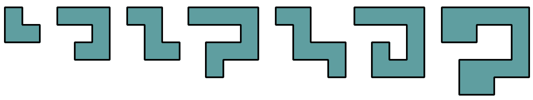


Fig. 4. Some grid n-gons with $|V_i| = 0$.

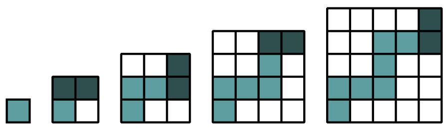
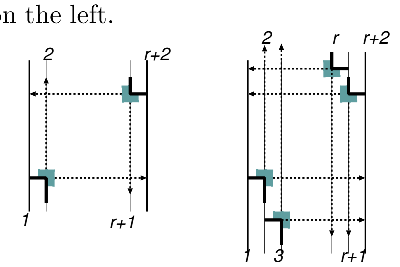


Fig. 5. Constructing the grid ogons of the smallest area, for $r = 0, 1, 2, 3, 4, \ldots$. The area is $2r + 1$.

Based on the proof of Proposition 9, we may prove the uniqueness of FATs and fully characterize them.

Proposition 12 *There is a single FAT n-ogon (except for symmetries of the grid) and its form is illustrated in Fig. 3.*

PROOF. We saw that FAT n-ogons must have a single reflex vertex in each vertical grid-line, for $x > 1$ and $x < \frac{n}{2}$. Also, the horizontal chords with origins at the reflex vertices that have $x = 2$ and $x = \frac{n}{2} - 1 = r + 1$, determine $2(r-1)$ internal points (by intersections with vertical chords). To achieve this value, they must be positioned as illustrated below on the left.



Moreover, the reflex vertices on the vertical grid-lines $x = 3$ and $x = r$ add $2(r-2)$ internal points. To achieve that, we may conclude by some simple case reasoning, that v_{L_2} must be below v_{L_1} and v_{R_2} must be above v_{R_1}, as shown above on the right. And, so forth… □

The area $A(P)$ of a grid n-ogon P is the number of grid cells in its interior. FATs are not the grid n-ogons that have the largest area, except for small values of n, as we may see in Fig 6.

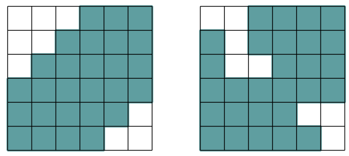


Fig. 6. On the left we see the FAT grid 14-ogon. It has area 27, whereas the grid 14-ogon on the right has area 28, which is the maximum.

Proposition 13 *Let P be any grid n-ogon with $n \geq 8$ and r reflex vertices ($r = \frac{n-4}{2}$, for all P). Then*

$$2r + 1 \leq A(P) \leq r^2 + 3$$

and there exist grid n-ogons having area $2r + 1$ (indeed, a single one except for symmetries) and grid n-ogons having area $r^2 + 3$.

PROOF. Our proof is strongly based on the INFLATE-PASTE method for generating grid ogons, that will be presented also at this workshop [3]. □

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